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Proof of an entropy conjecture for Bloch coherent spin states and its generalizations

by

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1. Introduction

Coherent states in the Hilbert space $\mathcal{H}=L^2(\mathbb{R}^n)$, introduced in the work of Schrödinger, Bargmann, Glauber and others, are certain normalized Gaussian functions parameterized by points in classical phase space, $(p,q)\in\mathbb{R}^n\times\mathbb{R}^n$. They are denoted by $|p,q\rangle$. See, e.g., [10] or §2 for definitions.

Given a density matrix ϱ on $\mathcal{H}$ (i.e., ϱ is a positive semi-definite operator with trace $\operatorname{Tr}\varrho=1$), its von Neumann entropy is $S(\varrho)=-\operatorname{Tr}\varrho\log\varrho$. This is always non-negative, but the usual classical Boltzmann type density, e.g., $f(p,q)=Z^{-1}\exp-\beta(p^2+V(q))$ can have an arbitrarily negative entropy $-\int_{\mathbb{R}^n\times\mathbb{R}^n} f(p,q)\log f(p,q)\,dp\,dq$. To remedy this and other problems with the classical approximation to entropy, A. Wehrl [22] used coherent states to propose another definition, as follows:

Define $\varrho^{\mathrm{cl}}(p,q):=\langle p,q|\,\varrho\,|p,q\rangle$. This function has several names. One is the Husimi Q-function [8]. Berezin [1] called it the covariant symbol. The name we shall use here is *the lower symbol* of ϱ (see [9]). Since $(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n}|p,q\rangle\langle p,q|\,dp\,dq$ is the unit operator on $\mathcal{H}$, we see that $\varrho^{\mathrm{cl}}(p,q)$ is a probability density with respect to the classical measure $(2\pi)^{-n}\,dp\,dq$. In [1] and [9] it is shown that the trace $\operatorname{Tr} f(\varrho)$ of a convex function f of ϱ is bounded below by the corresponding classical integral $(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} f(\varrho^{\mathrm{cl}}(p,q))\,dp\,dq$. Together with the corresponding upper bound for what is called the upper symbol (or contravariant symbol in [1]) they are often referred to as the Berezin–Lieb inequalities. The inequalities are, of course, reversed for concave functions. In this paper we shall not be concerned with the upper symbol.

Wehrl uses the lower symbol, to define

$$S^{\mathrm{cl}}(\varrho)=-(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n}\varrho^{\mathrm{cl}}(p,q)\log\varrho^{\mathrm{cl}}(p,q)\,dp\,dq. \tag{1}$$

Since $0 \leqslant \varrho^{\rm cl}(p,q) \leqslant 1$, we see that $S^{\rm cl}(\varrho) \geqslant 0$, as desired.

A question raised by Wehrl is which ϱ minimizes $S^{\rm cl}(\varrho)$, and he conjectured that this occurs exactly when ϱ is a 1-dimensional projection onto any coherent state, i.e., $\varrho = |p,q\rangle\langle p,q|$ for any choice of p and q. This conjecture was proved in [10]. Later Carlen [6] gave a different proof based on the log-Sobolev inequality; this proof included the uniqueness statement for the first time. A decade later Luo [15] gave a proof based on hypercontractivity, which is closely related to the log-Sobolev inequality.

In [10] a similar conjecture was made for the coherent states in the Hilbert spaces of the irreducible representations of SU(2). These *angular momentum coherent states* are very useful for the physics of quantum spin systems. They are usually called Bloch coherent states [2], [17] and will be the subject of this paper. *We shall prove the conjecture in* [10] *that the analog of Wehrl's conjecture holds here as well*, i.e., that the classical entropy is minimized by coherent states. We do not prove that coherent states alone minimize the entropy, however.

Previously, the conjecture had been established only for a few low-dimensional representations of SU(2). The 2-dimensional spin-$\frac{1}{2}$ case is simple as all pure states are coherent states. This was already pointed out in [10]. For the 3-dimensional case of spin 1 the conjecture was solved by Scutaru [19] and by Schupp [18] who also solved it for the 4-dimensional representation corresponding to spin $\frac{3}{2}$. Bodmann [3] proved a lower bound on the classical entropy which is asymptotically correct for large spin J.

We prove more than that $S^{\rm cl}(\varrho)$ is minimized when ϱ is a projection onto a coherent state. We prove this for *all concave functions* $f(t)$, not just $f(t) = -t\log t$. In fact, the original proof in [10] for the Gaussian Glauber states was for $f(t) = -t^p$, $p \geqslant 1$, and the proof for $-t\log t$ followed by taking the limit $p \to 1$. The extension to $f(t) = t^p$ for $0 < p < 1$ was given by Carlen [6]. To our knowledge it has not been proved for general concave functions. In Theorem 2.2 we show exactly that, by approximating Glauber coherent states by Bloch coherent states in an appropriate limit of large spin. The particulars of this approximation are in the appendix.

In order to prove the conjecture for all spin we utilize a generalization of coherent states, called *coherent operators* introduced by us in [13]. These are operators that map density matrices in one SU(2) space, characterized by an angular momentum (or "spin") J to a density matrix in a spin-K space. These maps are, in fact, *quantum channels*, i.e., completely positive trace-preserving maps, as will be made clear in Lemma 4.1. We shall also see that these quantum channels are equivalent to the *universal quantum cloning machines* later identified in [7] and [23]. The no-cloning theorem states that exact cloning of a quantum state is impossible. The universal quantum cloning machines are quantum channels that achieve the best degree of cloning for general input states. As explained in

Remark 4.2 below, the channels defined in (17), which we will show to be equivalent to the quantum coherent operators, are identical to the universal quantum cloning machines as represented in equation (4) in [23]. A particular realization of a universal quantum cloning channel appears in the context of quantum information in curved space times as the Unruh channel. It describes either the transformation of states prepared by an inertial observer as seen from an accelerated observer, or stimulated emissions from black holes; see, e.g., [4] and [5].

The Bloch coherent states map density matrices from J to functions on the classical phase space, i.e., the 2-sphere $\mathbb{S}^2$ (which can be thought of as $K=\infty$ [9]) via the lower symbol map mentioned above. We do this in small steps, so to speak, by going from J to $J+\frac{1}{2}$ to $J+1$, etc. For each finite K on the way we prove that projections onto coherent states in J minimize the von Neumann entropy of the lifted density matrix in K. In other words we determine the *minimal output entropy* of the quantum coherent operator channels to be the entropy of the output of a coherent state. We then show that after an appropriate scaling, the limit $K\to\infty$ gives us the desired classical (lower symbol) entropy, and thus we prove the conjecture for the entropy and for any concave function.

An important observation in our procedure is to note that the quantum coherent operator channels have a simple expression in terms of bosonic second quantization, i.e., bosonic creation and annihilation operators.

Coherent states can be generalized to any compact semi-simple Lie group, not just SU(2) (see [16] and [20]), and there we expect that a similar result holds. For any compact semi-simple Lie group, Sugita proved in [21] that coherent states minimize the classical Wehrl entropy corresponding to *Rényi entropies of integer order.*

In §2 we define Bloch coherent states, lower symbols, and discuss the corresponding Berezin–Lieb inequality. We also introduce the quantum coherent operator channels from J to K. (While we are interested here in $K>J$, the map is also defined when $K<J$.) In §3 we derive a more explicit formula for the quantum coherent operator channel which allows us, in §4, to give a bosonic second quantization representation of the channels. In fact, §3 is not important for our main conclusion as we could have defined the quantum channels from the second quantization formulation in §4. We include §3 in order to connect to our previous work in [13]. Following that, we show, in §5, that coherent states in J minimize the output von Neumann entropy in K or, more generally, the trace of any concave function. In particular, we have found the minimal output entropy of the universal quantum cloning machinces.

