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Dirac cohomology for graded affine Hecke algebras

by

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1. Introduction

This paper develops the theory of the Dirac operator for modules over a graded affine Hecke algebra $\mathbb{H}$. Our approach is influenced by the classical Dirac operator acting on sections of spinor bundles over a Riemannian symmetric space G/K (see, e.g., [AS], [P]). In that setting, the tangent space $T_{eK}(G/K)$ identifies with the (-1)-eigenspace of the Cartan involution for G acting on its Lie algebra $\mathfrak{g}$. In the present setting, $\mathbb{H}$ plays the role of the enveloping algebra of the complexification of $\mathfrak{g}$, the analogue of the Cartan involution is a canonical anti-involution $*$ of $\mathbb{H}$, and the role of $T_{eK}(G/K)$ is played by a subspace of the (-1)-eigenspace of $*$. Because $*$ is induced by the operation of inversion on a corresponding split p-adic group [BM1], [BM2], our setting can be thought of as a p-adic analogue of the Riemannian symmetric case.

Our Dirac operator acts on an $\mathbb{H}$-module X tensored with a space of spinors for an appropriate Clifford algebra. Motivated by the study of the index of the classical Dirac operator, we introduce the Dirac cohomology $H^D(X)$ of X. The cohomology space is a representation of a canonical double cover $\widetilde{W}$ of a relevant Weyl group. Our main result is a p-adic analogue of Vogan's conjecture for Harish-Chandra modules (proved by Huang and Pandžić [HP]). We prove that when X is irreducible, the $\widetilde{W}$-representation on $H^D(X)$ (when non-zero) determines the central character of X. It turns out that the central characters which arise in this way are closely related to the central characters of elliptic tempered representations of $\mathbb{H}$, in the sense of [A] and [OS]. Moreover, roughly speaking, $H^D(X)$ vanishes exactly when one would hope it would; in particular we know

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of no interesting unitary representation X (an isolated automorphic one, say) for which $H^D(X)=0$.

In more detail, fix a root system R as in [S, §1.3] but do not impose the crystallographic condition (RS3). (When we wish to impose (RS3), we shall refer to R as a crystallographic root system.) Let V_0 (resp. V) denote the real (resp. complex) span of R, write $V_0^\vee$ (resp. $V^\vee$) for the real (resp. complex) span of the coroots $R^\vee$, W for the Weyl group, and fix a W-invariant inner product $\langle\cdot,\cdot\rangle$ on $V_0^\vee$ (and hence V_0 as well). Extend $\langle\cdot,\cdot\rangle$ to a symmetric bilinear form on $V^\vee$ (and V). Let $\mathbb{H}$ denote the associated graded affine Hecke algebra with parameters defined by Lusztig [L2] (Definition 2.1). As a complex vector space, $\mathbb{H}\simeq\mathbb{C}[W]\otimes S(V^\vee)$. Lusztig proved that maximal ideals in the center of $\mathbb{H}$, and hence central characters of irreducible $\mathbb{H}$ modules, are parameterized by (W-orbits of) elements of V.

After introducing certain Casimir-type elements in §2, we then turn to the Dirac element in §3. Let $C(V_0^\vee)$ denote the corresponding Clifford algebra for the inner product $\langle\cdot,\cdot\rangle$. For a fixed orthonormal basis $\{\omega_i\}_{i=1}^n$ of $V_0^\vee$, the Dirac element is defined (Definition 3.1) as

$$\mathcal{D}=\sum_{i=1}^n \widetilde{\omega}_i\otimes\omega_i\in\mathbb{H}\otimes C(V_0^\vee),$$

where $\widetilde{\omega}_i\in\mathbb{H}$ is given by (2.13). The key point, as mentioned above, is that

$$\widetilde{\omega}_i^* = -\widetilde{\omega}_i;$$

see §2.5. In Theorem 3.1, we prove that $\mathcal{D}$ is roughly the square root of the Casimir element $\sum_{i=1}^n \omega_i^2\in\mathbb{H}$ (Definition 2.4).(1)

For a fixed space of spinors S for $C(V_0^\vee)$ and a fixed $\mathbb{H}$-module X, $\mathcal{D}$ acts as an operator D on $X\otimes S$. Since W acts by orthogonal transformation on $V_0^\vee$, we can consider its preimage $\widetilde{W}$ in $\mathsf{Pin}(V_0^\vee)$, see (3.4). By restriction, X is a representation of W, and so $X\otimes S$ is a representation of $\widetilde{W}$. Lemma 3.4 shows that $\mathcal{D}$ (and hence D) are approximately $\widetilde{W}$ invariant. Thus $\ker(D)$ is also a representation of $\widetilde{W}$. Corollary 3.6 shows that if X is irreducible and $\ker(D)$ is non-zero, then any irreducible representation of $\widetilde{W}$ occurring in $\ker(D)$ determines $\langle\nu,\nu\rangle$, where ν is the central character of X. This is an analogue of Parthasarathy's fundamental calculation [P, §3] (cf. [SV, §7]) for Harish-Chandra modules.

