Acta Math., 209 (2012), 323–395
DOI: 10.1007/s11511-012-0088-0

Non-realizability and ending laminations: Proof of the density conjecture

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1. Introduction

Throughout this paper, by a *Kleinian group* we mean a discrete and torsion-free subgroup of $\mathrm{PSL}_2(\mathbb{C})$ which is not virtually abelian. By definition, a Kleinian group is *geometrically finite* if its action on hyperbolic 3-space has a fundamental domain with finitely many sides. Equivalently, the Kleinian group Γ is geometrically finite if it is finitely generated and if the convex core $\mathrm{CC}(\mathbb{H}^3/\Gamma)$ of the associated hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$ has finite volume. Recall that the convex core is the quotient under Γ of the convex hull of the limit set of Γ.

It is well known that not all finitely generated Kleinian groups are geometrically finite. In fact, Greenberg [Gr] proved that certain Kleinian groups, shown to exist by Bers [Ber] and Bers–Maskit [BM], are not geometrically finite; in [Jø], Jørgensen gave concrete examples of such groups. The original examples of Greenberg are, by construction, algebraic limits of sequences of quasi-Fuchsian groups. Bers himself asked if Kleinian groups isomorphic to a surface group are always obtained by such a limiting process. This question was later modified by Sullivan and Thurston to cover all finitely generated Kleinian groups.

Density conjecture. *Every finitely generated Kleinian group is an algebraic limit of geometrically finite groups.*

The goal of this paper is to give a complete proof of this conjecture.

We fix from now on a finitely generated Kleinian group Γ. Following Thurston, let $\mathrm{AH}(\Gamma)$ be the set of all conjugacy classes of discrete and faithful representations

The first author was partially supported by the NSF grants DMS-0604111 and DMS-0852418. The second author was partially supported by the NSF grant DMS-0706878 and the Alfred P. Sloan Foundation

$\varrho\colon\Gamma\to\mathrm{PSL}_2(\mathbb{C})$ with $\varrho(\gamma)$ parabolic if $\gamma\in\Gamma$ is parabolic. A faithful and discrete representation $\varrho\in\mathrm{AH}(\Gamma)$ is said to be *geometrically finite* if the Kleinian group $\varrho(\Gamma)$ is; observe that any representation conjugated to a geometrically finite one is also geometrically finite. A sequence $\{[\varrho_i]\}_{i=1}^{\infty}$ in $\mathrm{AH}(\Gamma)$ converges *algebraically* to $[\varrho]\in\mathrm{AH}(\Gamma)$ if there are representatives $\varrho_i\in[\varrho_i]$ and $\varrho\in[\varrho]$ such that for all $\gamma\in\Gamma$ the sequence $\{\varrho_i(\gamma)\}_{i=1}^{\infty}$ converges to $\varrho(\gamma)$ in $\mathrm{PSL}_2(\mathbb{C})$. Abusing notation, we will not distinguish between discrete and faithful representations and the associated points in $\mathrm{AH}(\Gamma)$.

THEOREM 1.1. (Density conjecture) *If Γ is a finitely generated Kleinian group, then the set of geometrically finite points in* $\mathrm{AH}(\Gamma)$ *is dense in the algebraic topology. In other words, the density conjecture holds.*

Continuing with the same notation, let $\mathcal{X}(\Gamma,\mathrm{PSL}_2(\mathbb{C}))$ be the character variety of the group Γ. The relative character variety $\mathcal{X}_{\mathrm{rel}}(\Gamma,\mathrm{PSL}_2(\mathbb{C}))$ is the set of those characters corresponding to representations $\varrho\colon\Gamma\to\mathrm{PSL}_2(\mathbb{C})$ with $\mathrm{Tr}(\varrho(\gamma))^2=4$ for every γ which is contained in a non-cyclic abelian subgroup. Recall that $\mathrm{AH}(\Gamma)$ is a subset of the set of smooth points of the relative character variety $\mathcal{X}_{\mathrm{rel}}(\Gamma,\mathrm{PSL}_2(\mathbb{C}))$ of Γ (see for instance [Ka]). The algebraic topology on $\mathrm{AH}(\Gamma)$ is induced by the analytic topology of this variety. It is due to Sullivan [Su2] that the interior of $\mathrm{AH}(\Gamma)$, as a subset of $\mathcal{X}_{\mathrm{rel}}(\Gamma,\mathrm{PSL}_2(\mathbb{C}))$, consists of conjugacy classes of geometrically finite representations. Conversely, every geometrically finite point in $\mathrm{AH}(\Gamma)$ belongs to the closure of the interior of $\mathrm{AH}(\Gamma)$ by the work of Maskit [Mask] and Ohshika [Oh1]. We deduce hence from Theorem 1.1 the following result.

COROLLARY 1.2. *If Γ is a finitely generated Kleinian group, then* $\mathrm{AH}(\Gamma)$ *is the closure of its interior.*

Suppose from now on that Γ is a finitely generated Kleinian group, fix $\varrho\in\mathrm{AH}(\Gamma)$ and let $N_\varrho=\mathbb{H}^3/\varrho(\Gamma)$ be the associated oriented hyperbolic 3-manifold. It follows from the Margulis lemma that for every ε positive and smaller than a certain universal constant, the *Margulis constant*, every unbounded connected component of the set of points in N_ϱ where the injectivity radius is less than ε is homeomorphic either to $S^1\times\mathbb{R}\times(0,\infty)$, or to $S^1\times S^1\times(0,\infty)$. In addition, each such component, which we call an *ε-cusp* of N_ϱ, is a quotient of the interior of a horoball in $\mathbb{H}^3$ by a rank-1 or rank-2 abelian parabolic subgroup of $\varrho(\Gamma)$. It is due to Sullivan [Su1] that the number of ε-cusps is finite. Let N_ϱ^ε be the complement in N_ϱ of all the ε-cusps. It follows from the proof of the tameness theorem by Agol [Ag] and Calegari–Gabai [CG] that the manifold N_ϱ^ε admits a *standard compact core*. By this we mean a compact submanifold $M\subset N_\varrho^\varepsilon$ whose complement is homeomorphic to a product and such that the inclusion of $P=M\cap\partial N_\varrho^\varepsilon$ into $\partial N_\varrho^\varepsilon$ is a homotopy equivalence; the pair (M,P) is a *pared manifold*.

We denote by $\mathrm{AH}(M,P)$ the subset of $\mathrm{AH}(\Gamma)$ consisting of conjugacy classes of discrete and faithful representations of $\Gamma=\pi_1(M)$ into $\mathrm{PSL}_2(\mathbb{C})$ with the property that those elements whose conjugacy classes are represented by loops on P are mapped to parabolic elements. By construction, ϱ is a *minimally parabolic* element of $\mathrm{AH}(M,P)$, i.e. the image of an element is parabolic if and only if its conjugacy class is represented by a loop in P. In order to prove Theorem 1.1, we will show that that ϱ is an algebraic limit of geometrically finite, minimally parabolic, points in $\mathrm{AH}(M,P)$.

Each component of $\partial M\setminus P$ is called a *free side* of the pared manifold; ends of the manifold N_ϱ^ε are in one-to-one correspondence with free sides. Suppose that $\mathcal{E}$ is the end associated with the free side F. We say that $\mathcal{E}$ is *convex cocompact* if it has a neighborhood whose intersection with the convex core of N_ϱ is compact. The end invariant of the convex cocompact $\mathcal{E}$ is the point in the Teichmüller space of the free side F determined by the conformal structure at infinity. The geometry of convex cocompact ends is well understood by the work of Ahlfors, Bers, Kra, Marden, Maskit, Sullivan and others; we will refer to this as the theory of quasi-conformal deformations of Kleinian groups.

If the end $\mathcal{E}$ is not convex cocompact, then it is said to be *degenerate*. Every degenerate end has an associated *ending lamination* [Ca3]. In other words, the geometry of $\mathcal{E}$ determines a filling geodesic lamination on the free side F. The end invariant of $\mathcal{E}$ is, by definition, its ending lamination.

Canary [Ca3] proved that the end invariants of the ends of N_ϱ^ε satisfy the following two, rather mild, conditions:

(*) If M is an interval bundle over a compact (possibly unorientable) surface S and N_ϱ has no convex cocompact ends, then the projection of the ending laminations to S has transverse self-intersection.

(**) If a compressible component F of ∂M faces a degenerate end $\mathcal{E}$, then the ending lamination is the support of a Masur domain lamination. Equivalently, the support of the ending lamination is not contained in the Hausdorff limit of any sequence of meridians.

Note that by a *meridian* on a free side F, we mean a simple non-contractible loop on F which is homotopically trivial in M.

A collection of end invariants for (M,P), i.e. points in Teichmüller space for some free sides and filling laminations for others, is said to be *filling* if it satisfies the two preceding conditions (*) and (**). A spin-off of our proof of the density conjecture is that any filling collection of end invariants is in fact the set of end invariants of a hyperbolic 3-manifold.

THEOREM 1.3. *Let (M,P) be a pared 3-manifold. Given a filling collection of end invariants for (M,P), there exists a minimally parabolic representation $\varrho\in\mathrm{AH}(M,P)$ and an embedding $(M,P)\to(N_\varrho^\varepsilon,\partial N_\varrho^\varepsilon)$ in the homotopy class determined by ϱ, whose* *image is a standard compact core of N_ϱ and such that the end invariants of N_ϱ with respect to this standard compact core are the given end invariants in the beginning. Here $N_\varrho = \mathbb{H}^3/\varrho(\pi_1(M))$ is the hyperbolic 3-manifold determined by the representation ϱ.*

The ending lamination theorem, proved by Minsky [Mi2] and Brock–Canary–Minsky [BCM], asserts that the manifold N_ϱ is determined up to isometry by the topological type of a standard compact core together with the associated end invariants. As a result, the hyperbolic 3-manifold N_ϱ provided by Theorem 1.3 is unique up to isometry.

Using the same notation as above, assume that F is a free side of (M,P) and λ is a geodesic lamination on F. We say that λ is *realized* in N_ϱ if there exist a finite-area complete hyperbolic metric σ on F and a proper map $f\colon (F,\sigma)\to N$ which is totally geodesic on leaves of λ and which, on the level of fundamental groups, induces the same map as the composition of the embeddings $F\hookrightarrow M\to N_\varrho$. An important property of the ending lamination for an end of N_ϱ^ε is that it is not realized [Ca3]. Our proof of the density conjecture basically amounts to proving that in some sense this property identifies the ending lamination. This is in fact the most important contribution of this work to the proof of the density conjecture.

Theorem 1.4. *Let (M,P) be a pared manifold and $\varrho\in \mathrm{AH}(M,P)$. Let $(M',P')\subset (N_\varrho^\varepsilon, \partial N_\varrho^\varepsilon)$ be a relative compact core of the hyperbolic 3-manifold $N_\varrho = \mathbb{H}^3/\varrho(\pi_1(M))$ and $\phi\colon (M,P)\to(M',P')$ be in the homotopy class determined by ϱ. Suppose that λ is a filling Masur domain lamination on a free side F of (M,P) which is not realized in N_ϱ. Then ϕ is homotopic, relative to the complement of a regular neighborhood of F, to a map $\phi_1\colon (M,P)\to(M',P')$ such that*

- *the restriction of ϕ_1 to F is a homeomorphism to some free side F' of (M',P');*
- *the end of N_ϱ associated with F' is degenerate and has ending lamination $\phi_1(\lambda)$.*

To explain the relevance of Theorem 1.4, suppose that Γ is a finitely generated Kleinian group as above, $\varrho\in\mathrm{AH}(\Gamma)$ and (M,P) is a standard compact core for N_ϱ. One can use the quasi-conformal deformation theory and choose a sequence of geometrically finite elements of $\mathrm{AH}(M,P)$ with standard compact core (M,P) and so that the end invariants for this sequence "converge" to the end invariant of N_ϱ on (M,P). Convergence results for representations in $\mathrm{PSL}_2(\mathbb{C})$, that start from Thurston's double limit theorem and are extended and reproved by what is known as Morgan–Shalen theory, can be applied to show that a subsequence of this sequence converges to a minimally parabolic element ϱ' of $\mathrm{AH}(M,P)$. To prove the conjecture it is enough to show that ϱ and ϱ' are conjugate, i.e. represent the same point of $\mathrm{AH}(\Gamma)$. This follows form the ending lamination theorem if there is an embedding of (M,P) in $(N_{\varrho'}^\varepsilon, \partial N_{\varrho'}^\varepsilon)$ as a standard compact core in the homotopy class determined by ϱ' and the end invariants of N_ϱ' and

N_ϱ on (M,P) are the same. Previously known results guarantee that for every convex cocompact end of N_ϱ, there is a convex cocompact end of N'_ϱ with the same conformal structure at infinity, and also that the ending laminations of degenerate ends of N_ϱ are not realized in N'_ϱ. The above theorem is needed to show that, for every degenerate end of N_ϱ, there is a homeomorphic degenerate end of N'_ϱ with the same ending lamination. Once we know this, results from classical 3-dimensional topology can be used to finish the proof. The fact about the degenerate ends of $N_{\varrho'}$ and the above theorem is what was overlooked previously and is the original part of this article. Non-realizability of an ending lamination of N_ϱ in $N_{\varrho'}$ can be used to produce a sequence of hyperbolic surfaces and path-length preserving maps into $N_{\varrho'}$ whose images exit an end. If one knew that the maps from these surfaces to a neighborhood of this end was a homotopy equivalence, then we could prove the required claim for that end. In fact in many cases and in particular when the free sides of (M,P) are incompressible, this follows from elementary topological arguments and therefore the proof of the conjecture in those cases does not depend on our new results here. In other cases however, one has to rule out the possibility that the images of these surfaces are "twisted" in some non-trivial way. This is essentially the main objective of this paper.

Who has proved the density conjecture?

This paper concludes the proof of the density conjecture, but it would have been completely unconceivable without the proof of the tameness theorem by Agol and Calegari–Gabai and the proof of the ending lamination theorem by Brock–Canary–Minsky. It goes without saying that the more classical results of Ahlfors, Bers, Bonahon, Marden, Sullivan, Thurston and others are also basic. In some cases, the needed compactness theorems for sequences of representations can be obtained without making reference to actions on trees; however, in the general case, it seems that there is no way to make do without using the Morgan–Shalen machinery. In this paper, the needed compactness result relies directly on the work of Otal and Kleineidam–Souto but we could have chosen to use the more sophisticated theorem due to Kim–Lecuire–Ohshika. In fact, Ohshika has given alternative proofs to many of the results in this paper. This list is far from being complete. However, returning to the question preceding this list, we think that the appropriate answer is that the two *Gastarbeiter* who proved the last lemma had something to do with the proof, but that the proof is certainly not reduced to this last lemma.

Alternative approach

Before moving on we should mention that there is a different approach to the proof of the density conjecture. In [Bro] Bromberg proved Bers's original conjecture (for groups without parabolic elements) using the deformation theory of cone-manifolds. Later, Brock and Bromberg [BB] extended this result to prove that every finitely generated, freely indecomposable Kleinian group without parabolic elements is an algebraic limit of geometrically finite Kleinian groups. In their work, Brock and Bromberg avoid using most of the ending lamination theorem: they only need Minsky's original result [Mi1].

Bromberg and Souto have announced a complete proof of the density conjecture using the deformation theory of hyperbolic cone-manifolds and which does not contain any reference to the ending lamination theorem. Other ingredients of the proof given in this paper, for example the tameness theorem, are still absolutely crucial. In fact, in their work, Bromberg and Souto also need some simple consequences of the main new result of this paper, Theorem 1.4, such as the lack of unexpected parabolic elements.

Plan of the paper

After giving an outline of the proof of Theorem 1.1 in the case where Γ has no parabolics in §2, we recall in §3 some facts and definitions on pared manifolds; the only result of which we give a complete proof, Theorem 3.12, is an extension of a well-known result of Walshausen to the pared setting. In §4 we collect a few facts on hyperbolic 3-manifolds and on the basic deformation theory of Kleinian groups. In §5 we discuss laminations, measured laminations, currents and train-tracks. Laminations appear in the two subsequent sections as well: in §6 in the context of pleated surfaces and hyperbolic 3-manifolds and in §7 in the context of small actions of groups on trees. As the reader can see, §§3–7 of this paper are devoted in one form or another to recalling known facts; our own contributions are very minor.

In §8 we prove Theorem 8.1 ensuring that certain algebraic limits exist, have the expected conformal boundaries and where the ending laminations-to-be are not realized. In §9 we reduce the proof of Theorem 1.1 to proving Theorem 1.4. We also reduce to the case when the pared manifold in question is a compression body. The next two sections are devoted to the proof of Theorem 1.4 in the case of compression bodies. At this point, we will have proved all the results announced in the introduction.

In a final section, we add a few remarks and observations on other related results that we can prove using the same strategy.

Acknowledgements

The authors have profited from conversations and discussions with many people during the extended period of preparing and writing this article. We are grateful for these conversations which have helped and encouraged us in finalizing this project. The same is true for the many institutions which have supported and housed us one time or another. We want however to thank explicitly Ken Bromberg, Dick Canary and Yair Minsky; we owe much to them.

2. Outline of the proof of Theorem 1.1

Let Γ be a Kleinian group and $\varrho\in\mathrm{AH}(\Gamma)$. If the associated hyperbolic 3-manifold $N_\varrho=\mathbb{H}^3/\varrho(\Gamma)$ has finite volume, then it follows from the Mostow–Prasad rigidity theorem that $\mathrm{AH}(\Gamma)$ consists of two points, namely $\{\varrho,\bar{\varrho}\}$, where the hyperbolic manifold $N_{\bar{\varrho}}$ is isometric to N_ϱ via an orientation-reversing isometry. In particular both ϱ and $\bar{\varrho}$ are geometrically finite and we have nothing to prove. From now on we assume $\mathrm{vol}(N_\varrho)=\infty$.

It is a well-known feature of the deformation theory of Kleinian groups that most results which are true in the absence of parabolic elements are also true, at least in some form, in the general case. In fact, the proofs are often the same; however, the presence of parabolic elements causes additional technical difficulties. For the sake of readability, we will suppose until the end of this section that the Kleinian group $\varrho(\Gamma)$ has no parabolic elements. Using the notation introduced in the introduction, this means that $N_\varrho^\varepsilon=N_\varrho$ and that the pared locus P of the standard relative compact core (M,P) is empty.

Recall that ends of N_ϱ are in one-to-one correspondence with boundary components of M, the standard compact core of N_ϱ. In order to find a suitable sequence $\{\varrho_i\}_{i=1}^\infty$, we start choosing a convex cocompact representation $\varrho_0\in\mathrm{AH}(\Gamma)$ with N_{ϱ_0} homeomorphic to the interior of M. The existence of such a representation ϱ_0 is a consequence of Thurston's hyperbolization theorem. It follows from the quasi-conformal deformation theory of Kleinian groups that the connected component $\mathrm{QH}(\varrho_0)$ of the interior of $\mathrm{AH}(\Gamma)$ containing ϱ_0 is parameterized by the Teichmüller space $\mathcal{T}(\partial M)$ of ∂M (cf. Theorem 4.3); this can be seen as a special case of the ending lamination theorem. Endowing all the laminations in the list of end invariants of N_ϱ with a projective transverse measure, we can consider the tuple of end invariants as a point in the space of projective measured laminations on ∂M. Let $\{\varrho_i\}_{i=1}^\infty$ be any sequence in $\mathrm{QH}(\varrho_0)$ obtained by taking the image under the parametrization described above of a sequence in $\mathcal{T}(\partial M)$ which converges to the end invariants of N_ϱ; by construction ϱ_i is quasi-conformally conjugated to ϱ_0 and hence is convex cocompact for each i.

We claim that this sequence has a convergent subsequence. That this is the case is the first statement of Theorem 8.1 below. Before going any further, we add a few words on the proof of Theorem 8.1. By the work of Morgan and Shalen [MS], in order to show that $\{\varrho_i\}_{i=1}^{\infty}$ has a convergent subsequence, it suffices to show that certain small actions on real trees do not exist. We will achieve this combining Bestvina and Feighn's [BF] relative version of the Rips machine with previously known non-existence results for groups isomorphic to free products of surface groups and free groups [Sk2], [Ot3], [KS1]. Theorem 8.1 is also due to Ohshika who, in joint work with Kim and Lecuire [KLO], obtained a much more sophisticated compactness theorem.

Continuing with the sketch of the proof of Theorem 1.1 and keeping the same notation, let ϱ' be an accumulation point in $\mathrm{AH}(\Gamma)$ of the sequence $\{\varrho_i\}_{i=1}^{\infty}$. Recall that, by the ending lamination theorem, we need to prove that $N_{\varrho'}$ has the same topological type and ending invariants as N_ϱ. It is relatively easy to see that $N_{\varrho'}$ has the correct conformal boundary and that the ending laminations of N_ϱ are not realized in $N_{\varrho'}$ by any pleated surface; this is the content of the second and third statements of Theorem 8.1. We prove next that these non-realized laminations are indeed ending laminations of $N_{\varrho'}$. In particular, this shows that if a boundary component F of ∂M supports an ending lamination, then a standard compact core of $N_{\varrho'}$ has a boundary component homeomorphic to F, the associated end is geometrically infinite and has the same ending lamination as the end of N_ϱ associated with F.

In many cases, this fact can be easily deduced from earlier work. For instance, the boundary incompressible case is due to Thurston. The case where Γ is the free product of two surface groups is due to Otal, which unfortunately was never published. More generally, the case where M is not homeomorphic to a handlebody follows from [KS2]. In all these cases one can either use incompressibility or the fact that the second homology group of a certain cover is non-trivial. In the remaining case, if M is a handlebody, none of these are available. The following is a particular case of the needed result when M is a handlebody.

Theorem 2.1. *Let N be a hyperbolic manifold homeomorphic to the interior of a handlebody H of genus greater than* 1 *and suppose that $\lambda\subset\partial H$ is a filling Masur domain lamination which is not realized in N. Then there is a homeomorphism $\phi\colon H\to H$ homotopic to the identity such that $\phi(\lambda)$ is the ending lamination of N. In particular, N does not have cusps.*

The above theorem is a particular case of the more general Theorem 1.4. We prove the latter using an argument which is in spirit close to Bonahon's [Bo2] proof that incompressible degenerate ends of hyperbolic 3-manifolds are tame and have an ending lamination. Theorem 1.4 is the main novel result of this paper.

Continuing with the discussion above, it follows from Theorem 1.4 and the previously collected facts that for every end of N_ϱ, the algebraic limit $N_{\varrho'}$ has a homeomorphic end with the same end invariant. Since N_ϱ and $N_{\varrho'}$ are homotopy equivalent, it follows from a well-known generalization of a classical result of Waldhausen that N_ϱ and $N_{\varrho'}$ are homeomorphic. At this point we will have proved that the original manifold N_ϱ and the algebraic limit $N_{\varrho'}$ are homeomorphic and have the same end invariants. By the ending lamination theorem, the representatives ϱ and ϱ' are conjugated. Hence ϱ is an algebraic limit of geometrically finite points in $\mathrm{AH}(\Gamma)$. This concludes the outline of the proof of Theorem 1.1.

3. Pared manifolds

In this section we recall a few facts and definitions on pared manifolds. Most of the material is well known and we would humbly suggest the reader to skip this section in a first reading. The only result of which we give a rather complete proof is Theorem 3.12, which essentially gives sufficient conditions for a homotopy equivalence between pared manifolds which maps boundary-to-boundary, to be homotopic, through maps which map boundary-to-boundary, to a homeomorphism. This extends to the pared setting a well-known theorem due to Waldhausen.

Pared manifolds are special types of 3-manifolds with boundary patterns. See Johannson [Jo] and Canary–McCullough [CM] for a complete discussion of 3-manifolds with boundary patterns. All the results discussed in this section are well known for manifolds without boundary patterns; the proofs in the pared setting can be either found in the aforementioned references or are only minimal modifications of the proofs in the traditional setting. See [H] and [Ja] for basic facts on 3-manifolds.

3.1. Pared manifolds

Let M be a compact, oriented, irreducible and atoroidal 3-manifold with non-empty boundary. Assume that M is neither a 3-ball nor a solid torus and let $P \subset \partial M$ be a compact subsurface. We say that (M, P) is a *pared* 3-manifold (see Morgan [Mo]) if the following three conditions hold:

- every component of P is an incompressible torus or annulus;
- every non-cyclic abelian subgroup of $\pi_1(M)$ is conjugated into the fundamental group of a component of P;

- every map $c: (S^1 \times I, S^1 \times \partial I) \to (M, P)$ which induces an injection on fundamental groups is homotopic, as a map of pairs, to a map whose image is contained in P.

If (M, P) is a pared manifold, then the components of $\partial M \setminus P$ are the *free sides* of (M, P). In order to avoid unnecessary details, we will refer to the closure of a free side as a free side as well. Accordingly, we will also denote the interior of P by P.

We say that a pared manifold (N, Q) is a *pared submanifold* of a pared manifold (M, P) if $N \subset M$, $Q \subset P$ and every component of Q is essential in the corresponding component of P. The pared submanifold (N, Q) is *properly embedded* if the interior of each one of its free sides is either contained in or disjoint of the union of the free sides of (M, P). Two pared submanifolds which are isotopic through pared submanifolds are *pared isotopic*.