In the last §6 we study the classical limit $K\to\infty$ and use the Berezin–Lieb inequality to prove the conjecture about the classical entropy.

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2. Basic definitions and main results

For all integer or half-integer J we let $\mathcal{H}_J$ denote the spin-J representation space of SU(2), i.e., $\mathcal{H}_J=\mathbb{C}^{2J+1}$. The corresponding classical phase space is $\mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$. For each point $\omega\in\mathbb{S}^2$ we have the 1-dimensional coherent state projection $P^J_\omega=|\omega\rangle_J\,{}_J\langle\omega|$ projecting $\mathcal{H}_J$ onto the subspace of maximal spin in the direction ω, i.e., the 1-dimesional subspace of $\mathcal{H}_J$ corresponding to the eigenspace of $\omega\cdot\mathbf{S}_J$ with eigenvalue J. Here $\mathbf{S}_J$ is the vector of spin operators, i.e, the representation on $\mathcal{H}_J$ of the standard generators $\mathbf{S}=(S_x,S_y,S_z)$ of SU(2). The vector $|\omega\rangle_J$ is only defined up to a phase, but this will not play a role here as only the projection P^J_ω is important. We will use the notation that $\uparrow,\downarrow\in\mathbb{S}^2$ are respectively the north and south pole.

The coherent state transform is based on the identity that

$$\frac{2J+1}{4\pi}\int_{\mathbb{S}^2}|\omega\rangle_J\,{}_J\langle\omega|\,d\omega=\frac{2J+1}{4\pi}\int_{\mathbb{S}^2}P^J_\omega\,d\omega=I_J, \tag{2}$$

where I_J is the identity on $\mathcal{H}_J$. If ϱ is a density matrix on $\mathcal{H}_J$ its *lower symbol* is the function on $\mathbb{S}^2$ given by

$$\Phi^\infty(\varrho)(\omega)={}_J\langle\omega|\,\varrho\,|\omega\rangle_J=\mathrm{Tr}_J(P^J_\omega\,\varrho), \tag{3}$$

where Tr_J is the trace on $\mathcal{H}_J$. The *classical entropy* of ϱ is

$$S^{\mathrm{cl}}(\varrho)=-\frac{2J+1}{4\pi}\int_{\mathbb{S}^2}\Phi^\infty(\varrho)(\omega)\log(\Phi^\infty(\varrho)(\omega))\,d\omega.$$

We are using the notation Φ^∞ for the lower symbol since we shall consider it as the natural classical limit $k\to\infty$ of the quantum channels Φ^k defined below.

The Berezin–Lieb [1], [9] inequality for the lower symbol states that for any concave function $f\colon[0,1]\to\mathbb{R}$ we have

$$\operatorname{Tr}_J f(\varrho)\leqslant\frac{2J+1}{4\pi}\int_{\mathbb{S}^2} f(\Phi^\infty(\varrho)(\omega))\,d\omega. \tag{4}$$

The inequality follows from (2) as a consequence of Jensen's inequality.

The conjecture from [10] that we shall prove here is that S^{cl} is minimized when the density matrix is any coherent state projection, e.g., $\varrho=|\uparrow\rangle_{J\,J}\langle\uparrow|$. In this case the lower symbol is $\Phi^\infty(|\uparrow\rangle_{J\,J}\langle\uparrow|)(\omega)=\left|{}_J\langle\omega|\uparrow\rangle_J\right|^2$. In fact, we shall prove the more general statement that the same is true if the function $-t\log t$ is replaced by any concave funtion. Our main theorem is the following.

THEOREM 2.1. (Lower symbols of Bloch coherent states minimize concave averages) *Let $f\colon[0,1]\to\mathbb{R}$ be a concave function.*(1) *Then for any density matrix ϱ on $\mathcal{H}_J$ we have*

$$\frac{2J+1}{4\pi}\int_{\mathbb{S}^2} f\left({}_J\langle\omega|\,\varrho\,|\omega\rangle_J\right)d\omega\geqslant\frac{2J+1}{4\pi}\int_{\mathbb{S}^2} f\left(\left|{}_J\langle\omega|\uparrow\rangle_J\right|^2\right)d\omega. \tag{5}$$

By SU(2) *invariance* $\uparrow$ *could be replaced by any other point on* $\mathbb{S}^2$.

The following analogous result for the Glauber coherent states is proved by an easy limiting argument which we give in the appendix.

THEOREM 2.2. (Lower symbols of Glauber coherent states minimize concave averages) *Let $f\colon[0,1]\to\mathbb{R}$ be a continuous concave function with $f(0)=0$. Then for any density matrix ϱ on $L^2(\mathbb{R}^n)$ we have*

$$(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} f(\varrho^{\mathrm{cl}}(p,q))\,dp\,dq\geqslant(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} f\left(\left|\langle p_0,q_0|p,q\rangle\right|^2\right)dp\,dq \tag{6}$$

for all $p_0,q_0\in\mathbb{R}^n$.

Remark 2.3. As in the main Theorem 2.1, we could have allowed $\lim_{t\to1^-}f(t)=-\infty$ (see footnote (1)). We could, in fact, also allow $f(0)\neq0$, but in this case the integrals on both sides of (6) are either both $+\infty$ or both $-\infty$. Even if $f(0)=0$ the integrals may still be $+\infty$, but the inequality holds in the sense that either both sides are $+\infty$ or the right-hand side is finite.

Using the fact that the Glauber coherent state $|p,q\rangle\in L^2(\mathbb{R}^n)$ is explicitly given by

$$\pi^{-n/4}\exp\left(-\frac{(x-q)^2}{2}+ipx\right),$$

(1) It is, in fact, enough to assume that $f\colon[0,1)\to\mathbb{R}$, i.e., to allow that $\lim_{t\to0^-}f(t)=-\infty$. Only coherent state projections have lower symbols that attain the value 1. If ϱ is not a coherent state projection we can find a concave function $\tilde f\colon[0,1]\to\mathbb{R}$ such that $\tilde f\geqslant f$ and $\tilde f({}_J\langle\omega|\varrho|\omega\rangle_J)=f({}_J\langle\omega|\varrho|\omega\rangle_J)$.

we have

$$\langle p,q|\psi\rangle = \pi^{-n/4}\int \exp\Big(-\frac{(x-q)^2}{2}-ipx\Big)\psi(x)\,dx$$

for $\psi\in L^2(\mathbb{R}^n)$. The inequality (6) for the rank-1 state $|\psi\rangle\langle\psi|$ then states that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} f\big(\big|\langle p,q|\psi\rangle\big|^2\big)\,dp\,dq$$

is minimized for concave f when ψ is a Glauber coherent state.

We now define the quantum coherent operator channels. We refer to [13] for details. For fixed K and J we let P_- be the projection in $\mathcal{H}_J\otimes\mathcal{H}_K$ onto the minimal total spin $|K-J|$, i.e., onto the unique copy of $\mathcal{H}_{|K-J|}\subseteq\mathcal{H}_J\otimes\mathcal{H}_K$ on which the tensor product representation acts irreducibly.(2) For simplicity we omit in our notation the dependence of P_- on K and J.

