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Every finite group is the group of self-homotopy equivalences of an elliptic space

by

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1. Introduction

For simply connected CW-complexes X of finite type, we are interested in the group of homotopy classes of self-homotopy equivalences, $\mathcal{E}(X)$, and the realizability problem for groups. Namely, if a given group G can appear as the group $\mathcal{E}(X)$ for some space X. This problem has been placed as the first to solve in [3], being around for over fifty years and recurrently appearing in surveys and lists of open problems about self-homotopy equivalences [2], [14], [20], [21], [26]. The difficulty of this question relies on the fact that techniques used so far are specific to certain groups [6], [7], [12], [22], [24], and have not proved fruitful when addressing this problem in general.

Apart from the group of automorphisms of a group π, $\mathrm{Aut}(\pi)$, which is isomorphic to $\mathcal{E}(K(\pi,n))$ for an Eilenberg–MacLane space $K(\pi,n)$, there is no global picture in this context. A special mention deserves the cyclic group of order 2, which is the group of automorphisms of the cyclic group of order 3, and hence it can be realized as $\mathcal{E}(K(\mathbb{Z}_3,n))$. Arkowitz and Lupton show that, moreover, it is the group of self-homotopy equivalences of a rational space, pointing out the surprising appearance of a finite group in rational homotopy theory, and raising the question of when finite groups can be realized by rational spaces [4].

In this paper, we give a complete answer to the realizability problem for finite groups.

Theorem 1.1. *Every finite group G can be realized as the group of self-homotopy equivalences of infinitely many* (*non-homotopy-equivalent*) *rational elliptic spaces X.*

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To build up those spaces, we introduce a general method which we hope can be useful for obtaining examples with interesting properties in subjects of different nature. For instance, it appears to produce differential manifolds related to a question of Gromov, as mentioned below (see also §3). Indeed, we construct a contravariant functor from a subcategory of finite graphs to the homotopy category of differential graded commutative algebras whose cohomology is 1-connected and of finite type. Then, the geometric realization functor of Sullivan [27] gives the equivalence of categories between the homotopy category of minimal Sullivan algebras and the homotopy category of rational simply connected spaces of finite type.

We remark that by dropping the requirement on the finiteness type of the differential graded algebras, our method can be extended to infinite, locally finite graphs. This is a subtle and technical point that is handled in [10], where this extended version of our techniques is used to obtain an isomorphism criteria for a large class of groups, having thus consequences in representation theory.

In this paper, we prove the following theorem.

Theorem 1.2. *Let $\mathcal{G}$ be a finite connected graph with more than one vertex. Then, there exists an elliptic minimal Sullivan algebra $\mathcal{M}_{\mathcal{G}}$ such that the group of automorphisms of $\mathcal{G}$ is realizable by the group of self-homotopy equivalences of $\mathcal{M}_{\mathcal{G}}$.*

Our idea of using graphs has its origin on the following classical result ([16], [17]).

Theorem 1.3. (Frucht, 1939) *Given a finite group G, there exist infinitely many non-isomorphic connected (finite) graphs $\mathcal{G}$ whose automorphism group is isomorphic to G.*

Because of the equivalence given by the geometric realization functor of Sullivan, Theorem 1.1 follows directly from Theorems 1.2 and 1.3 (see Proposition 2.7). Applying Theorem 1.1 to the trivial group, we supply a partial answer to Problem 3 in [21]. This problem consists on determining spaces, which were thought to be quite rare [20], with a trivial group of self-homotopy equivalences, the so-called homotopically rigid spaces.

Corollary 1.4. *There exist infinitely many rational spaces that are homotopically rigid.*

Recall that in homotopy theory, naive dichotomy [13] classifies spaces in either elliptic or hyperbolic. Ellipticity is a very severe restriction on a space X, and it is remarkable that many of the spaces which play an important role in geometry are rationally elliptic. In particular the rational cohomology of X satisfies Poincaré duality [19] and, with extra hypothesis on the dimension of the fundamental class, X has the rational homotopy type of a simply connected manifold ([5], [27]). Indeed, the spaces in Theorem 1.1 can be chosen to have rational homotopy type of a special class of simply connected manifolds called inflexible. A manifold M is *inflexible* if all its self-maps have degree -1, 0 or 1. The work of Crowley–Löh [11] relates the existence of inflexible d-manifolds with the existence of functorial semi-norms on singular homology in degree d that are positive and finite on certain homology classes of simply connected spaces, solving in the negative a question raised by Gromov [18]. Following those ideas we prove the following.