3.2. JSJ-splitting

Let (M, P) be a pared 3-manifold and F be the union of its free sides. By an *essential disk* in (M, P) we mean an inclusion $(D, \partial D) \hookrightarrow (M, F)$ which is not homotopic to a map whose image is contained in F. A *meridian* is a simple closed curve which bounds an essential disk. It follows from Dehn's lemma that a simple closed curve is a meridian if and only if it is homotopically trivial in M. Also, by the loop theorem, a free side F of (M, P) contains a meridian if and only if the homomorphism $\pi_1(F) \to \pi_1(M)$ is not injective. A free side which contains a meridian is said to be *compressible*; a pared manifold without compressible free sides is said to have *incompressible boundary.*

We say that a pared manifold (M, P) is an *interval bundle* if there is a compact surface F such that M is the total space of a bundle

$$I = [0, 1] \longrightarrow M \longrightarrow F$$

and P is the preimage of ∂F in M. Observe that, because of our restriction to orientable manifolds, every topological surface is the base of a single I-bundle with orientable total space. The bundle is trivial if F is orientable and twisted otherwise. Before going further, we recall Waldhausen's characterization of trivial interval bundles.

Cobordism theorem. (Waldhausen) *Let (M, P) be a pared manifold which has two distinct, incompressible free sides F_1 and F_2 which are properly homotopic to each other. Then (M, P) is an interval bundle with orientable base.*

Before moving on, we would also like to mention the following useful, and well known, fact.

Lemma 3.1. *Suppose that* $\phi:(M,P)\to(M',P')$ *is a, necessarily finite, (pared) cover between orientable pared manifolds. If* (M,P) *is an interval bundle, then so is* (M',P').

An *essential annulus* in (M,P) is an embedding $(A,\partial A)\hookrightarrow(M,\partial M\setminus P)$ which is not properly homotopic to a map whose image is contained in either a free side of (M,P) or a component of P. By the annulus theorem, any two disjoint homotopically essential simple curves in $\partial M\setminus P$ which are freely homotopic in M bound an embedded annulus. A pared manifold (M,P) with incompressible boundary which does not contain any essential annuli is said to be *acylindrical*.

The JSJ-splitting, which we briefly describe now, is a canonical decomposition of a pared manifold with incompressible boundary along essential annuli into acylindrical pared manifolds, interval bundles and solid tori.

Theorem 3.2. (JSJ-splitting) *Let* (M,P) *be a pared manifold with incompressible boundary. Then there is a collection* $\mathcal{A}$ *of disjoint properly embedded annuli in* (M,P) *such that the following holds*:

(1) *If* U *is a connected component of the manifold obtained by cutting* M *along* $\mathcal{A}$, *then*

- *either* U *is a solid torus and* $(P\cup\mathcal{A})\cap U$ *is a collection of parallel non-meridional annuli in* ∂U, *or*
- $(U,(P\cup\mathcal{A})\cap U)$ *is an interval bundle*, *or*
- $(U,(P\cup\mathcal{A})\cap U)$ *is an acylindrical pared manifold.*

(2) *Any essential annulus and any properly embedded Möbius band in* (M,P) *can be properly isotoped into one of the components of* $M\setminus\mathcal{A}$.

Moreover, if $\mathcal{A}$ *is chosen to be minimal with respect to these properties, then* $\mathcal{A}$ *is unique up to pared isotopy.*

We refer to the decomposition of (M,P) given by Theorem 3.2 as the JSJ-splitting of (M,P). A free side F of (M,P) is *small* if it does not intersect any of the annuli in the collection $\mathcal{A}$ provided by Theorem 3.2; naturally, a free side which is not small is *large*. The following observation will play a role below when we ensure that certain sequences of representations have a convergent subsequence.

Lemma 3.3. *Let* (M,P) *be a pared manifold with incompressible boundary and suppose that it is not an interval bundle. Then every small free side* F *is either contained in one of the acylindrical pieces of the JSJ-splitting of* (M,P) *or can be homotoped, but not properly homotoped, within* M *to a subsurface of a large free side.*

3.3. Pared compression bodies

A pared manifold (C,P) is a *pared compression body* if there is a free side $\partial_e(C,P)$ such that the homomorphism

$$\pi_1(\partial_e(C,P)) \longrightarrow \pi_1(M)$$

is surjective. The free side $\partial_e(C,P)$ is the *distinguished free side* of (C,P) and we often denote it simply by $\partial_e C$; the remaining free sides, i.e. the connected components of $C\setminus(P\cup\partial_e(C,P))$, are the *constituents* of (C,P).

A pared compression body is *trivial* if $\partial_e(C,P)$ is incompressible; equivalently, the inclusion of $\partial_e(C,P)$ into C is a homotopy equivalence. If (C,P) is a trivial pared compression body, then there is a compact orientable surface F and an annular subsurface $A\subset F$ such that (C,P) and $(F\times I,(\partial F\times I)\cup(A\times\{0\}))$ are pared compression bodies homeomorphic to each other. Observe that interval bundles with orientable base are examples of trivial pared compression bodies. On the other hand, twisted interval bundles over a closed non-orientable surface are not compression bodies.

Assume now that (C,P) is a non-trivial pared compression body with distinguished free side $\partial_e(C,P)$. Then there is a properly embedded disk $(D,\partial D)\subset(C,\partial_e(C,P))$. Let C' be the manifold obtained from C by cutting along D and observe that $P\subset\partial C'$ and $\partial_e(C,P)\setminus\partial D\subset\partial C'$. Let F' be the subsurface of $\partial C'$ obtained by gluing the two copies of D to $\partial_e(C,P)\setminus\partial D$. It is easy to see that each component of (C',P) is either a solid torus, which possibly contains an annular component of P, or a pared compression body whose distinguished free side is a connected component of F'. Observe also that F' has larger Euler characteristic than $\partial_e(C,P)$. In particular, after repeating this process finitely many times, we obtain that (C,P) can be cut open along disks into solid tori with or without marked essential primitive annuli and trivial pared compression bodies. In other words, we have the following: *A pared manifold (C,P) is a pared compression body if and only if it is obtained from a finite collection of trivial pared compression bodies and solid tori, each possibly containing a marked essential primitive annulus, by attaching finitely many 1-handles to the boundaries of the tori and to the distinguished free sides of the trivial pared compression bodies.*

Note that any π_1-injective surface $(F,\partial F)\subset(C,P)=C$ in a compression body can be made disjoint of any given finite set of properly embedded disks $(D,\partial D)\subset(C,\partial_e C)$. In particular, if (C,P) has no constituents, there is no such surface. More generally, we have the following result.

LEMMA 3.4. *Let (C,P) be a pared compression body and $(F,\partial F)\to(C,P)$ be a proper π_1-injective immersion of the surface F with the property that the image of no non-peripheral homotopically essential simple closed curve in F can be homotoped within*

C to a curve in P. The immersion $(F,\partial F)\to(C,P)$ is properly homotopic to a cover of a constituent of (C,P).

In this article we are mostly concerned with compression bodies without constituents. Our characterization above tells us that such a compression body is constructed from a number of solid tori, each possibly containing a marked essential primitive annulus, and attaching finitely many 1-handles to the complements of the marked annuli. The underlying manifold for any such pared compression body is a handlebody and the annuli in the pared locus represent conjugacy classes of elements in a subset of a generating set for the fundamental group. In particular the homeomorphism type of the pared compression body is determined by the genus of the handlebody and the number of components of the pared locus. As a simple consequence we have the following lemma.

LEMMA 3.5. *Let (C_1,P_1) and (C_2,P_2) be pared compression bodies. Suppose that (C_1,P_1) has no constituents, and let $f:(C_1,P_1)\to(C_2,P_2)$ be a π_1-isomorphism such that every component of P_2 contains the image of a component of P_1. Then f is homotopic to a homeomorphism through maps of pared manifolds.*

Proof. Let $g: C_2\to C_1$ be the homotopy inverse of the map f, i.e. $f\circ g$ and $g\circ f$ are homotopic to the identity. Because of the assumption that every component of P_2 contains the f-image of a component of P_1, we may assume that $g:(C_2,P_2)\to(C_1,P_1)$ is a pared map. If (C_2,P_2) had a constituent then the restriction of g to this constituent would contradict the conclusion of Lemma 3.4. Hence (C_2,P_2) is also a pared compression body without constituents and f induces a bijection between the components of P_1 and P_2. Now our discussion above shows that f is homotopic to a homeomorphism through maps of pared manifolds. □

3.4. Relative compression bodies

Let F be a compressible free side of a pared manifold (M,P). Following Bonahon [Bo1] and Canary–McCullough [CM], we now define the *relative compression body neighborhood* of F to be any properly embedded pared submanifold (C,Q) of (M,P) with $F\subset C$ satisfying the following conditions:

- (C,Q) is a pared compression body with distinguished free side F;
- each constituent F_i of (C,Q) is incompressible in M;
- if a constituent F_i of (C,Q) is properly isotopic in (M,P) to a free side F' of (M,P), then $F_i=F'$;
- no non-peripheral homotopically essential simple closed curve in the constituents of (C,Q) can be freely homotoped into P within M.

Our definition is slightly different from that in [CM], since we only needed to deal with pared manifolds. But the next proposition follows from the same arguments as in [CM].

PROPOSITION 3.6. *Let* (M,P) *be a compact orientable irreducible pared* 3*-manifold and* F *be a compressible free side. Then* F *has a relative compression body neighborhood* (C,Q) *and any two such neighborhoods are isotopic through properly embedded pared submanifolds of* (M,P). *Moreover, the relative compression body neighborhoods of different compressible free sides of* M *can be isotoped to be disjoint.*

Sketch of the proof of the existence part. Fix a maximal collection of disjoint, non-parallel properly embedded disks in (M,P) with boundary on F. Let C_1 be a regular neighborhood of the union of F with these disks, and C_2 be the union of C_1 and those balls in M which are bounded by spheres contained in ∂C_1. Let Q_2 be the intersection $\partial C_2 \cap P$ and observe that every component of $\partial C_2 \setminus P$ distinct from F is incompressible in M. Consider now a maximal collection of disjoint, non-parallel properly embedded annuli $(A,\partial_1 A,\partial_2 A) \subset (M\setminus C_2, \partial(M\setminus C_2)\setminus P, \partial(M\setminus C_2)\cap P)$ and let C_3 be a regular neighborhood of the union of C_2 with these annuli; set $Q_3 = \partial C_3 \cap P$. If some component Z of $\partial C_3 \setminus Q_3$ is an annulus, then there is $Z' \subset P$ with $\partial Z = \partial Z'$. In particular, $Z \cup Z'$ bounds a solid torus; let C_4 be the union of C_3 and all so obtained solid tori. As above, set $Q_4 = C_4 \cap P$. If some constituent F' of (C_4, Q_4) is properly isotopic in (M,P) to a free side $\overline{F}'$ of (M,P), then there is a pared trivial interval bundle in (M,P) homeomorphic to $(F' \times [0,1], \partial F' \times [0,1])$ with free sides F' and $\overline{F}'$. Let C be the union of C_4 and all these trivial interval bundles and $Q = C \cap P$; (C,P) is the desired relative compression body neighborhood of F. $\square$

Let (M,P) be a pared manifold and $F_1, ..., F_k$ be the collection of its compressible free sides. By the last claim of Proposition 3.6, we may assume that the relative pared compression body neighborhoods (C_i, Q_i) in M of the sides F_i are disjoint. Their complement $\left(M \setminus \bigcup_{i=1}^k C_i, P \setminus \bigcup_{i=1}^k Q_i\right)$ is a (possibly disconnected) pared manifold with incompressible boundary. Following [CM], we refer to $\left(M \setminus \bigcup_{i=1}^k C_i, P \setminus \bigcup_{i=1}^k Q_i\right)$ as the *incompressible core* of (M,P).

3.5. Homeomorphisms and homotopy equivalences between pared manifolds

In this section we discuss the relation between homotopy equivalences and homeomorphisms of pared manifolds. Let (M,P) and (M',P') be pared manifolds. We say that a map of pairs $\phi: (M,P) \to (M',P')$ is a *homotopy equivalence* if $\phi_*: \pi_1(M) \to \pi_1(M')$ is an isomorphism; in other words, f is a homotopy equivalence of the underlying manifold. The following important observation follows directly from the definition of pared manifolds.

LEMMA 3.7. *Let* $\phi: (M, P) \to (M', P')$ *be a homotopy equivalence between pared manifolds and let* $\overline{P}'$ *be the union of those components of* P' *which contain the image of a component of* P. *Then the restriction* $\phi|_P$ *of* ϕ *to* P *is a homotopy equivalence onto* $\overline{P}'$.

A homotopy equivalence $\phi: (M, P) \to (M', P')$ is *type preserving* if every closed curve on a free side of (M, P) whose image under ϕ can be homotoped into P', can be itself homotoped into P. Finally, we say that ϕ *maps boundary-to-boundary* if the image under ϕ of every free side of (M, P) is contained in some free side of (M', P'). Observe that this implies that if F and F' are free sides of (M, P) and (M', P'), respectively, with $f(F) \subset F'$, then the map

$$f: F \longrightarrow F'$$

is proper and hence has a well-defined degree $\deg(f_F)$. Here we have endowed ∂M and $\partial M'$ with the induced orientations.

Before going any further, we give some examples showing that in general homotopy equivalences are not properly homotopic to homeomorphisms. It is helpful to keep these examples in mind when reading this paper.

Example 3.8. Let F_1, F_2, F_3 and F_4 be compact orientable surfaces with pairwise different genera, each one with a single boundary component, and let X be the 2-complex obtained by identifying the boundary of F_i with $\mathbb{S}^1$ via a homeomorphism for $i=1, ..., 4$. Then the complex X is homotopy equivalent to three different manifolds M_1, M_2 and M_3 with incompressible boundary. In particular, there are homotopy equivalences

$$(M_1, \varnothing) \longrightarrow (M_2, \varnothing)$$

which are not homotopic to any homeomorphism. This is Canary's *oil drum* example.

Example 3.9. Let F be a closed orientable surface and $M = F \times I$. The map

$$\phi: (M, \varnothing) \longrightarrow (M, \varnothing)$$

given by $\phi(x, t) = (x, 0)$ is a type-preserving homotopy equivalence which maps boundary-to-boundary but which is not homotopic to a homeomorphism via maps which map boundary-to-boundary.

Example 3.10. Let F be a compact orientable surface with boundary and $M = F \times I$; observe that M is a handlebody. The map $\phi: (M, \varnothing) \to (M, \varnothing)$ given by $\phi(x, t) = (x, 0)$ is a type-preserving homotopy equivalence which maps boundary-to-boundary but whose restriction $\partial M \to \partial M$ to the only boundary component is not even π_1-injective.

After these examples recall the following positive result which is a generalization of a theorem by Waldhausen to the case of ∂-reducible 3-manifolds. (Cf. Tucker [Tu] and Jaco [Ja] for discussion and proof.)

THEOREM 3.11. *Assume that* $f:(M,\partial M)\to(M',\partial M')$ *is a homotopy equivalence between compact manifolds* M *and* M', *where* M' *is a Haken manifold and* f *induces a homeomorphism* $\partial M\to\partial M'$. *Then* f *is homotopic to a homeomorphism with a homotopy that remains constant on* ∂M.

Minimally extending Theorem 3.11, we describe the possible π_1-injective pared maps between pared manifolds. Essentially the outcome is that, as long as we consider maps which map boundary-to-boundary, any such map is homotopic to a covering or is one of the Examples 3.9 and 3.10.

THEOREM 3.12. *Suppose that* (M,P) *and* (M',P') *are pared manifolds and that* $f:(M,P)\to(M',P')$ *maps boundary-to-boundary and induces an injective map on the level of fundamental groups. Then there is a homotopy, through maps* $(M,P)\to(M',P')$ *which map boundary-to-boundary, from* f *to a map* g *such that one of the following mutually exclusive alternatives is satisfied*:

(1) f *maps a meridian in* (M,P) *to a homotopically trivial curve in* $\partial M'\setminus P'$. *In this case* (M,P) *is a non-trivial pared compression body without constituents and one has* $g(M)\subset\partial M'$. *Moreover*, $\deg(f|_{\partial_e(M,P)})=0$.

(2) f *maps two distinct free sides* F_1 *and* F_2 *of* (M,P) *to the same free side* F *of* (M',P') *in such a way that* $\deg(f|_{F_1})\geqslant 0$ *and* $\deg(f|_{F_2})\leqslant 0$. *In this case* (M,P) *is a trivial interval bundle and* $g(M)\subset\partial M'$.

(3) *Neither* (1) *nor* (2) *are satisfied and* g *is a covering map of finite degree.*

In the course of the proof of Theorem 3.12 we will make use several times of the following observation.

LEMMA 3.13. *Let* $(S,\partial S)$ *and* $(S',\partial S')$ *be compact surfaces of negative Euler characteristic. If* $f:(S,\partial S)\to(S',\partial S')$ *is a proper map which does not map any essential simple closed curve to a homotopically trivial one*, *then* f *is homotopic via a map of pairs to a map* $g:(S,\partial S)\to(S',\partial S')$ *satisfying one of the following*:

- *either* g *is a branched cover*, *or*
- *the image of* g *is contained in* $\partial S'$.

If the surfaces S and S' in the statement of Lemma 3.13 are closed, then the claim follows directly from the first (easy) part of the proof of the simple loop theorem [Ga]. We assume that the proof in the closed case can be modified to the general case. We prefer however to give an analytic argument using harmonic maps; observe that this argument also applies in the closed case.

Proof. Assuming that the second alternative in Lemma 3.13 does not hold, we claim that f is homotopic to a branched cover. To begin with observe that the assumptions in the lemma imply that f is not homotopic to either a constant map nor a map whose image is a closed essential curve in S'. Endow (the interior of) S' with a fixed complete hyperbolic metric ϱ_0 with finite area. The assumption that f is not homotopic to a map whose image is contained in $\partial S'$ implies [Co] that for every finite-type conformal structure σ on (the interior of) S the map f is properly homotopic to a harmonic map

$$f_\sigma: (S, \sigma) \longrightarrow (S', \varrho_0).$$

Denote by $E(f_\sigma, \sigma)$ the energy of f_σ with respect to σ and ϱ_0.

The assumption that f does not map any essential simple loop to a homotopically trivial curve implies [SU] that there is some conformal structure σ_0 on S with

$$E(f_{\sigma_0}, \sigma_0) \leqslant E(f_\sigma, \sigma)$$

for every other choice of σ. This implies that the map f_{σ_0} is conformal with respect to the Riemann-surface structure induced by ϱ_0 on S' [SU]. Since conformal maps between surfaces are branched covers and since f is properly homotopic to $g{=}f_{\sigma_0}$, the claim follows. □

Before launching the proof of Theorem 3.12, we state concretely the incarnation of Lemma 3.13 needed below. Observe that it follows from Dehn's lemma and Lemma 3.7 that, under the assumptions of Theorem 3.12, any essential simple closed curve in $\partial M \setminus P$ whose image under f is homotopically trivial in $\partial M' \setminus P'$ is in fact a meridian. In particular, *assuming that no meridian in (M, P) is mapped to a homotopically trivial curve in $\partial M' \setminus P'$ implies that the restriction of f to $\partial M \setminus P$ is homotopic to a branched cover.* We start now the proof of Theorem 3.12.

Proof of Theorem 3.12. We start proving that if (1) and (2) are not satisfied then we are in case (3).

Claim 1. *Suppose that*

- *f does not map any meridian in (M, P) to a homotopically trivial curve in $\partial M' \setminus P'$, and that*
- *there are no two distinct free sides F_1 and F_2 of (M, P) which are mapped to the same free side F of (M', P') in such a way that $\deg(f|_{F_1}) \geqslant 0$ and $\deg(f|_{F_2}) \leqslant 0$.*

Then f is homotopic, through maps $(M,P)\to(M',P')$ which map boundary-to-boundary, to a finite covering $g\colon(M,P)\to(M',P')$.

Proof of Claim 1. Let F and F' be free sides of (M,P) and (M',P'), respectively, with $f(F)\subset F'$. As remarked above, the assumption that f does not map any meridian in F to a homotopically trivial curve in F' implies that the restriction of f to F is homotopic to a branched cover. From now on we assume that the restriction of f to any free side is a branched cover. Observe at this point that this implies that the restriction of f to any free side F of (M,P) is either orientation preserving or reversing. The second assumption in the claim implies that any two free sides of (M,P) which are mapped to the same free side of (M',P'), are mapped with the same orientation.

Let $\widetilde{M}'$ be the cover of M' corresponding to the image of $\pi_1(M)$ under the homomorphism induced by f, and denote by $\widetilde{P}'\subset\widetilde{M}'$ the preimage of P'. The map f lifts to a homotopy equivalence $\tilde{f}\colon M\to\widetilde{M}'$; we claim that it is homotopic to a homeomorphism. We first prove that $\tilde{f}$ is onto. Recall that, by Lemma 3.7, the restriction of $\tilde{f}$ to P is homotopic to a homeomorphism onto its image in $\widetilde{P}$. In particular, we can homotope $\tilde{f}$, through maps which map boundary-to-boundary, to a map whose restriction to P is a homeomorphism onto its image and whose restriction to every free side is a branched cover; we assume that $\tilde{f}$ had this property to begin with. Since no two free sides are mapped to the same free side with different orientations, it follows that the restriction of $\tilde{f}$ to ∂M is altogether a branched cover onto its image. As the degree of the restriction of $\tilde{f}$ to any two boundary components of ∂M which are mapped to the same component of $\partial M'$ has the same sign, it follows that the image of ∂M is a non-trivial 2-cycle in $H_2(\partial\widetilde{M}')$. This implies that the induced map $H_3(M,\partial M)\to H_3(\widetilde{M}',\partial\widetilde{M}')$ is an injective homomorphism, and hence shows that $\tilde{f}$ is onto.

Since M is compact, we deduce that $\widetilde{M}'$ is compact as well; hence the cover $\widetilde{M}'\to M'$ is finite. It remains to prove that $\tilde{f}$ is homotopic to a homeomorphism. In the light of Theorem 3.11, it suffices to prove that the restriction of f to the boundary of M is a homeomorphism to the boundary of $\widetilde{M}'$. In order to see that this is the case, we observe that

$$\chi(M)=\frac{\chi(\partial M)}{2}=\frac{1}{2}\sum_{S\subset\partial M}\chi(S)\leqslant\frac{1}{2}\sum_{S\subset\partial\widetilde{M}'}\chi(\tilde{f}(S))=\frac{\chi(\partial\widetilde{M}')}{2}=\chi(\widetilde{M}'),$$

where the inequality holds because the restriction of f to every component of ∂M is a branched cover and because f is surjective. In particular, equality holds if and only if the restriction of f to every component S of ∂M is a homeomorphism. Since M and $\widetilde{M}'$ are homotopy equivalent $\chi(M)=\chi(\widetilde{M}')$, and hence equality must hold. We have proved that the restriction of $\tilde{f}$ to ∂M is a homeomorphism. As mentioned above, it follows from

Theorem 3.11 that $\tilde{f}$ is homotopic to a homeomorphism, proving that Theorem 3.12 (3) holds. This concludes the proof of Claim 1. □

We suppose now that we are not in case (1) but that the assumption of case (2) is satisfied.

Claim 2. *Suppose that f does not map any meridian in (M,P) to a homotopically trivial curve in $\partial M'\setminus P'$, and that there are two distinct free sides F_1 and F_2 of (M,P) whose images are contained in the same free side F' of (M',P') and such that $\deg(f|_{F_1})\geqslant 0$ and $\deg(f|_{F_2})\leqslant 0$. Then (M,P) is a trivial interval bundle and f is homotopic, through maps $(M,P)\to(M',P')$ which map boundary-to-boundary, to a map g with $g(M)\subset F'$.*

In order to prove Claim 2, we will need the following observation that, lacking a better name, we state as a lemma.

Lemma 3.14. *Let (M,P) be a pared manifold and F be a free side of (M,P). If for every simple loop γ on F there is a non-zero multiple γ^m which is freely homotopic into another free side of (M,P), then F is incompressible.*

Proof of Claim 2. As in the proof of Claim 1, let $(\widetilde{M}',\widetilde{P}')$ be the cover of (M',P') corresponding to the image of $\pi_1(M)$ under the homomorphism induced by f and denote a lift of f by $\tilde{f}$. Again as in the proof of Claim 1, we may assume that the restriction of f to any free side of (M,P) is a branched cover onto a free side of (M',P').

A priori, it could be that there are no two free sides F_1 and F_2 of (M,P) which are mapped under $\tilde{f}$ to the same component of $\partial\widetilde{M}'\setminus\widetilde{P}'$ with degrees of distinct sign. Suppose for a moment that we are in this situation. Then, the argument used in the proof of Claim 1 shows that the image of ∂M is a non-trivial 2-cycle in $H_2(\partial\widetilde{M}')$, that the induced map $H_3(M,\partial M)\to H_3(\widetilde{M}',\partial\widetilde{M}')$ is an injective homomorphism, that $\tilde{f}$ is onto and that $\tilde{f}$ is homotopic to a homeomorphism. Hence, f was to begin with homotopic to a covering, but this contradicts the assumption that the different free sides are mapped to the same free side with degrees of distinct sign.