In the language of elementary quantum mechanics, a particle of angular momentum K and one of angular momentum J can combine in exactly one way to produce a composite particle of angular momentum $|K-J|$. The Hilbert space of this composite particle is the subspace $\mathcal{H}_{|K-J|}\subseteq\mathcal{H}_J\otimes\mathcal{H}_K$.

If we let $k=2K-2J\in\mathbb{Z}$, we consider the map Φ^k from operators on $\mathcal{H}_J$ to operators on $\mathcal{H}_K$ defined by the partial trace

$$\Phi^k(\varrho)=\frac{2J+1}{2|K-J|+1}\,\mathrm{Tr}_J\,P_-(\varrho\otimes I_K). \tag{7}$$

This is a trace-preserving completely positive map (see also (16) and Lemma 4.1), i.e., using the language of quantum information theory it is a *quantum channel*. The reader might find it useful at this point to look at the formulation of this map in terms of boson creation and annihilation operators in §4. The trace-preserving property is easily seen, since the partial traces $\mathrm{Tr}_J\,P_-$ and $\mathrm{Tr}_K\,P_-$ are both proportional to the identities. In particular, Φ^k maps density matrices to density matrices. In the notation we have for simplicity omitted the dependence of Φ^k on K and J and only kept the dependence on the difference in dimension $k=2K-2J$.

Our main result about these channels is that they are majorized by coherent states in the following sense.

THEOREM 2.4. (Coherent states majorize Φ^k) *For a density matrix ϱ on $\mathcal{H}_J$ and $k=2(K-J)$ the sequence of eigenvalues of the density matrix $\Phi^k(\varrho)$ is majorized by the sequence of eigenvalues of $\Phi^k(|\omega\rangle\langle\omega|)$, which by* SU(2)*-invariance is independent of $\omega\in\mathbb{S}^2$.*

(2) Strictly speaking the isometric imbedding of $\mathcal{H}_{|K-J|}$ into $\mathcal{H}_J\otimes\mathcal{H}_K$ is given uniquely only up to a phase.

To say that a finite real sequence $a_1 \geqslant a_2 \geqslant ... \geqslant a_M$ *majorizes* another real sequence $b_1 \geqslant b_2 \geqslant ... \geqslant b_M$, written $(a_1, ..., a_M) \succ (b_1, ..., b_M)$, means that

$$\begin{aligned} a_1 &\geqslant b_1, \\ a_1+a_2 &\geqslant b_1+b_2, \\ &\vdots \\ a_1+...+a_{M-1} &\geqslant b_1+...+b_{M-1}, \\ a_1+...+a_M &= b_1+...+b_M. \end{aligned} \tag{8}$$

Note the equality in the last condition (8).

It is a fact that $(a_1, ..., a_M) \succ (b_1, ..., b_M)$ if and only if

$$\sum_{j=1}^{M} f(a_j) \leqslant \sum_{j=1}^{M} f(b_j)$$

for all concave functions $f: \mathbb{R} \to \mathbb{R}$. This is often called Karamata's theorem,(3) cf. [12, Remark 4.7 after equation (4.5.4)]. It is, in fact, enough that the concave function is defined on the interval $[a_M, a_1]$.

If A and B are two Hermitian matrices of the same size we write $A \succ B$ if the eigenvalue sequence of A majorizes the eigenvalue sequence of B. The notion of majorization of sequences can be easily generalized to infinite summable sequences and trace-class operators but we will not need this here.

It is an easy exercise, using the variational principle, to prove that if A, B and C are Hermitian matrices such that $A \succ B$ and $A \succ C$, then

$$A \succ \lambda B + (1-\lambda) C \tag{9}$$

for all $0 \leqslant \lambda \leqslant 1$.

As a consequence it follows from Theorem 2.4 that the minimal output (von Neumann) entropy of the channel Φ^k is achieved for a coherent state, i.e.,

$$\min_{\varrho} S(\Phi^k(\varrho)) = S(\Phi^k(|\omega\rangle\langle\omega|)).$$

More generally, the output of coherent states minmize the trace of concave functions.

COROLLARY 2.5. (Minimization of the trace of concave functions) *If* $f: [0,1] \to \mathbb{R}$ *is concave,* ϱ *is a density matrix on* $\mathcal{H}_J$ *and* $k = 2(K-J)$, *then*

$$\operatorname{Tr}_K f(\Phi^k(\varrho)) \geqslant \operatorname{Tr}_K f(\Phi^k(|\omega\rangle\langle\omega|))$$

for all $\omega \in \mathbb{S}^2$.

Of course, the inequalities about concave functions are reversed for convex functions.

(3) In [12] the concave function is assumed to be monotone increasing. With the assumption of equality in (8) the assumption of monotonicity is not required.

3. A formula for P_-

Our goal here is to find an explicit formula for the projection P_- that projects $\mathcal{H}_J\otimes\mathcal{H}_K$ onto the subspace $\mathcal{H}_{K-J}$, under the assumption that $K\geqslant J$.

We start by choosing the standard preferred basis

$$|M\rangle_L,\quad M=-L,...,L,$$

in $\mathcal{H}_L$, in which S_z is diagonal and S_x is real. This specifies the basis up to an over-all phase. We introduce the *anti*-unitary map $U_L:\mathcal{H}_L\to\mathcal{H}_L$ given by

$$U_L\sum_{M=-L}^{L}\alpha_M|M\rangle_L=\sum_{M=-L}^{L}(-1)^{L-M}\bar{\alpha}_M|-M\rangle_L.$$

This map has the property that

$$U_L^{-1}\mathbf{S}_LU_L=-\mathbf{S}_L. \tag{10}$$

It is the unitary operator $e^{i\pi S_y}$ followed by complex conjugation in the preferred basis. Any anti-unitary operator satisfying (10) agrees with U_L up to an over-all phase. Note that $U_L^{-1}=(-1)^{2L}U_L$ and hence

$$\langle U_L\phi|\psi\rangle=\langle U_L\psi|U_LU_L\phi\rangle=(-1)^{2L}\langle U_L\psi|\phi\rangle \tag{11}$$

for all $\phi,\psi\in\mathcal{H}_L$.

If $K\geqslant J$ then $\mathcal{H}_K\subseteq\mathcal{H}_{K-J}\otimes\mathcal{H}_J$ (recall that $\mathcal{H}_{K-J}\subseteq\mathcal{H}_J\otimes\mathcal{H}_K$ as we said in the beginning). We thus have a sesquilinear map

$$\mathcal{H}_J\times\mathcal{H}_K\ni(\psi,\phi)\longmapsto {}_J\langle\psi\|\phi\rangle_K\in\mathcal{H}_{K-J},$$

where the *partial* inner product ${}_J\langle\psi\|\phi\rangle_K$ is defined by the inner product in $\mathcal{H}_{K-J}$ as follows

$${}_{K-J}\langle\eta|\,{}_J\langle\psi\|\phi\rangle_K\rangle_{K-J}={}_{(K-J)\otimes J}\langle\eta\otimes\psi|\phi\rangle_{(K-J)\otimes J}$$

for all $\eta\in\mathcal{H}_{K-J}$, where the last inner product is in $\mathcal{H}_{K-J}\otimes\mathcal{H}_J$.