We then define the Dirac cohomology of X as

$$H^D(X)=\frac{\ker(D)}{\ker(D)\cap\operatorname{im}(D)}$$

(1) Note that if we replace $*$ by some other anti-involution, Definition 3.1 suggests how to construct a corresponding Dirac element. But these other Dirac elements will not be interesting from the point of view of p-adic group representations, and we therefore do not consider them here.

in Definition 4.1. (For representations which are unitary with respect to $*$, one has $H^D(X)=\ker(D)$.) Once again $H^D(X)$ is a representation of $\widetilde{W}$. One is naturally led to the following version of Vogan's conjecture: if X is irreducible and $H^D(X)$ is non-zero, then any irreducible representation of $\widetilde{W}$ occurring in $H^D(X)$ determines the central character ν of X, not just $\langle \nu, \nu \rangle$. This is the content of Theorem 4.4 below. For algebras $\mathbb{H}$ attached to crystallographic root systems and equal parameters, a much more precise statement is given in Theorem 5.8. As explained in Remark 5.9, the proof of Theorem 5.8 also applies for the special kinds of unequal parameters arising in Lusztig's geometric theory [L1], [L3].

To make Theorem 5.8 precise, we need a way of passing from an irreducible $\widetilde{W}$-representation to a central character, i.e. an element of V. This is a fascinating problem in its own right. The irreducible representations of $\widetilde{W}$—the so-called spin representations of W—have been known for a long time from the work of Schur, Morris, Reade and others. But only recently has a uniform parametrization of them in terms of nilpotent orbits emerged [C]. This parametrization (partly recalled in Theorem 5.1) provides exactly what is needed for the statement of Theorem 5.8.

One of the main reasons for introducing the Dirac operator (as in the real case) is to study unitary representations. Here, as above, unitarity of $\mathbb{H}$-modules is defined with respect to $*$ (§2.5). When R is crystallographic and the parameters are taken to be constant, [BM1] and [BM2] show that unitary $\mathbb{H}$-modules (with "real" central character in V_0) correspond to unitary representation of the split adjoint p-adic group attached to R.

Corollary 5.4 and Remarks 5.5 and 5.6 contain powerful general statements about unitary representations of $\mathbb{H}$, and hence of split p-adic groups. In particular, we obtain an elegant uniform bound on the "spectral gap" measuring the degree to which the trivial representation is isolated in the unitary dual of $\mathbb{H}$. (The terminology "spectral gap" is used since the representation-theoretic statement immediately gives a lower bound on the first non-trivial eigenvalue of a relevant Laplacian acting on certain spaces of automorphic forms.) Given the machinery of the Dirac operator, the proofs of these results are very simple.

As an example we include the following remarkable consequence of Theorem 5.8 (from which the estimate on the spectral gap follows immediately).

Theorem 1.1. *Suppose that $\mathbb{H}$ is attached to a crystallographic root system with equal parameters* (*Definition* 2.1), *and let $\mathfrak{g}$ denote the corresponding complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}\simeq V$. Let X be an irreducible $\mathbb{H}$-module with central character χ_ν for $\nu\in V\simeq\mathfrak{h}$* (*as in Definition* 2.2). *Suppose further that X is unitary with respect to $*$ as in* §2.5, *and that the kernel of the Dirac operator for X is non-zero*

(*Definition* 3.1). *Then there exists an* $\mathfrak{sl}_2$*-triple* $\{e,h,f\}\subset\mathfrak{g}$ *such that the centralizer in* $\mathfrak{g}$ *of* e *is solvable and such that*

$$\nu=\tfrac{1}{2}h.$$

In particular, if R *is irreducible of rank at least* 2, *and* X *is not equal to the trivial or Steinberg representation, then* $\langle\nu,\nu\rangle$ *is at most* $\frac{1}{4}\langle h_{\mathsf{sr}},h_{\mathsf{sr}}\rangle$, *where* h_{sr} *is the middle element of an* $\mathfrak{sl}_2$*-triple for the subregular nilpotent orbit in* $\mathfrak{g}$.