It follows that $\tilde{f}$ maps two free sides, which we may assume to be F_1 and F_2, of (M,P) to the same free side F' of (M',P') in such a way that $\deg(f|_{F_1})\geqslant 0$ and $\deg(f|_{F_2})\leqslant 0$. Since the restriction of $\tilde{f}$ to F_i is a branched cover, it follows that the images of $\pi_1(F_1)$ and $\pi_1(F_2)$ have finite index in $\pi_1(F')$. Since $\tilde{f}$ is an isomorphism on π_1, it follows that $\pi_1(F_1)$ and $\pi_1(F_2)$ in $\pi_1(M)$ have finite-index subgroups which are conjugate within $\pi_1(M)$. Lemma 3.14 shows that F_1 and F_2 are incompressible. Passing to a finite sheeted cover $(\widetilde{M},\widetilde{P})$ of (M,P), the free sides F_1 and F_2 lift to incompressible boundary components $\widetilde{F}_1$ and $\widetilde{F}_2$ which are homotopic in the cover. Waldhausen's cobordism theorem shows that $(\widetilde{M}, \widetilde{P})$ has to be a trivial interval bundle. By Lemma 3.1, (M, P) is also an interval bundle, which is trivial because it has two distinct boundary components. We have proved the first part of the claim.

Consider now the commuting diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{\tilde{f}|_{F_1}} & F' \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{\tilde{f}} & \widetilde{M}', \end{array}$$

where the vertical arrows are inclusions. The inclusion of F_1 into M and the map

$$\tilde{f}: M \longrightarrow \widetilde{M}'$$

are both homotopy equivalences. It follows that the restriction $\tilde{f}|_{F_1}$ is π_1-injective. Being a π_1-injective branched cover, it follows that $\tilde{f}|_{F_1}$ is a homeomorphism and hence a homotopy equivalence. This proves that the inclusion of F' into $\widetilde{M}'$ is a homotopy equivalence. Hence, F' is a strong deformation retract of $\widetilde{M}'$. This implies that $\tilde{f}$ is homotopic, through maps mapping $\partial M \setminus P = F_1 \cup F_2$ to $\widetilde{F}'$, to a map whose image is contained in F'. This homotopy descends to the desired homotopy of f. This concludes the proof of Claim 2. □

At this point it remains to prove that whenever a meridian α in a free side F of (M, P) is mapped to a homotopically trivial curve in a free side F' of (M', P'), then (M, P) is a non-trivial pared compression body without constituents and f is homotopic to a map g with $g(M) \subset \partial M'$. In order to see that this is the case we will argue by induction on $\chi(\partial M)$. Observe that whenever $\chi(\partial M) = 0$ then $\partial M = P$. Hence ∂M is incompressible and therefore the base case of the induction is trivially satisfied.

Suppose now that F and F' are free sides of (M, P) and (M', P'), respectively, with $f(F) \subset F'$, and that f maps a meridian $\alpha \subset F$ to a homotopically trivial curve in F'. By Dehn's lemma, there is a properly embedded disk $(D, \partial D) \subset (M, F)$ with $\partial D = \alpha$. Up to homotopy constant on ∂M, we may assume that $f(D) \subset F'$. Let (M_1, P_1) be the pair obtained by cutting (M, P) along the disk D. Every component (N, Q) of (M_1, P_1) is either a solid torus, with at most a single primitive annulus from P_1, or a pared manifold. In the latter case, observe that the map f induces a π_1-injective map

$$f_1: (N, Q) \longrightarrow (M', P')$$

and that $\chi(\partial N) > \chi(\partial M)$. Arguing by induction, we may assume that the components of (M_1, P_1) are either solid tori or satisfy Theorem 3.12. Then the following claim is immediate.

CLAIM 3. *Suppose that every component* (N,Q) *of* (M_1,P_1) *which is not a solid torus satisfies one of the outcomes* (1) *and* (2) *of Theorem* 3.12. *Then*

$$f:(M,P)\longrightarrow(M',P')$$

is homotopic, *through maps which map boundary-to-boundary*, *to a map*

$$g:(M,P)\longrightarrow(M',P')$$

with $g(M)\subset\partial M'$.

Now, we are ready to rule out outcomes (2) and (3) for every pared manifold component (N,Q) of (M_1,P_1) which is not a solid torus. The map f_1 cannot be homotopic, through maps mapping boundary-to-boundary, to a finite cover $g_1:(N,Q)\rightarrow(M',P')$ because then the image of $\pi_1(N)$ would have finite index in $\pi_1(M')$; but $\pi_1(N)$ has infinite index in $\pi_1(M)$ and hence the map f could not have been π_1-injective. This proves that f_1 does not satisfy the conclusion of (3) in the statement of Theorem 3.12. Suppose now that (2) holds for f_1. In other words, there is a compact oriented surface F_1 with $(N,Q)=(F_1\times I,\partial F_1\times I)$ and the map f_1 induces an injective homomorphism from $\pi_1(F_1)$ to $\pi_1(M')$, and therefore to $\pi_1(F')$. This implies that the proper map $f_1|_{F_1}:F_1\rightarrow F'$ is homotopic to a covering and therefore the image of $\pi_1(N)$ has finite index in $\pi_1(F')$. But $\pi_1(N)$ has infinite index in $\pi_1(M)$. By Claim 3, f is homotopic to the map g whose image is contained in F', and we have again a contradiction with the π_1-injectivity of f.

We have proved that Theorem 3.12 (1) holds for every component (N,Q) of (M_1,P_1) which is not a solid torus. In particular, any such (N,Q) is a non-trivial pared compression body without constituents. It follows that (M,P) itself is a non-trivial pared compression body without constituents. The claim that f is homotopic, via maps mapping boundary-to-boundary, to a map g with $g(M)\subset\partial M'$ follows directly from Claim 3.

This concludes the proof of Theorem 3.12. □

We state here the following consequence of Theorem 3.12.

LEMMA 3.15. *Let* $f:(C,P)\rightarrow(C',P')$ *be a* π_1*-injective map between pared compression bodies* (C,P) *and* (C',P') *which takes* ∂_eC *to* ∂_eC'. *Also assume that* (C,P) *is non-trivial and that the* f*-image of no non-peripheral loop in a constituent of* (C,P) *is homotopic into a component of* P'. *Then* f *is homotopic*, *through maps* $(C,P)\rightarrow(C',P')$ *which map* ∂_eC *to* ∂_eC', *to a map* g *such that either*

(a) g *is a covering of finite degree*, *or*

(b) $g(C)\subset\partial_eC'$, (C,P) *has no constituents and* f *maps a meridian of* (C,P) *to a homotopically trivial curve on* ∂_eC'.

Proof. In the light of Theorem 3.12, it suffices to show that f is homotopic, through maps $(C,P)\to(C',P')$ which map $\partial_e C$ to $\partial_e C'$, to a map which maps boundary-to-boundary. Clearly, it suffices to prove that if F is any constituent of (C,P), then the restriction $f|_F$ of f to F is properly homotopic to a map into a constituent of (C',P'). That this is the case follows immediately from Lemma 3.4. □

3.6. Mapping class group

Recall that the *mapping class group* $\mathrm{Mod}(M,P)$ of a pared manifold (M,P) is the group of all pared isotopy classes of pared self-homeomorphisms of (M,P). We denote by $\mathrm{Mod}_0(M,P)$ (resp. $\mathrm{Mod}_0^+(M,P)$) the subgroup consisting of those mapping classes represented by elements which are pared homotopic to the identity (resp. pared homotopic to the identity and orientation preserving). At this point we would like to observe that if the free sides of (M,P) are incompressible, then $\mathrm{Mod}_0^+(M,P)$ is trivial. Under the same assumption, $\mathrm{Mod}_0(M,P)$ has at most order 2 and this happens only when (M,P) is an interval bundle. On the other hand, both $\mathrm{Mod}_0(M,P)$ and $\mathrm{Mod}_0^+(M,P)$ are infinite groups if (M,P) has a compressible free side.

4. Hyperbolic manifolds

Throughout this section let N be an oriented hyperbolic 3-manifold with finitely generated non-abelian fundamental group. In other words, N is a complete Riemannian manifold isometric to the quotient $\mathbb{H}^3/\Gamma$, where Γ is a discrete and torsion-free finitely generated subgroup of $\mathrm{PSL}_2(\mathbb{C})=\mathrm{Isom}_+(\mathbb{H}^3)$. We will be exclusively interested in those hyperbolic 3-manifolds which have infinite volume.

See [MT], [BP], [Ka] and [Mar] for basic facts on hyperbolic 3-manifolds and Kleinian groups.

4.1. Thick-thin decomposition

For $x\in N$, let $\mathrm{inj}_N(x)$ be the injectivity radius of N in x, i.e. half of the length of the shortest homotopically essential loop in N based at x. It follows from the thick-thin decompositon theorem that for every positive ε smaller than the 3-dimensional Margulis constant, the closure of every component U of the subset

$$N^{<\varepsilon}=\{x\in N:\mathrm{inj}_N(x)<\varepsilon\}$$

has one of the following forms:

(a) U is a regular neighborhood of a short closed geodesic;

(b) U is isometric to the quotient of a horoball under a rank-1 parabolic subgroup; in particular U is homeomorphic to $[0,\infty)\times\mathbb{S}^1\times\mathbb{R}$;

(c) U is isometric to the quotient of a horoball under a rank-2 parabolic subgroup; in particular U is homeomorphic to $\mathbb{S}^1\times\mathbb{S}^1\times[0,\infty)$.

The components of type (a) are called the *Margulis tubes*; the components of types (b) and (c) are respectively rank-1 and rank-2 cusps. The assumption that the fundamental group of N is finitely generated implies that N contains only finitely many cusps [Su1].

Denote by N^ε the closure of the complement of the union of the cusps of N and notice that N is homeomorphic to the interior of N^ε.

4.2. Pared manifold associated with a hyperbolic 3-manifold

Still under the assumption that N is a hyperbolic 3-manifold with finitely generated fundamental group, it follows from the *tameness theorem* by Agol [Ag] and Calegari–Gabai [CG] that N is homeomorphic to the interior of a compact manifold. More precisely, we have the following result.

TAMENESS THEOREM. (Agol, Calegari–Gabai) *Suppose that N is a hyperbolic 3-manifold with finitely generated fundamental group and let ε be positive and smaller than the Margulis constant. There is a compact 3-manifold M whose boundary ∂M contains a subsurface P consisting of all toroidal components of ∂M and a possibly empty collection of annuli such that N^ε is homeomorphic to the complement in M of $\partial M\setminus P$.*

Continuing with the same notation as in the tameness theorem, it is well known that (M,P) is a pared manifold. Moreover, (M,P) is unique up to pared homeomorphisms; in particular, (M,P) does not depend on the concrete choice of ε. It is hence justified to refer to (M,P) as the pared manifold associated with N.

An immediate consequence of the tameness theorem is the existence of what we refer to as a *standard* (*relative*) *compact core* of $(N^\varepsilon,\partial N^\varepsilon)$. This is a compact submanifold $(M',P')\subset(N^\varepsilon,\partial N^\varepsilon)$ homeomorphic to (M,P) and such that $(N^\varepsilon\setminus M',\partial N^\varepsilon\setminus P')$ is homeomorphic to $(\partial M'\setminus P)\times\mathbb{R}$. Observe that while every standard compact core (M',P') is homeomorphic to the pared manifold (M,P) associated with N, this identification is far from being canonical. Even if we do not distinguish between isotopic identifications, we must still take into account the effect of precomposing the embedding of (M,P) with self-homeomorphisms of (M,P).

4.3. Convex core and conformal boundary

We are still assuming that $N=\mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold with finitely generated non-virtually abelian fundamental group. Let $\Lambda_\Gamma\subset\partial_\infty\mathbb{H}^3$ be the limit set of Γ, i.e. the minimal closed Γ-invariant subset of the boundary at infinity $\partial_\infty\mathbb{H}^3$ of hyperbolic space. The convex hull $\mathrm{CH}(\Gamma)$ of the limit set Λ_Γ is the minimal, closed, Γ-invariant convex subset of $\mathbb{H}^3$. The *convex core* $\mathrm{CC}(N)=\mathrm{CH}(\Gamma)/\Gamma$ of N, i.e. the projection of the convex hull to N, is a totally convex submanifold of N. Unless Γ is Fuchsian, the boundary of the convex hull is, with respect to the induced path metric, a complete hyperbolic surface without boundary [EM]. In particular, the boundary $\partial\mathrm{CC}(N)$ also has a natural hyperbolic metric; it is known that with respect to this metric $\partial\mathrm{CC}(N)$ has finite area.

Continuing with the same notation, let $\Omega_\Gamma=\partial_\infty\mathbb{H}^3\setminus\Lambda_\Gamma$ be the *discontinuity domain* of Γ. The action of the group Γ on $\mathbb{H}^3\cup\Omega_\Gamma$ is properly discontinuous and free. In particular, $N\cup\partial_cN=(\mathbb{H}^3\cup\Omega_\Gamma)/\Gamma$ is a 3-manifold with boundary. It is known that every connected component of ∂_cN has negative Euler characteristic.

The boundary at infinity of the hyperbolic space $\partial_\infty\mathbb{H}^3$ can be identified with the complex projective line $\mathbb{C}P^1$. This identification is compatible with the actions $\mathrm{PSL}_2(\mathbb{C})\curvearrowright\mathbb{H}^3$ by orientation-preserving isometries and $\mathrm{PSL}_2(\mathbb{C})\curvearrowright\mathbb{C}P^1$ by Möbius transformations. As a consequence of the Ahlfors finiteness theorem, we see that the surface $\partial_cN=\partial_c(\mathbb{H}^3/\Gamma)=\Omega_\Gamma/\Gamma$ is naturally endowed with the structure of a Riemann surface of finite conformal type, and hence with a finite-volume hyperbolic metric. The Riemann surface $\partial_c(\mathbb{H}^3/\Gamma)$ is the *conformal boundary* of $\mathbb{H}^3/\Gamma$.

The convex projection $\varkappa\colon N\to\mathrm{CC}(N)$ extends continuously to the conformal boundary ∂_cN. The following result due to Canary asserts that the restriction of $\varkappa$ to ∂_cN is Lipschitz when we endow ∂_cN with the canonical hyperbolic metric.

PROPOSITION 4.1. (Canary [Ca1]) *For every $\varepsilon>0$ there exists $K>0$ such that the following holds. Let M be a hyperbolic 3-manifold, with finitely generated fundamental group, such that every closed geodesic in the conformal boundary which is homotopically trivial in M has at least length ε with respect to the canonical hyperbolic metric of ∂_cN. Then the convex projection $\varkappa\colon\partial_cM\to\partial\mathrm{CC}(M)$ is K-Lipschitz.*

4.4. Ends

A *geometric end* of N is an end of N^ε. Observe that whenever (M,P) is a standard compact core of $(N^\varepsilon,\partial N^\varepsilon)$, then we have a bijection between the ends of N^ε and the free sides of (M,P). A geometric end $\mathcal{E}$ is *convex cocompact* if it has a neighborhood which is disjoint from $\mathrm{CC}(N)\cap N^\varepsilon$. If every geometric end of N is convex cocompact then we say that N is *geometrically finite*. The following is a well-known fact.

Proposition 4.2. *Suppose that N is geometrically finite and let $\mathcal{N}(\mathrm{CC}(N))$ be a regular neighborhood of the convex core of N. Then*

$$(\mathcal{N}(\mathrm{CC}(N))\cap N^{\varepsilon}, \mathcal{N}(\mathrm{CC}(N))\cap\partial N^{\varepsilon})$$

is a standard compact core of $(N^{\varepsilon}, \partial N^{\varepsilon})$. Moreover, the union $N\cup\partial_c N$ of N and its conformal boundary is homeomorphic to $N^{\varepsilon}\setminus\partial N^{\varepsilon}$.

Geometric ends $\mathcal{E}$ which are not convex cocompact are said to be *degenerate*. We will discuss a few properties of degenerate ends in §6.

4.5. AH(M,P)

If (M,P) is a pared manifold, let $\mathrm{AH}(M,P)$ be the set of conjugacy classes of discrete and faithful representations $\varrho\colon\pi_1(M)\to\mathrm{PSL}_2(\mathbb{C})$ which map conjugacy classes represented by curves in P to parabolic elements.

Given $[\varrho]\in\mathrm{AH}(M,P)$ choose a representative $\varrho\in[\varrho]$ and consider the hyperbolic 3-manifold $N_\varrho=\mathbb{H}^3/\varrho(\pi_1(M))$. We say that ϱ is *geometrically finite* if N_ϱ is. Similarly, we say that ϱ is *minimally parabolic* if every element in $\pi_1(M)$ whose image under ϱ is parabolic, is represented by a curve in P. Conjugated representations give rise to isometric hyperbolic 3-manifolds. In particular, if a representation is conjugated to a geometrically finite (resp. minimally parabolic) one, then it is itself geometrically finite (resp. minimally parabolic). Abusing notation and terminology, we will not make any distinction between representations and the corresponding points in $\mathrm{AH}(M,P)$.

Given $\varrho\in\mathrm{AH}(M,P)$, let N_ϱ be the associated hyperbolic 3-manifold. Let (M',P') be the pared manifold corresponding to N_ϱ and identify (M',P') with a standard compact core of $(N_\varrho^{\varepsilon},\partial N_\varrho^{\varepsilon})$. The representation ϱ can be interpreted as an isomorphism between $\pi_1(M)$ and $\pi_1(N_\varrho)\simeq\pi_1(M')$. In particular, ϱ determines, up to homotopy, a homotopy equivalence $\phi\colon M\to M'$. The assumption that ϱ maps elements represented by curves in P to parabolic elements implies that ϕ is homotopic to a map of pairs

$$(M,P)\longrightarrow(M',P').$$

Any map of pairs in this homotopy class is said to be in the *homotopy class determined by ϱ*.

In general ϱ also maps some elements which are not represented by curves in P to parabolic elements; if this is the case, then the pared manifolds (M,P) and (M',P') are not homeomorphic. However, much more dramatic events can occur, since it may well be that M and M' themselves are not homeomorphic to each other. Compare with Example 3.8 above.

From now on, we will consider $\mathrm{AH}(M,P)$ with the *algebraic topology*: a sequence of conjugacy classes of representations $\{[\varrho_i]\}_{i=1}^{\infty}$ converges *algebraically* to $[\varrho]$ if there are representatives $\varrho_i\in[\varrho_i]$ and $\varrho\in[\varrho]$ such that for each $\gamma\in\pi_1(M)$ the sequence $\{\varrho_i(\gamma)\}_{i=1}^{\infty}$ converges to $\varrho(\gamma)$.

At this point we would like to point out that the behavior of the hyperbolic manifolds associated with an algebraically convergent sequence of representations is all but transparent. For example, it was observed by Anderson and Canary [AC1] that the homeomorphism type of the quotient manifold can be constant along the sequence and change at the limit.

4.6. Quasi-conformal deformations of Kleinian groups

Continuing with the same notation, fix $\varrho\in\mathrm{AH}(M,P)$ geometrically finite and minimally parabolic and such that (M,P) is the pared manifold associated with the hyperbolic 3-manifold N_ϱ. The existence of such a representation ϱ is ensured by Thurston's hyperbolization theorem [Ot4], [Ka]. Identify (M,P) with a standard compact core of $(N_\varrho^\varepsilon,\partial N_\varrho^\varepsilon)$; doing so we obtain an identification, well defined up to isotopy, between the conformal boundary $\partial_c(N_\varrho)$ of N_ϱ and $\partial M\setminus P$, the union of the free sides of (M,P). In particular, we may consider the Riemann surface $\partial_c N_\varrho$ as a point in the Teichmüller space $\mathcal{T}(\partial M\setminus P)$.

We say that two representations ϱ' and ϱ'' of the same group G are *quasi-conformally conjugated* if there is a quasi-conformal homeomorphism $f\colon\partial_\infty\mathbb{H}^3\to\partial_\infty\mathbb{H}^3$ with

$$\varrho'(\gamma)\circ f=f\circ\varrho''(\gamma)\quad\text{for all }\gamma\in G.$$

Continuing with the same notation as above, let $\mathrm{QH}(\varrho)$ be the subset $\mathrm{AH}(M,P)$ consisting of representations which are quasi-conformally conjugated to ϱ.

Suppose that $\varrho'\in\mathrm{QH}(\varrho)$ and let $f\colon\partial_\infty\mathbb{H}^3\to\partial_\infty\mathbb{H}^3$ be a conjugating map. The map f extends to a bi-Lipschitz diffeomorphism $F\colon\mathbb{H}^3\to\mathbb{H}^3$ conjugating the actions of $\varrho(\pi_1(M))$ and $\varrho'(\pi_1(M))$ on $\mathbb{H}^3$. In particular, N'_ϱ is homeomorphic to N_ϱ and ϱ is geometrically finite and minimally parabolic. It follows that (M,P) is the pared manifold associated with $N_{\varrho'}$, and hence that we can again consider the conformal boundary $\partial_c N_{\varrho'}$ as a point in the Teichmüller space $\mathcal{T}(\partial M\setminus P)$. More precisely, it is a major consequence of the theory of quasi-conformal deformations of Kleinian groups developed by Ahlfors, Bers, Maskit, Kra, Marden and Sullivan that $\mathrm{QH}(\varrho)$ is parameterized by $\mathcal{T}(M\setminus P)$. We refer to this as the *Ahlfors–Bers parametrization*.

THEOREM 4.3. (Ahlfors–Bers parametrization) *There is a covering map*

$$\pi_{\mathrm{AB}}\colon\mathcal{T}(\partial M\setminus P)\longrightarrow\mathrm{QH}(\varrho)$$

with covering group $\mathrm{Mod}_0^+(M,P)$*. Moreover, for all* $X\in\mathcal{T}(\partial M\setminus P)$*, the hyperbolic manifold associated with the representation* $\pi_{\mathrm{AB}}(X)$ *has associated pared manifold* (M,P) *and conformal boundary bi-holomorphic to* X*.*

Before going further, we prove the following technical result which we will need below.

PROPOSITION 4.4. *Let* (M,P) *be a pared manifold with free sides* $F_1,...,F_s$ *and let* $\varrho_0\in\mathrm{AH}(M,P)$ *be a geometrically finite representation such that the hyperbolic manifold* $N_{\varrho_0}=\mathbb{H}^3/\varrho_0(\pi_1(M))$ *has associated pared manifold* (M,P)*. Assume that we have a sequence* $(X_1^n,...,X_s^n)$ *in* $\mathcal{T}(\partial M\setminus P)$ *such that*

- *for* $i=1,...,r$ *we have* $X_i^n=X_i^1$ *for all* n;
- *the images* ϱ_n *of the points* $(X_1^n,...,X_s^n)$ *under the Ahlfors–Bers parametrization* $\pi_{\mathrm{AB}}:\mathcal{T}(\partial M\setminus P)\to\mathrm{QH}(\varrho_0)$ *converge to some* $\varrho\in\mathrm{AH}(M,P)$*.*

If (M',P') *is the pared manifold associated with the manifold* $N_\varrho=\mathbb{H}^3/\varrho(\pi_1(M))$*, then there is a map* $\phi:(M,P)\to(M',P')$ *in the homotopy class determined by* ϱ *which maps the free sides* $F_1,...,F_r$ *homeomorphically to free sides* $F_1',...,F_r'$ *of* (M',P') *facing convex cocompact ends of* N_ϱ*. Moreover, for* $i=1,...,r$*,* X_i^1 *is the point of* $\mathcal{T}(F_i)$ *obtained by identifying* F_i' *with the corresponding components of* $\partial_c N_\varrho$ *and pulling back the structure by* $\phi|_{F_i}$*.*

Proof. The claim is well known if ϱ is Fuchsian. We assume that this is not the case and observe that this implies that, up to forgetting finitely many terms of the sequence, none of the representations ϱ_n is Fuchsian either.

Abusing notation slightly, let ϱ_n and ϱ be actual representations such that $\varrho_n(\gamma)$ converges to $\varrho(\gamma)$ for all $\gamma\in\pi_1(M)$. Fixing a base point $p_{\mathbb{H}^3}$ (or more precisely a base frame), we obtain base points $p_n\in N_n$ and $p_\varrho\in N_\varrho$, respectively. It is then standard to see that the sequence of pointed manifolds (N_n,p_n) converge, up to passing to a subsequence, in the geometric topology (pointed Gromov–Hausdorff topology) to a complete hyperbolic 3-manifold N_G. Equivalently, the groups $\varrho_n(\pi_1(M))$ converge in the Chabauty topology to a discrete faithful group G with $N_G=\mathbb{H}^3/G$; observe that $\varrho(\pi_1(M))\subset G$.

Given $i\in\{1,...,r\}$, we know that the geometric end associated with $X_i^n=X_i^1$ of N_n is convex cocompact and the corresponding component of $\partial\mathrm{CC}(N_n)$, with the induced path metric, is a complete hyperbolic structure on the interior of F_i. We denote this component of $\partial\mathrm{CC}(N_n)$ by Y_i^n. Even more by Canary's Theorem 4.1, the convex projection from X_i^1 to Y_i^n is Lipschitz, with a Lipschitz constant independent of n. In particular, there is $\varepsilon>0$ such that for every $i=1,...,r$ and n, the surface obtained by removing the ε-cusps of Y_i^n is ε-thick and has diameter bounded independently of n. Our first claim is to show that the distance between p_n and the ε-thick part of Y_i^n is bounded independently of n.