THEOREM 3.1. (Formula for P_-) *If $K\geqslant J$ then we have for all $\psi\in\mathcal{H}_J$ and $\phi\in\mathcal{H}_K$*

$$P_-(\psi\otimes\phi)=\mu\,{}_J\langle U_J\psi\|\phi\rangle_K, \tag{12}$$

where $\mu\in\mathbb{C}$ satisfies

$$|\mu|^2=\frac{2(K-J)+1}{2K+1}.$$

Proof. Formula (12) above for $P_-(\psi\otimes\phi)$ is clearly a bilinear map in ψ and ϕ. It is thus enough to prove the above formula for a linear spanning set for ϕ and ψ. Such spanning sets are provided by the coherent states $|\omega\rangle_J$ and $|\omega\rangle_K$ for $\omega\in\mathbb{S}^2$. It is thus enough to prove the formula for $\psi=|\omega'\rangle_J$ and $\phi=|\omega\rangle_K$. For simplicity we will write $U_J|\omega'\rangle_J=|U_J\omega'\rangle_J$, where U_J is the anti-unitary operator defined above.

In the following we let $\mathbf{S}_J$ and $\mathbf{S}_K$, respectively, denote the spin operators on $\mathcal{H}_J$ and $\mathcal{H}_K$, respectively, and we let $\mathbf{S}$ be the total spin operator on $\mathcal{H}_J\otimes\mathcal{H}_K$, i.e.,

$$\mathbf{S}=\mathbf{S}_J\otimes I_K+I_J\otimes\mathbf{S}_K.$$

Since $\omega\cdot\mathbf{S}_K|\omega\rangle_K=K|\omega\rangle_K$, and since $\eta=|\omega'\rangle_J-|U_J\omega\rangle_J\,{}_J\langle U_J\omega|\omega'\rangle_J$ has no component in the subspace $\omega\cdot\mathbf{S}_J=-J$, it is clear that $P_-\eta\otimes|\omega\rangle_K=0$, for otherwise the total $\omega\cdot\mathbf{S}$ component of this vector would be bigger than the maximal possible namely $K-J$. Hence

$$P_-|\omega'\rangle_J\otimes|\omega\rangle_K={}_J\langle U_J\omega|\omega'\rangle_J P_-|U_J\omega\rangle_J\otimes|\omega\rangle_K. \tag{13}$$

We have $\omega\cdot\mathbf{S}|U_J\omega\rangle_J\otimes|\omega\rangle_K=(K-J)|U_J\omega\rangle_J\otimes|\omega\rangle_K$ and thus, since P_- commutes with $\omega\cdot\mathbf{S}$,

$$P_-|U_J\omega\rangle_J\otimes|\omega\rangle_K=\mu'|\omega\rangle_{K-J}=\mu'\,{}_J\langle\omega\|\omega\rangle_K \tag{14}$$

for some complex scalar μ'. Here we have used that ${}_J\langle\omega\|\omega\rangle_K=|\omega\rangle_{K-J}$ in $\mathcal{H}_{K-J}\otimes\mathcal{H}_J$, since $|\omega\rangle_K=|\omega\rangle_{K-J}\otimes|\omega\rangle_J$. Inserting (14) into (13), we obtain

$$P_-|\omega'\rangle_J\otimes|\omega\rangle_K=\mu'\,{}_J\langle U_J\omega|\omega'\rangle_J\,{}_J\langle\omega\|\omega\rangle_K.$$

Since $U_J^2=(-1)^{2J}$ and U_J is anti-unitary, we have $U_J(|\omega'\rangle_J-\eta)=(-1)^{2J}\,{}_J\langle\omega'|U_J\omega\rangle_J|\omega\rangle_J$. Moreover, $U_J\eta$ has no component in the space $\omega\cdot\mathbf{S}_J=J$, hence ${}_J\langle U_J\eta\|\omega\rangle_K=0$ and thus

$$P_-|\omega'\rangle_J\otimes|\omega\rangle_K=\mu'\,{}_J\langle U_J\omega|\omega'\rangle_J\,{}_J\langle\omega\|\omega\rangle_K=\mu\,{}_J\langle U_J\omega'\|\omega\rangle_K,$$

with $\mu=(-1)^{2J}\mu'$, which is what we wanted to prove.

We can find the modulus of μ from the fact that Φ^{-k}, with $k=2(K-J)$, is trace-preserving:

$$\begin{aligned}|\mu|^2&=({}_J\langle U_J\omega|\otimes{}_K\langle\omega|)P_-(|U_J\omega\rangle_J\otimes|\omega\rangle_K)=\operatorname{Tr}_J\operatorname{Tr}_K(P_-|\omega\rangle_K\,{}_K\langle\omega|)\\&=\frac{2(K-J)+1}{2K+1}\operatorname{Tr}_J\Phi^{-k}(|\omega\rangle_K\,{}_K\langle\omega|)=\frac{2(K-J)+1}{2K+1}.\end{aligned}$$ □

If $k=2(K-J)\geqslant 0$, we therefore have

$$ {}_K\langle\phi|\Phi^k(|\psi\rangle_J\,{}_J\langle\psi|)|\phi\rangle_K=\frac{2J+1}{2K+1}\big\|{}_J\langle U_J\psi\|\phi\rangle_K\big\|_{K-J}^2. \tag{15}$$

If we introduce the channel

$$\tilde{\Phi}^k(\varrho) = \Phi^k(U_J \varrho U_J^{-1}), \tag{16}$$

we see that

$$_K\langle\phi|\tilde{\Phi}^k(|\psi\rangle_{J\,J}\langle\psi|)|\phi\rangle_K = \frac{2J+1}{2K+1}\big\|_J\langle\psi\|\phi\rangle_K\big\|^2_{K-J},$$

or equivalently

$$\tilde{\Phi}^k(\varrho) = \frac{2J+1}{2K+1} P_K(I_{K-J}\otimes\varrho)P_K, \tag{17}$$

where P_K is the projection onto $\mathcal{H}_K$ in $\mathcal{H}_{K-J}\otimes\mathcal{H}_J$. In particular, $\tilde{\Phi}^0$ is the identity map.

If $K \leqslant J$ the corresponding result is that

$$P_-\psi\otimes\phi = \mu'\,_K\langle U_K\phi\|\psi\rangle_J \quad \text{and} \quad |\mu'|^2 = \frac{2(J-K)+1}{2J+1}. \tag{18}$$

In particular, in this case,

$$_K\langle\phi|\Phi^{-|k|}(|\psi\rangle_{J\,J}\langle\psi|)|\phi\rangle_K = \big\|_K\langle U_K\phi\|\psi\rangle_J\big\|^2_{J-K}. \tag{19}$$

Hence, if $K \leqslant J$ and we now set $\tilde{\Phi}^{-|k|}(\varrho) = U_K^{-1}\Phi^k(\varrho)U_K$, we obtain

$$_K\langle\phi|\tilde{\Phi}^{-|k|}(|\psi\rangle_{J\,J}\langle\psi|)|\phi\rangle_K = \big\|_K\langle\phi\|\psi\rangle_J\big\|^2_{J-K}.$$

4. The Bosonic formulation

The space $\mathcal{H}_J$ may be identified with the completely symmetric subspace $\bigotimes_{\text{sym}}^{2J}\mathcal{H}_{1/2}$ of the tensor product $\bigotimes^{2J}\mathcal{H}_{1/2}$.