We remark that while this paper is inspired by the ideas of Huang–Pandžić, Kostant, Parthasarathy, Schmid and Vogan, it is essentially self-contained. There are two exceptions, and both arise during the proof of the sharpened Theorem 5.8. We have already mentioned that we use the main results of [C] in §5. The other non-trivial result we need is the classification and W-module structure of certain tempered $\mathbb{H}$-modules ([KL], [L2], [L3]). These results (in the form we use them) are not available at arbitrary parameters. This explains the crystallographic condition and restrictions on parameters in the statement of Theorem 5.8 and in Remark 5.9. For applications to unitary representations of p-adic groups, these hypotheses are natural. Nonetheless we expect a version of Theorem 5.8 to hold for arbitrary parameters and non-crystallographic root systems.

The results of this paper suggest generalizations to other types of related Hecke algebras. They also suggest possible generalizations along the lines of [K] for a version of Kostant's cubic Dirac operator. Finally, in [EFM] and [CT] (and also in an unpublished work of Hiroshi Oda) functors between Harish-Chandra modules and modules for associated graded affine Hecke algebras are introduced. It would be interesting to understand how these functors relate Dirac cohomology in the two categories.

2. Casimir operators

2.1. Root systems

Fix a root system $\Phi=(V_0,R,V_0^\vee,R^\vee)$ over the real numbers. In particular, $R\subset V_0\setminus\{0\}$ spans the real vector space V_0; $R^\vee\subset V^\vee\setminus\{0\}$ spans the real vector space $V_0^\vee$; there is a perfect bilinear pairing

$$(\,\cdot\,,\cdot\,)\colon V_0\times V_0^\vee\longrightarrow\mathbb{R};$$

and there is a bijection between R and $R^\vee$ denoted $\alpha\mapsto\alpha^\vee$ such that $(\alpha,\alpha^\vee)=2$ for all α. Moreover, for $\alpha\in R$, the reflections

$$\begin{aligned} s_\alpha\colon V_0\longrightarrow V_0, &\qquad s_\alpha(v)=v-(v,\alpha^\vee)\alpha,\\ s_\alpha^\vee\colon V_0^\vee\longrightarrow V_0^\vee, &\qquad s_\alpha^\vee(v')=v'-(\alpha,v')\alpha^\vee, \end{aligned}$$

leave R and $R^\vee$ invariant, respectively. Let W be the subgroup of $\mathrm{GL}(V_0)$ generated by $\{s_\alpha:\alpha\in R\}$. The map $s_\alpha\mapsto s_\alpha^\vee$ gives an embedding of W into $\mathrm{GL}(V_0^\vee)$ such that

$$(v, wv') = (wv, v') \tag{2.1}$$

for all $v\in V_0$ and $v'\in V_0^\vee$.

We will assume that the root system Φ is reduced, meaning that $\alpha\in R$ implies that $2\alpha\notin R$. However, initially we do not need to assume that Φ is crystallographic, meaning that for us $(\alpha,\beta^\vee)$ need not always be an integer. We will fix a choice of positive roots $R^+\subset R$, let Π denote the corresponding simple roots in R^+, and let $R^{\vee,+}$ denote the corresponding positive coroots in $R^\vee$. Often we will write $\alpha>0$ or $\alpha<0$ in place of $\alpha\in R^+$ or $\alpha\in -R^+$, respectively.

We fix, as we may, a W-invariant inner product $\langle\cdot,\cdot\rangle$ on $V_0^\vee$. The constructions in this paper of the Casimir and Dirac operators depend on the choice of this inner product. Using the bilinear pairing $(\cdot,\cdot)$, we define a dual inner product on V_0 as follows. Let $\{\omega_i\}_{i=1}^n$ and $\{\omega^i\}_{i=1}^n$ be $\mathbb{R}$-bases of $V_0^\vee$ which are in duality; i.e. such that $\langle\omega_i,\omega^j\rangle=\delta_{i,j}$, the Kronecker delta. Then for $v_1, v_2\in V_0$, set

$$\langle v_1, v_2\rangle = \sum_{i=1}^n (v_1,\omega_i)(v_2,\omega^i). \tag{2.2}$$

(Since the inner product on $V_0^\vee$ is also denoted $\langle\cdot,\cdot\rangle$, this is an abuse of notation. But it causes no confusion in practice.) Then (2.2) defines an inner product on V_0 which once again is W-invariant. It does not depend on the choice of bases $\{\omega_i\}_{i=1}^n$ and $\{\omega^i\}_{i=1}^n$. If v is a vector in V or in $V^\vee$, we set $|v|:=\langle v,v\rangle^{1/2}$.

Finally, we extend $\langle\cdot,\cdot\rangle$ to a symmetric bilinear form on the complexification

$$V^\vee := V_0^\vee\otimes_{\mathbb{R}}\mathbb{C}$$

(and $V=V_0\otimes_{\mathbb{R}}\mathbb{C}$).