THEOREM 1.5. *Any finite group G can be realized by the group of self-homotopy equivalences of the rationalization of an inflexible manifold M.*

COROLLARY 1.6. *For every $n\in\mathbb{N}$, $n>1$, there are functorial semi-norms on singular homology in degree $d=415+160n$ that are positive and finite on certain homology classes of simply connected spaces.*

This paper is organized as follows. In §2, for any finite connected graph $\mathcal{G}$, we construct an elliptic minimal Sullivan algebra $\mathcal{M}_{\mathcal{G}}$ such that its group of self-homotopy equivalences, $\mathcal{E}(\mathcal{M}_{\mathcal{G}})$, is isomorphic to the automorphim group of the graph, $\operatorname{Aut}(\mathcal{G})$. The construction, restricted to a suitable category of graphs, gives a contravariant faithful functor which is injective on objects (see Remark 2.8). This algebra $\mathcal{M}_{\mathcal{G}}$ is inspired by [4], where some examples of minimal Sullivan algebras, verifying that the monoid of homotopy classes of self-maps is neither trivial nor infinite, are constructed, thus disproving a conjecture of Copeland–Shar [9]. Our construction gives infinitely many examples of this nature (see Theorem 2.6). In §3, we upgrade our construction in order for it to be the rational homotopy type of an inflexible manifold M.

For the basic facts about graphs, we refer to [8]. Only simple graphs $\mathcal{G}=(V,E)$ are considered. This means that they do not have loops and they are not directed, that is, for any vertex u in V, the edge (u,u) is not in E and, if an edge (v,w) is in E, then (w,v) is also in E. We refer to [15] for basic facts in rational homotopy theory. Only simply connected $\mathbb{Q}$-algebras of finite type are considered. If W is a graded rational vector space, we write ΛW for the free commutative graded algebra on W. This is a symmetric algebra on W^{even} tensored with an exterior algebra on W^{odd}. A *Sullivan algebra* is a commutative differential graded algebra which is free as a commutative graded algebra on a simply connected graded vector space W of finite dimension in each degree. It is *minimal* if in addition $d(W)\subset\Lambda^{\geqslant 2}W$. A Sullivan algebra is *pure* if $d=0$ on W^{even} and $d(W^{\text{odd}})\subset W^{\text{even}}$.

2. From graphs to elliptic Sullivan algebras

Ellipticity for a Sullivan algebra $(\Lambda W, d)$ means that both W and $H^*(\Lambda W)$ are finite-dimensional. Hence, the cohomology is a Poincaré duality algebra [19]. One can easily compute the degree of its fundamental class (a fundamental class of a Poincaré duality algebra $H=\sum_{i=0}^{n} H^i$ is a generator of H^n, n is called the formal dimension of the algebra) by the formula

$$\sum_{i=1}^{p} \deg x_i - \sum_{j=1}^{q} (\deg y_j - 1), \tag{1}$$

where $\deg x_i$ are the degrees of the elements of a basis of W^{odd} and $\deg y_j$ of a basis of W^{even}.

Definition 2.1. For a finite connected graph $\mathcal{G}=(V,E)$ with more than one vertex, we define the minimal Sullivan algebra associated with $\mathcal{G}$ as

$$\mathcal{M}_{\mathcal{G}} = (\Lambda(x_1, x_2, y_1, y_2, y_3, z) \otimes \Lambda(x_v, z_v \mid v \in V), d),$$

where degrees and differentials are described by

$$\begin{aligned}
\deg x_1 &= 8, & d(x_1) &= 0,\\
\deg x_2 &= 10, & d(x_2) &= 0,\\
\deg y_1 &= 33, & d(y_1) &= x_1^3 x_2,\\
\deg y_2 &= 35, & d(y_2) &= x_1^2 x_2^2,\\
\deg y_3 &= 37, & d(y_3) &= x_1 x_2^3,\\
\deg x_v &= 40, & d(x_v) &= 0,\\
\deg z &= 119, & d(z) &= y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15} + x_2^{12},\\
\deg z_v &= 119, & d(z_v) &= x_v^3 + \sum_{(v,w)\in E} x_v x_w x_2^4.
\end{aligned}$$

Lemma 2.2. *The constructed $\mathcal{M}_{\mathcal{G}}=(\Lambda W, d)$ is an elliptic minimal Sullivan algebra of formal dimension $n=208+80|V|$, where $|V|$ is the order of the graph.*

Proof. We need to prove that the cohomology of $(\Lambda W, d)$ is finite-dimensional. Instead, we prove that the cohomology of the pure Sullivan algebra associated with $(\Lambda W, d)$ is finite-dimensional, which is an equivalent condition [15, Proposition 32.4].

The pure Sullivan algebra associated with $(\Lambda W, d)$, and denoted by $(\Lambda W, d_\sigma)$, is determined by its differential which is described by

$$\begin{aligned}
&d_\sigma(x_1)=0, && d_\sigma(y_1)=x_1^3x_2, && \\
&d_\sigma(x_2)=0, && d_\sigma(y_2)=x_1^2x_2^2, && d_\sigma(z)=x_1^{15}+x_2^{12},\\
&d_\sigma(x_v)=0,\ v\in V, && d_\sigma(y_3)=x_1x_2^3, && d_\sigma(z_v)=x_v^3+\sum_{(v,w)\in E} x_vx_wx_2^4,\ v\in V.
\end{aligned}$$