As before, let $i\in\{1,...,r\}$ and let $\gamma_1,...,\gamma_m$ be a finite set of elements in $\pi_1(M)$ such that the subgroup of $\pi_1(M)$ generated by these elements is conjugate to the image of $\pi_1(F_i)$ in $\pi_1(M)$. Since $\{\varrho_n(\gamma_j)\}_{n=1}^{\infty}$ converge to $\varrho(\gamma_j)$ for every $j=1,...,m$, there is an upper bound for the translation length of $\varrho_n(\gamma_j)$ independent of n. However, since the map from X_i^1 to Y_i^n is uniformly Lipschitz, we know that the conjugacy classes of $\gamma_1,...,\gamma_m$ can also be represented by loops on the ε-thick part of Y_i^n with a base point on the ε-thick part of Y_i^n and lengths bounded independently of n. But the group generated by $\varrho_n(\gamma_1),...,\varrho_n(\gamma_m)$ is non-elementary and in particular for every n, there are two elements, say γ_1 and γ_2, such that $\varrho_n(\gamma_1)$ and $\varrho_n(\gamma_2)$ generate a free group. By discreteness, the axes of these two elements cannot be too close to be parallel, and therefore the locus in $\mathbb{H}^3$ where both elements have simultaneously bounded translation length is uniformly bounded. This bounds the distance between the base point $p_{\mathbb{H}^3}$ and a lift of the base point on the ε-thick part of Y_i^n. Hence, the distance between p_n and the base points on the ε-thick part of Y_i^n must be bounded independently of n.

Once we know this, and after passing to a subsequence, we may assume that the sequence of surfaces $\{Y_i^n\}_{n=1}^{\infty}$ converges to a surface Y_i in the geometric limit N_G. Since Y_i^n is embedded for all n and we are assuming that the algebraic limit ϱ is not Fuchsian, we deduce that Y_i is also embedded in the geometric limit N_G. Even more, the surface Y_i separates a geometric end of N_G, homeomorphic to $F_i\times\mathbb{R}$, from a convex subset of N_G. In fact, we can use the convergence of arbitrarily large neighborhoods of the ε-thick parts of the surfaces Y_i^n to see that this geometric end of N_G is convex cocompact. On the other hand, the fundamental group of this end is generated by

$$\lim_{n\to\infty}\varrho_n(\gamma_1)=\varrho(\gamma_1),\quad ...,\quad \lim_{n\to\infty}\varrho_n(\gamma_m)=\varrho(\gamma_m).$$

So this end lifts homeomorphically to a geometric end of the algebraic limit N_ϱ. Hence, for every $i=1,...,r$, (M',P') has a free side homeomorphic to F_i and N_ϱ has a convex cocompact geometric end associated with this free side. Even more, the ends of the manifolds N_n associated with F_i converge geometrically to this end. From here on, one can repeat the proof in [Oh2], where it is shown that algebraic and geometric convergence together imply convergence in the sense of Carathéodory. This immediately implies that the conformal structure at infinity associated with the F_i-end of N_ϱ is X_i^1. □

5. Laminations, currents and train-tracks

In this section we recall some facts about laminations, currents and train-tracks on a surface. We refer to [Th1], [CB], [FLP], [PH] and [Bo3] for more on these topics.

5.1. Laminations and currents

Let $S=\mathbb{H}^2/\Gamma$ be a complete hyperbolic surface with finite area. A *geodesic lamination* λ is a closed subset of S which is foliated by complete geodesics. The path-connected components of λ coincide with the leaves of the associated foliation; we refer to them as the *leaves* of the lamination. We endow the set $\mathcal{L}(S)$ of geodesic laminations on S with the Hausdorff topology on the set of compact subsets of S.

Suppose now that λ is a geodesic lamination on S and denote by $\tilde{\lambda}$ its lift to the universal cover $\mathbb{H}^2$. We consider $\tilde{\lambda}$ as a Γ-invariant subset of the set $\mathcal{G}(\mathbb{H}^2)$ of all (unoriented) geodesics on $\mathbb{H}^2$. Endowing the set of closed subsets of $\mathcal{G}(\mathbb{H}^2)$ with the Hausdorff topology, we have that the lifting map $\lambda\mapsto\tilde{\lambda}$ from $\mathcal{L}(S)$ is a homeomorphism onto its image.

A (geodesic) *current* on $S=\mathbb{H}^2/\Gamma$ is a non-trivial, locally finite, Γ-invariant, Borel measure on $\mathcal{G}(\mathbb{H}^2)$. We endow the set $\mathcal{C}(S)$ of all currents on S with the weak*-topology. The group $\mathbb{R}_+$ acts on $\mathcal{C}(S)$ by scaling the measure. We endow the quotient, the space of *projective measured currents* $P\mathcal{C}(S)=\mathcal{C}(S)/\mathbb{R}_+$, with the quotient topology. It is known (cf. [Bo3, Corollary 5]) that $P\mathcal{C}(S)$ is Hausdorff and compact.

We may associate with each primitive closed geodesic γ in S a current as follows: the measure of a subset K of $\mathcal{G}(\mathbb{H}^2)$ is the number of lifts of γ to $\mathbb{H}^2$ which belong to K. The map which associates with any two geodesics in S their geometric intersection number extends continuously to the *geometric intersection form*

$$\iota: \mathcal{C}(S)\times\mathcal{C}(S)\longrightarrow\mathbb{R}_+.$$

The geometric intersection form is defined so that it is multiplicative under scaling any of the two entries. In particular, it is unambiguous to write $\iota(\lambda,\mu)=0$ for $\lambda,\mu\in P\mathcal{C}(S)$.

The *support* of a current μ is the smallest closed Γ-invariant subset of $\mathcal{G}(\mathbb{H}^2)$ whose complement has 0-measure. We will write $\operatorname{supp}(\mu)$ for the projection of the support to S and, abusing terminology, also refer to it as the support of μ. A *measured lamination* is a geodesic current λ with $\operatorname{supp}(\lambda)\in\mathcal{L}(S)$. It is known that, as long as $\operatorname{supp}(\lambda)$ is compact, λ is a measured lamination if and only if $\iota(\lambda,\lambda)=0$. Denote by $\mathcal{ML}(S)$ the set of measured laminations on S and by $\mathcal{PML}(S)$ its projectivization; $\mathcal{PML}(S)$ is compact.

A measured lamination $\lambda\in\mathcal{PML}(S)$ is *filling* if $\iota(\lambda,\gamma)\neq0$ for every simple closed geodesic γ in S. It is known that λ is filling if and only if every complementary component of the support of λ is an ideal polygon or a once punctured ideal polygon.

The subset of $\mathcal{PML}(S)$ consisting of measured laminations which are supported by a simple closed curve is dense. Before moving on, suppose now that $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence of simple closed geodesics in S which converges in $\mathcal{PML}(S)$ to some measured lamination λ. Passing to a subsequence, we may assume that $\{\gamma_n\}_{n=1}^{\infty}$ converges in

$\mathcal{L}(S)$ to some geodesic lamination $\lambda_{\mathcal{L}}$ as well. Clearly, $\operatorname{supp}(\lambda)\subset\lambda_{\mathcal{L}}$, but in general $\lambda_{\mathcal{L}}$ may be larger than $\operatorname{supp}(\lambda)$; sometimes much larger. However, if λ is a filling measured lamination, then any geodesic lamination $\lambda_{\mathcal{L}}$ which contains the support of λ is equal to the union of $\operatorname{supp}(\lambda)$ and a finite collection of isolated non-compact leaves.

5.2. Independence of the hyperbolic structure

Suppose now that $S=\mathbb{H}^2/\Gamma$ and $S'=\mathbb{H}^2/\Gamma'$ are two complete hyperbolic surfaces, $f\colon S\to S'$ is a bi-Lipschitz homeomorphism and

$$\tilde{f}\colon \mathbb{H}^2 \longrightarrow \mathbb{H}^2$$

is a lift of f. Since $\tilde{f}$ is also bi-Lipschitz, the restriction of $\tilde{f}$ to any geodesic in $\mathbb{H}^2$ is a quasi-geodesic. Recall that if X is a metric space, a map $\alpha\colon\mathbb{R}\to X$ is a (K,A)-quasi-geodesic if

$$\frac{|s-t|}{K}-A\leqslant d_X(\alpha(s),\alpha(t))\leqslant K|s-t|+A \tag{5.1}$$

for all $s,t\in\mathbb{R}$. It is known that the image of a quasi-geodesic $\alpha\colon\mathbb{R}\to\mathbb{H}^2$ is at bounded distance of a geodesic, where the bound depends only on the quasi-geodesic constants. In particular, the bi-Lipschitz map $\tilde{f}\colon\mathbb{H}^2\to\mathbb{H}^2$ induces a homeomorphism

$$\bar{f}\colon \mathcal{G}(\mathbb{H}^2) \longrightarrow \mathcal{G}(\mathbb{H}^2)$$

which conjugates the actions of the groups Γ and Γ'. It follows that $\bar{f}$ induces homeomorphisms $\bar{f}_*$ between the spaces of geodesic laminations, currents and measured laminations associated with the surfaces S and S'.

5.3. Train-tracks

A *train track* on $S=\mathbb{H}^2/\Gamma$ is an embedded 1-complex $\tau\subset S$ without dead-ends whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. The branches which are incident to a switch are divided into two non-empty subsets: "incoming" and "outgoing" branches according to their inward pointing tangent at the switch. Moreover, for each component R of $\Sigma\setminus\tau$, the double of R along the interiors of edges of ∂R must have negative Euler characteristic.

A *route* of the train track τ is the image of a monotonic map $h\colon\mathbb{R}\to\tau$ parameterized by arc-length and such that whenever it arrives at a vertex through an incoming (resp. outgoing) branch, it leaves through an outgoing (resp. incoming) branch. It is important to observe that in our definitions the routes are neither parameterized nor have a preferred direction.

A geodesic lamination λ is *carried* by the train track τ, if there is a map $S\to S$ homotopic to the identity such that the image of any leaf of λ is a route in τ. It is an observation of Thurston that if λ is carried by τ and $\{\gamma_n\}_{n=1}^{\infty}$ is any sequence of curves which converges in the Hausdorff topology to λ, then for all sufficiently large n, γ_n is carried by τ as well.

Let $\tau\subset S$ be a train-track and denote by $\tilde{\tau}$ the preimage of τ in $\mathbb{H}^2$. We denote by $\mathcal{R}(\tau)$ the set of all routes carried by τ. It is known that every route in $\mathcal{R}(\tau)$ has two distinct well-determined endpoints in $\partial_\infty\mathbb{H}^2$ and that two routes with the same endpoints agree. In particular, we may associate with each route in $\mathcal{R}(\tau)$ a geodesic in $\mathbb{H}^2$ and the so-obtained map $\mathcal{R}(\tau)\to\mathcal{G}(\mathbb{H}^2)$ is injective; we endow $\mathcal{R}(\tau)$ with the subspace topology of its image in $\mathcal{G}(\mathbb{H}^2)$. Observe that a lamination λ is carried by τ if and only if each leaf of $\tilde{\lambda}$, the lift of λ to $\mathbb{H}^2$, is contained in the image of $\mathcal{R}(\tau)$.

Continuing with the same notation, let $\mathcal{C}(\mathcal{R}(\tau))\subset\mathcal{C}(S)$ be the closed set consisting of currents supported on the image of $\mathcal{R}(\tau)$ in $\mathcal{G}(\mathbb{H}^2)$. To conclude this section, we state a technical lemma; the proof is identical to the proof of the fact that a map between spaces of currents, induced by a bi-Lipschitz map, is a homeomorphism.

LEMMA 5.1. *Let $S=\mathbb{H}^2/\Gamma$ and $S'=\mathbb{H}^2/\Gamma'$ be two hyperbolic surfaces, $\tau\subset S$ be a train-track and $f: S\to S'$ be continuous. Lift f to a map $\tilde{f}: \mathbb{H}^2\to\mathbb{H}^2$ and suppose that there are A and K such that the restriction of $\tilde{f}$ to each route in $\mathcal{R}(\tau)$ is an (A,K)-quasi-isometry. Then the map $\bar{f}: \mathcal{R}(\tau)\to\mathcal{G}(\mathbb{H}^2)$, which associates with each route $\alpha\in\mathcal{R}(\tau)$ the unique geodesic in $\mathbb{H}^2$ which is at bounded distance of $\tilde{f}(\alpha)$, is continuous. Hence, the induced map $\bar{f}_*: \mathcal{C}(\mathcal{R}(\tau))\to\mathcal{C}(S')$ is also continuous.*

6. Masur domain and ending laminations

In this section we recall some facts about the Masur domain and on the ending laminations of degenerate ends of hyperbolic 3-manifolds.

6.1. Masur domain

Recall that the pared manifold $C=(C,P)$ is a pared compression body if there exists a free side, called the distinguished free side and denoted by $\partial_e C=\partial_e(C,P)$, so that the induced homomorphism from $\pi_1(\partial_e C)$ to $\pi_1(C)$ is surjective. Also recall that C is a trivial pared compression body if this homomorphism is also injective. Finally we say that C is *small* if it contains a single separating, properly embedded essential disk; in particular, C is the connected sum along the boundary of two pieces where each of them is either a trivial pared compression body or is a solid torus containing a single component of P.

By $\mathcal{M}$, we denote the set of isotopy classes of meridians, i.e. essential simple closed curves on $\partial_e C$ which bound disks in C. We identify $\mathcal{M}$ with a subset of $\mathcal{PML}(\partial_e C)$, the space of projective classes of measured laminations on $\partial_e C$. Let $\mathcal{M}'$ be the closure of $\mathcal{M}$ in $\mathcal{PML}(\partial_e C)$.

If C is neither trivial nor small, then we define

$$\mathcal{O}(C) = \{\lambda \in \mathcal{PML}(\partial_e C) : \iota(\lambda, \mu) > 0 \text{ for all } \mu \in \mathcal{M}'\},$$

where $\iota(\cdot, \cdot)$ is the geometric intersection number. For a small compression body assume that m is the unique separating meridian and define

$$\mathcal{M}'' = \{\lambda \in \mathcal{PML}(\partial_e C) : \iota(\lambda, m) = 0\},$$

and then

$$\mathcal{O}(C) = \{\lambda \in \mathcal{PML}(\partial_e C) : \iota(\lambda, \mu) > 0 \text{ for all } \mu \in \mathcal{M}''\}.$$

Finally, if (C, P) is a trivial compression body, meaning that $\partial_e C$ is incompressible, then $\mathcal{M}=\varnothing$ and we define $\mathcal{O}(C)=\mathcal{PML}(\partial_e C)$. The set $\mathcal{O}(C)$ is the *Masur domain* of the pared compression body $C=(C, P)$.

If F is a free side of a pared manifold (M, P), then the *Masur domain* $\mathcal{O}(F)$ is defined to be the Masur domain of the relative compression body in (M, P) with distinguished free side F; see §3.4 for the definition of relative compression bodies.

In this article we will only deal with filling Masur domain laminations. There is a more straightforward characterization of filling laminations that do not belong to the Masur domain which could in fact be considered as the definition. We state this characterization in the following lemma.

LEMMA 6.1. *Assume that $\lambda \in \mathcal{PML}(\partial_e C)$ is filling. Either λ belongs to the Masur domain or there is a sequence of meridians $\{m_i\}_{i=1}^{\infty}$ which converges in $\mathcal{PML}(\partial_e C)$ to a measured lamination μ with the same support as λ. These two alternatives are mutually exclusive.*

Before going any further, a brief historical remark on the Masur domain. The Masur domain was defined by Masur is [Masu] and then extensively studied by Otal [Ot1]. See also Kerckhoff [Ke], Canary [Ca2], [Ca3] and Kleineidam–Souto [KS1], [KS2]. Recently, Lecuire [L2] has defined and studied a certain extension of the Masur domain. Working in the Masur domain is still technically simpler and it fully suffices for our purposes;

it should be remarked that a filling lamination is contained in Lecuire's domain if and only if it is contained in the Masur domain. Previous authors have restricted themselves to the study of the Masur domain in the case when the pared locus of (C,P) is empty. However, the proofs remain word-by-word the same in our slightly more general situation. In particular, Masur and Otal proved that when C is a non-trivial compression body, $\mathrm{Mod}(C)$ acts properly discontinuously on the Masur domain of C. This can be generalized to show that if (C,P) is a non-trivial pared compression body, $\mathrm{Mod}(C,P)$ acts properly discontinuously on $\mathcal{O}(\partial_e C)$. This is crucial in our generalizations of known facts about the Masur domain.

6.2. Realizing laminations in $\mathcal{O}(C)$

Still with the same notation as above, let (C,P) be a pared compression body and $\varrho\in\mathrm{AH}(C,P)$ be a discrete and faithful representation of $\pi_1(C)$ in $\mathrm{PSL}_2(\mathbb{C})$ which maps elements represented by curves in P to parabolic elements. Let N be the hyperbolic manifold $\mathbb{H}^3/\varrho(\pi_1(C))$ and recall that the identification of $\pi_1(C)$ with $\pi_1(N)=\varrho(\pi_1(C))$ determines a homotopy class of maps

$$j\colon (C,P)\longrightarrow (N^\varepsilon,\partial N^\varepsilon)$$

which, after forgetting the pared locus, is a homotopy equivalence. We can extend the restriction of j to $\partial_e C$ to a proper map from the interior of $\partial_e C$ to N by extending images of annular neighborhoods of boundary components of $\partial_e C$ into cusps of N. With abuse of notation, we still refer to this map as $j\colon\partial_e C\to N$. We will say that a map from the interior of $\partial_e C$ to N is in *the correct homotopy class* if it is properly homotopic to the restriction of j to $\partial_e C$.

A geodesic lamination λ on $\partial_e C$ is *realized* in N if there exists a complete, finite-area hyperbolic metric σ on $\partial_e C$ and a *pleated map* $f\colon(\partial_e C,\sigma)\to N$ in the correct homotopy class which is totally geodesic on leaves of λ. Recall that a *pleated map* is a map f which sends geodesic segments to rectifiable paths of the same length and also every point is contained in a geodesic segment whose image under f is a geodesic segment; see [CEG] for basic facts on pleated maps.

Remark. When working with pleated maps we will often drop any reference to the underlying hyperbolic metric on the domain. Besides reducing the almost unbearable burden of notation, this is justified by the fact that the metric is uniquely determined by the map.

Assume that a lamination λ in the Masur domain $\mathcal{O}(C)$ is realized by a pleated map $f\colon\partial_e C\to N$, identify $\mathbb{H}^3$ with the universal cover of N, let $\partial_e C'$ be the cover of $\partial_e C$

associated with the subgroup $\mathrm{Ker}(\pi_1(\partial_e C)\to\pi_1(C))$ and let $f'\colon\partial_e C'\to\mathbb{H}^3$ be a lift of f. By definition, the map f' maps every leaf of the preimage of λ' in $\partial_e C'$ to a geodesic in $\mathbb{H}^3$. The following result of Otal [Ot1] asserts that the induced map is injective.

THEOREM 6.2. (Otal) *If two leaves l_1 and l_2 of λ' are such that $f'(l_1)=f'(l_2)$, then*

$$l_1=l_2.$$

Following Kleineidam–Souto [KS2], let $\mathcal{O}_\varrho(C)$ be the subset of $\mathcal{O}(C)$ consisting of those Masur domain laminations λ which intersect every non-peripheral simple closed curve α on $\partial_e C$ whose image under ϱ is parabolic, $\iota(\lambda,\alpha)>0$. Note that a filling lamination in $\mathcal{O}(C)$ is always in $\mathcal{O}_\varrho(C)$.

LEMMA 6.3. ([KS2, Lemma 4.5]) *For every lamination λ in $\mathcal{O}_\varrho(C)$, there exists a sequence of multicurves $\{\gamma_i\}_{i=1}^\infty\subset\mathcal{O}_\varrho(C)$, where $\{\gamma_i\}_{i=1}^\infty$ converges in $\mathcal{PML}(\partial_e C)$ to a lamination that contains λ. Even more, when λ is not filling, we may assume that there exists a multicurve γ with $\gamma\subset\gamma_i$ for every i.*

Let now λ be a lamination in $\mathcal{O}_\varrho(C)$ and $\{\gamma_i\}_{i=1}^\infty$ be a sequence obtained from the above lemma. The following lemma asserts that the curves γ_i are realized in N for all i.

LEMMA 6.4. ([KS2, Lemma 4.1]) *Every finite lamination which contains a multicurve in $\mathcal{O}_\varrho(C)$ is realized in N.*

In fact, the proof of the above lemma in [KS2] proves more. It shows that if no component of a multicurve α in $\mathcal{O}(C)$ is mapped by ϱ to parabolics, then every geodesic lamination which is a finite extension of α by non-compact leaves is realized. This we will use in §12.3.

Continuing with the same notation, let $f_i\colon\partial_e C\to N$ be a sequence of pleated surfaces realizing the curves γ_i. The following compactness result for pleated surfaces gives a condition for the lamination λ to be realized in terms of the images $f_i(\partial_e C)$ of the maps f_i.

PROPOSITION 6.5. *A lamination $\lambda\in\mathcal{O}_\varrho(C)$ is realized in N if there is a sequence γ_i of multicurves in $\mathcal{O}_\varrho(C)$ converging to λ in $\mathcal{PML}(\partial_e C)$ and a compact set $K\subset N$ such that each γ_i is realized by a pleated surface $f_i\colon\partial_e C\to N$ with $f_i(\partial_e C)\cap K\neq\varnothing$. In particular, every non-filling lamination $\lambda\in\mathcal{O}_\varrho(C)$ is realized in N.*

The last statement is a consequence of the first part of the proposition and the fact that, by Lemma 6.3, we may choose γ_i in such a way that they all contain a multicurve γ. Proposition 6.5 is a version of Otal's compactness theorem [Ot1] for pleated surfaces, which is in turn an extension of Thurston's compactness theorem for pleated surfaces (see

[CEG]). The proof of Proposition 6.5 is, word-by-word, the same as the proof of [KS2, Proposition 4.3], where the pared locus was assumed to be empty but where the image of ϱ was allowed to have parabolic elements. Before moving on, observe that Lemmas 6.3 and 6.4 and Proposition 6.5 imply the following result essentially due to Otal [Ot1].

COROLLARY 6.6. *If N is geometrically finite, then every lamination which contains the support of a filling Masur domain lamination is realized in N.*

Besides the statement of Proposition 6.5, we will also need the following ingredient of its proof. See [KS2, Lemmas 4.5 and 4.6] for details.

LEMMA 6.7. *Suppose that λ is a filling lamination in the Masur domain $\mathcal{O}(C)$, let $\{\gamma_i\}_{i=1}^{\infty}$ be a sequence of multicurves in $\mathcal{O}_{\varrho}(C)$ converging to λ and, for each i, let $f_i: X_i \to N$ be a pleated surface realizing γ_i. Then there is a constant $\varepsilon > 0$ such that the following holds for all i:*

- *for every meridian η in $\partial_e C$ we have $l_{X_i}(\eta) \geqslant \varepsilon$;*
- *for every essential simple closed curve η in $\partial_e C$, with $\varrho(\eta)$ parabolic, we have $l_{X_i}(\eta) \geqslant \varepsilon$, where $l_{X_i}(\cdot)$ denotes the length in the hyperbolic metric induced by f_i on $\partial_e C$.*

6.3. More train-tracks

Suppose now that N is a hyperbolic 3-manifold and let L be large and δ small. We say that a train-track $\tau \subset \Sigma$ is (L, δ)-*realized* by some map $\phi: \Sigma \to N$ if

(i) ϕ maps monotonically each branch of τ to a geodesic segment of length $\geqslant L$;

(ii) if v_1 and v_2 are incoming and outgoing half-branches at x_0, respectively, then $\phi(v_1)$ and $\phi(v_2)$ have at least angle $\pi - \delta$.

It is known that any path in the hyperbolic space, which consists of sufficiently long geodesic segments meeting with a sufficiently large angle, is a (K, A)-quasi-geodesic, where K and A depend only on the meaning of "sufficiently". In particular, if a train-track τ is (L, δ)-realized in N with L large and δ small, then routes go to quasi-geodesics.

LEMMA 6.8. *There are universal constants L, δ, K and A such that if Σ is a surface of finite type, N is a hyperbolic 3-manifold and the train track $\tau \subset \Sigma$ is (L, δ)-realized by some map $f: \Sigma \to N$, then the restriction of $\tilde{f}: \widetilde{\Sigma} \to \widetilde{N} = \mathbb{H}^3$ to any route of $\tilde{\tau}$ is a (K, A)-quasi-geodesic. Here $\widetilde{\Sigma}$ and $\widetilde{N}$ are the universal covers of Σ and N, $\tilde{\tau}$ is the preimage of τ in $\mathbb{H}^2$ and $\tilde{f}$ is a lift of f.*

Suppose now that (M, P) is a pared manifold and $\varrho \in \mathrm{AH}(M, P)$. Let also $\bar{\Sigma}$ be the closure of a free side of (M, P). Finally, let λ be a geodesic lamination in $\Sigma = \bar{\Sigma} \setminus \partial \bar{\Sigma}$ which contains the support of a filling Masur domain measured lamination. It is well known that if λ is realized in $N_\varrho=\mathbb{H}^3/\varrho(\pi_1(M))$ by some pleated map in the correct homotopy class, then λ is carried by some train-track τ which is (L,δ)-realized in N with large L and small δ. In fact, if N is the algebraic limit of a sequence of manifolds, then τ can be chosen to be (L,δ)-realized in the approximates as well. More precisely, we have the following result.