2.2. The graded affine Hecke algebra

Fix a root system Φ as in the previous section. Set $V=V_0\otimes_{\mathbb{R}}\mathbb{C}$ and $V^\vee=V_0^\vee\otimes_{\mathbb{R}}\mathbb{C}$. Fix a W-invariant "parameter function" $c: R\to\mathbb{R}_{>0}$, and set $c_\alpha=c(\alpha)$.

Definition 2.1. ([L2, §4]) The graded affine Hecke algebra $\mathbb{H}=\mathbb{H}(\Phi,c)$ attached to the root system Φ, and with parameter function c, is the complex associative algebra with unit generated by the symbols $\{t_w:w\in W\}$ and $\{t_f: f\in S(V^\vee)\}$, subject to the following relations:

(1) The linear map from the group algebra $\mathbb{C}[W]=\bigoplus_{w\in W}\mathbb{C}w$ to $\mathbb{H}$ taking w to t_w is an injective map of algebras.

(2) The linear map from the symmetric algebra $S(V^\vee)$ to $\mathbb{H}$ taking an element f to t_f is an injective map of algebras.

We will often implicitly invoke these inclusions and view $\mathbb{C}[W]$ and $S(V^\vee)$ as subalgebras of $\mathbb{H}$. As is customary, we write f instead of t_f in $\mathbb{H}$. The following is the final relation:

(3) For all $\alpha\in\Pi$ and $\omega\in V^\vee$ we have

$$\omega t_{s_\alpha} - t_{s_\alpha} s_\alpha(\omega) = c_\alpha(\alpha,\omega), \tag{2.3}$$

where $s_\alpha(\omega)$ is the element of $V^\vee$ obtained by s_α acting on ω.

[L2, Proposition 4.5] says that the center $Z(\mathbb{H})$ of $\mathbb{H}$ is $S(V^\vee)^W$. Therefore maximal ideals in $Z(\mathbb{H})$ are parameterized by W-orbits in V.

Definition 2.2. For $\nu\in V$ we write χ_ν for the homomorphism from $Z(\mathbb{H})$ to $\mathbb{C}$ whose kernel is the maximal ideal parameterized by the W-orbit of ν. By a version of Schur's lemma, $Z(\mathbb{H})$ acts in any irreducible $\mathbb{H}$-module X by a scalar $\chi: Z(\mathbb{H})\to\mathbb{C}$. We call χ the central character of (π, X). In particular, there exists $\nu\in V$ such that $\chi=\chi_\nu$.

2.3. The Casimir element of $\mathbb{H}$

Definition 2.3. Let $\{\omega_i\}_{i=1}^n$ and $\{\omega^i\}_{i=1}^n$ be dual bases of $V_0^\vee$ with respect to $\langle\cdot,\cdot\rangle$. Define

$$\Omega = \sum_{i=1}^n \omega_i\omega^i \in \mathbb{H}. \tag{2.4}$$

It follows from a simple calculation that Ω is well defined, independent of the choice of bases.

LEMMA 2.4. *The element Ω is central in $\mathbb{H}$.*

Proof. To see that Ω is central, in light of Definition 2.1, it is sufficient to check that $t_{s_\alpha}\Omega=\Omega t_{s_\alpha}$ for every $\alpha\in\Pi$. Using (2.3) twice and the fact that $(\alpha, s_\alpha(\omega))=-(\alpha,\omega)$ (as follows from (2.1)), we find that

$$t_{s_\alpha}(\omega_i\omega^i) = (s_\alpha(\omega_i)s_\alpha(\omega^i))t_{s_\alpha} + c_\alpha(\alpha,\omega^i)s_\alpha(\omega_i) + c_\alpha(\alpha,\omega_i)\omega^i. \tag{2.5}$$

Therefore, we have

$$\begin{aligned} t_{s_\alpha}\Omega &= \sum_{i=1}^n s_\alpha(\omega_i)s_\alpha(\omega^i)t_{s_\alpha} + c_\alpha\sum_{i=1}^n(\alpha,\omega^i)s_\alpha(\omega_i) + c_\alpha\sum_{i=1}^n(\alpha,\omega_i)\omega^i \\ &= \Omega t_{s_\alpha} + c_\alpha\sum_{i=1}^n(\alpha, s_\alpha(\omega^i))\omega_i + c_\alpha\sum_{i=1}^n(\alpha,\omega_i)\omega^i \\ &= \Omega t_{s_\alpha} - c_\alpha\sum_{i=1}^n(\alpha,\omega^i)\omega_i + c_\alpha\sum_{i=1}^n(\alpha,\omega_i)\omega^i. \end{aligned} \tag{2.6}$$