Therefore, the cohomology of $(\Lambda W, d_\sigma)$ is finite-dimensional because

$$d_\sigma(zx_1^2-y_2x_2^{10})=x_1^{17} \quad \text{and} \quad d_\sigma(zx_2-y_1x_1^{12})=x_2^{13},$$

and the cohomology class

$$[x_v^3]^4=\Big[-\sum_{(v,w)\in E} x_vx_wx_2^4\Big]^4=0.$$

Now, the formal dimension of $(\Lambda W, d)$ is immediately obtained by (1). □

Our next step is to describe $\operatorname{Hom}(\mathcal{M}_\mathcal{G},\mathcal{M}_\mathcal{G})$. Actually, it is the most demanding task in this paper. Recall that an automorphism of $\mathcal{G}$ is a permutation σ on V with $(v,w)\in E$ if and only if $(\sigma(v),\sigma(w))\in E$ for every $(v,w)\in E$. The following is a straightforward result.

LEMMA 2.3. *Every $\sigma\in\operatorname{Aut}(\mathcal{G})$ induces an automorphism f_σ of $\mathcal{M}_\mathcal{G}$.*

Proof. Take $f_\sigma\colon\mathcal{M}_\mathcal{G}\to\mathcal{M}_\mathcal{G}$ defined by

$$\begin{aligned}
f_\sigma(\omega)&=\omega && \text{for } \omega\in\{x_1,x_2,y_1,y_2,y_3,z\},\\
f_\sigma(x_v)&=x_{\sigma(v)} && \text{for } v\in V,\\
f_\sigma(z_v)&=z_{\sigma(v)} && \text{for } v\in V.
\end{aligned}$$

□

LEMMA 2.4. *For every $f\in\operatorname{Hom}(\mathcal{M}_\mathcal{G},\mathcal{M}_\mathcal{G})$ one of the following holds.*

(1) *If f is an automorphism, then there exists $\sigma\in\operatorname{Aut}(\mathcal{G})$ such that*

$$\begin{aligned}
f(\omega)&=f_\sigma(\omega) && \text{for } \omega\in\{x_1,x_2,y_1,y_2,y_3,x_v\mid v\in V\},\\
f(z)&=f_\sigma(z)+d(m_z) && \text{for } m_z\in\mathcal{M}_\mathcal{G}^{118},\\
f(z_v)&=f_\sigma(z_v)+d(m_{z_v}) && \text{for } v\in V \text{ and } m_{z_v}\in\mathcal{M}_\mathcal{G}^{118}.
\end{aligned}$$

(2) *If f is not an automorphism, then there exist $s\in\{0,1\}$ and $f_s\in\operatorname{Hom}(\mathcal{M}_\mathcal{G},\mathcal{M}_\mathcal{G})$ defined by*

$$\begin{aligned}
f_s(\omega)&=s\omega && \text{for } \omega\in\{x_1,x_2,y_1,y_2,y_3,z\},\\
f_s(x_v)&=0 && \text{for } v\in V,\\
f_s(z_v)&=0 && \text{for } v\in V,
\end{aligned}$$

such that

$$\begin{aligned} f(\omega) &= f_s(\omega) && \textit{for } \omega \in \{x_1, x_2, y_1, y_2, y_3, x_v \mid v \in V\}, \\ f(z) &= f_s(z) + d(m_z) && \textit{for } m_z \in \mathcal{M}_{\mathcal{G}}^{118}, \\ f(z_v) &= f_s(z_v) + d(m_{z_v}) && \textit{for } v \in V \textit{ and } m_{z_v} \in \mathcal{M}_{\mathcal{G}}^{118}. \end{aligned}$$

Proof. For $f \in \mathrm{Hom}(\mathcal{M}_{\mathcal{G}}, \mathcal{M}_{\mathcal{G}})$, by degree-reasoning we write

$$\begin{aligned} f(x_1) &= a_1 x_1, \\ f(x_2) &= a_2 x_2, \\ f(y_1) &= b_1 y_1, \\ f(y_2) &= b_2 y_2, \\ f(y_3) &= b_3 y_3, \\ f(x_v) &= \sum_{w \in V} a(v,w) x_w + a_1(v) x_1^5 + a_2(v) x_2^4, \quad v \in V, \\ f(z) &= cz + \sum_{w \in V} c(w) z_w + \alpha_1 y_1 x_1^2 x_2^7 + \beta_1 y_2 x_1^3 x_2^6 + \gamma_1 y_3 x_1^4 x_2^5 \\ &\quad + \alpha_2 y_1 x_1^7 x_2^3 + \beta_2 y_2 x_1^8 x_2^2 + \gamma_2 y_3 x_1^9 x_2 \\ &\quad + \sum_{w \in V} x_w (\alpha_3(w) y_1 x_1^2 x_2^3 + \beta_3(w) y_2 x_1^3 x_2^2 + \gamma_3(w) y_3 x_1^4 x_2), \\ f(z_v) &= e(v) z + \sum_{w \in V} c(v,w) z_w + \alpha_1(v) y_1 x_1^2 x_2^7 + \beta_1(v) y_2 x_1^3 x_2^6 + \gamma_1(v) y_3 x_1^4 x_2^5 \\ &\quad + \alpha_2(v) y_1 x_1^7 x_2^3 + \beta_2(v) y_2 x_1^8 x_2^2 + \gamma_2(v) y_3 x_1^9 x_2 \\ &\quad + \sum_{w \in V} x_w (\alpha_3(v,w) y_1 x_1^2 x_2^3 + \beta_3(v,w) y_2 x_1^3 x_2^2 + \gamma_3(v,w) y_3 x_1^4 x_2), \quad v \in V. \end{aligned} \tag{2}$$