A particularly simple way to see this is to use the Schwinger representation of spin operators in terms of creation and annihilation operators. Let $\mathcal{H}_{1/2}$ be the one-particle space and let $a_\uparrow^*$ and $a_\downarrow^*$ be the creation operators corresponding to spin up and down, respectively. They are the operators which, for all positive integers ℓ, map $\bigotimes_{\text{sym}}^{\ell}\mathcal{H}_{1/2}$ to $\bigotimes_{\text{sym}}^{\ell+1}\mathcal{H}_{1/2}$, such that, for $\psi\in\bigotimes_{\text{sym}}^{\ell}\mathcal{H}_{1/2}$,

$$a_\uparrow^*\psi = \sqrt{\ell+1}\,P_{\text{sym}}(|\uparrow\rangle_{1/2}\otimes\psi),$$

and likewise for $a_\downarrow^*$, where P_{sym} is the projection onto the symmetric space $\bigotimes_{\text{sym}}^{\ell+1}\mathcal{H}_{1/2}$. The annihilation operators $a_\uparrow$ and $a_\downarrow$ are the adjoints of $a_\uparrow^*$ and $a_\downarrow^*$.

The symmetric subspace of $\bigotimes^{2J}\mathcal{H}_{1/2}$ is the subspace corresponding to $2J$ bosonic particles, i.e., the subspace

$$a_\uparrow^* a_\uparrow + a_\downarrow^* a_\downarrow = 2J.$$

On this $(2J+1)$-dimensional subspace we observe that the operators

$$S_x = \frac{1}{2}(a_\uparrow^* a_\downarrow + a_\downarrow^* a_\uparrow), \quad S_y = \frac{1}{2i}(a_\uparrow^* a_\downarrow - a_\downarrow^* a_\uparrow) \quad \text{and} \quad S_z = \frac{1}{2}(a_\uparrow^* a_\uparrow - a_\downarrow^* a_\downarrow)$$

satisfy the correct commutation relations and $S_x^2 + S_y^2 + S_z^2 = J(J+1)$.

The spin representation on $\mathcal{H}_J$ may then be identified with the space of $2J$ bosons over a 2-dimensional 1-particle space. In particular, the coherent state $|\omega\rangle_J \in \mathcal{H}_J$ is in the bosonic language the pure condensate wave function $(2J)!^{-1/2}(a_\omega^*)^{2J}|0\rangle$, where $|0\rangle$ is the vacuum state (i.e., the state of zero particles) and a_ω^* is the creation of a particle in the state $|\omega\rangle_{1/2}$, i.e., $a_\omega^* = {}_{1/2}\langle\uparrow|\omega\rangle_{1/2} a_\uparrow^* + {}_{1/2}\langle\downarrow|\omega\rangle_{1/2} a_\downarrow^*$. We will use the canonical commutation relations, that all creation operators commute and $[a_{\omega'}, a_\omega^*] = {}_{1/2}\langle\omega'|\omega\rangle_{1/2}$. This gives, in particular,

$$a_\uparrow a_\uparrow^* + a_\downarrow a_\downarrow^* = a_\uparrow^* a_\uparrow + a_\downarrow^* a_\downarrow + 2 = 2J+2 \tag{20}$$

on $\mathcal{H}_J$.

The channel $\tilde{\Phi}^k$, defined in (17), has a simple form in terms of creation and annihilation operators.

LEMMA 4.1. (The channel $\tilde{\Phi}^k$ in second quantization) *If ϱ is a density matrix on $\mathcal{H}_J = \bigotimes_{\mathrm{sym}}^{2J} \mathcal{H}_{1/2}$ and $K = J + \frac{1}{2}k$ for some integer $k \geqslant 0$ then*

$$\tilde{\Phi}^k(\varrho) = \frac{(2J+1)!}{(2K+1)!} \sum_{i_1,\ldots,i_k=\uparrow,\downarrow} a_{i_k}^* \ldots a_{i_1}^* \varrho a_{i_1} \ldots a_{i_k}. \tag{21}$$

If $K = J - \frac{1}{2}k$ with $k \geqslant 0$ we have

$$\tilde{\Phi}^{-k}(\varrho) = \frac{(2K)!}{(2J)!} \sum_{i_1,\ldots,i_k=\uparrow,\downarrow} a_{i_k} \ldots a_{i_1} \varrho a_{i_1}^* \ldots a_{i_k}^*. \tag{22}$$

Proof. We use the expression (17) for $\tilde{\Phi}^k$. The space $\mathcal{H}_K = \bigotimes_{\mathrm{sym}}^{2K} \mathcal{H}_{1/2}$ is the totally symmetric subspace of $\big(\bigotimes_{\mathrm{sym}}^{2(K-J)} \mathcal{H}_{1/2}\big) \otimes \big(\bigotimes_{\mathrm{sym}}^{2J} \mathcal{H}_{1/2}\big)$. Thus, by the definition of the creation and annihilation operators,[4]

$$\begin{aligned}
\tilde{\Phi}^k(\varrho) &= \frac{2J+1}{2K+1} \sum_{i_1,\ldots,i_k=\uparrow,\downarrow} P_K |i_k\rangle_{1/2} \otimes \ldots \otimes |i_1\rangle_{1/2} \otimes \varrho \otimes {}_{1/2}\langle i_1| \otimes \ldots \otimes {}_{1/2}\langle i_k| P_K & (23)\\
&= \frac{2J+1}{2K+1} \frac{(2J)!}{(2K)!} \sum_{i_1,\ldots,i_k=\uparrow,\downarrow} a_{i_k}^* \ldots a_{i_1}^* \varrho a_{i_1} \ldots a_{i_k}. & (24)
\end{aligned}$$

The case $K < J$ follows in the same way. □

([4]) The meaning of the operator $|\psi\rangle \otimes \varrho \otimes \langle\psi|$ in (23) is clear if ϱ is a rank-1 projection $|\phi\rangle\langle\phi|$ and it is defined in general by linearity.

Remark 4.2. As explained in the proof, P_K is the projection onto the totally symmetric subspace. This makes it clear that the expression (17) for the channel $\tilde{\Phi}^k$ is identical to equation (4) in [23] for the universal quantum cloner that takes $2J$ identical (pure) qubit states and produces $2K$ approximate clones. Here $2J$ identical pure states mean the vector state corresponding to a tensor product of $2J$ identical $\mathcal{H}_{1/2}$ vectors, i.e., exactly a coherent state. The output state is not a pure tensor product state, it is not even a pure state, but it has the smallest entropy among all output states. This is a consequence of our Corollary 2.5.

Remark 4.3. Note that (21) and (22) are the Kraus representations of the completely positive trace-preserving maps $\tilde{\Phi}^k$.

We shall use the representation (21) to calculate the action of $\tilde{\Phi}^k$ on coherent states.