But the last two terms cancel (which can be seen by taking $\{\omega_i\}_{i=1}^n$ to be a self-dual basis, for example). So indeed $t_{s_\alpha}\Omega = \Omega t_{s_\alpha}$. □

LEMMA 2.5. *Let* (π, X) *be an irreducible* $\mathbb{H}$*-module with central character* χ_ν *for* $\nu \in V$ *(as in Definition 2.2). Then*

$$\pi(\Omega) = \langle \nu, \nu \rangle \operatorname{Id}_X .$$

Proof. Since Ω is central (by Lemma 2.4) and since X is assumed to have central character χ_ν, it follows from Definition 2.2 that Ω acts by the scalar

$$\sum_{i=1}^n (\nu, \omega_i)(\nu, \omega^i).$$

According to (2.2), this is simply $\langle \nu, \nu \rangle$, as claimed. □

2.4. We will need the following formula. To simplify notation, we define

$$t_{w\beta} := t_w t_{s_\beta} t_{w^{-1}} \quad \text{for } w \in W \text{ and } \beta \in R. \tag{2.7}$$

LEMMA 2.6. *For* $w \in W$ *and* $\omega \in V^\vee$,

$$t_w \omega t_w^{-1} = w(\omega) + \sum_{\substack{\beta>0 \\ w\beta<0}} c_\beta (\beta, \omega) t_{w\beta}. \tag{2.8}$$

Proof. The formula holds if $w = s_\alpha$, for $\alpha \in \Pi$, by (2.3):

$$t_{s_\alpha} \omega t_{s_\alpha} = s_\alpha(\omega) + c_\alpha (\alpha, \omega) t_{s_\alpha}. \tag{2.9}$$

We now proceed by induction on the length of w. Suppose that the formula holds for w and let α be a simple root such that $s_\alpha w$ has strictly greater length. Then

$$\begin{aligned} t_{s_\alpha} t_w \omega t_{w^{-1}} t_{s_\alpha} &= t_{s_\alpha} \Bigg(w(\omega) + \sum_{\beta : w\beta<0} c_\beta (\beta, \omega) t_{w\beta} \Bigg) t_{s_\alpha} \\ &= s_\alpha w(\omega) + c_\alpha (\alpha, \omega) t_{s_\alpha} + \sum_{\beta : w\beta<0} c_\beta (\beta, \omega) t_{s_\alpha} t_{w\beta} t_{s_\alpha} \\ &= s_\alpha w(\omega) + c_\alpha (\alpha, \omega) t_{s_\alpha} + \sum_{\beta : w\beta<0} c_\beta (\beta, \omega) t_{s_\alpha w\beta}. \end{aligned} \tag{2.10}$$

The claim follows. □

2.5. The $*$-operation, Hermitian and unitary representations

The algebra $\mathbb{H}$ has a natural conjugate linear anti-involution defined on generators as follows ([BM2, §5]):

$$\begin{aligned} t_w^* &= t_{w^{-1}}, && w \in W, \\ \omega^* &= -\omega + \sum_{\beta>0} c_\beta(\beta,\omega) t_{s_\beta}, && \omega \in V_0^\vee. \end{aligned} \tag{2.11}$$

In general there are other conjugate linear anti-involutions on $\mathbb{H}$, but this one is distinguished by its relation to the canonical notion of unitarity for p-adic group representations [BM1], [BM2].

An $\mathbb{H}$-module (π, X) is said to be $*$-Hermitian (or just Hermitian) if there exists a Hermitian form $(\cdot,\cdot)_X$ on X which is invariant in the sense that

$$(\pi(h)x, y)_X = (x, \pi(h^*)y)_X \quad \text{for all } h \in \mathbb{H} \text{ and } x, y \in X. \tag{2.12}$$

If such a form exists which is also positive definite, then X is said to be $*$-unitary (or just unitary).

Because the second formula in (2.11) is complicated, we need other elements which behave more simply under $*$. For every $\omega \in V^\vee$, define

$$\widetilde{\omega} = \omega - \frac{1}{2} \sum_{\beta>0} c_\beta(\beta,\omega) t_{s_\beta} \in \mathbb{H}. \tag{2.13}$$

Then it follows directly from the definitions that $\omega^* = -\omega$ when $\omega \in V_0^\vee$. Thus, if (π, X) is a Hermitian $\mathbb{H}$-module and $\omega \in V_0^\vee$, then

$$(\pi(\widetilde{\omega})x, \pi(\widetilde{\omega})x)_X = (\pi(\widetilde{\omega}^*)\pi(\widetilde{\omega})x, x)_X = -(\pi(\widetilde{\omega}^2)x, x)_X. \tag{2.14}$$

If we further assume that X is unitary, then

$$(\pi(\widetilde{\omega}^2)x, x)_X \leqslant 0 \quad \text{for all } x \in X \text{ and } \omega \in V_0^\vee. \tag{2.15}$$

For each ω and x, this is a necessary condition for a Hermitian representation X to be unitary. It is difficult to apply because the operators $\pi(\widetilde{\omega}^2)$ are intractable in general. Instead we introduce a variation on the Casimir element of Definition 2.3 whose action in an $\mathbb{H}$-module will be seen to be tractable.