Since $df(y_i) = f(dy_i)$ for $i = 1, 2, 3$, we obtain

$$b_1 = a_1^3 a_2, \quad b_2 = a_1^2 a_2^2 \quad \text{and} \quad b_3 = a_1 a_2^3. \tag{3}$$

As $df(z) = f(dz)$, the two expressions below must be equal:

$$\begin{aligned} df(z) &= c(x_1^4 x_2^2 y_1 y_2 - x_1^5 x_2 y_1 y_3 + x_1^6 y_2 y_3 + x_1^{15} + x_2^{12}) \\ &\quad + \sum_{w \in V} c(w) \left(x_w^3 + \sum_{(w,u) \in E} x_w x_u x_2^4 \right) + (\alpha_1 + \beta_1 + \gamma_1) x_1^5 x_2^8 + (\alpha_2 + \beta_2 + \gamma_2) x_1^{10} x_2^4 \\ &\quad + \sum_{w \in V} (\alpha_3(w) + \beta_3(w) + \gamma_3(w)) x_w x_1^5 x_2^4, \\ f(dz) &= b_1 b_2 a_1^4 a_2^2 y_1 y_2 x_1^4 x_2^2 - b_1 b_3 a_1^5 a_2 y_1 y_3 x_1^5 x_2 + b_2 b_3 a_1^6 y_2 y_3 x_1^6 + a_1^{15} x_1^{15} + a_2^{12} x_2^{12}. \end{aligned}$$

Hence, we obtain

$$c = a_1^{15} = a_2^{12} = b_1 b_2 a_1^4 a_2^2 = b_1 b_3 a_1^5 a_2 = b_2 b_3 a_1^6 \tag{4}$$

and

$$\begin{aligned} c(w) &= 0 \quad \text{for all } w \in V, \\ \alpha_i + \beta_i + \gamma_i &= 0 \quad \text{for } i = 1, 2, \\ \alpha_3(w) + \beta_3(w) + \gamma_3(w) &= 0 \quad \text{for all } w \in V. \end{aligned}$$

Equations (3) and (4) are the same as in [4, Example 5.1]. Therefore

$$a_1 = a_2 = b_1 = b_2 = b_3 = c = s \quad \text{and} \quad s \in \{0, 1\}.$$

This yields

$$f(x_1) = sx_1, \quad f(x_2) = sx_2, \quad f(y_1) = sy_1, \quad f(y_2) = sy_2, \quad f(y_3) = sy_3$$

and

$$\begin{aligned} f(z) = sz &+ d(\beta_1 y_1 y_2 x_2^5 + \gamma_1 y_1 y_3 x_1 x_2^4) + d(\beta_2 y_1 y_2 x_1^5 x_2 + \gamma_2 y_1 y_3 x_1^6) \\ &+ \sum_{w \in V} d(\beta_3(w) y_1 y_2 x_w x_2 + \gamma_3(w) y_1 y_3 x_w x_1). \end{aligned}$$

Assume first that $s=1$. As $df(z_v)=f(dz_v)$, the following two expressions must be equal:

$$\begin{aligned} df(z_v) = e(v)&(y_1 y_2 x_1^4 x_2^4 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15} + x_2^{12}) \\ &+ \sum_{w \in V} c(v, w) \left(x_w^3 + \sum_{(w,u) \in E} x_w x_u x_2^4 \right) \\ &+ (\alpha_1(v) + \beta_1(v) + \gamma_1(v)) x_1^5 x_2^8 + (\alpha_2(v) + \beta_2(v) + \gamma_2(v)) x_1^{10} x_2^4 \\ &+ \sum_{w \in V} (\alpha_3(v, w) + \beta_3(v, w) + \gamma_3(v, w)) x_w x_1^5 x_2^4 \end{aligned} \tag{5}$$

and

$$\begin{aligned} f(dz_v) = &\left(\sum_{w \in V} a(v, w) x_w + a_1(v) x_1^5 + a_2(v) x_2^4 \right)^3 \\ &+ \sum_{(v,r) \in E} \left(\sum_{w \in V} a(v, w) x_w + a_1(v) x_1^5 + a_2(v) x_2^4 \right) \\ &\qquad \times \left(\sum_{u \in V} a(r, u) x_u + a_1(r) x_1^5 + a_2(r) x_2^4 \right) x_2^4. \end{aligned} \tag{6}$$