Lemma 6.9. *Let (M,P) be a pared manifold and $\bar\Sigma$ be a free side of (M,P). Let also λ be a lamination on Σ which contains the support of a filling Masur domain measured lamination. Assume that $\{\varrho_n\}_{n=1}^\infty$ is a sequence in $\mathrm{AH}(M,P)$ which converges algebraically to some ϱ_∞ such that λ is realized in $N_{\varrho_\infty}=\mathbb{H}^3/\varrho_\infty(\pi_1(M))$ by a pleated surface $f\colon(\Sigma,\sigma)\to N_\varrho$ in the correct homotopy class. Then, for all L and δ, there are n_0 and a train-track $\tau\subset\Sigma$ carrying λ such that*

- *for all $n=n_0,...,\infty$ the train-track τ is (L,δ)-realized in N_{ϱ_n} by some map*

$$\phi_n\colon\Sigma\longrightarrow N_{\varrho_n}$$

in the correct homotopy class;

- *the pleated map f and the map ϕ_∞ realizing λ and τ in N_{ϱ_∞}, respectively, are homotopic to f by a homotopy whose tracks have at most length* 1.

Combining Lemmas 6.8 and 6.9 we deduce the following result.

Proposition 6.10. *Suppose that we are in the situation described in Lemma 6.9. Then there is a constant L such that for any sequence $\{\gamma_n\}_{n=1}^\infty$ of curves which converges to λ in the Hausdorff topology and for any $n\geqslant n_0$ we have*

$$\frac{1}{L}\leqslant\frac{l_{\mathbb{H}^3}(\varrho_n(\gamma_n))}{l_{(\Sigma,\sigma)}(\gamma_n)}\leqslant L.$$

Here $l_{\mathbb{H}^3}(\varrho_n(\gamma_n))$ is the translation length in $\mathbb{H}^3$ of $\varrho_n(\gamma_n)$ and $l_{(\Sigma,\sigma)}(\gamma_n)$ is the length in (Σ,σ) of the geodesic freely homotopic to γ_n.

Proposition 6.10 holds when $n=\infty$; in this case we also have the following easy consequence of Lemma 6.9.

Corollary 6.11. *Let $\varrho\in\mathrm{AH}(M,P)$ for a pared manifold (M,P) and $\overline\Sigma$ be a free side of (M,P). Also assume that λ is lamination on $\Sigma=\overline\Sigma\setminus\partial\overline\Sigma$ which contains the support of a filling measured lamination and is realized by a pleated surface $f\colon(\Sigma,\sigma)\to N_\varrho$ in the correct homotopy class for $N_\varrho=\mathbb{H}^3/\varrho(\pi_1(M))$. Given any sequence $\{\gamma_n\}_{n=1}^\infty$ of simple closed curves which converges to λ in the Hausdorff topology, there is n_0 such that, for any $n\geqslant n_0$, $f(\gamma_n)$ is homotopically non-trivial in N_ϱ and its geodesic representative is in the* 1*-neighborhood of the image of f.*

See [Br1, §4] for a proof of Lemma 6.9 and Proposition 6.10; see also [Ot2]. It should be remarked that we have chosen to state the results in this section only for filling laminations because the statements in the more general case are slightly more involved. Remark for instance that Lemma 6.9, as stated, does not hold if λ is a simple closed curve.

6.4. Simply degenerate ends and ending laminations

Suppose now that N is a hyperbolic 3-manifold with associated pared manifold (M,P), and let F be a free side of (M,P). Identify (M,P) with a standard compact core of $(N^\varepsilon,\partial N^\varepsilon)$ in the correct homotopy class and let $\mathcal{E}_F$ be the component of $N^\varepsilon\setminus M$ facing F. The component $\mathcal{E}_F$ corresponds to a unique geometric end of N which we again denote by $\mathcal{E}_F$. Suppose that $\mathcal{E}_F$ is not convex cocompact. It follows from the tameness theorem together with previous work by Bonahon [Bo2] and Canary [Ca3] that $\mathcal{E}_F$ is *simply degenerate*. This means that there is a sequence of homotopically essential simple closed curves γ_i in F which cannot be homotoped into P and whose geodesic representatives γ_i^* in N exit the end $\mathcal{E}_F$ and are homotopic to γ_i therein. Up to passing to a subsequence, we may assume that the curves γ_i converge to a measured lamination $\lambda\in\mathcal{PML}(F)$. It is due to Thurston [Th1], Bonahon [Bo2] and Canary [Ca3] that λ is filling and that any two so obtained measured laminations have the same support. In particular, the support of λ, the *ending lamination* of the end $\mathcal{E}_F$, depends only on the manifold M. The following result can be understood as a stronger version of the uniqueness of the ending lamination [Bo2], [Ca3].

Proposition 6.12. *Let N be a hyperbolic 3-manifold with the associated pared manifold (M,P) and F be a free side of (M,P) facing a degenerate geometric end $\mathcal{E}_F$ of N. There exists a unique filling unmeasured lamination λ such that whenever $\{\gamma_i^*\}_{i=1}^\infty$ is a sequence of closed geodesics in $\mathcal{E}_F\cap N^\varepsilon$, exiting $\mathcal{E}_F$ and homotopic within $\mathcal{E}_F$ to (possibly non-simple) curves $\{\gamma_i\}_{i=1}^\infty$ in F, then every convergent subsequence of the sequence $\{\gamma_i\}_{i=1}^\infty$, in the space $P\mathcal{C}(F)$ of projective geodesic currents, converges to a current supported on λ.*

The following result, due to Canary, asserts that the ending lamination of a degenerate end belongs to the Masur domain of the associated free side.

Theorem 6.13. ([Ca3, Corollary 10.2]) *Let N be a hyperbolic 3-manifold with the associated pared manifold (M,P) and F be a free side of (M,P) facing a degenerate geometric end $\mathcal{E}$ of N. Then any measured lamination supported on the ending lamination of the end $\mathcal{E}$ belongs to the Masur domain $\mathcal{O}(F)$.*

Moreover, if (M,P) is an interval bundle with base S and all the ends of N are degenerate, then the projections of the ending laminations of the ends of N to S is not a lamination.

In [Ca3], the first part of Theorem 6.13 was only proved in the absence of parabolics. The proof in the pared setting is identical and, actually, the only fact on the Masur domain needed in the proof is Lemma 6.1 above. The second statement in Theorem 6.13, which is made uncomfortably cumbersome by the possibility that (M,P) is a twisted interval bundle, should probably be attributed to Thurston [Th1].

Note now that Proposition 6.12, together with Corollary 6.11, easily implies that ending laminations are not realized. Before stating this fact as a corollary, we would like to remark that our main technical result, Theorem 1.4, asserts that under suitable conditions the converse is also true.

Corollary 6.14. *Let N be a hyperbolic 3-manifold with associated pared manifold (M,P) and F be a free side of (M,P) facing a degenerate geometric end with ending lamination λ. Then λ is non-realizable in N.*

Before moving on to other topics, we state here the following technical consequence of Corollary 6.14, Theorem 6.13 and Proposition 6.5.

Proposition 6.15. *Let N be a hyperbolic 3-manifold with associated pared manifold (M,P) and F be a free side facing a degenerate geometric end with ending lamination λ. Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence of simple curves on F converging to a projective measured lamination supported on λ. Then, for n sufficiently large, γ_n is homotopically non-trivial in N and the geodesic representatives γ_n^* of the curves γ_n exit the end $\mathcal{E}_F$ as $n\to\infty$.*

6.5. Two great theorems

In this section we recall the statements of the covering theorem and the ending lamination theorem.

Covering theorem. (Thurston, Canary) *Let N and N' be hyperbolic 3-manifolds with finitely generated fundamental group and infinite volume, suppose that $\pi: N\to N'$ is a Riemannian covering and let $\mathcal{E}$ be a degenerate end of N. Then, there are standard compact cores (M,P) and (M',P') of N^ε and $(N')^\varepsilon$ such that, if F is the free side of (M,P) facing $\mathcal{E}$, then the following hold:*

- *$\pi(F)=F'$ is a free side of (M',P');*
- *the restriction of π to the component of $N^\varepsilon\setminus M$ facing F is a finite-to-one covering map onto the component of $(N')^\varepsilon\setminus M'$ facing F';*

• *the end $\mathcal{E}'$ of N' facing F' is degenerate*;

• *the finite covering $\pi|_F: F \to F'$ maps the ending lamination of $\mathcal{E}$ to the ending lamination of $\mathcal{E}'$.*

The covering theorem was proved by Thurston in [Th1] in the case where N is a trivial interval bundle. From Thurston's result it is standard how to deduce the general case that the pared manifold associated with N has incompressible boundary. The compressible case is due to Canary [Ca3].

The second important theorem that we mention here is the ending lamination theorem. This result will play a paramount role in this paper.

ENDING LAMINATION THEOREM. (Minsky [Mi2], Brock–Canary–Minsky [BCM]) *Let N and N' be hyperbolic 3-manifolds with finitely generated fundamental groups. Let also (M,P) and (M',P') be standard compact cores of N^ε and $(N')^\varepsilon$ and suppose that $\phi: (M,P) \to (M',P')$ is a homeomorphism satisfying the following conditions*:

• *if $F \subset \partial M \setminus P$ faces a convex cocompact end of N, then $\phi(F)$ also faces a convex cocompact end $\mathcal{E}'$ of N'; moreover, the induced map $\partial_c \mathcal{E} \to \partial_c \mathcal{E}'$ between the conformal boundaries is homotopic to a bi-holomorphic map*;

• *if $F \subset \partial M \setminus P$ faces a degenerate end $\mathcal{E}$ of N with ending lamination $\lambda_{\mathcal{E}}$, then $\phi(F)$ also faces a degenerate end $\mathcal{E}'$ of N' with ending lamination $\phi(\lambda_{\mathcal{E}})$.*

Then, there is an isometry $\Phi: N \to N'$ in the isotopy class determined by ϕ.

The origins of the ending lamination theorem can be found in the work of Ahlfors and Bers (cf. Theorem 4.3) who proved it in the geometrically finite case. In its final form, the ending lamination theorem was conjectured to be true. In [Mi1] Minsky proved the ending lamination theorem for those hyperbolic 3-manifolds homeomorphic to the interior of a trivial interval bundle and which have non-vanishing injectivity radius. From this first work of Minsky, the general statement can be deduced if N is tame and has non-vanishing injectivity radius. The assumption that N is tame was made superfluous by the proof of the tameness theorem by Agol [Ag] and Calegari–Gabai [CG]. Following work of Masur–Minsky [MM1], [MM2], Minsky [Mi2] completed the construction of a Lipschitz model for the general hyperbolic 3-manifolds homeomorphic to the interior of an interval bundle. Brock–Canary–Minsky [BCM] proved that the model was actually a bi-Lipschitz model, concluding the proof of the ending lamination theorem for manifolds homeomorphic to the interior of a trivial interval bundle. The general case can be treated as in [BCM] and is promised in a forthcoming work of Brock, Canary and Minsky.

7. Trees

By a *real tree* we understand a path metric space T such that every two points $x, y \in T$ are joined by a unique injective continuous path $[x, y]$ isometric to an interval.

7.1. Dual trees

In order to give an example, and also for later use, we recall now briefly the construction of the dual tree of a filling measured lamination. Suppose that λ is a filling measured lamination on a hyperbolic surface Σ. Identify $\mathbb{H}^2$ with the universal cover of Σ and let $\tilde{\lambda}$ be the lift of λ to $\mathbb{H}^2$. For $x, y \in \mathbb{H}^2$ we define $d_\lambda(x, y)$ to be the λ-measure of the geodesic segment $[x, y]$. Observe that the assumption that λ is filling implies that the function $(x, y) \mapsto d_\lambda(x, y)$ is continuous. Moreover, $d_\lambda(\cdot\,, \cdot)$ is symmetric and satisfies the triangle inequality. In particular, if we consider the quotient T_λ of $\mathbb{H}^2$ under the equivalence relation

$$x \sim y \quad \text{if } d_\lambda(x, y) = 0,$$

we obtain a metric space. The fibers of the projection $\mathbb{H}^2 \to T_\lambda$ are the leaves and complementary components of λ. It follows that T_λ is a real tree.

LEMMA 7.1. *T_λ is a real tree. Moreover, the action $\pi_1(\Sigma) \curvearrowright \mathbb{H}^2$ descends to a free isometric action $\pi_1(\Sigma) \curvearrowright T_\lambda$.*

The tree T_λ is the *dual tree* to the lamination λ. See Skora [Sk1], [Sk2] and Otal [Ot3] for details.

Remark. For the sake of simplicity, we have discussed only the dual tree associated with a minimal measured lamination. It should be mentioned that one can modify the construction given here and associate with every measured lamination λ on S a dual tree T_λ which admits a *small* minimal action $\pi_1(S) \curvearrowright T_\lambda$; recall that an action of a group on a tree is *small* if the stabilizers of non-degenerate segments are virtually abelian. Conversely, it is a result of Skora [Sk2] that any tree admitting a small minimal action of $\pi_1(S)$ is dual to a lamination. As stated, this last statement is only true if S is a closed surface but it can be modified to allow for surfaces of finite topological type as well.

7.2. Morgan–Shalen theory

Suppose now that (M, P) is a pared manifold and let $\{\varrho_n\}_{n=1}^\infty$ be a sequence in $\mathrm{AH}(M, P)$. It is due to Morgan and Shalen [MS] (see also [P] and [Bes]) that, up to conjugation, the sequence $\{\varrho_n\}_{n=1}^\infty$ satisfies one of the following two alternatives:

(1) $\{\varrho_n\}_{n=1}^\infty$ contains a subsequence which converges in $\mathrm{AH}(M, P)$;

(2) there is a positive sequence $\varepsilon_n \to 0$ and a subsequence of $\{\varrho_n\}_{n=1}^{\infty}$, say the whole sequence, such that the actions $\pi_1(M) \curvearrowright_{\varrho_n} \varepsilon_n \mathbb{H}^3$ converge in the pointed equivariant Gromov–Hausdorff topology to an isometric action $\pi_1(M) \curvearrowright T$ on a real tree T such that

- any element represented by a curve in P has a fixed point in T;
- the stabilizer of any non-degenerate arc in T is virtually abelian; in other words, the action $\pi_1(M) \curvearrowright T$ is small;
- for all fixed $\gamma \in \pi_1(M)$ we have

$$l_T(\gamma) = \lim_{n\to\infty} \varepsilon_n l_{\mathbb{H}^3}(\varrho_n(\gamma)),$$

where $l_T(\cdot)$ is the translation length in T;

- the tree T is not reduced to a point and the action $\pi_1(M) \curvearrowright T$ is *minimal*, meaning that T does not contain any proper invariant subtree.

It is known that groups which do not split as amalgamated products or HNN-extensions over virtually abelian groups do not admit small actions on trees without global fixed points. This, and other similarly powerful results, follow from the so-called Rips machine. Bestvina and Feighn obtained a relative version of the Rips machine. More precisely, it follows directly from [BF, Theorem 9.6] together with the definition of pared manifolds, that in order to prove that a small minimal action $\pi_1(M) \curvearrowright T$ is trivial, it suffices to show that each one of the boundary groups has a fixed point; compare also with [Th2]. Since an action of a group in a tree has a global fixed point if and only if $l_T(\gamma)=0$ for all elements γ, we deduce the following result.

THEOREM 7.2. *Let (M,P) be a pared manifold and $\pi_1(M) \curvearrowright T$ be a small action on a real tree. Suppose that every element $\gamma \in \pi_1(M)$ represented by a curve in a free side of (M,P) is such that $l_T(\gamma)=0$. Then the whole group $\pi_1(M)$ fixes some point in T.*

Assume now that (M,P) is a pared manifold such that each free side F is incompressible. If (M,P) is also acylindrical, then we know that every small action $\pi_1(M) \curvearrowright T$ on a real tree such that the elements in P have fixed points in T, has a global fixed point. If (M,P) is not acylindrical this is no longer true. However we have the following fact [Ka] relating the JSJ-splitting of (M,P) and the possible small actions of $\pi_1(M)$ on real trees.

THEOREM 7.3. *Let (M,P) be a pared manifold and $\pi_1(M) \curvearrowright T$ be a small action such that every element in $\pi_1(M)$ represented by a curve in P fixes a point in T. We have $l_T(\gamma)=0$ for every element $\gamma \in \pi_1(M)$ which is represented by a curve which can be homotoped into either one of the annuli $\mathcal{A}$ provided by Theorem 3.2 or into one of the acylindrical components of the complement of $\mathcal{A}$.*

Note that if the tree T arises as a limit of representations in $\mathrm{AH}(M,P)$, then Theorem 7.3 follows from Thurston's *only windows break theorem* [Th2].

7.3. Continuity

Continuing with the same notation, let (M,P) be a pared manifold and $\pi_1(M)\curvearrowright T$ be a small action without global fixed point on a real tree. Suppose that every element in $\pi_1(M)$, which is represented by a curve in P, fixes a point in T. Finally let F be a free side of (M,P) and let λ be a measured lamination on F.

Denote by $\widetilde{F}$ the universal cover of F and let $\tilde{\lambda}$ be the preimage of λ in $\widetilde{F}$. Composing the homomorphism $\pi_1(F)\to\pi_1(N)$ with the action on T, we obtain an action $\pi_1(F)\curvearrowright T$. Following Otal [Ot1], [Ot3], we say that λ is *realized* in T if there is a continuous $\pi_1(F)$-equivariant map

$$\phi\colon \tilde{\lambda}\longrightarrow T$$

which is strictly monotonic when restricted to any leaf of $\tilde{\lambda}$. Note that, if λ is a simple closed curve, then this in particular means that $l_T(\lambda)>0$, where $l_T(\lambda)$ denotes the translation length of every element of $\pi_1(F)$ whose conjugacy class is represented by λ.

This notion will be important for us because of the following theorem, which is due to Otal [Ot1], [Ot3].

THEOREM 7.4. (Continuity theorem) *Let (M,P) be a pared manifold and $\{\varrho_n\}_{n=1}^{\infty}$ be a sequence of representations in $\mathrm{AH}(N,P)$. Suppose that, after scaling, the actions $\varrho_n(\pi_1(M))\curvearrowright\mathbb{H}^3$ converge in the equivariant Gromov–Hausdorff topology to a small action $\pi_1(M)\curvearrowright T$ without global fixed points. Suppose also that F is a free side of (M,P) and let $\lambda\in\mathcal{O}(F)$ be a filling Masur domain lamination on F.*

If λ is realized in T, then, for any sequence of simple closed curves $\{\gamma_n\}_{n=1}^{\infty}$ in F which converge to λ in $\mathcal{PML}(F)$, we have

$$\lim_{n\to\infty}\frac{l_{\mathbb{H}^3}(\varrho_n(\gamma_n))}{l_X(\gamma)}=\infty,$$

where $l_{\mathbb{H}^3}(\cdot)$ is the translation length in $\mathbb{H}^3$ and $l_X(\cdot)$ is the length with respect to some arbitrarily fixed finite-area hyperbolic metric on F.

Theorem 7.4 is useless unless one can ensure that in certain situations the lamination λ is realized. Not surprisingly, the first of these realization results is due to Otal [Ot3].

THEOREM 7.5. ([Ot3]) *Suppose that F is a hyperbolic surface with finite area, $\lambda,\mu\in\mathcal{PML}(F)$ and $\pi_1(F)\curvearrowright T$ is a small action without global fixed point on a real tree such that every element in $\pi_1(F)$ which is represented by peripheral curve in F fixes a point* *in T. Suppose that λ is filling, that μ is not realized in T and that $\iota(\lambda,\mu)>0$. Then λ is realized in T.*

Theorem 7.5, a consequence of the result of Skora [Sk2] mentioned at the end of §7.1, is stated in [Ot3] under the assumption that both λ and μ are filling; however, the proof applies word-by-word. Before going any further, observe that the following is a direct consequence of Theorem 7.5.

COROLLARY 7.6. *Let (M,P) be a twisted trivial interval bundle over a surface F and let $\lambda\in\mathcal{PML}(\partial M\setminus P)$ be a filling lamination whose projection to F is not a lamination on F. If $\pi_1(M)\curvearrowright T$ is a small action without global fixed point on a real tree such that every element in $\pi_1(M)$ which is represented by a curve in P fixes a point in T, then λ is realized in T.*

In [KS1] a version of Theorem 7.5 was proved for filling Masur domain laminations on the distinguished free side of compression bodies with empty pared locus. The proof in the general case is due to Lecuire [L1].

THEOREM 7.7. ([KS1], [L1]) *Let (C,P) be a non-trivial pared compression body with distinguished free side F and $\pi_1(C)\curvearrowright T$ be a small action without global fixed point on a real tree such that every element in $\pi_1(C)$ which is represented by a curve in P fixes a point in T. Then every filling Masur domain lamination $\lambda\in\mathcal{O}(C)$ is realized in T.*

8. Compactness theorem

In the proof of the density conjecture in the next section, the existence of certain algebraic limits will be crucial. The goal of this section is to ensure that such limits exist.

Fix some pared manifold (M,P) and a discrete and faithful representation

$$\varrho_0\colon \pi_1(M)\longrightarrow \mathrm{PSL}_2(\mathbb{C})$$

such that the associated hyperbolic manifold N_{ϱ_0} is geometrically finite and has associated pared manifold (M,P). The existence of such a representation is ensured by Thurston's hyperbolization theorem. Label the free sides of (M,P) by $F_1,...,F_s$ and suppose that we fix a tuple

$$\mathcal{I}=(X_1,...,X_r,\lambda_{r+1},...,\lambda_s), \tag{8.1}$$

where $X_i\in\mathcal{T}(F_i)$ is a point in the Teichmüller space of $\mathcal{T}(F_i)$ for $i=1,...,r$ and where $\lambda_i\in\mathcal{PML}(F_i)$ is a filling measured lamination for $i=r+1,...,s$. Fix also arbitrary hyperbolic metrics on the surfaces $F_{r+1},...,F_s$.

Then we say that the tuple $\mathcal{I}$ is *filling* if the following two conditions are satisfied:

(*) if (M, P) is an interval bundle over a surface S and $r=0$, then the union of the projection of the laminations $\lambda_1, ..., \lambda_s$ to S is not a lamination;

(**) if $i \geqslant r+1$ and F_i is a compressible free side of (M, P), then the lamination λ_i is a Masur domain lamination.

Theorem 6.13 asserts that the ending invariants of a hyperbolic 3-manifold form a filling tuple.

Continuing with the same notation, let $\pi_{\mathrm{AB}}: \mathcal{T}(\partial M \setminus P) \to \mathrm{QH}(\varrho_0)$ be the covering provided by Theorem 4.3; in other words, π_{AB} is the Ahlfors–Bers parametrization of the subset $\mathrm{QH}(\varrho_0)$ of $\mathrm{AH}(M, P)$ consisting of (conjugacy classes of) representations which are quasi-conformally conjugated to ϱ_0. Observe that the Teichmüller space $\mathcal{T}(\partial M \setminus P)$ is the cartesian product of the Teichmüller spaces of the free sides of (M, P). We say that a sequence

$$(X_1^n, ..., X_s^n) \in \mathcal{T}(F_1) \times ... \times \mathcal{T}(F_s) = \mathcal{T}(\partial M \setminus P)$$

is *filling* if, after possibly relabeling the free sides of (M, P), there is an associated filling tuple $\mathcal{I}$ as in (8.1) such that the following holds:

- for all n and all $i \leqslant r$ we have $X_i^n = X_i$;
- for all $i \geqslant r+1$ there is a sequence of simple closed curves γ_i^n which converge to λ_i in $\mathcal{PML}(F_i)$ such that

$$\lim_{n \to \infty} \frac{l_{X_i^n}(\gamma_n)}{l_{X_i^1}(\gamma_n)} = 0,$$

where $l_{X_i^n}(\gamma_n)$ is the length of the geodesic freely homotopic to γ_n in X_i^n.

Remark. Thurston showed how the Teichmüller space of a surface of finite type F can be compactified naturally by the space of projective measured laminations on F. It follows (see [FLP]) that the second condition above is satisfied whenever the sequence $\{X_i^n\}_{n=1}^{\infty}$ converges to the point λ_i in Thurston's compactification of $\mathcal{T}(F_i)$.

The main compactness result that we will need is the following.

THEOREM 8.1. *Let (M, P) be a pared manifold and $\varrho_0: \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ be a discrete and faithful representation such that the associated hyperbolic manifold N_{ϱ_0} is geometrically finite and has associated pared manifold (M, P). Denote by*

$$\pi_{\mathrm{AB}}: \mathcal{T}(\partial M \setminus P) \longrightarrow \mathrm{QH}(\varrho_0)$$

the Ahlfors–Bers parametrization of $\mathrm{QH}(\varrho_0)$ and label $F_1, ..., F_s$ the free sides of (M, P). If $(X_1^n, ..., X_s^n)$ is a filling sequence with associated filling tuple $(X_1, ..., X_r, \lambda_{r+1}, ..., \lambda_s)$ then, up to passing to a subsequence, the sequence $\varrho_n = \pi_{\mathrm{AB}}(X_1^n, ..., X_s^n)$ converges in $\mathrm{AH}(M, P)$ to some discrete and failthful representation ϱ.