Definition 2.7. Let $\{\omega_i\}_{i=1}^n$ and $\{\omega^i\}_{i=1}^n$ be dual bases of $V_0^\vee$ with respect to $\langle\cdot,\cdot\rangle$. Define

$$\widetilde{\Omega} = \sum_{i=1}^n \widetilde{\omega}_i \widetilde{\omega}^i \in \mathbb{H}. \tag{2.16}$$

It will follow from Theorem 2.11 below that $\widetilde{\Omega}$ is independent of the bases chosen.

If we sum (2.15) over a self-dual orthonormal basis of $V_0^\vee$, we immediately obtain the following necessary condition for unitarity.

PROPOSITION 2.8. *A Hermitian $\mathbb{H}$-module (π, X) with invariant form $(\cdot,\cdot)_X$ is unitary only if*

$$(\pi(\widetilde{\Omega})x, x)_X \leqslant 0 \quad \text{for all } x \in X. \tag{2.17}$$

The remainder of this section will be aimed at computing the action of $\widetilde{\Omega}$ in an irreducible $\mathbb{H}$-module as explicitly as possible (so that the necessary condition of Proposition 2.8 becomes as effective as possible). Since $\widetilde{\Omega}$ is no longer central, nothing as simple as Lemma 2.5 is available. But Proposition 2.10 (2) below immediately implies that $\widetilde{\Omega}$ is invariant under conjugation by t_w for $w\in W$. It therefore acts by a scalar on each W-isotypic component of an $\mathbb{H}$-module. We compute these scalars next, the main results being Theorem 2.11 and Corollary 2.12.

To get started, set

$$T_\omega = \omega - \widetilde{\omega} = \frac{1}{2}\sum_{\beta>0} c_\beta(\beta,\omega)t_{s_\beta} \in \mathbb{H} \tag{2.18}$$

and

$$\Omega_W = \frac{1}{4}\sum_{\substack{\alpha,\beta>0\\ s_\alpha(\beta)<0}} c_\alpha c_\beta \langle\alpha,\beta\rangle t_{s_\alpha} t_{s_\beta} \in \mathbb{C}[W]. \tag{2.19}$$

Note that Ω_W is invariant under the conjugation action of W. As we shall see, Ω_W plays the role of a Casimir element for $\mathbb{C}[W]$.

LEMMA 2.9. *If $\omega_1,\omega_2\in V^\vee$, then*

$$[T_{\omega_1}, T_{\omega_2}] = \frac{1}{4}\sum_{\substack{\alpha,\beta>0\\ s_\alpha(\beta)<0}} c_\alpha c_\beta((\alpha,\omega_1)(\beta,\omega_2)-(\beta,\omega_1)(\alpha,\omega_2))t_{s_\alpha}t_{s_\beta}.$$

Proof. From the definition (2.18), we see that

$$[T_{\omega_1}, T_{\omega_2}] = \frac{1}{4}\sum_{\alpha,\beta>0} c_\alpha c_\beta((\alpha,\omega_1)(\beta,\omega_2)-(\beta,\omega_1)(\alpha,\omega_2))t_{s_\alpha}t_{s_\beta}.$$

Assume that $\alpha,\beta>0$ are such that $s_\alpha(\beta)>0$. Notice that if $\gamma=s_\alpha(\beta)$, then $t_{s_\gamma}t_{s_\alpha}=t_{s_\alpha}t_{s_\beta}$. Also, it is elementary to verify (by a rank-2 reduction to the span of $\alpha^\vee$ and $\beta^\vee$, for instance) that

$$(s_\alpha(\beta),\omega_1)(\alpha,\omega_2)-(\alpha,\omega_1)(s_\alpha(\beta),\omega_2) = -((\alpha,\omega_1)(\beta,\omega_2)-(\beta,\omega_1)(\alpha,\omega_2)).$$

Since c is W-invariant, this implies that the contributions of the pairs of roots $\{\alpha,\beta\}$ and $\{s_\alpha(\beta),\alpha\}$ (when $s_\alpha(\beta)>0$) cancel out in the above sum. The claim follows. □

Proposition 2.10. *Fix $w\in W$ and $\omega,\omega_1,\omega_2\in V^\vee$. The elements defined in* (2.13) *have the following properties*:

(1) $\widetilde{\omega}^*=-\widetilde{\omega}$;

(2) $t_w\widetilde{\omega}t_{w^{-1}}=\widetilde{w(\omega)}$;

(3) $[\widetilde{\omega}_1,\widetilde{\omega}_2]=-[T_{\omega_1},T_{\omega_2}]$.