Close examination of equations (5) and (6) yields the following remarks. Firstly, remark that in (5) there is no summand containing $x_v x_w x_u$ for $v \neq w \neq u \neq v$. This forces (6) to have, at most, two non-trivial coefficients $a(v,w)$. Observe now that in (5) neither there is a summand containing $x_w^2 x_v$, so in (6) only one non-trivial coefficient $a(v,w)$ can exist. Therefore, in (6), there is at most a unique summand containing x_w^3 and, in (5), a unique non-trivial coefficient $c(v,w)$. Secondly, comparing in both equations the coefficients of $y_1 y_2 x_1^4 x_2^4$ and x_1^{15}, we obtain

$$e(v) = a_1(v) = 0.$$

Now, there is no term of type $x_w^2 x_2^4$ in (5) (the graph does not contain any loop) so we deduce that

$$a_2(v) = 0.$$

Finally, comparing the coefficients of $x_1^5 x_2^8$, $x_1^{10} x_2^4$ and $x_w x_1^5 x_2^4$, we obtain that

$$\begin{aligned} \alpha_i(v) + \beta_i(v) + \gamma_i(v) = 0 \quad \text{for } i = 1, 2, \\ \alpha_3(v,w) + \beta_3(v,w) + \gamma_3(v,w) = 0. \end{aligned}$$

Summarizing,

$$\begin{aligned} f(x_v) &= a(v,\sigma(v)) x_{\sigma(v)}, \\ f(z_v) &= c(v,\sigma(v)) z_{\sigma(v)} + d(\beta_1(v) x_2^5 y_1 y_2 + \gamma_1(v) x_1 x_2^4 y_1 y_3) \\ &\quad + d(\beta_2(v) x_1^5 x_2 y_1 y_2 + \gamma_2(v) x_1^6 y_1 y_3) \\ &\quad + \sum_{w \in V} d(\beta_3(v,w) x_w x_2 y_1 y_2 + \gamma_3(v,w) x_w x_1 y_1 y_3), \end{aligned}$$

where σ is a self-map of V and

$$\begin{aligned} c(v,\sigma(v)) &= a(v,\sigma(v))^3 && \text{for all } v \in V, \\ c(v,\sigma(v)) &= a(v,\sigma(v)) a(w,\sigma(w)) && \text{for all } (v,w) \in E. \end{aligned}$$

Therefore $a(v,\sigma(v))^2 = a(w,\sigma(w))$ if $(v,w) \in E$. Since $\mathcal{G}$ is not a directed graph (which implies that if $(v,w) \in E$ then $(w,v) \in E$ too) we deduce that $a(w,\sigma(w))^2 = a(v,\sigma(v))$, and hence $a(v,\sigma(v))^4 = a(v,\sigma(v))$. Moreover, since $\mathcal{G}$ is connected, one of the following holds:

(1) $a(v,\sigma(v)) = c(v,\sigma(v)) = 0$ for all $v \in V$, which proves Lemma 2.4 (2) for $s = 1$.

(2) $a(v,\sigma(v)) = c(v,\sigma(v)) = 1$ for all $v \in V$. Then,

$$\begin{aligned} f(x_v) &= x_{\sigma(v)}, \\ f(z_v) &= z_{\sigma(v)} + d(\beta_1(v) x_2^5 y_1 y_2 + \gamma_1(v) x_1 x_2^4 y_1 y_3) + d(\beta_2(v) x_1^5 x_2 y_1 y_2 + \gamma_2(v) x_1^6 y_1 y_3) \\ &\quad + \sum_{w \in V} d(\beta_3(v,w) x_w x_2 y_1 y_2 + \gamma_3(v,w) x_w x_1 y_1 y_3). \end{aligned}$$

The self-map $\sigma\colon V\to V$ is, in fact, an element in $\mathrm{Aut}(\mathcal{G})$. We first show that $\sigma\in\mathrm{Hom}(\mathcal{G},\mathcal{G})$, that is, $(v,w)\in E$ if and only if $(\sigma(v),\sigma(w))\in E$. Indeed, $(v,w)\in E$ if and only if there is a summand $x_vx_wx_2^4$ in $d(z_v)$, and hence if and only if there is a summand $x_{\sigma(v)}x_{\sigma(w)}x_2^4$ in $f(dz_v)=df(z_v)=d(z_{\sigma(v)})$, that is, if and only if $(\sigma(v),\sigma(w))\in E$. Now, since for every $v\in V$, $f(dz_v)=d(z_{\sigma(v)})$, σ is one-to-one on the neighborhood of every vertex. Therefore $\sigma\in\mathrm{Aut}(\mathcal{G})$ [23, Lemma 1], which proves Lemma 2.4 (1).