Let N_ϱ be the hyperbolic 3-manifold associated with the limiting representation and (M', P') be its associated pared manifold. There is a map $\phi: (M, P) \to (M', P')$ in the homotopy class determined by ϱ such that

(1) *ϕ maps F_i homeomorphically onto a free side of (M', P') for all $i=1, ..., r$; moreover, for any such i, the end of N_ϱ associated with $\phi(F_i)$ is convex cocompact and has conformal boundary $\phi_*(X_i)$;*

(2) *for $i=r+1, ..., s$, the lamination λ_i is not realized in N_ϱ by any map in the homotopy class of the restriction of ϕ to F_i.*

For the sake of simplicity, we will prove Theorem 8.1 in the case when (M, P) is not an interval bundle. The proof for interval bundles follows along the same lines, actually allowing for some simplifications; compare for example with [Ot3].

Before going any further we state two lemmas which play an important role in the proof of Theorem 8.1. The assumption that the conformal structures on the compressible components of $\partial M \setminus P$ are either fixed or tend to a Masur domain lamination implies the following result.

LEMMA 8.2. *There is some $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ and $i=1, ..., s$, the meridians in X_i^n have at least length ε.*

See for example [KS1] for a proof of Lemma 8.2. Now, combining Lemma 8.2, Proposition 4.1 and the assumption that for $i=r+1, ..., s$ the conformal structures X_i^n converge to λ_i, we deduce the following lemma.

LEMMA 8.3. *There is some $K > 0$ such that the following holds*:

- *if γ is a non-peripheral curve in F_i for $i=1, ..., r$, then $l_{\mathbb{H}^3}(\varrho_n(\gamma)) \leqslant K l_{X_i}(\gamma)$;*
- *for $i > r$ there is a sequence $\{\gamma_i^n\}_{n=1}^\infty$ of simple closed curves in F_i converging in $\mathcal{PML}(F_i)$ to λ_i and such that*

$$\lim_{n\to\infty} \frac{l_{\mathbb{H}^3}(\varrho_n(\gamma_i^n))}{l_{X_i^1}(\gamma_i^n)} = 0.$$

We are now ready to launch the proof of Theorem 8.1.

Proof of Theorem 8.1. Suppose that we are in the situation described by Theorem 8.1. We start proving that the sequence $\{\varrho_n\}_{n=1}^\infty$ has a convergent subsequence.

CLAIM 1. *Up to conjugacy, the sequence $\{\varrho_n\}_{n=1}^\infty$ has a convergent subsequence.*

Seeking a contradiction, assume that no subsequence of the sequence $\{\varrho_n\}_{n=1}^\infty$ converges after conjugacy. Then, by the results of Morgan and Shalen mentioned in the previous section we may pass to a subsequence and conjugate so that there is a positive sequence $\varepsilon_n \to 0$ such that the actions $\varrho_n(\pi_1(M)) \curvearrowright \varepsilon_n \mathbb{H}^3$ converge in the pointed equivariant Gromov–Hausdorff topology to a small action $\pi_1(M) \curvearrowright T$ without global fixed point on a real tree. Recall that any element in $\pi_1(M)$ represented by a curve in P has a fixed point in T. We will obtain a contradiction showing that $\pi_1(M)$ has a global fixed point in T. In the light of Theorem 7.2, it suffices to prove that every element in $\pi_1(M)$ which is represented by a curve in a free side of (M,P) has vanishing translation length in T.

To begin with, consider those elements represented by curves $\gamma \subset F_1 \cup ... \cup F_r$; say $\gamma \subset F_1$. By the first statement of Lemma 8.3, the elements $\varrho_n(\gamma)$ have uniformly bounded translation length $l_{\mathbb{H}^3}(\varrho_n(\gamma))$. It follows that

$$l_T(\gamma) = \lim_{n\to\infty} \varepsilon_n l_{\mathbb{H}^3}(\varrho_n(\gamma)) = 0.$$

In other words, γ has a fixed point in T.

Suppose now that $i \geqslant r+1$ and that F_i is compressible. Let $(C,Q) \subset (M,P)$ be the relative pared compression body associated with the free side F_i and suppose that the restriction of the action $\pi_1(M) \curvearrowright T$ to $\pi_1(C)$ does not have a global fixed point. By Theorem 7.7, the lamination λ_i is realized in T. In particular, it follows from Theorem 7.4 that

$$\lim_{n\to\infty} \frac{l_{\mathbb{H}^3}(\varrho_n(\gamma_n))}{l_{X_i^1}(\gamma_n)} = \infty$$

for any sequence $\{\gamma_n\}_{n=1}^{\infty}$ of simple closed curves on F_i which converges to λ_i. This contradicts the second claim of Lemma 8.3.

It remains to consider the case where $i \geqslant r+1$ and F_i is incompressible. If F_i is a free side of the relative compression body neighborhood (C,Q) associated with some compressible side F_i', then the group $\pi_1(F_i)$ is a subgroup of $\pi_1(C)$ and hence, by the cases treated so far, has a global fixed point in T. Suppose that F_i does not belong to the union of the relative compression body neighborhoods associated with the compressible free sides of (M,P). In other words, F_i is a free side of the incompressible core (M'',P'') of (M,P). Since we are assuming that (M,P) is not a trivial interval bundle (and that F_i is a free side of both (M,P) and (M'',P'')), it follows that (M'',P'') is not a trivial interval bundle either. By Lemma 3.3, F_i is either large or at least can be homotoped into a large free side of (M'',P''). Hence, we may assume that F_i is large itself. By definition, this means that F_i contains some simple closed essential curve μ which is either contained in one of the annuli in the JSJ-splitting in Theorem 3.2 or in one of the acylindrical pieces. By Theorem 7.3, $l_T(\mu)=0$. Restrict the action of $\pi_1(M)$ on T to the action $\pi_1(F_i) \curvearrowright T$ and suppose that the latter action is non-trivial. As $l_T(\mu)=0$, we have that the curve μ is not realized in T. Since λ_i is filling, it follows from Theorem 7.5 that

λ_i is realized in T. We can now obtain a contradiction as in the compressible case. This concludes the proof of Claim 1. □

Continuing with the proof of Theorem 8.1, let ϱ be the limiting representation for a convergent subsequence of the sequence $\{\varrho_n\}_{n=1}^\infty$ and N_ϱ be the associated hyperbolic 3-manifold. Let also (M', P') be the pared manifold associated with N_ϱ and let

$$\phi: (M, P) \longrightarrow (M', P')$$

be a map in the homotopy class determined by ϱ. By Proposition 4.4, we may assume that ϕ maps, for $i=1, ..., r$, the free side F_i homeomorphically to a free side of (M', P') and that the end of N_ϱ associated with $\phi(F_i)$ is convex cocompact with conformal boundary $\phi_*(X_i)$. In other words, (1) holds.

It remains to prove that the laminations $\lambda_{r+1}, ..., \lambda_s$ are not realized in N_ϱ.

Claim 2. *For $i=r+1, ..., s$, the lamination λ_i is not realized in N_ϱ by any map in the homotopy class of the restriction of ϕ to F_i.*

Suppose that λ_i is realized in N_ϱ. Since λ_i fills F_i, every extension of λ_i is a union of λ_i and finitely many non-compact leaves, and it follows that every such extension is also realized. In particular, if $\{\gamma_n\}_{n=1}^\infty$ is a sequence of simple closed curves in F_i which converges to λ_i in $\mathcal{PML}(F_i)$, then every Hausdorff limit of $\{\gamma_n\}_{n=1}^\infty$ is also realized.

Then, by Proposition 6.10, there is some ε with

$$\liminf_{n\to\infty} \frac{l_{\mathbb{H}^3}(\varrho_n(\gamma_n))}{l_{X_i^1}(\gamma_n)} \geqslant \varepsilon,$$

which contradicts Lemma 8.3.

This concludes the proof of Claim 2 and hence of Theorem 8.1. □

Before moving on, we would like to observe that in [Oh3], Ohshika has given a proof of Theorem 8.1 without using the Rips machine. Other sophisticated compactness results can be found in [KLO].

9. The density conjecture

In this section we reduce the proof of the density conjecture to proving Theorem 1.4.

Theorem 1.1. (Density conjecture) *If Γ is a finitely generated Kleinian group, then the set of geometrically finite points in* $\mathrm{AH}(\Gamma)$ *is dense in the algebraic topology. In other words, the density conjecture holds.*

Before moving on, we recall that, according to the convention we have followed so far, Kleinian groups are automatically torsion free. In the final section of this paper we will prove that Theorem 1.1 remains valid for groups with torsion.

Proof. From now on, let Γ be as in the statement of Theorem 1.1. Fix $\bar{\varrho}\in\mathrm{AH}(\Gamma)$, consider the hyperbolic 3-manifold $N=\mathbb{H}^3/\bar{\varrho}(\Gamma)$, let (M,P) be the associated pared manifold and identify (M,P) with a standard compact core of $(N^\varepsilon,\partial N^\varepsilon)$.

To begin with, fix a geometrically finite, minimally parabolic representation $\varrho_0\in\mathrm{AH}(M,P)$ such that the hyperbolic 3-manifold $N_{\varrho_0}=\mathbb{H}^3/\varrho_0(\Gamma)$ has associated pared manifold (M,P); recall that the existence of ϱ_0 follows from Thurston's hyperbolization theorem [Mo], [Ot4], [Ka]. We are going to find a sequence $\{\varrho_n\}_{n=1}^\infty\subset\mathrm{QH}(\varrho_0)$ of quasi-conformal deformations of ϱ_0 with

$$\bar{\varrho}=\lim_{n\to\infty}\varrho_n.$$

We begin by labeling the free sides $F_1,...,F_s$. Up to relabeling, we may assume that the ends of N associated with $F_1,...,F_r$ are convex cocompact and those associated with $F_{r+1},...,F_s$ are degenerate. For $i=1,...,r$ let $X_i\in\mathcal{T}(F_i)$ be the point in the Teichmüller space of F_i representing the conformal boundary of the end facing F_i. Similarly, for $i=r+1,...,s$ let $\lambda_i\in\mathcal{PML}(F_i)$ be a measured lamination supported on the ending lamination of the end facing F_i. Recall that, by Theorem 6.13, the measured lamination λ_i is filling and belongs to the Masur domain $\mathcal{O}(F_i)$ of the free side F_i. In particular, the tuple $(X_1,...,X_r,\lambda_{r+1},...,\lambda_s)$ is filling. Let $(X_1^n,...,X_s^n)\in\mathcal{T}(\partial M\setminus P)$ be a filling sequence with associated filling tuple $(X_1,...,X_r,\lambda_{r+1},...,\lambda_s)$ and let

$$\varrho_n=\pi_{\mathrm{AB}}(X_1^n,...,X_s^n),$$

where $\pi_{\mathrm{AB}}\colon\mathcal{T}(\partial M\setminus P)\to\mathrm{QH}(\varrho_0)$ is the Ahlfors–Bers parametrization (Theorem 4.3). It follows from Theorem 8.1 that, up to passing to a subsequence, the sequence $\{\varrho_n\}_{n=1}^\infty$ converges to some $\varrho\in\mathrm{AH}(M,P)$ such that the limiting hyperbolic 3-manifold $N_\varrho=\mathbb{H}^3/\varrho(\Gamma)$ satisfies the following property:

(*) Let (M',P') be the pared manifold associated with N_ϱ which we again identify with a standard relative compact core of N_ϱ. There is a map $\phi\colon(M,P)\to(M',P')$ compatible with the the representation ϱ such that

(1) ϕ maps F_i homeomorphically onto a free side of (M',P') for all $i=1,...,r$; moreover, for any such i, the end of N_ϱ associated with $\phi(F_i)$ is convex cocompact and has conformal boundary $\phi_*(X_i)$;

(2) for $i=r+1,...,s$ the lamination λ_i is not realized in N_ϱ by any map in the homotopy class of the restriction of ϕ to F_i.

In this and the next two sections we will prove the following result.

THEOREM 1.4 *Let* (M,P) *be a pared manifold and* $\varrho\in\mathrm{AH}(M,P)$. *Let also*

$$(M',P')\subset(N_\varrho^\varepsilon,\partial N_\varrho^\varepsilon)$$

be a standard relative compact core of the hyperbolic 3*-manifold* $N_\varrho=\mathbb{H}^3/\varrho(\pi_1(M))$, *and* $\phi\colon(M,P)\to(M',P')$ *be in the homotopy class determined by* ϱ. *Suppose that* λ *is a filling Masur domain lamination on a free side* F *of* (M,P) *which is not realized in* N_ϱ. *Then* ϕ *is homotopic, relative to the complement of a regular neighborhood of* F, *to a map* $\phi_1\colon(M,P)\to(M',P')$ *such that*

- *the restriction of* ϕ_1 *to* F *is a homeomorphism to some free side* F' *of* (M',P');
- *the end of* N_ϱ *associated with* F' *is degenerate and has ending lamination* $\phi_1(\lambda)$.

We continue with the proof of Theorem 1.1. It follows from (*) and applying Theorem 1.4 for every free side F_i of (M,P), $i=r+1,...,s$, that the following holds:

(**) Let (M',P') be the pared manifold associated with N_ϱ. There is a map $\phi\colon(M,P)\to(M',P')$ compatible with the isomorphism $\pi_1(M)\to\pi_1(M')=\pi_1(N_\varrho)$ induced by the representation ϱ such that

(1) ϕ maps F_i homeomorphically onto a free side of (M',P') for all $i=1,...,r$; moreover, for any such i, the end of N_ϱ associated with $\phi(F_i')$ is convex cocompact and has conformal boundary $\phi_*(X_i)$;

(2) for $i=r+1,...,s$ the restriction of ϕ to F_i is a homeomorphism onto a free side of (M',P') and the end of N associated with $\phi(F_i)$ is degenerate with ending lamination $\phi(\lambda_i)$.

Observe now that the restriction of the map ϕ provided by (**) to any free side of (M,P) is a homeomorphism onto its image. In particular, we are not in case (1) of Theorem 3.12. We claim that case (2) does not occur either because, unless $\bar\varrho$ is Fuchsian, even if (M,P) is a trivial interval bundle, the end invariants associated with the different free sides are different. (Obviously a Fuchsian representation is geometrically finite.) We then deduce from Theorem 3.12 that ϕ is homotopic, via maps which map boundary-to-boundary, to a homeomorphism

$$\phi_1\colon(M,P)\longrightarrow(M',P')$$

which maps ending laminations to ending laminations and is holomorphic on the conformal boundary. It follows from the ending lamination theorem that there is an isometry $\Psi\colon N\to N_\varrho$ in the isotopy class determined by ϕ_1.

At this point, it is possible that Ψ is orientation reversing. However, if this is the case, we can replace each geometrically finite representation ϱ_n by another one which provides the same geometrically finite hyperbolic 3-manifold with the opposite orientation. The new sequence will still be convergent and the limiting representation provides a hyperbolic 3-manifold which is isometric to N with an orientation-preserving isometry.

Hence, we may assume that $\Psi: N \to N_\varrho$ is an orientation-preserving isometry in the isotopy class determined by ϕ_1. In other words, the representations $\bar{\varrho}$ and ϱ are conjugated in $\mathrm{PSL}_2(\mathbb{C})$. Since by construction ϱ is the limit of a sequence of the geometrically finite representations $\{\varrho_n\}_{n=1}^{\infty}$, it follows that after suitably conjugating $\{\varrho_n\}_{n=1}^{\infty}$ we obtain a geometrically finite sequence converging to $\bar{\varrho}$.

As $\bar{\varrho}$ was arbitrary, this proves that every point in $\mathrm{AH}(\Gamma)$ is an algebraic limit of geometrically finite representations and hence ends the proof of Theorem 1.1. □

The remaining part of the paper , but for the last section, is devoted to the proof of Theorem 1.4. Before going any further, we reduce it to a more concrete problem about compression bodies. In the course of the next two sections we are going to prove the following theorem.

THEOREM 9.1. *Let $C=(C,P)$ be a non-trivial pared compression body with distinguished free side $\partial_e C$. Suppose that $\varrho \in \mathrm{AH}(C,P)$ is such that there is no non-peripheral simple closed curve γ on a constituent of (C,P), with $\varrho(\gamma)$ parabolic, and let*

$$N_\varrho = \mathbb{H}^3/\varrho(\pi_1(C))$$

be the associated hyperbolic 3-manifold. Also assume that $(M,Q) \subset (N_\varrho^\varepsilon, \partial N_\varrho^\varepsilon)$ is a standard compact core of N_ϱ and that λ is a filling Masur domain lamination on $\partial_e C$ which is not realized in N_ϱ. Then the following hold:

- *there is a pared homeomorphism $G:(C,P) \to (M,Q)$ in the proper homotopy class determined by ϱ; in particular, (M,Q) is a pared compression body*;
- *the image of λ under G is supported on the ending lamination of the end of N_ϱ associated with the distinguished free side of (M,Q).*

Assuming Theorem 9.1, we prove Theorem 1.4.

Proof of Theorem 1.4. First we use Theorem 9.1 to show that ϕ is homotopic, relative to the complement of a regular neighborhood of F, to a map $\phi_1:(M,P) \to (M',P')$ such that the restriction of ϕ_1 to F is a finite sheeted covering to a free side F' of (M',P'), and also that the end of N associated with F' must be degenerate with ending lamination $\phi_1(\lambda)$. Then we use the fact that ϕ_1 is a homotopy equivalence to prove that the restriction to F must indeed be a homeomorphism.

If F is incompressible, then the restriction of ϱ to $\pi_1(F)$ induces an element of $\mathrm{AH}(F,\partial F)$. The hyperbolic manifold $\widetilde{N} = \mathbb{H}^3/\varrho(\pi_1(F))$ covers N_ϱ. Note that λ cannot be realized in this cover, otherwise the projection to N_ϱ would give a realization of λ in N. Work of Bonahon [Bo2] shows that $\widetilde{N}$ is homeomorphic to a trivial interval bundle over the interior of F. Thurston [Th1] showed that for hyperbolic 3-manifolds homeomorphic to a trivial interval bundle over the interior of a surface F, a filling lamination is not realized if and only if it is the ending lamination of a geometric end of the hyperbolic manifold. Hence the relative compact core of $\widetilde{N}$ has a free side that can be identified with F in such a way that the induced map from the inclusion of this side to $\widetilde{N}$ is in the homotopy class determined by ϱ. Furthermore, the end of $\widetilde{N}$ associated with this free side is degenerate with ending lamination λ. Once we know this, the covering theorem shows that the covering map $\widetilde{N}\to N_\varrho$ is finite-to-one restricted to this end and projects to a degenerate end of N_ϱ. In particular, if F' is the free side of (M',P') associated with this end, we can homotope the restriction of ϕ to $(F,\partial F)$, through maps of pairs $(F,\partial F)\to(M',P')$, to a covering map onto $(F',\partial F')$. Moreover, the ending lamination for the considered end of $\widetilde{N}$ projects to the ending lamination for the considered end of N_ϱ, and therefore it is the projection of λ under the covering map.

If F is compressible, then the relative pared compression body $C=(C,P)$ associated with F is non-trivial. The representation ϱ induces a representation in $\mathrm{AH}(C,P)$ which we call ϱ'. Let $\gamma_1,...,\gamma_l$ be a maximal collection of pairwise disjoint non-parallel and non-peripheral simple closed curves in the constituents of (C,P) with $\varrho'(\gamma_i)$ parabolic for each i. Let then P' be the union of P with a regular neighborhood of $\gamma_1\cup...\cup\gamma_l$ in $\partial C\setminus P$. Observe that the representation ϱ' belongs not only to $\mathrm{AH}(C,P)$, but also to $\mathrm{AH}(C,P')$. As a representation in $\mathrm{AH}(C,P')$, the representation ϱ' satisfies the hypothesis of Theorem 9.1.

Let $\widetilde{N}$ be the cover of N_ϱ associated with the image of ϱ'. Theorem 9.1 shows that there is a pared homeomorphism from (C,P') to the relative compact core of $\widetilde{N}$ in the homotopy class determined by ϱ' and that the end of $\widetilde{N}$ associated with F is degenerate with ending lamination λ. From here on, the argument is similar to the previous case. Using Canary's covering theorem, the projection $\widetilde{N}\to N_\varrho$ is finite-to-one restricted to this end, and maps this end to a degenerate end of N_ϱ which is associated with a free side F' of (M',P'). This immediately implies again that the restriction of ϕ to $(F,\partial F)$ (through maps of pairs $(F,\partial F)\to(M',P')$) is homotopic to a covering map onto $(F',\partial F')$ and that the ending lamination of the end associated with F' is the image of λ.

At this stage, we are almost done with the proof of Theorem 1.4. We have namely constructed $\phi_1\colon(M,P)\to(M',P')$ satisfying the conclusion of the theorem except that the restriction of ϕ_1 to F may be a non-trivial covering map to the free side F' of (M',P'). We claim that this is impossible knowing that ϕ_1 is a homotopy equivalence.

First consider the case where F is not closed and meets the pared locus P. It follows from Lemma 3.7 that, if E_1 and E_2 are distinct components of ∂F, their images by a homotopy equivalence have to be distinct, unless E_1 and E_2 bound an annular component of P. Even more, if the homotopy equivalence identifies E_1 and E_2, the induced map from E_1 to E_2 is orientation reversing with respect to the orientations induced by F on E_1 and E_2. This shows that such a map cannot be a covering map and the cover $F\to F'$ has to be trivial outside of a compact set in F; clearly, this implies that the whole cover is trivial.

Suppose now that F, and hence F', are closed surfaces. It is well known that the kernels of the maps

$$H_1(F,\mathbb{R}) \longrightarrow H_1(M,\mathbb{R}) \quad \text{and} \quad H_1(F',\mathbb{R}) \to H_1(M',\mathbb{R})$$

are Lagrangian subspaces of the symplectic vector spaces $H_1(F,\mathbb{R})$ and $H_1(F',\mathbb{R})$. On the other hand, the kernel of the homomorphism

$$H_1(F,\mathbb{R}) \longrightarrow H_1(F',\mathbb{R})$$

induced by a non-trivial cover is a non-trivial symplectic subspace. This implies that, if the cover $F\to F'$ is non-trivial, then there is $[\alpha]\in\operatorname{Ker}(H_1(F,\mathbb{R})\to H_1(F',\mathbb{R}))$ with $[\alpha]\notin\operatorname{Ker}(H_1(F,\mathbb{R})\to H_1(M,\mathbb{R}))$. Hence, the homomorphism

$$H_1(M,\mathbb{R}) \longrightarrow H_1(M',\mathbb{R})$$

induced by ϕ has non-trivial kernel, contradicting the assumption that ϕ is a homotopy equivalence. □

10. Reducing Theorem 9.1 to a topological problem

From now on, assume that we are in the setting of Theorem 9.1. In other words, we have a pared compression body $C=(C,P)$ with distinguished free side $\partial_e C=\partial_e(C,P)$, a discrete and faithful representation $\varrho\colon\pi_1(C)\to\mathrm{PSL}_2(\mathbb{C})$ mapping elements in P to parabolic elements and a filling Masur domain lamination $\lambda\in\mathcal{O}(\partial_e C)$ which is not realized in the hyperbolic 3-manifold $N_\varrho=\mathbb{H}^3/\varrho(\pi_1(C))$ associated with ϱ. We also assume that there is no simple closed non-peripheral curve γ contained in a constituent of (C,P) with $\varrho(\gamma)$ parabolic.

In order to save notation, set $N=N_\varrho$.

10.1. A proper map

Our first goal is to approximate λ by a sequence of multicurves which are realized in a controlled way. It follows from the work of Klarreich [Kl] that the lamination λ represents a point in the Gromov boundary of the curve complex of $\partial_e C$. With the curve complex in mind (compare with for example [Br2]), we can now find a sequence of pants decompositions γ_n of $\partial_e C$ converging to λ in the curve complex, and such that, for all n, the pants decompositions γ_n and γ_{n+1} are at distance 1 in the pants complex. In more concrete terms, this last condition means that, for all n, there are two curves α and β with

$$\gamma_n = \alpha \cup (\gamma_n \cap \gamma_{n+1}) \quad \text{and} \quad \gamma_{n+1} = \beta \cup (\gamma_n \cap \gamma_{n+1})$$

and such that α and β intersect minimally; this means that $\iota(\alpha, \beta) = 2$ if α is separating in $\partial_e C \setminus (\gamma_n \cap \gamma_{n+1})$ and $\iota(\alpha, \beta) = 1$ otherwise. Fix from now on such a sequence γ_n of pants decompositions and choose for all n a lamination μ_n that contains γ_n and is *maximal*, i.e. all the complementary regions are ideal triangles. Note that such a lamination consist of components of γ_n and a finite number of non-compact leaves spiraling about a component of γ_n.

It follows from the work of Klarreich [Kl] that any accumulation point of the sequence $\{\gamma_n\}_{n=1}^{\infty}$ in $\mathcal{PML}(\partial_e C)$ has the same support as λ, and hence belongs to $\mathcal{O}_\varrho(C)$ by Lemma 6.1. Lemma 6.3 implies now that for all sufficiently large n, say for all n, γ_n belongs to $\mathcal{O}_\varrho(C)$ as well. In particular, it follows from Lemma 6.4 that the laminations μ_n are realized in N by pleated surfaces $f_n: (\partial_e C, \sigma_n) \to N$ in the correct homotopy class. Moreover, Lemma 6.7 implies that the surfaces $(\partial_e C, \sigma_n)$ do not contain short essential curves which are either compressible in C or homotopic into P. Finally, it follows from Proposition 6.5 and the assumption that λ is not realized, that for every compact set K there is some n_K such that $f_n(\partial_e C) \cap K \neq \varnothing$ for all $n \geqslant n_K$. We summarize all this in the following lemma.