LEMMA 4.4. (The action of $\tilde{\Phi}^k$ on coherent states) *If* $K=J+\frac{1}{2}k$ *for some integer* $k\geqslant 0$, *then* $\tilde{\Phi}^k(|\uparrow\rangle_{J\,J}\langle\uparrow|)$ *has the orthonormalized eigenfunctions*

$$\phi_j^{C,k}=j!^{-1/2}(2J+k-j)!^{-1/2}(a_\downarrow^*)^j(a_\uparrow^*)^{2J+k-j}|0\rangle,\quad j=0,\dots,2J+k=2K,$$

with corresponding eigenvalues

$$\lambda_j^{C,k}=\frac{2J+1}{2J+k+1}\frac{k!(2J+k-j)!}{(2J+k)!(k-j)!}=\frac{2J+1}{2K+1}\frac{(2(K-J))!(2K-j)!}{(2K)!(2(K-J)-j)!}$$

for $j=0,\dots,k$ *and zero for* $j>k$. *Note that the eigenvalues are listed in decreasing order.*

Proof. This follows immediately from Lemma 4.1, since

$$\tilde{\Phi}^k(|\uparrow\rangle_{J\,J}\langle\uparrow|)=\frac{(2J+1)!}{(2J+k+1)!}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{(2J)!}(a_\downarrow^*)^j(a_\uparrow^*)^{2J+k-j}|0\rangle\langle 0|a_\uparrow^{2J+k-j}a_\downarrow^j. \qquad \square$$

An important ingredient in the proof of our main Theorem 2.4 below is to study the operator

$$\Gamma_m^{C,k+1}=\sum_{j=0}^{m}a_\uparrow|\phi_j^{C,k+1}\rangle\langle\phi_j^{C,k+1}|a_\uparrow^*+a_\downarrow|\phi_j^{C,k+1}\rangle\langle\phi_j^{C,k+1}|a_\downarrow^*$$

for $m\leqslant k$. Using the fact that $a_\downarrow\phi_j^{C,k+1}=\sqrt{j}\,\phi_{j-1}^{C,k}$ for $j\geqslant 1$, $a_\downarrow\phi_0^{C,k+1}=0$ and

$$a_\uparrow\phi_j^{C,k+1}=\sqrt{2J+k+1-j}\,\phi_j^{C,k}\quad\text{for } j=0,\dots,m,$$

we find that

$$\Gamma_m^{C,k+1}=\sum_{j=0}^{m-1}(2J+k+2)|\phi_j^{C,k}\rangle\langle\phi_j^{C,k}|+(2J+k+1-m)|\phi_m^{C,k}\rangle\langle\phi_m^{C,k}|. \tag{25}$$

Remark 4.5. The expression in Lemma 4.1 for the channel $\tilde{\Phi}^k$ may be generalized to define analogous channels between bosonic many-particle spaces where the 1-particle space instead of being 2-dimensional, as $\mathcal{H}_{1/2}$, could be of arbitrary finite dimension. We conjecture that Theorem 2.4 holds also in this case in the sense that pure condensates majorize these channels.

5. Proof of the main theorem for the channels Φ^k

Proof of Theorem 2.4. If $k=2(K-J)\leqslant 0$ then $\Phi^k(|\omega\rangle\langle\omega|)$ has rank 1 and the result is obvious. We now consider the case $k=2(K-J)>0$. We first point out that from (9) it is enough to consider the rank-1 case, i.e., $\varrho=|\psi\rangle\langle\psi|$ for some $\psi\in\mathcal{H}_J$. Since I_{K-J} has rank

$$2(K-J)+1=k+1$$

it is clear from (17) that $\Phi^k(|\psi\rangle\langle\psi|)$ has rank at most $k+1$. It is of course equivalent to consider the channel $\tilde{\Phi}^k$. Let $\lambda_j^k(\psi)$, $j=0,1,...,2J+k$, be the eigenvalues of $\tilde{\Phi}^k(|\psi\rangle\langle\psi|)$ in decreasing order and counted with multiplicity. Let ϕ_j^k, $j=0,1,...,2J+k$, be the corresponding orthonormalized eigenvectors. For all $m\geqslant k$ we have

$$\sum_{j=0}^{m}\lambda_j^k(\psi)=\operatorname{Tr}\,\tilde{\Phi}^k(|\psi\rangle\langle\psi|)=1.$$

The claim is that, moreover,

$$\sum_{j=0}^{m}\lambda_j^k(\psi)\leqslant\sum_{j=0}^{m}\lambda_j^{C,k} \tag{26}$$

for $m=0,...,k-1$, where $\lambda_j^{C,k}$ are the eigenvalues for the coherent states, which were given in Lemma 4.4.

We shall prove (26) by induction on m. For $m=0$, this is easy since we clearly have from (17) that

$$\lambda_0^k(\psi)\leqslant\frac{2J+1}{2K+1}=\lambda_0^{C,k}.$$

Let us now assume that we have proved (26) for all integers up to $m-1$ for some $m\geqslant 1$. We want to prove it for m. We shall do this by induction on k. For $k\leqslant m$, (26) is an equality since both sides are 1. Let us assume, therefore, that we have proved (26) up to some $k\geqslant m$. We want to prove it for $k+1$.

Since $\phi_0^{k+1},\ldots,\phi_m^{k+1}\in\mathcal{H}_{J+(k+1)/2}$ are the orthonormal eigenvectors corresponding to the m top eigenvalues of $\tilde{\Phi}^{k+1}(|\psi\rangle\langle\psi|)$ we have

$$\begin{aligned}\sum_{j=0}^{m}\lambda_j^{k+1}(\psi) &= \frac{(2J+1)!}{(2J+k+2)!}\sum_{j=0}^{m}\sum_{i_1,\ldots,i_{k+1}=\uparrow,\downarrow}\langle\phi_j^{k+1}|a_{i_{k+1}}^*\ldots a_{i_1}^*|\psi\rangle\langle\psi|a_{i_1}\ldots a_{i_{k+1}}|\phi_j^{k+1}\rangle\\ &= \frac{(2J+1)!}{(2J+k+2)!}\sum_{i_1,\ldots,i_k=\uparrow,\downarrow}\mathrm{Tr}(\Gamma a_{i_k}^*\ldots a_{i_1}^*|\psi\rangle\langle\psi|a_{i_1}\ldots a_{i_k})\\ &= \frac{1}{(2J+k+2)}\,\mathrm{Tr}(\Gamma\tilde{\Phi}^k(|\psi\rangle\langle\psi|))\\ &= \frac{1}{(2J+k+2)}\sum_{j=0}^{2J+k}\lambda_j^k(\psi)\langle\phi_j^k|\Gamma|\phi_j^k\rangle,\end{aligned}\tag{27}$$

where

$$\Gamma=\sum_{j=0}^{m}(a_\uparrow|\phi_j^{k+1}\rangle\langle\phi_j^{k+1}|a_\uparrow^*+a_\downarrow|\phi_j^{k+1}\rangle\langle\phi_j^{k+1}|a_\downarrow^*)$$

is an operator on the space $\mathcal{H}_{J+k/2}$. Observe that since ϕ_j^{k+1} are $2J+k+1$ particle states we have

$$\mathrm{Tr}\,\Gamma=\sum_{j=0}^{m}\langle\phi_j^{k+1}|a_\uparrow^*a_\uparrow+a_\downarrow^*a_\downarrow|\phi_j^{k+1}\rangle=(m+1)(2J+k+1).$$

Likewise, we have from (20) with $2J$ replaced by $2J+k$, the operator inequalities

$$0\leqslant\Gamma=a_\uparrow\sum_{j=0}^{m}|\phi_j^{k+1}\rangle\langle\phi_j^{k+1}|a_\uparrow^*+a_\downarrow\sum_{j=0}^{m}|\phi_j^{k+1}\rangle\langle\phi_j^{k+1}|a_\downarrow^*\leqslant a_\uparrow a_\uparrow^*+a_\downarrow a_\downarrow^*=(2J+k+2)I_{\mathcal{H}_{J+k/2}}.$$