Assume now that $s=0$. Then

$$f(dz_v)=\biggl(\sum_{w\in V}a(v,w)x_w+a_1(v)x_1^5+a_2(v)x_2^4\biggr)^3. \tag{7}$$

Since $df(z_v)=f(dz_v)$, an argument similar to the one above, comparing (5) and (7), yields

$$\begin{aligned}
f(x_v)&=0,\\
f(z_v)&=0+d(\beta_1(v)x_2^5y_1y_2+\gamma_1(v)x_1x_2^4y_1y_3)+d(\beta_2(v)x_1^5x_2y_1y_2+\gamma_2(v)x_1^6y_1y_3)\\
&\quad+\sum_{w\in V}d(\beta_3(v,w)x_wx_2y_1y_2+\gamma_3(v,w)x_wx_1y_1y_3),
\end{aligned}$$

which proves Lemma 2.4 (2) for $s=0$. □

As mentioned in §1, isomorphism classes of minimal Sullivan algebras whose cohomology is 1-connected and of finite type are in bijection with rational homotopy types for simply connected spaces with rational homology of finite type. Also, the homotopy classes of morphisms of the corresponding minimal Sullivan algebras are in bijection with the homotopy classes of maps between the corresponding rational homotopy types. Recall that two morphisms from a Sullivan algebra to an arbitrary commutative cochain algebra, $\phi_0,\phi_1\colon(\Lambda W,d)\to(A,d)$, are *homotopic* if there exists

$$H\colon(\Lambda W,d)\longrightarrow(A,d)\otimes(\Lambda(t,dt),d)$$

such that $(\mathrm{id}\cdot\varepsilon_i)H=\phi_i$, $i=0,1$, where $\deg t=0$, $\deg dt=1$, and d is the differential sending $t\mapsto dt$. The augmentations $\varepsilon_0,\varepsilon_1\colon\Lambda(t,dt)\to\mathbb{Q}$ are defined by $\varepsilon_0(t)=0$ and $\varepsilon_1(t)=1$.

LEMMA 2.5. *For any $f\in\mathrm{Hom}(\mathcal{M}_\mathcal{G},\mathcal{M}_\mathcal{G})$, one of the following holds*:

(1) *There exists an automorphism f_σ, as in Lemma 2.4 (1), such that f is homotopic to f_σ.*

(2) *There exists f_s, as in Lemma 2.4 (2), such that f is homotopic to f_s.*

Proof. This follows directly from Lemma 2.4. □

Gathering Lemmas 2.3–2.5, we have proved the following result from which we deduce Theorem 1.2 as a corollary.

THEOREM 2.6. *Let $\mathcal{G}$ be a finite connected graph with more than one vertex. Then, there exists an elliptic minimal Sullivan algebra $\mathcal{M}_{\mathcal{G}}$ such that the monoid of homotopy classes of self-maps is*

$$[\mathcal{M}_{\mathcal{G}}, \mathcal{M}_{\mathcal{G}}] \cong \mathrm{Aut}(\mathcal{G}) \sqcup \{f_s : s = 0, 1\}.$$

Therefore $\mathrm{Aut}(\mathcal{G}) \cong \mathcal{E}(\mathcal{M}_{\mathcal{G}})$.

We finish this section with some comments on the properties of the construction above. The following, together with Theorem 1.3, justifies the infinitely many rational spaces X from Theorem 1.1 (see also Remark 2.9).

PROPOSITION 2.7. *Let $\mathcal{G}_1 = (V, E)$ and $\mathcal{G}_2 = (V', E')$ be two non-isomorphic graphs. Then, $\mathcal{M}_{\mathcal{G}_1}$ and $\mathcal{M}_{\mathcal{G}_2}$ are non-isomorphic minimal Sullivan algebras.*

Proof. Assume that $\mathcal{M}_{\mathcal{G}_1}$ and $\mathcal{M}_{\mathcal{G}_2}$ are isomorphic, and let f denote such an isomorphism. Since

$$|V| + 2 = \dim \mathcal{M}^{40}_{\mathcal{G}_1} = \dim \mathcal{M}^{40}_{\mathcal{G}_2} = |V'| + 2,$$

we have $|V| = |V'|$ and, without loss of generality, we may assume that $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same set of vertices V. Then, the isomorphism f is described by the system of equations (2). Reproducing the same steps as in the proof of Lemma 2.4, we get that f is homotopic to f_σ, where σ is a permutation of V such that $(v, w) \in E$ if and only if $(\sigma(v), \sigma(w)) \in E'$. That is, σ induces an isomorphism between $\mathcal{G}_1$ and $\mathcal{G}_2$. □

The construction of $\mathcal{M}$ is functorial when considering the appropriate category of graphs. Recall that given $\mathcal{G}_1 = (V, E)$ and $\mathcal{G}_2 = (V', E')$, a morphism $\sigma: \mathcal{G}_1 \to \mathcal{G}_2$ is said to be *full* if for every pair of vertices $v, w \in V$, $(v, w) \in E$ if and only if $(\sigma(v), \sigma(w)) \in E'$.