Lemma 10.1. *There is n_0 such that for every $n \geqslant n_0$ the lamination μ_n is realized by some pleated surface $f_n: (\partial_e C, \sigma_n) \to N$ in the correct homotopy class. Moreover, there is ε such that for all $n \geqslant n_0$ every essential non-peripheral curve $\eta \subset \partial_e C$ which either bounds a disk in C or whose image under ϱ is parabolic has at least length $l_{\sigma_n}(\eta) \geqslant \varepsilon$. Finally, if $K \subset N$ is a compact set, then there is n_K with $K \cap f_n(\partial_e C) = \varnothing$ for all $n \geqslant n_K$.*

Forgetting the first terms of the sequence, we may assume that $n_0 = 0$. Below we are going to prove the following result.

Lemma 10.2. *There is a constant d such that for all n the maps f_n and f_{n+1} are homotopic by a homotopy whose image is contained in the radius-d neighborhood of $f_n(\partial_e C) \cup f_{n+1}(\partial_e C)$.*

Assuming Lemma 10.2, let

$$F:\partial_e C\times[0,\infty)\longrightarrow N \tag{10.1}$$

be the map with $F(x,n)=f_n(x)$ for all $n\in\mathbb{N}$ and $x\in\partial_e C$, and whose restriction to

$$\partial_e C\times[n,n+1]$$

is the homotopy provided by Lemma 10.2.

As in the statement of Theorem 9.1, let (M,Q) be the pared manifold associated with the hyperbolic manifold N and identify (M,Q) with a standard relative compact core of $(N^\varepsilon,\partial N^\varepsilon)$. It follows from Lemma 10.1 that there is n_1 such that $f_n(\partial_e C)$ does not intersect the radius-d neighborhood of M for all $n\geqslant n_1$. In particular,

$$F(\partial_e C\times[n_1,\infty))\cap M=\varnothing.$$

The collar lemma and the fact that there are no short compressible or parabolic curves in $(\partial_e C,\sigma_n)$ imply that there is n_2 such that $N^\varepsilon\cap(F(\partial_e C\times[n_2,\infty))$ is contained in a single connected component of $N^\varepsilon\setminus M$. Assuming Lemma 10.2, we have proved the following result.

Proposition 10.3. *There is some n_0 such that the map $F:\partial_e C\times[0,\infty)\to N$ defined in* (10.1) *satisfies the following properties*:

- $F(x,n)=f_n(x)$ *for all* $n\in\mathbb{N}$ *and* $x\in\partial_e C$;
- $F(\partial_e C\times[n_0,\infty))\cap M=\varnothing$;
- $N^\varepsilon\cap F(\partial_e C\times[n_0,\infty))$ *is contained in a single connected component of* $N^\varepsilon\setminus M$.

Now we prove Lemma 10.2.

Proof of Lemma 10.2. In the argument below it will be important to differentiate between the open distinguished free side $\partial_e C$ and the compact free side $\overline{\partial_e C}$. Suppose that $\gamma_n=\alpha\cup(\gamma_n\cap\gamma_{n+1})$ and $\gamma_{n+1}=\beta\cup(\gamma_n\cap\gamma_{n+1})$, and let $Y\subset\overline{\partial_e C}$ be the compact surface, obtained by cutting $\overline{\partial_e C}$ along $\gamma_n\cap\gamma_{n+1}$, that contains α and β. We know that Y is either a 4-holed sphere or a 1-holed torus. First we construct a triangulation $\mathcal{T}$ on $\overline{\partial_e C}$ as follows. Restricted to Y, we assume that $\mathcal{T}$ is one of the triangulations in Figure 1. Then extend this to a triangulation of the entire surface $\overline{\partial_e C}$ in such a way that all the vertices are on components of $\gamma_n\cap\gamma_{n+1}$ or on components of $\partial\overline{\partial_e C}$. What is important for us about this triangulation is that γ_n and γ_{n+1} are homotopic to subgraphs of the 1-skeleton of $\mathcal{T}$ and every triangle on Y has at least one vertex on ∂Y.

For any choice for images of vertices of $\mathcal{T}$ on geodesic representative of $\varrho(\gamma_n)$ (resp. $\varrho(\gamma_{n+1})$) that preserves their ordering, we can construct a simplicial hyperbolic map

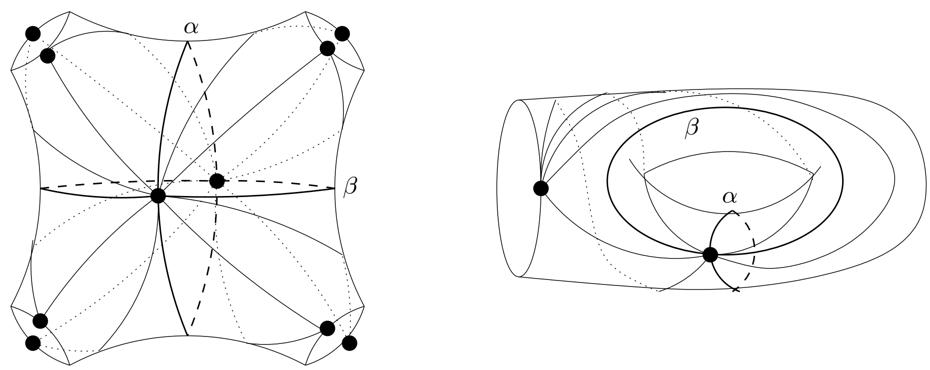

Figure 1. The triangulation on Y.

$\partial_e C \to N$ with associated triangulation $\mathcal{T}$ that realizes γ_n (resp. γ_{n+1}) and maps every simplex of $\mathcal{T}$ contained in $\partial_e C$ to a totally geodesic simplex in N. Intuitively, vertices and edges contained in the boundary of $\overline{\partial_e C}$ are mapped to cusps. The construction is standard; see [Ca3] for details.

Using an idea of Thurston, we can construct a continuous family g_n^t (resp. g_{n+1}^t of simplicial hyperbolic surfaces as above that converge to f_n (resp. f_{n+1}) uniformly on compact subsets of $\partial_e C$. This is possible by starting from one such map g_n^0 (resp. g_{n+1}^0) and for each component γ of γ_n (resp. γ_{n+1}) continuously twist the images of vertices on γ about the geodesic representative of γ in the direction that non-compact leaves of μ_n (resp. μ_{n+1}) spiral about γ when approaching γ, and then construct the simplicial hyperbolic surface g_n^t (resp. g_{n+1}^t) as above. (Here we are assuming that the image of γ_n (resp. γ_{n+1}) is fixed and we are twisting the vertices of the triangulation about the components.) The maps g_n^t (resp. g_{n+1}^t) converge uniformly on every compact set, as $t \to \infty$, and obviously the limit is a map $\partial_e C \to N$ that realizes μ_n (resp. μ_{n+1}). However μ_n (resp. μ_{n+1}) is maximal and therefore every two realizations of μ_n (resp. μ_{n+1}) are the same up to precomposition with a self-homeomorphism of $\partial_e C$ that is isotopic to the identity. After such a precomposition, we have that g_n^t (resp. g_{n+1}^t) converges uniformly to f_n (resp. f_{n+1}) when $t \to \infty$. In particular given $\varepsilon > 0$, we can choose t large enough so that the homotopy between f_n and g_n^t (resp. f_{n+1} and g_{n+1}^t) has tracks $\leqslant 1$ for all $x \in \partial_e C$.

Let $h_n = g_n^t$ and $h_{n+1} = g_{n+1}^t$. It will be enough to show the existence of a homotopy between h_n and h_{n+1} whose image is contained in a uniformly bounded neighborhood of $h_n(\partial_e C) \cup h_{n+1}(\partial_e C)$. First of all, we precompose h_n or h_{n+1} with a self-homeomorphism of $\partial_e C$ isotopic to the identity to make h_n and h_{n+1} identical restricted to $\gamma_n \cap \gamma_{n+1}$. We know that there is a homotopy between h_n and h_{n+1} and we can consider this as a

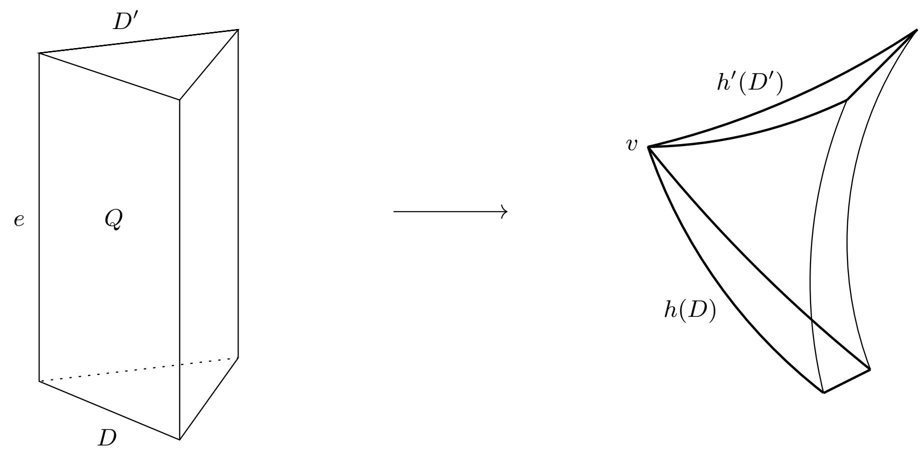

Figure 2. The image of a prism.

proper map from $\partial_e C \times [0,1] \to N$, where restricted to $\partial_e C \times \{0\}$ and $\partial_e C \times \{1\}$ the map induces h_n and h_{n+1}. Similar to our construction of the simplicial hyperbolic surfaces, we consider the maps h_n and h_{n+1} as maps of $\partial_e C$ with the triangulation $\mathcal{T}$.

The simplicial structure of h_n and h_{n+1} makes $\overline{\partial_e C} \times \{0,1\}$ triangulated with two triangulations which are isotopic to $\mathcal{T}$ on $\overline{\partial_e C}$. Extend this to a triangulation of $\overline{\partial_e C} \times [0,1]$, first connecting every vertex on $\overline{\partial_e C} \times \{0\}$ to the corresponding vertex on $\overline{\partial_e C} \times \{1\}$. Then add faces homeomorphic to rectangles, where two opposite sides of the rectangle are corresponding edges of the triangulations on $\overline{\partial_e C} \times \{0\}$ and $\overline{\partial_e C} \times \{1\}$. Finally, we are left with regions that are homeomorphic to a triangle times an interval, we call them *prisms*, and simply divide each of these into three tetrahedrons arbitrarily. Now we may assume that the homotopy is totally geodesic restricted to the 1-skeleton and 2-skeleton of the constructed triangulation and extend it to the 3-skeleton (the prisms) by coning off from a vertex of each tetrahedron and map every line segment geodesically.

It will be enough to show that the image of every prism stays in a bounded diameter neighborhood of $h_n(\partial_e C) \cup h_{n+1}(\partial_e C)$. In fact, it is enough to do this for faces of the prisms. Every prism Q has two triangular faces D and D' which we call *horizontal*, and we call the other faces and edges that connect these horizontal faces *vertical*. The image of horizontal faces are contained in $h_n(\partial_e C) \cup h_{n+1}(\partial_e C)$. In our construction of the triangulation $\mathcal{T}$, each triangle has at least one vertex on either γ_n, γ_{n+1} or the boundary of $\partial_e C$, and the image of the vertical edge e associated with this vertex will be a single point v on the geodesic representative of $\gamma_n \cap \gamma_{n+1}$ or an ideal point. A picture of the image of a prism is suggested in Figure 2.

Every point in the image of the prism is contained in a triangle with a vertex v and two sides on $h_n(D)$ and $h_{n+1}(D')$, and from this and hyperbolicity of N we can see that it has to have bounded distance from $h_n(D) \cup h_{n+1}(D')$, and therefore from $h_n(\partial_e C) \cup h_{n+1}(\partial_e C)$. This concludes the proof of Lemma 10.2. □

We should remark that Brock [Br2] used a very similar construction of triangulations and interpolations between pleated surfaces to control volume. In the above lemma we also need to control the homotopy tracks and therefore we have explained the construction of the interpolations, otherwise this is just the same construction.

10.2. Proof of Theorem 9.1

Continuing with the same notation as above, let $F: \partial_e C \times [0, \infty) \to N$ be the map defined in (10.1). Recall also that (M, Q) is the pared manifold associated with N and that we have identified (M, Q) with a standard relative compact core of $(N^\varepsilon, \partial N^\varepsilon)$.

Up to replacing t by $t+n_0$, where n_0 is the constant provided by Proposition 10.3, we may assume that the image of F is disjoint from M, and that the image of F intersects a single connected component $\mathcal{E}$ of $N^\varepsilon \setminus M$. Denote the free side of (M, Q) facing $\mathcal{E}$ by R and, from now on, consider $\mathcal{E}$ as a geometric end of N. The assumption that (M, Q) is a standard relative compact core implies that $\mathcal{E}$ is homeomorphic to $R \times [0, \infty)$. Even more, the sequence $f_n(\gamma_n) = F(\gamma_n, n)$ is a sequence of closed geodesics in $\mathcal{E}$ that leave every compact set. So the geometric end $\mathcal{E}$ has to be simply degenerate; let $\mu_{\mathcal{E}}$ be the unmeasured lamination on R which is the ending lamination of $\mathcal{E}$. Also let

$$\pi: \mathcal{E} \longrightarrow R$$

be the projection induced by such a product structure. We denote by F_0 the restriction of F to $\partial_e C \times \{0\} \simeq \partial_e C$ and set

$$g = \pi \circ F_0: \partial_e C \longrightarrow R.$$

In particular, for every n, the closed geodesic $F(\gamma_n, n)$ is homotopic to $g(\gamma_n)$ within the end $\mathcal{E}$. By Proposition 6.12, every accumulation point of the sequence $\{g(\gamma_n)\}_{n=1}^\infty$ in the space $P\mathcal{C}(R)$ of projective currents is a current supported on $\mu_{\mathcal{E}}$. Since this fact is going to play a crucial role, we state it as a lemma.

Lemma 10.4. *Every accumulation point of the sequence $\{g(\gamma_n)\}_{n=1}^\infty$ in the space $P\mathcal{C}(R)$ of projective currents is a current μ supported on the ending lamination $\mu_{\mathcal{E}}$ of the end of N facing R.*

The map $F_0\colon \partial_e C \to N$ is in the homotopy class determined by the composition of ϱ and the induced map by the inclusion $\partial_e C \hookrightarrow C$, and hence it extends to a proper map $(C,P) \to (N^\varepsilon, \partial N^\varepsilon)$ which induces ϱ after identifying $\pi_1(M)$ with $\pi_1(N)=\varrho(\pi_1(C))$ and choosing suitable base points. Since π is a homotopy equivalence, the same argument applies again showing that g extends to a map $G\colon (C,P)\to(M,Q)$ which induces an isomorphism on fundamental groups. Being a compression body, $\pi_1(\partial_e C)$ surjects onto $\pi_1(C)$. In particular $g_*(\pi_1(\partial_e C))\subset\pi_1(R)$ surjects onto $\pi_1(M)$. This proves that (M,Q) is a pared compression body with exterior boundary $R=\partial_e(M,Q)$.

We denote by Q' the union of those components of Q which contain the image under G of a component of P. Clearly, (M,Q') is a pared compression body and we can consider g and G as maps $g\colon \partial_e C\to\partial_e(M,Q')$ and $G\colon (C,P)\to(M,Q')$, respectively. Observe that, because G is a homotopy equivalence and because of the choice of Q', no essential non-peripheral curve in a constituent of $C=(C,P)$ is mapped under G to a curve which can be homotoped within M to a curve in Q'. In particular, if (C,P) has constituents then we can apply Lemma 3.15 and deduce that G is homotopic, through maps mapping $\partial_e C$ to $\partial_e(M,Q')$, to a homeomorphism $(C,P)\to(M,Q')$. In the case where (C,P) has no constituents, it follows from Lemma 3.5 that G is again homotopic to a homeomorphism $(C,P)\to(M,Q')$. However, it should be recalled that this time and at this point, we still have to face the possibility that perhaps the homotopy cannot be chosen to take place through maps mapping $\partial_e C$ to $\partial_e(M,Q')$; by Lemma 3.15, this is only the case if $g\colon \partial_e C\to R$ is not π_1-injective (compare with Example 3.10).

Once we know that G is pared homotopic to a homeomorphism between (C,P) and (M,Q') through maps that take $\partial_e C$ to $\partial_e(M,Q')$, it follows from the assumption in Theorem 9.1 that there is no non-peripheral simple closed curve γ on a constituent of (C,P) with $\varrho(\gamma)$ parabolic and that $Q'=Q$, and therefore $\partial_e(M,Q')=R$. Keeping the same notation, we can sum up what we have as follows.

Lemma 10.5. *The map $g\colon \partial_e C\to R$ extends to a map of pairs $G\colon (C,P)\to(M,Q')$ homotopic to a homeomorphism. Moreover, the homotopy can be chosen to be through maps mapping $\partial_e C$ to R if either (C,P) has constituents or if g is π_1-injective.*

In the light of Lemma 10.5, we can identify (C,P) and (M,Q'). Hence, we consider g and G as self maps of $\partial_e C$ and (C,P), respectively; observe that G is then homotopic to the identity. Since R is an essential subsurface of $\partial_e(M,Q')$, the natural map from the space $P\mathcal{C}(R)$ of projective currents to $P\mathcal{C}(\partial_e(M,Q'))$ is an embedding. In particular, we can consider $\mu_{\mathcal{E}}$, the ending lamination associated with the geometric end of N facing R, as an unmeasured lamination μ on $\partial_e C$, which may be supported on a proper subsurface of $\partial_e C$. Even more, Lemma 10.4 can be restated as asserting that every accumulation point of the sequence $\{g(\gamma_n)\}_{n=1}^{\infty}$ in $P\mathcal{C}(\partial_e C)$ is a current supported on μ. On the other hand, recall that, by construction, the curves γ_n converge in $\mathcal{PML}(\partial_e C)$ to the filling Masur domain lamination λ. Below we are going to prove the following result.

PROPOSITION 10.6. *Let $\lambda\in\mathcal{O}(C,P)$ be a filling Masur domain lamination on the distinguished free side $\partial_e C$ of the pared compression body (C,P) and $g\colon\partial_e C\to\partial_e C$ be a continuous map which extends to a self-map of (C,P) that is homotopic to the identity through maps $(C,P)\to(C,P)$. Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence of simple closed curves on $\partial_e C$ converging to λ in $\mathcal{PML}(\partial_e C)$. If the sequence of closed curves $\{g(\gamma_n)\}_{n=1}^{\infty}$ converges* (*in the space of projective currents*) *to a projective current supported on a lamination μ, then g is homotopic to a homeomorphism.*

Assuming Proposition 10.6, observe that since g is homotopic to a homeomorphism, it is in particular π_1-injective. Hence, by Lemma 10.5, it follows that the map G is in all cases homotopic to a homeomorphism via maps which map the distinguished free side $\partial_e C$ of $C=(C,P)$ to the distinguished free side R of (M,Q). This proves the first claim of Theorem 9.1.

In order to prove the second one, we observe that a homeomorphism between surfaces induces a homeomorphism between the associated spaces of projective measure laminations. In particular,

$$\lim_{n\to\infty} g(\gamma_n)=g\Big(\lim_{n\to\infty}\gamma_n\Big)=g(\lambda)$$

is supported on the ending lamination $\mu_{\mathcal{E}}$ associated with the distinguished free side of (M,Q). Up to proving Proposition 10.6, this concludes the proof of Theorem 9.1.

11. Proof of Proposition 10.6

Before launching the proof of Proposition 10.6, we describe briefly the strategy. To begin with we may assume, passing to a subsequence, that the curves γ_n converge in the Hausdorff topology to some lamination on $\partial_e C$; for typographical reasons, we will abuse notation and denote this lamination by λ as well. Fixing some hyperbolic metric on $\partial_e C$, we will first find a train-track τ carrying λ and such that the restriction of g to all the routes in τ is a quasi-geodesic with constants independent of the particular route. In particular, g induces a continuous map from the set of currents whose support is carried by τ to the set of currents on $\partial_e C$. Hence, the image of λ under this map is carried by the support of μ. This implies that, up to homotopy, g maps leaves of λ to leaves of μ. We are going to show that g also maps complementary regions of λ to complementary regions of μ. At this point we see that g induces an equivariant map from the tree T_λ

dual to λ to the tree T_μ dual to μ. This map is continuous and we will show that it is injective. This implies that g induces an isomorphism

$$g_*\colon \pi_1(\partial_e C) \to \pi_1(\partial_e C),$$

and hence that it is properly homotopic to a homeomorphism.

Assume from now on that we are in the situation described in the statement of Proposition 10.6. Let also N be a geometrically finite hyperbolic manifold with associated pared manifold (C, P). From now on, we identify $C \setminus P$ with the convex core $\mathrm{CC}(N)$. Hence we have a topological identification between $\partial_e C$ and the compressible component of $\partial \mathrm{CC}(N)$.

Recall that, by Corollary 6.6, the lamination λ is realized in N by some pleated surface

$$f\colon (\partial_e C, \sigma) \to \mathrm{CC}(N).$$

From now on, we consider $\partial_e C$ to be endowed with the metric σ.

Since λ is realized, it follows from Lemmas 6.8 and 6.9 that λ is carried by some train-track $\tau \subset \partial_e C$ which admits a realization

$$\phi\colon \tau \longrightarrow \mathrm{CC}(N)$$

homotopic to the inclusion $\tau \hookrightarrow \partial_e C \subset \partial \mathrm{CC}(N)$ and such that the induced map between universal covers is uniformly quasi-geodesic on each route.

11.1. Routes go to quasi-geodesics

Let C' be the universal cover of C, P' be the preimage of P under the covering map and $\partial_e C'$ be the preimage of $\partial_e C$. Here we use $\partial_e C$ to denote the interior of the distinguished free side of (C, P). The identification of $C \setminus P$ with the convex core of N yields an embedding $C' \hookrightarrow \mathbb{H}^3$. The image of C' is the convex hull of the limit set of the Kleinian group $\pi_1(N)$ and, from this point of view, $\partial_e C'$ is the compressible boundary component of the convex hull. Observe that $\partial_e C'$ covers $\partial_e C$ but is not simply connected. We identify the universal cover of $\partial_e C$ with $\mathbb{H}^2$. The extension $G\colon (C, P) \to (C, P)$ of g lifts to $G'\colon (C', P') \to (C', P')$. In particular, g lifts to $g'\colon \partial_e C' \to \partial_e C'$. Denote by $\tilde{g}\colon \mathbb{H}^2 \to \mathbb{H}^2$ a lift to the universal cover. Denote by τ' the preimage of the train-track τ in $\partial_e C'$ and by $\tilde{\tau}$ the preimage in $\mathbb{H}^2$. The realization ϕ of τ lifts to a map $\phi'\colon \tau' \to C' \setminus P'$. Putting together all these covers and maps, we obtain the following diagram which commutes, up to homotopy, with tracks of uniformly bounded length:

$$
\begin{tikzcd}
\tilde{\tau} \arrow[r] \arrow[d, dotted] & \mathbb{H}^2 \arrow[rr] \arrow[d, dotted] & & \mathbb{H}^2 \arrow[d, dotted] & \\
\tau' \arrow[r] \arrow[drr] \arrow[dd, dotted] & \partial_e C' \arrow[rr, "g'"] \arrow[dr] \arrow[dd, dotted] & & \partial_e C' \arrow[dr] \arrow[dd, dotted] & \\
& & C' \setminus P' \arrow[rr] \arrow[dd, dotted] & & C' \setminus P' \arrow[dd, dotted] \\
\tau \arrow[r] \arrow[drr, "\phi"'] & \partial_e C \arrow[rr] \arrow[dr] & & \partial_e C \arrow[dr] & \\
& & C \setminus P \arrow[rr, "G"] & & C \setminus P.
\end{tikzcd}
\tag{11.1}
$$

For the convenience of the reader we have only labeled some of the arrows in diagram (11.1). The dotted vertical arrows are all coverings. Two arrows which differ by a vertical translation are related by a covering and all the other arrows are either inclusions or are maps induced by g. We also remind the reader that we have identified $C \setminus P$ with the convex core of some fixed geometrically finite hyperbolic 3-manifold N.

The map G is homotopic to the identity and it is elementary to change the homotopy in such a way that its restriction to $C \setminus P$ has bounded tracks (with respect to the metric induced by the identification of $C \setminus P$ and the convex core of N). We deduce that the lift $G' : C' \to C'$ restricts to a quasi-isometry $C' \setminus P' \to C' \setminus P'$. In particular, there are K' and A' such that the restriction of the composition $G' \circ \phi'$ to any route of τ' is a (K', A')-quasi-geodesic. The restrictions of g' and $G' \circ \phi'$ to τ' are homotopic in $C' \setminus P'$ by a homotopy with tracks of bounded length. Hence, there are K'' and A'' such that the g'-image of any route of τ' is a (K'', A'')-quasi-geodesic in $C' \setminus P'$. Obviously this implies that the restriction of the lift $\tilde{g}$ of g' to any route of $\tilde{\tau}$ is a quasi-geodesic with uniform constants. We have proved the following lemma.