Proof. As remarked above, property (1) is obvious from (2.11). For (2), using Lemma 2.6, we have

$$\begin{aligned} t_w\widetilde{\omega}t_{w^{-1}} &= t_w\omega t_{w^{-1}}-\frac{1}{2}\sum_{\beta>0}c_\beta(\beta,\omega)t_wt_{s_\beta}t_{w^{-1}} \\ &= w(\omega)+\sum_{\substack{\beta>0\\ w\beta<0}}c_\beta(\beta,\omega)t_{w\beta}-\frac{1}{2}\sum_{\beta>0}c_\beta(\beta,\omega)t_{w\beta} \\ &= w(\omega)+\frac{1}{2}\sum_{\substack{\beta>0\\ w\beta<0}}c_\beta(\beta,\omega)t_{w\beta}-\frac{1}{2}\sum_{\substack{\beta>0\\ w\beta>0}}c_\beta(\beta,\omega)t_{w\beta} \\ &= w(\omega)-\frac{1}{2}\sum_{\beta'>0}c_{\beta'}(w^{-1}\beta',\omega)t_{s_{\beta'}} \\ &= \widetilde{w(\omega)}. \end{aligned} \tag{2.20}$$

For the last step, we set $\beta'=-w\beta$ in the first sum and $\beta'=w\beta$ in the second sum, and also used that $c_{\beta'}=c_\beta$, since c is W-invariant.

Finally, we verify (3). We have

$$[\widetilde{\omega}_1,\widetilde{\omega}_2]=[\omega_1-T_{\omega_1},\omega_2-T_{\omega_2}]=[T_{\omega_1},T_{\omega_2}]-([T_{\omega_1},\omega_2]+[\omega_1,T_{\omega_2}]).$$

We do a direct calculation:

$$[T_{\omega_1},\omega_2]=\frac{1}{2}\sum_{\alpha>0}c_\alpha(\alpha,\omega_1)(t_{s_\alpha}\omega_2t_{s_\alpha}-\omega_2)t_{s_\alpha}.$$

Applying Lemma 2.6, we get

$$\begin{aligned} [T_{\omega_1},\omega_2] &= \frac{1}{2}\sum_{\alpha>0}c_\alpha(\alpha,\omega_1)(s_\alpha(\omega_2)-\omega_2)t_{s_\alpha}+\frac{1}{2}\sum_{\substack{\alpha,\beta>0\\ s_\alpha(\beta)<0}}c_\alpha c_\beta(\alpha,\omega_1)(\beta,\omega_2)t_{s_\alpha(\beta)}t_{s_\alpha} \\ &= -\frac{1}{2}\sum_{\alpha>0}c_\alpha(\alpha,\omega_1)(\alpha,\omega_2)\alpha^\vee t_{s_\alpha}+\frac{1}{2}\sum_{\substack{\alpha,\beta>0\\ s_\alpha(\beta)<0}}c_\alpha c_\beta(\alpha,\omega_1)(\beta,\omega_2)t_{s_\alpha}t_{s_\beta}. \end{aligned}$$

From this and Lemma 2.9, it follows immediately that $[T_{\omega_1},\omega_2]+[\omega_1,T_{\omega_2}]=2[T_{\omega_1},T_{\omega_2}]$. This completes the proof of (3). □

THEOREM 2.11. *Let $\widetilde{\Omega}$ be the W-invariant element of $\mathbb{H}$ from Definition 2.7. Recall the notation of (2.18) and (2.19). Then*

$$\widetilde{\Omega}=\Omega-\Omega_W. \tag{2.21}$$

Proof. From Definition 2.7, we have

$$\widetilde{\Omega}=\sum_{i=1}^n \omega_i\omega^i-\sum_{i=1}^n(\omega_i T_{\omega^i}+T_{\omega_i}\omega^i)+\sum_{i=1}^n T_{\omega_i}T_{\omega^i}. \tag{2.22}$$

On the other hand, we have $\widetilde{\omega}=\frac{1}{2}(\omega-\omega^*)$, and so

$$\widetilde{\omega}_i\widetilde{\omega}^i=\tfrac{1}{4}(\omega_i\omega^i+\omega_i^*\omega^{i^*})-\tfrac{1}{4}(\omega_i\omega^{i^*}+\omega_i^*\omega^i). \tag{2.23}$$