To get an upper bound to the expression in (27) we optimize the expression with the restrictions that

$$\langle\phi_j^k|\Gamma|\phi_j^k\rangle\leqslant 2J+K+2\quad\text{and}\quad\sum_{j=0}^{2J+k}\langle\phi_j^k|\Gamma|\phi_j^k\rangle=(m+1)(2J+k+1).$$

The optimizer is easily seen to correspond to the first $m-1$ values of $\langle\phi_j^k|\Gamma|\phi_j^k\rangle$ being maximal and the mth value chosen so as to give the correct trace. This is a special case of the bathtub principle, see [11]. We would therefore get an upper bound to the expression in (27) if Γ is replaced by

$$\sum_{j=0}^{m-1}(2J+k+2)|\phi_j^k\rangle\langle\phi_j^k|+(2J+k+1-m)|\phi_m^k\rangle\langle\phi_m^k|.\tag{28}$$

This gives the bound

$$\begin{aligned}\sum_{j=0}^{m}\lambda_j^{k+1}(\psi)&\leqslant\frac{2J+k+1-m}{2J+k+2}\lambda_m^k(\psi)+\sum_{j=0}^{m-1}\lambda_j^k(\psi)\\&=\frac{2J+k+1-m}{2J+k+2}\sum_{j=0}^{m}\lambda_j^k(\psi)+\frac{m+1}{2J+k+2}\sum_{j=0}^{m-1}\lambda_j^k(\psi).\end{aligned}$$

We conclude from the induction hypotheses on both m and k that

$$\sum_{j=0}^{m}\lambda_j^{k+1}(\psi)\leqslant\frac{2J+k+1-m}{2J+k+2}\sum_{j=0}^{m}\lambda_j^{C,k}+\frac{m+1}{2J+k+2}\sum_{j=0}^{m-1}\lambda_j^{C,k}=\sum_{j=0}^{m}\lambda_j^{C,k+1}(\psi).$$

That the last recursive identity holds for the coherent eigenvalues follows since for coherent states we know from (25) that Γ is, in fact, equal to the optimizing expression (28). This can also be seen from the explicit formulas in Lemma 4.4 (e.g. using induction). The induction is thus complete. □

Note that as a special case we have seen in the proof that, for $k=2(K-J)\geqslant 0$,

$$\|\Phi^k(\varrho)\|\leqslant\frac{2J+1}{2K+1}. \tag{29}$$

6. The classical limit of the channels Φ^k

In this section we consider the limit as the dimension of the output space tends to infinity, i.e., $K\to\infty$.

Lemma 6.1. (The large K limit of coherent state outputs) *Let $f\colon[0,1]\to\mathbb{R}$ be a continuous function. Then*

$$\lim_{K\to\infty}\frac{2J+1}{2K+1}\operatorname{Tr}_K f\left(\frac{2K+1}{2J+1}\Phi^k(|\uparrow\rangle_{J\,J}\langle\uparrow|)\right)=\frac{2J+1}{4\pi}\int_{\mathbb{S}^2}f\left(\left|{}_J\langle\omega|\uparrow\rangle_J\right|^2\right)d\omega. \tag{30}$$

Proof. We may of course replace the channel Φ^k by $\tilde{\Phi}^k$. This a simple calculation based on the explicit expressions in Lemma 4.4. If θ denotes the polar angle of ω, i.e., $\cos\theta$ is the z-component of ω, then $\left|{}_J\langle\omega|\uparrow\rangle_J\right|^2=\cos^{4J}\left(\frac{1}{2}\theta\right)$. The integral over the sphere may hence be rewritten as

$$\frac{2J+1}{4\pi}\int_{\mathbb{S}^2}f\left(\left|{}_J\langle\omega|\uparrow\rangle_J\right|^2\right)d\omega=(2J+1)\int_0^1 f(t^{2J})\,dt.$$

On the other hand, the explicit eigenvalues in Lemma 4.4 give

$$\frac{2J+1}{2K+1}\operatorname{Tr}_K f\left(\frac{2K+1}{2J+1}\Phi^k(|\uparrow\rangle_{J\,J}\langle\uparrow|)\right)=\frac{2J+1}{2K+1}\sum_{j=0}^{2(K-J)} f\left(\left(\frac{2K+1}{2J+1}\right)\lambda_j^C\right)$$
$$=\frac{2J+1}{2K+1}\sum_{j=0}^{2(K-J)} f\left(\frac{(2(K-J))!(2K-j)!}{(2K)!(2(K-J)-j)!}\right).$$

It is an easy exercise which we leave to the reader to show that this converges to the above integral in the limit as $K\to\infty$. □

It can be proved that the same limiting equality (30) holds even if $|\uparrow\rangle_{J\,J}\langle\uparrow|$ is replaced by any density matrix ϱ on $\mathcal{H}_J$, at least for a large class of functions f. We will not do this here. Instead, we shall restrict ourselves to an inequality similar to (30) for concave functions f. To do this, we shall use the Berezin–Lieb inequality (4).

LEMMA 6.2. (The classical integral dominate the trace of concave functions of Φ) *Let $f\colon[0,1]\to\mathbb{R}$ be a concave function. Then for any density matrix ϱ on $\mathcal{H}_J$ we have, for all integers $k=2(K-J)\geqslant 0$,*

$$\frac{2J+1}{2K+1}\operatorname{Tr}_K f\left(\frac{2K+1}{2J+1}\Phi^k(\varrho)\right)\leqslant\frac{2J+1}{4\pi}\int_{\mathbb{S}^2} f({}_J\langle\omega|\,\varrho\,|\omega\rangle_J)\,d\omega.$$

Proof. Again we consider the equivalent channel $\tilde\Phi^k$. The result follows from the Berezin–Lieb inequality (4) if we can show that the lower symbol of $\tilde\Phi^k(\varrho)$ satisfies

$${}_K\langle\omega|\,\tilde\Phi^k(\varrho)\,|\omega\rangle_K=\frac{2J+1}{2K+1}\,{}_J\langle\omega|\,\varrho\,|\omega\rangle_J.$$

This is straightforward from (17). □

We are now in a position to prove the main result Theorem 2.1. In fact it is the analog of the main Theorem 2.4, or rather the equivalent formulation Corollary 2.5 for the classical map Φ^∞ from density matrices on $\mathcal{H}_J$ to functions on (the classical phase space) $\mathbb{S}^2$. For classical functions the trace is replaced with the integral over phase space.