Remark 2.8. Let $\mathcal{G}raph_{fm}$ be the category whose objects are finite graphs with more than one vertex, and whose morphisms are full graph monomorphisms. Then, the construction $\mathcal{M}$ provides a contravariant faithful functor which is injective on objects (an embedding) from $\mathcal{G}raph_{fm}$ to the category of Sullivan algebras. Let $\mathcal{G}_1 = (V, E)$ and $\mathcal{G}_2 = (V', E')$ be graphs, and $\mathcal{M}_{\mathcal{G}_1}$ and $\mathcal{M}_{\mathcal{G}_2}$ be the associated minimal Sullivan algebras provided by Theorem 2.6. If $\sigma: \mathcal{G}_1 \to \mathcal{G}_2$ in $\mathcal{G}raph_{fm}$, then there is a morphism of minimal Sullivan algebras $\mathcal{M}(\sigma): \mathcal{M}_{\mathcal{G}_2} \to \mathcal{M}_{\mathcal{G}_1}$ given by

$$\mathcal{M}(\sigma)(\omega) = \omega \qquad \text{for } \omega \in \{x_1, x_2, y_1, y_2, y_3, z\}$$

$$\mathcal{M}(\sigma)(x_{v'}) = \begin{cases} x_v, & \text{if } \sigma(v) = v', \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{M}(\sigma)(z_{v'}) = \begin{cases} z_v, & \text{if } \sigma(v) = v', \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathcal{G}_1 = \mathcal{G}_2$, then $\sigma \in \mathrm{Aut}(\mathcal{G}_1)$ and $\mathcal{M}(\sigma) = f_{\sigma^{-1}}$ as described in Lemma 2.3.

We finish this section illustrating other possible constructions of minimal Sullivan algebras for a given graph $\mathcal{G}$.

Remark 2.9. The construction of $\mathcal{M}_\mathcal{G}$ is not unique, that is, given a finite connected graph $\mathcal{G}$ with more than one vertex, there exist infinitely many non-isomorphic minimal Sullivan algebras whose group of self-homotopy equivalences is isomorphic to $\mathrm{Aut}(\mathcal{G})$. In fact, given a non-trivial vector $(u_1,u_2)\in\mathbb{Q}^2$, it is possible to construct a minimal Sullivan algebra $(\mathcal{M}_{(u_1,u_2)}, d_{(u_1,u_2)})$ having the same generators as $\mathcal{M}_\mathcal{G}$, and $d_{(u_1,u_2)}$ equal to d in every generator but in

$$d_{(u_1,u_2)}(z_v)=x_v^3+\sum_{(v,w)\in E} x_v x_w(u_1x_1^5+u_2x_2^4).$$

3. From graphs to inflexible manifolds

Inflexibility for an oriented compact closed manifold M means that the set of mapping degrees ranging over all continuous self-maps is finite. By composition of self-maps, it is obvious that it is equivalent to demanding that all its self-maps have degree -1, 0 or 1. For an elliptic (and hence Poincaré duality) Sullivan algebra $(\Lambda W,d)$ of formal dimension n, inflexibility means that, for every $f\in\mathrm{Hom}((\Lambda W,d),(\Lambda W,d))$, and for a representative x of the fundamental class in $H^n(\Lambda W,d)$, the equality $[f(x)]=a[x]$ holds for $a\in\{-1,0,1\}$.

Proposition 3.1. *Let $\mathcal{A}=(\Lambda W,d)$ be a 1-connected elliptic Sullivan algebra of formal dimension $2n$. Choose $x\in\mathcal{A}^{2n}$ representing the fundamental class in $H^{2n}(\mathcal{A})$. Define the Sullivan algebra $\tilde{\mathcal{A}}=(\Lambda W\otimes\Lambda(y),\tilde{d})$ with $\tilde{d}|_W=d$, $\deg y=2n-1$ and $\tilde{d}(y)=x$. Then, $\tilde{\mathcal{A}}$ is a 1-connected elliptic Sullivan algebra of formal dimension $4n-1$. Moreover, if we choose $z\in\mathcal{A}^{4n-1}$ such that $d(z)=x^2$, then $xy-z$ is a representative of the fundamental class in $H^{4n-1}(\tilde{\mathcal{A}})$.*

Proof. First notice that, since $(W\oplus\mathbb{Q}y)^{\mathrm{even}}=W^{\mathrm{even}}$, every element in $H^*(\tilde{\mathcal{A}})$ is nilpotent because every element in $H^*(\mathcal{A})$ is nilpotent. Hence $\tilde{\mathcal{A}}$ is elliptic and the formal dimension is easily obtained by (1).

Now, $\tilde{d}(xy-z)=x\tilde{d}(y)-d(z)=0$. Let us see that it is not a boundary. Assume that $xy-z=\tilde{d}(\omega)$ for $\omega=\omega_1y+\omega_2\in\tilde{\mathcal{A}}^{4n-2}$, $\omega_1,\omega_2\in\mathcal{A}$. Then, $xy-z=d(\omega_1)y+\omega_1x+d(\omega_2)$. Since $z,\omega_1x,d(\omega_2)\in\mathcal{A}$, we deduce that $xy=d(\omega_1)y$, and so $x=d(\omega_1)$. This contradicts the fact that x is a representative of the fundamental class. □

Lemma 3.2. *The elliptic Sullivan algebra $\tilde{\mathcal{A}}$ is inflexible if $\mathcal{A}$ is inflexible. Moreover, $[\mathcal{A},\mathcal{A}]\cong[\tilde{\mathcal{A}},\tilde{\mathcal{A}}]$ as monoids and, in particular, $\mathcal{E}(\mathcal{A})\cong\mathcal{E}(\tilde{\mathcal{A}})$.*