LEMMA 11.1. *There exist constants A and K such that the restriction of $\tilde{g}$ to any route of $\tilde{\tau}$ is a (K, A)-quasi-geodesic in $\mathbb{H}^2$.*

Remark. At this point we would like to point out that it is due to Lecuire [L1] that there is some geometrically finite hyperbolic manifold N with $\mathrm{CC}(N)$ homeomorphic to $C \setminus P$ and with the property that the boundary $\partial \mathrm{CC}(N)$ of the convex core is a pleated surface realizing λ. Using this result, the proof of Lemma 11.1 becomes marginally simpler. Since however the basic idea does not change, we decided to use the much simpler realization result stated in Corollary 6.6.

Recall that the train-track τ carries λ, assume that l_1 and l_2 are leaves of the preimage $\tilde{\lambda}$ of λ to $\mathbb{H}^2$ with projections l_1' and l_2' in $\partial_e C'$, and let $\hat{l}_1$ and $\hat{l}_2$ be the corresponding routes of $\tilde{\tau}$. If $\hat{l}_1$ and $\hat{l}_2$ are mapped by $\tilde{g}$ at bounded distance, then the images of the corresponding routes of τ' under g' are also at bounded distance from each other. In particular, the images of l_1' and l_2' under the lift $f'\colon \partial_e C' \to C' \setminus P'$ of the pleated map realizing λ' are also close to each other. Theorem 6.2 implies that we must have $l_1' = l_2'$. In other terms, we have the following result.

Corollary 11.2. *Let l_1 and l_2 be two leaves of $\tilde{\lambda}$ and let $\hat{l}_1$ and $\hat{l}_2$ be the associated routes of $\tilde{\tau}$. If $\tilde{g}(\hat{l}_1)$ and $\tilde{g}(\hat{l}_2)$ are at bounded distance from each other, then there is $\gamma \in \operatorname{Ker}(\pi_1(\partial_e C) \to \pi_1(C))$ with $\gamma l_1 = l_2$.*

Note that in the above corollary we are using the fact that $\partial_e C'$ is the cover of $\partial_e C$ associated with the subgroup $\operatorname{Ker}(\pi_1(\partial_e C) \to \pi_1(C))$ of $\pi_1(\partial_e C)$.

Let $\mathcal{R}(\tilde{\tau})$ be the set of routes of $\tilde{\tau}$ and $\mathcal{G}(\mathbb{H}^2)$ the set of geodesics in $\mathbb{H}^2$. Since the map $\tilde{g}\colon \tilde{\tau} \to \mathbb{H}^2$ maps routes to quasi-geodesics, we deduce from Lemma 5.1 that the induced map

$$\bar{g}\colon \mathcal{R}(\tilde{\tau}) \longrightarrow \mathcal{G}(\mathbb{H}^2)$$

is continuous. As in §5, denote by $\mathcal{CR}(\tilde{\tau})$ and $\mathcal{C}(\mathbb{H}^2)$ the sets of (projectiviced) currents supported by $\mathcal{R}(\tilde{\tau})$ and $\mathcal{G}(\mathbb{H}^2)$, respectively. Again, by Lemma 5.1, the induced map

$$\bar{g}_*\colon \mathcal{CR}(\tilde{\tau}) \longrightarrow \mathcal{C}(\mathbb{H}^2)$$

is continuous. The continuity of this map and the assumption that every limit (in $P\mathcal{C}(\partial_e C)$) of a subsequence of $\{g(\gamma_n)\}_{n=1}^{\infty}$ is supported on the unmeasured lamination μ implies that $\bar{g}_*(\lambda)$ is supported on μ. In particular $\bar{g}$ maps (routes of τ corresponding to) leaves of $\tilde{\lambda}$ to leaves of $\tilde{\mu}$.

Observe also that $\tilde{g}$ maps half-routes of $\tilde{\tau}$ to quasi-geodesic rays in $\mathbb{H}^2$ and that this map from the set of half-routes to the set of (K, A)-quasi-geodesic rays is continuous. In particular, $\bar{g}$ maps asymptotic half-leaves of $\tilde{\lambda}$ to asymptotic half-leaves of $\tilde{\mu}$. Hence, $\bar{g}$ induces a map from the set of all complementary regions of $\tilde{\lambda}$ to the set of complementary regions of $\tilde{\mu}$. Moreover, the fact that every half-leaf of λ is dense implies that the support of μ is connected. Also, μ is not a closed curve since otherwise the image of λ under the pleated map $f\colon \partial_e C \to N$ would be contained in a single closed geodesic contradicting Theorem 6.2. We summarize our findings in the following lemma.

Lemma 11.3. *The measured lamination μ has connected support and is not a closed geodesic. Moreover, the map $\bar{g}$ maps leaves and complementary regions of $\tilde{\lambda}$ to leaves and complementary regions of $\tilde{\mu}$.*

Before going further, observe that we do not yet know that μ is filling. In particular, we do not know if $\tilde{\mu}$ is connected.

11.2. The induced map between dual trees

We recall now briefly the definition of the dual tree of a lamination. So far we have considered μ as an unmeasured lamination. From now on we consider a measured lamination which is supported on this unmeasured lamination and with abuse of notation denote it by μ. Suppose that ν is either $\tilde{\lambda}$ or $\tilde{\mu}$ and recall that none of these two laminations has closed leaves. For $x, y \in \mathbb{H}^2$ we define $d_\nu(x, y)$ to be the ν-measure of the geodesic segment $[x, y]$; $d_\nu(\cdot, \cdot)$ is symmetric and satisfies the triangle inequality. In particular, if we consider the quotient T_ν of $\mathbb{H}^2$ under the equivalence relation

$$x \sim y \quad \text{if } d_\nu(x, y) = 0,$$

we obtain a metric space. The fibers of the projection $\mathbb{H}^2 \to T_\nu$ are the leaves and closures of the complementary components of ν. It follows that T_ν is a real tree. See Skora [Sk1], [Sk2] and Otal [Ot3] for details.

By Lemma 11.3, the map $\bar{g}$ maps leaves and complementary regions of $\tilde{\lambda}$ to leaves and complementary regions of $\tilde{\mu}$. In particular, we obtain an equivariant map

$$\hat{g}: T_\lambda \longrightarrow T_\mu.$$

Observe that the continuity of $\hat{g}$ follows, as everything else, from the continuity of

$$\bar{g}: \mathcal{R}(\tilde{\tau}) \longrightarrow \mathcal{G}(\mathbb{H}^2),$$

together with the fact that μ does not contain closed leaves.

Lemma 11.4. *$\hat{g}$ is continuous.*

We claim now that $\hat{g}$ is locally injective.

Lemma 11.5. *There is ε such that for any two distinct leaves l_1 and l_2 of $\tilde{\lambda}$ with $d_{T_\lambda}(l_1, l_2) \leqslant \varepsilon$ we have $\bar{g}(l_1) \neq \bar{g}(l_2)$.*

Proof. Seeking a contradiction, assume there are sequences $\{l_1^n\}_{n=1}^\infty$ and $\{l_2^n\}_{n=1}^\infty$ of leaves of $\tilde{\lambda}$, with $d_{T_\lambda}(l_1^n, l_2^n) \to 0$ and with $l_1^n \neq l_2^n$, but such that $\bar{g}(l_1^n) = \bar{g}(l_2^n)$ for all n. Up to conjugating by an element of the fundamental group, we may assume that the sequences $\{l_1^n\}_{n=1}^\infty$ and $\{l_2^n\}_{n=1}^\infty$ converge to leaves l_1 and l_2, respectively. Observe that either $l_1 = l_2$ or that l_1 and l_2 are contained in the boundary of some fixed complementary region. We first assume that we are in the first case; the second case being treated in a similar way at the end; let $l=l_1=l_2$.

By Corollary 11.2, we have for every n some $\phi_n\in\operatorname{Ker}(\pi_1(\partial_e C)\to\pi_1(C))$ such that

$$\phi_n(l_2^n)=l_1^n. \tag{11.2}$$

Let $\varkappa$ be a small transversal arc to l. After passing to a subsequence, we assume that for every n, l_1^n and l_2^n intersect $\varkappa$ in x_1^n and x_2^n, respectively. We also denote by $[x_1^n,x_2^n]$ the subarc of $\varkappa$ with endpoints x_1^n and x_2^n. Obviously as $n\to\infty$ the points x_1^n and x_2^n converge to $x_0=\varkappa\cap l$ and the transverse measure left on $\varkappa_n$ by $\tilde{\lambda}$, i.e. $\iota(\varkappa_n,\tilde{\lambda})$, converges to zero.

By (11.2) we know that $\phi_n(x_2^n)\in l_1^n$ and therefore the curve $[x_1^n,x_2^n]\cup[x_1^n,\phi_n(x_2^n)]$, which is the result of the concatenation of $[x_1^n,x_2^n]$ and the subarc of l_1^n enclosed by x_1^n and $\phi_n(x_2^n)$, projects to a homotopically essential closed curve γ_n on F. Furthermore, γ_n is a representative of the conjugacy class of ϕ_n, and hence it is homotopically trivial in C (since $\phi_n\in\operatorname{Ker}(\pi_1(\partial_e C)\to\pi_1(C))$). Also note that

$$i(\gamma_n,\lambda)\leqslant i([x_1^n,x_2^n],\tilde{\lambda})\to 0$$

as $n\to\infty$.

If γ_n is simple for infinitely many n, we already have a contradiction; because every limit of the sequence of these meridians in $\mathcal{PML}$ would have zero intersection with λ, and this contradicts the fact that λ is in $\mathcal{O}(C,P)$.

Even when γ_n is not simple for n sufficiently large, by using the loop theorem, we know that there exists an essential simple closed curve γ_n' obtained by doing some surgery on γ_n and such that γ_n' is also homotopically trivial in C. Since γ_n' is obtained by surgery on γ_n, we see that

$$i(\gamma_n',\lambda)\leqslant i(\gamma_n,\lambda)\to 0$$

as $n\to\infty$, and in a similar way to the above we have a contradiction.

If $l_1\neq l_2$ are distinct sides of a complementary region, we choose an arc $\varkappa$ that is transversal to both l_1 and l_2. Let x_1^n and x_2^n be intersections of $\varkappa$ with l_1^n and l_2^n, respectively, and γ_n be the closed curve that is the projection of the concatenation $[x_1^n,x_2^n]\cup[x_1^n,\phi_n(x_2^n)]$ similar to the above construction. A similar observation shows that $i(\gamma_n,\lambda)\to 0$ as $n\to\infty$, and we get a contradiction in the same way as above. □

The above lemma immediately implies the following result.

Corollary 11.6. *The map* $\hat{g}\colon\mathcal{T}_\lambda\to\mathcal{T}_\mu$ *is locally injective.*

The intermediate-value theorem shows that any continuous locally injective map between real trees is injective. We have proved the following result.

Proposition 11.7. *The map* $\hat{g}:\mathcal{T}_\lambda\to\mathcal{T}_\mu$ *is injective.*

As an immediate consequence of Proposition 11.7, we obtain the following result.

Lemma 11.8. *The homomorphism* $g_*:\pi_1(\partial_e C)\to\pi_1(\partial_e C)$ *is injective.*

Proof. Assume that g_* is not injective and let γ be a non-trivial element in the kernel. Note that γ cannot be peripheral since the g-image of every component of the boundary of $\partial_e C$ is homotopically non-trivial.

We claim that a non-trivial element $\gamma\in\pi_1(\partial_e C)$ has no fixed point on T_λ unless γ is peripheral. Assume that $\gamma x=x$ for $x\in T_\lambda$. The point x cannot represent a leaf of λ, since λ does not have any closed leaves. The point x cannot represent a finite-sided complementary component of λ either, because only elliptic isometries of $\mathbb{H}^2$ preserve such a polygon. Finally if x represents an infinite-sided complementary component of λ, one can see that it is associated with a fixed point of a parabolic element of $\pi_1(\partial_e C)$ and γ must be peripheral.

Hence, if γ is in the kernel of g_*, it is non-peripheral and has no fixed point on T_λ. The equivariance of $\hat{g}$ shows that

$$\hat{g}(\gamma x)=g_*(\gamma)\hat{g}(x)=\hat{g}(x),$$

contradicting Proposition 11.7. $\square$

Recall now that a proper π_1-injective map between two surfaces is properly homotopic to a homeomorphism. Hence, it follows from Lemma 11.8 that the proper map $g:\partial_e C\to\partial_e C$ is homotopic to a homeomorphism. This concludes the proof of Proposition 10.6.

12. Some remarks

In this final section we add a few remarks on other density results which either follow directly from Theorem 1.1 or from its proof.

12.1. Groups with torsion

We extend now Theorem 1.1 to the case of finitely generated groups with torsion and prove the general form of the density conjecture. As in the case of torsion-free groups, we define $\mathrm{AH}(\Gamma)$ to be the set of all conjugacy classes of discrete and faithful representations of a finitely generated group Γ possibly with torsion. We define the topology on $\mathrm{AH}(\Gamma)$ as in the introduction and we say that a representation $\varrho:\Gamma\to\mathrm{PSL}_2(\mathbb{C})$ is geometrically finite if it has a fundamental domain with finitely many sides.

Theorem 12.1. *Every finitely generated discrete subgroup of* $\mathrm{PSL}_2(\mathbb{C})$ *is an algebraic limit of geometrically finite groups.*

Proof. Let $\varrho\in\mathrm{AH}(\Gamma)$ be given. It follows from Selberg's lemma that Γ contains a finite-index, torsion-free, normal subgroup Γ'. Observe that the finite group $G=\Gamma/\Gamma'$ acts by isometries on the 3-manifold $N'=\mathbb{H}^3/\varrho(\Gamma')$. In particular, G acts on the set of ending invariants of N'. As in the beginning of the proof of Theorem 1.1, we choose a sequence ϱ'_n of geometrically finite representations of Γ' converging algebraically to the representation ϱ' which is the restriction of ϱ to Γ'. The G-equivariance of the ending invariants of N' imply that we can choose ϱ'_n in such a way that the ending invariants of $N'_n=\mathbb{H}^3/\varrho'_n(\Gamma')$ are also G-equivariant. We can use this to extend ϱ'_n by G to an action of Γ on $\mathbb{H}^3$ which extends to a conformal action on the domain of discontinuity of $\varrho'_n(\Gamma')$. It is standard to see that this is in fact a representation of Γ into $\mathrm{PSL}_2(\mathbb{C})$, the group of isometries of $\mathbb{H}^3$. In other words, the representations ϱ'_n extend to faithful and discrete representations ϱ_n of Γ in $\mathrm{AH}(\Gamma)$. It is elementary to see that every finite extension of a geometrically finite Kleinian group is geometrically finite and therefore ϱ_n is geometrically finite for every n.

For every element $\alpha\in\Gamma$, there is an integer $k>0$ such that $\alpha^k\in\Gamma'$ and therefore the sequence $\varrho'_n(\alpha^k)=\varrho_n(\alpha^k)=[\varrho_n(\alpha)]^k$ converges to $\varrho'(\alpha^k)=\varrho(\alpha^k)=[\varrho(\alpha)]^k$ as $n\to\infty$. Since the kth roots of isometries of $\mathbb{H}^3$ are unique, this immediately implies that the sequence $\{\varrho_n(\alpha)\}_{n=1}^{\infty}$ converges to $\varrho(\alpha)$ and we have shown that the representations $\{\varrho_n\}_{n=1}^{\infty}$ converge algebraically to ϱ.

This proves the theorem. □

From now on we return to the realm of torsion-free groups.

12.2. Strong topology

Recall that a representation $\varrho\colon\Gamma\to\mathrm{PSL}_2(\mathbb{C})$ is *type preserving* if for all $\gamma\in\Gamma$ we have that $\varrho(\gamma)$ is parabolic if and only if γ is parabolic. Observe that the approximating sequence constructed in the proof of Theorem 1.1 is type preserving. Hence we have the following result.

Corollary 12.2. *Every finitely generated Kleinian group* Γ *is the algebraic limit of a type-preserving sequence of discrete and faithful geometrically finite representations.*

An algebraically convergent sequence of discrete and faithful representations $\varrho_i\Gamma$ into $\mathrm{PSL}_2(\mathbb{C})$ with algebraic limit ϱ converges *strongly* if the associated groups $\varrho_i(\Gamma)$ converge in the Chabauty topology to $\varrho(\Gamma)$. Anderson–Canary [AC2] proved that every algebraically convergent and type-preserving sequence converges strongly. Hence, from Corollary 12.2, we deduce directly the following result.

COROLLARY 12.3. *Every finitely generated Kleinian group* Γ *is the algebraic limit of a sequence of discrete and faithful geometrically finite representations whose images converge also in the Chabauty topology to* Γ. *In other words, the geometrically finite representations are dense in the strong topology.*

12.3. Length function and realizability

In Thurston's original work on hyperbolic manifolds homotopy equivalent to a surface, the notion of non-realizability was very closely related to having zero length. In fact, if a minimal measured lamination is realizable, then one can use a pleated surface that realizes it to define the length. In this case one can define the length to be zero for a minimal lamination which is not realized and define the length for a general measured lamination to be the sum of the lengths of its components. It was conjectured by Thurston that this length function is continuous with respect to the algebraic topology. The proof of this by Brock [Br1] uses delicate arguments that require a careful analysis of geometric limits and appearance of new parabolics in the limit.

An important feature of this length function in the case of surface groups is that by work of Thurston and Bonahon, a lamination has zero length if and only if it is a union of ending laminations and accidental parabolics.

In the more general case of $\varrho\in\mathrm{AH}(M,P)$ for a pared manifold (M,P), one can again define the length for minimal laminations which are realized. Then one can ask how this function can be extended continuously and what the zero-locus of the length function represents. Obviously the zero-locus will be a subset of non-realizable laminations. The statements for the case of surface groups easily generalize to the case where (M,P) is a pared manifold with incompressible free sides. In the more general case, things are more difficult for laminations outside of the Masur domain and in fact the above definition may not be the correct way of defining the length. Still one expect similar results when one restricts the length to the Masur domain, by which we mean the union of Masur domains of the free sides of (M,P). In particular the next theorem shows that non-realizable or zero-length laminations will consist of accidental parabolics and ending laminations. We point out that a similar statement should hold for a slightly larger domain (cf. Lecuire [L2]). Let $\mathcal{O}(F)=\mathcal{O}(C_F)$ denote the Masur domain of the relative compression body associated with F. Also note that $\lambda\subset F$ is realized for $\varrho\in\mathrm{AH}(M,P)$ if and only if it is realized for the restriction of ϱ to the subgroup associated with C_F.

Theorem 12.4. *Let (M,P) be a pared manifold, $\varrho\in\mathrm{AH}(M,P)$ and $\lambda\in\mathcal{O}(F)$ be a minimal measured lamination in the Masur domain of a free side F of (M,P). If λ is non-realizable in N_ϱ then either*

(1) *λ is a closed curve which is mapped by ϱ to a parabolic*, *or*

(2) *there exist a pared locus $Q\supset P$, a free side $E\subset F$ of (M,Q) and a map*

$$\phi:(M,Q)\longrightarrow(M',P)$$

in the homotopy class of ϱ, where (M',P') is a relative compact core for $(N_\varrho^\varepsilon,\partial N_\varrho^\varepsilon)$, such that restricted to E, ϕ is a homeomorphism to a free side F' of (M',Q') and $\phi(\lambda)$ is the ending lamination for the end of N_ϱ corresponding to F'.

Note that the second case in particular implies that λ fills F' and every component of F' is mapped to a parabolic by ϱ.

Proof. First note that, when λ is a simple loop, it follows from our remark after Lemma 6.4 that if the ϱ-image of λ is not parabolic, then it is realizable. So the statement follows in that case, and therefore we assume that λ is infinite.

Let $E\subset F$ be the smallest essential subsurface that contains λ. In particular $E=F$ if λ is filling. Assuming that λ is not realizable, we first show that every component of ∂E is mapped to a parabolic. Otherwise let $\gamma\subset\partial E$ be a component whose ϱ-image is not a parabolic. Obviously γ is not parallel to P and therefore it is a non-peripheral loop on F. Choose a sequence $\{\alpha_i\}_{i=1}^\infty$ in $F\setminus\gamma$ that converges in $\mathcal{PML}(F)$ to λ. Since $\mathcal{O}(F)$ is open, we may assume that α_i, and therefore $\alpha_i\cup\gamma$, is in $\mathcal{O}(F)$ for every i. It is a consequence of Sullivan's finiteness theorem of cusps and properties of the Masur domain that $\varrho(\alpha_i)$ is not parabolic for i sufficiently large (cf. [KS2, Lemma 4.5]). Then again we can use our remark after Lemma 6.4 to show that, for i sufficiently large, there exists a pleated surface $f_i: F\to N_\varrho$ which realizes $\alpha_i\cup\gamma$. Note that the sequence $\{\alpha_i\cup\gamma\}_{i=1}^\infty$ converges in $\mathcal{PML}(F)$ to a lamination which is supported on $\lambda\cup\gamma$. Since all these pleated surfaces realize γ, they intersect a compact set $K\subset N_\varrho$ and, by Proposition 6.5, $\lambda\cup\gamma$ and therefore λ is realized.

So we may assume that ϱ maps every component of ∂E to a parabolic. We can add annular neighborhoods of those to P, to obtain a new pared locus $Q\supset P$, and consider ϱ as an element of $\mathrm{AH}(M,Q)$. If we prove the theorem for $\varrho\in\mathrm{AH}(M,Q)$ and λ, then the theorem for the case of (M,P) also follows. So it suffices to prove the theorem in the case where λ fills F. But in that case the theorem is nothing but Theorem 1.4 and we have proved the theorem. $\square$

We expect the length function to be continuous restricted to

$$\mathrm{AH}(M,P)\times\mathcal{O}(M,P),$$

where $\mathcal{O}(M,P)$ is the union of Masur domains of the free sides of (M,P).

12.4. Service

As we mentioned in the introduction, Bromberg and Souto have announced a proof of the density conjecture which completely avoids using the ending lamination theorem. As we also mentioned in the introduction, their work relies partly on a consequence of the main result of this paper. We discuss this consequence here.

COROLLARY 12.5. *Let (M,P) be a pared manifold and $\varrho_0:\pi_1(M)\to\mathrm{PSL}_2(\mathbb{C})$ be a discrete and faithful representation such that the associated hyperbolic manifold N_{ϱ_0} is geometrically finite and has associated pared manifold (M,P). Denote the Ahlfors–Bers parametrization of $\mathrm{QH}(\varrho_0)$ by*

$$\pi_{\mathrm{AB}}:\mathcal{T}(\partial M\setminus P)\longrightarrow \mathrm{QH}(\varrho_0)$$

and label the free sides of (M,P) by $F_1,...,F_s$. If $(X_1^n,...,X_s^n)$ is a filling sequence with associated filling tuple $(X_1,...,X_r,\lambda_{r+1},...,\lambda_s)$, then the following holds:

(1) *up to passing to a subsequence, the sequence $\varrho_n=\pi_{\mathrm{AB}}(X_1^n,...,X_s^n)$ converges strongly to some discrete and failthful representation ϱ;*

(2) *if (M',P') is the pared manifold associated with the hyperbolic 3-manifold N_ϱ corresponding to the limiting representation ϱ, then there is a homeomorphism*

$$\phi:(M,P)\longrightarrow(M',P')$$

in the homotopy class determined by ϱ which maps the filling tuple

$$(X_1,...,X_r,\lambda_{r+1},...,\lambda_s)$$

to the ending invariants of N_ϱ.

In some sense, the proof of Corollary 12.5 summarizes this whole paper. Also, Corollary 12.5 implies directly Theorem 1.1 if we allow ourselves to use the ending lamination theorem, as we have done in this paper.

Proof. From Theorem 8.1 we know that the sequence $\{\varrho_i\}_{i=1}^\infty$ converges algebraically, up to passing to a subsequence, to some representation ϱ, and that the limiting hyperbolic 3-manifold N_ϱ satisfies (*) in §9. In particular, N_ϱ has the expected conformal boundary. At this point we argue as in the proof of Theorem 1.1 in §9 and deduce from Theorem 1.4 that the limiting manifold N_ϱ must also satisfy (**). Finally our argument at the end of the proof of Theorem 1.1 prove that the pared manifold (M',P') associated with N_ϱ

is homeomorphic to (M, P) and that there is a homeomorphism $\phi: (M, P) \to (M', P')$ in the homotopy class determined by ϱ which maps the filling tuple $(X_1, ..., X_r, \lambda_{r+1}, ..., \lambda_s)$ to the ending invariants of N_ϱ.

This implies that the only curves in N_ϱ which are homotopic to parabolic elements are those which can be homotoped into P. In other words, the convergence of ϱ_i to ϱ is type preserving. As above, we deduce from this fact and from the work of Anderson–Canary [AC2] that the sequence $\{\varrho_i\}_{i=1}^{\infty}$ actually converges strongly to ϱ. This concludes the proof of Corollary 12.5. $\square$

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Received April 17, 2010
Received in revised form May 8, 2012