Proof of Theorem 2.1. From Corollary 2.5 and Lemma 6.2 we have, for all integers $k=2(K-J)\geqslant 0$,

$$\frac{2J+1}{2K+1}\operatorname{Tr}_K f\left(\frac{2K+1}{2J+1}\Phi^k(|\uparrow\rangle_{J\,J}\langle\uparrow|)\right)\leqslant\frac{2J+1}{2K+1}\operatorname{Tr}_K f\left(\frac{2K+1}{2J+1}\Phi^k(\varrho)\right)$$
$$\leqslant\frac{2J+1}{4\pi}\int_{\mathbb{S}^2} f({}_J\langle\omega|\,\varrho\,|\omega\rangle_J)\,d\omega.$$

The result now follows from Lemma 6.1. □

Appendix A. Proof of the generalized Wehrl conjecture (Theorem 2.2)

It is enough to prove Theorem 2.2 for $n=1$. The general case follows by induction as follows. Assume that we have proved it for $n-1$. Then for each $(p',q')\in\mathbb{R}^{n-1}$ we define an operator $\tilde{\varrho}_{p',q'}$ on $L^2(\mathbb{R})$ by

$$\langle\phi|\,\tilde{\varrho}_{p',q'}|\psi\rangle=\langle\phi|\otimes\langle p',q'|\,\varrho\,|p',q'\rangle\otimes|\psi\rangle.$$

Then

$$\varrho_{p',q'}=(\operatorname{Tr}_{L^2(\mathbb{R})}\tilde{\varrho}_{p',q'})^{-1}\tilde{\varrho}_{p',q'}$$

is a density matrix on $L^2(\mathbb{R})$ and we get from the inequality for $n=1$ that

$$(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} f(\varrho^{\mathrm{cl}}(p,q))\,dp\,dq\geqslant(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} f\big(\operatorname{Tr}_{L^2(\mathbb{R})}\tilde{\varrho}_{p',q'}\big|\langle 0,0|p_n,q_n\rangle\big|^2\big)\,dp\,dq.$$

We have, however, that $\operatorname{Tr}_{L^2(\mathbb{R})}\tilde{\varrho}_{p',q'}=\langle p',q'|\operatorname{Tr}_n\varrho|p',q'\rangle$, where $\operatorname{Tr}_n\varrho$ is the density matrix on $L^2(\mathbb{R}^{n-1})$ obtained by taking the partial trace on the nth variable. Thus, from the induction hypothesis, we find that

$$\begin{aligned}(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} f(\varrho^{\mathrm{cl}}(p,q))\,dp\,dq&\geqslant(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} f\big(\big|\langle 0,0|p',q'\rangle\big|^2\big|\langle 0,0|p_n,q_n\rangle\big|^2\big)\,dp\,dq\\&=(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} f\big(\big|\langle 0,0|p,q\rangle\big|^2\big)\,dp\,dq.\end{aligned}$$

It remains to prove (6) in the case $n=1$. We first observe that, by possibly replacing $f(t)$ by $f(t)+at$, we may assume that f is non-negative. Moreover, using the monotone convergence theorem, we may assume that f is piecewise linear. In this case we have the inequality

$$|f(x)-f(y)|\leqslant C|x-y|$$

for some $C>0$ and all $x,y\in[0,1]$. Hence for all density matrices ϱ_1 and ϱ_2 we have

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n}|f(\varrho_1^{\mathrm{cl}}(p,q))-f(\varrho_2^{\mathrm{cl}}(p,q))|\,dp\,dq&\leqslant C\int_{\mathbb{R}^n\times\mathbb{R}^n}|\varrho_1^{\mathrm{cl}}(p,q)-\varrho_2^{\mathrm{cl}}(p,q)|\,dp\,dq\\&\leqslant C\int_{\mathbb{R}^n\times\mathbb{R}^n}\langle p,q|\,|\varrho_1-\varrho_2|\,|p,q\rangle\,dp\,dq\\&=C\|\varrho_1-\varrho_2\|_1,\end{aligned}$$

where the the norm on the right is the trace norm. Hence it is enough to prove (6) for a subset of density matrices that is dense in trace norm.

Let $|n\rangle$ denote the eigenfunctions of the harmonic oscillator

$$\left(-\frac{d^2}{dq^2}+q^2\right)|n\rangle=(2n+1)|n\rangle.$$

We will prove (6) for the dense family of ϱ satisfying the property that there exists a positive integer N such that

$$\langle n|\,\varrho\,|m\rangle = 0 \quad \text{if } n>N \text{ or } m>N. \tag{31}$$

We shall apply the convenient complex notation, where

$$z=2^{-1/2}(q+ip) \quad \text{and} \quad \bar z = 2^{-1/2}(q-ip).$$

The Glauber coherent states may then be written as

$$|p,q\rangle = |z\rangle = \sum_{n=0}^{\infty} e^{-|z|^2/2}\frac{z^n}{\sqrt{n!}}|n\rangle.$$

We identify the subspace $\mathcal{H}_J$ with $\operatorname{span}\{|0\rangle,...,|2J\rangle\}$ in $L^2(\mathbb{R})$ in such a way that $|M\rangle_J = |M+J\rangle$ for $M=-J,...,J$. Moreover, we also identify the 2-sphere with the complex plane through stereographic projection, such that the measure is $\bigl(1+\frac14|z|^2\bigr)^{-2}\,d^2z$. With these identifications we can conveniently write the Bloch coherent states (see [9]) as

$$|z\rangle_J = \sum_{n=0}^{2J}\binom{2J}{n}^{1/2}\left(1+\frac{|z|^2}{4}\right)^{-J}\left(\frac{\bar z}{2}\right)^n|n\rangle.$$

It is now straightforward to see that if ϱ satisfies assumption (31) then

$$\langle z|\,\varrho\,|z\rangle = \lim_{J\to\infty}\,{}_J\!\left\langle\left(\frac{2}{J}\right)^{1/2}\bar z\,\middle|\,\varrho\,\middle|\left(\frac{2}{J}\right)^{1/2}\bar z\right\rangle_J.$$

Using the fact that $(1+|z|^2/2J)^{-2J}\leqslant(1+|z|^2/2K)^{-2K}$ for all $z\in\mathbb{C}$ and all $J\geqslant K$, we easily see that, for $J\geqslant N+2$,

$${}_J\!\left\langle\left(\frac{2}{J}\right)^{1/2}\bar z\,\middle|\,\varrho\,\middle|\left(\frac{2}{J}\right)^{1/2}\bar z\right\rangle_J \leqslant C_{\varrho,N}\left(1+\frac{|z|^2}{2(N+2)}\right)^{-2}$$

if ϱ satisfies (31). Since f is non-negative and bounded above by $t\mapsto at$ for some $a>0$ we immediately find from dominated convergence that

$$\begin{aligned}\int_{\mathbb{C}} f(\langle z|\,\varrho\,|z\rangle)\,d^2z &= \lim_{J\to\infty}\int_{\mathbb{C}} f\left({}_J\!\left\langle\left(\frac{2}{J}\right)^{1/2}\bar z\,\middle|\,\varrho\,\middle|\left(\frac{2}{J}\right)^{1/2}\bar z\right\rangle_J\right)\left(1+\frac{|z|^2}{2J}\right)^{-2}d^2z\\ &= \lim_{J\to\infty}\frac{J}{2}\int_{\mathbb{C}} f({}_J\langle z|\,\varrho\,|z\rangle_J)\left(1+\frac{|z|^2}{4}\right)^{-2}d^2z.\end{aligned} \tag{32}$$

Our main Theorem 2.1 and the observation that $|0\rangle = |0\rangle_J$ implies that

$$\int_{\mathbb{C}} f({}_J\langle z|\,\varrho\,|z\rangle_J)\left(1+\frac{|z|^2}{4}\right)^{-2}d^2z \geqslant \int_{\mathbb{C}} f\bigl(\bigl|{}_J\langle z|0\rangle\bigr|^2\bigr)\left(1+\frac{|z|^2}{4}\right)^{-2}d^2z.$$

Since the density matrix $|0\rangle\langle 0|$ clearly satisfies (31), we see from our main result and (32) that (6) holds for all ϱ satisfying (31) and hence by approximation for all density matrices on $L^2(\mathbb{R})$.

Note added in proof. The method used in this paper does not transport easily to SU(N). However, we have recently been able to make such an extension [14].

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