Proof. For $\tilde{f}\in\mathrm{Hom}(\tilde{\mathcal{A}},\tilde{\mathcal{A}})$, let f denote $\tilde{f}|_{\mathcal{A}}$. Then $\tilde{f}(xy-z)=f(x)\tilde{f}(y)-f(z)$. Since $\mathcal{A}$ is inflexible, $f(x)=ax+d(m_x)$ with $a\in\{1,0,-1\}$. Applying d to $f(z)$, and using that $d(z)=x^2$, a straightforward calculation shows that

$$f(z)=a^2z+(2axm_x+m_xd(m_x))+d(\gamma).$$

Applying now $\tilde{d}$ to $\tilde{f}(y)$, and using that $\tilde{d}(y)=x$, again a straightforward calculation shows that

$$\tilde{f}(y)=ay+m_x+d(\gamma').$$

Hence $[\tilde{f}(xy-z)]=a^2[xy-z]$, proving that $\tilde{\mathcal{A}}$ is inflexible.

Observe now that any $\tilde{f}\in\mathrm{Hom}(\tilde{\mathcal{A}},\tilde{\mathcal{A}})$ is determined, up to homotopy, by its restriction to $\mathcal{A}$. The only undetermined term appears when $\tilde{f}(y)$ is computed. This means that if $\tilde{f}_1|_{\mathcal{A}}$ and $\tilde{f}_2|_{\mathcal{A}}$ are equal, then $\tilde{f}_1(y)-\tilde{f}_2(y)=d(\gamma_1'-\gamma_2')$. Hence $\tilde{f}_1$ and $\tilde{f}_2$ are homotopic. In the same way, any $f\in\mathrm{Hom}(\mathcal{A},\mathcal{A})$ can be extended to $\mathrm{Hom}(\tilde{\mathcal{A}},\tilde{\mathcal{A}})$ in a unique way, up to homotopy. □

We can now prove Theorem 1.5.

Proof of Theorem 1.5. Let G be a finite group. There exists a finite and connected graph $\mathcal{G}=(V,E)$ such that $\mathrm{Aut}(\mathcal{G})\cong G$ (by Theorem 1.3). Associated with the graph $\mathcal{G}$ of order n, there exists a 1-connected elliptic minimal Sullivan algebra $\mathcal{M}_{\mathcal{G}}$ (of formal dimension $208+80n$) such that $\mathrm{Aut}(\mathcal{G})\cong\mathcal{E}(\mathcal{M}_{\mathcal{G}})$ (by Theorem 2.6). We modify $\mathcal{M}_{\mathcal{G}}$ into an elliptic minimal Sullivan algebra $\widetilde{\mathcal{M}}_{\mathcal{G}}$ of formal dimension $(416+160n)-1$ (by Proposition 3.1) which, by Lemma 3.2, is inflexible since $\mathcal{M}_{\mathcal{G}}$ is inflexible. This is clear since $[\mathcal{M}_{\mathcal{G}},\mathcal{M}_{\mathcal{G}}]\cong G\sqcup\{f_0,f_1\}$ is finite, and because of the multiplicativity of the mapping degree.

Now, since $415+160n\not\equiv 0 \pmod 4$, the theorems of Sullivan [27, Theorem (13.2)] and Barge [5, Théorème 1] give a sufficient condition for the realization of $\widetilde{\mathcal{M}}_{\mathcal{G}}$ by a simply connected manifold M.

Finally, again by Lemma 3.2, $\mathcal{E}(\mathcal{M}_{\mathcal{G}})\cong\mathcal{E}(\widetilde{\mathcal{M}}_{\mathcal{G}})$. Hence, putting the isomorphisms of groups together, we get

$$G\cong\mathrm{Aut}(\mathcal{G})\cong\mathcal{E}(\mathcal{M}_{\mathcal{G}})\cong\mathcal{E}(\widetilde{\mathcal{M}}_{\mathcal{G}})\cong\mathcal{E}(M_0),$$

where M_0 is the rational homotopy type of M. □

The question of whether certain orientation-reversing maps on manifolds exist is treated in the literature (see for example [25] and [1]). Examples of such manifolds are provided by Theorem 1.5.

Corollary 3.3. *For any $n>1$, there exists a simply connected manifold M of dimension $415+160n$ that does not admit an orientation-reversing self-map.*

Proof. The existence of such a manifold M is given by Theorem 1.5 for a graph $\mathcal{G}$ of order n, with $\widetilde{\mathcal{M}}_{\mathcal{G}}$ being the minimal Sullivan algebra of M. Now, any self-map of $\widetilde{\mathcal{M}}_{\mathcal{G}}$, is shown to satisfy $\deg(\tilde{f})=\deg(\tilde{f}|_{\mathcal{M}_{\mathcal{G}}})^2$ (by the proof of Lemma 3.2). Therefore, any self-map of M has either degree 0 or 1. □

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