Summing (2.23) over i from 1 to n, we find that

$$\begin{aligned}\widetilde{\Omega}&=\sum_{i=1}^n\frac{\omega_i\omega^i+\omega_i^*\omega^{i^*}}{4}-\sum_{i=1}^n\frac{\omega_i\omega^{i^*}+\omega_i^*\omega^i}{4}\\ &=\frac{1}{2}\sum_{i=1}^n\omega_i\omega^i-\frac{1}{4}\sum_{i=1}^n[\omega_i(-\omega^i+2T_{\omega^i})+(-\omega_i+2T_{\omega_i})\omega^i]\\ &=\sum_{i=1}^n\omega_i\omega^i-\frac{1}{2}\sum_{i=1}^n(\omega_iT_{\omega^i}+T_{\omega_i}\omega^i).\end{aligned} \tag{2.24}$$

We conclude from (2.22) and (2.24) that

$$\sum_{i=1}^n T_{\omega_i}T_{\omega^i}=\frac{1}{2}\sum_{i=1}^n(\omega_iT_{\omega^i}+T_{\omega_i}\omega^i) \tag{2.25}$$

and

$$\widetilde{\Omega}=\Omega-\frac{1}{2}\sum_{i=1}^n(\omega_iT_{\omega^i}+T_{\omega_i}\omega^i)=\Omega-\sum_{i=1}^n T_{\omega_i}T_{\omega^i}. \tag{2.26}$$

This is the first assertion of the theorem. For the remainder, write out the definition of T_{ω_i} and T_{ω^i}, and use (2.2):

$$\sum_{i=1}^n T_{\omega_i}T_{\omega^i}=\frac{1}{4}\sum_{\alpha,\beta>0}c_\alpha c_\beta\langle\alpha,\beta\rangle t_{s_\alpha}t_{s_\beta}=\frac{1}{4}\sum_{\substack{\alpha,\beta>0\\ s_\alpha(\beta)<0}}c_\alpha c_\beta\langle\alpha,\beta\rangle t_{s_\alpha}t_{s_\beta}, \tag{2.27}$$

with the last equality following as in the proof of Lemma 2.9. □

Corollary 2.12. *Retain the setting of Proposition 2.8 but further assume that (π, X) is irreducible and unitary with central character χ_ν with $\nu \in V$ (as in Definition 2.2). Let (σ, U) be an irreducible representation of W such that $\mathrm{Hom}_W(U, X) \neq 0$. Then*

$$\langle \nu, \nu \rangle \leqslant c(\sigma), \tag{2.28}$$

where

$$c(\sigma) = \frac{1}{4} \sum_{\alpha>0} c_\alpha^2 \langle \alpha, \alpha \rangle + \frac{1}{4} \sum_{\substack{\alpha,\beta>0 \\ \alpha\neq\beta \\ s_\alpha(\beta)<0}} c_\alpha c_\beta \langle \alpha, \beta \rangle \frac{\mathrm{tr}_\sigma(s_\alpha s_\beta)}{\mathrm{tr}_\sigma(1)}. \tag{2.29}$$

is the scalar by which Ω_W acts in U and tr_σ denotes the character of σ.

Proof. The result follows from the formula in Theorem 2.11 by applying Proposition 2.8 to a vector x in the σ-isotypic component of X. □

Theorem 2.11 will play an important role in the proof of Theorem 3.5 below.

3. The Dirac operator

Throughout this section we fix the setting of §2.1.

3.1. The Clifford algebra

Denote by $C(V_0^\vee)$ the Clifford algebra defined by $V_0^\vee$ and $\langle \cdot, \cdot \rangle$. More precisely, $C(V_0^\vee)$ is the quotient of the tensor algebra of $V_0^\vee$ by the ideal generated by

$$\omega \otimes \omega' + \omega' \otimes \omega + 2\langle \omega, \omega' \rangle, \quad \omega, \omega' \in V_0^\vee.$$

Equivalently, $C(V_0^\vee)$ is the associative algebra with unit generated by $V_0^\vee$ with relations

$$\omega^2 = -\langle \omega, \omega \rangle \quad \text{and} \quad \omega\omega' + \omega'\omega = -2\langle \omega, \omega' \rangle. \tag{3.1}$$

Let $\mathsf{O}(V_0^\vee)$ denote the group of orthogonal transformations of $V_0^\vee$ with respect to $\langle \cdot, \cdot \rangle$. This acts by algebra automorphisms on $C(V_0^\vee)$, and the action of $-1 \in \mathsf{O}(V_0^\vee)$ induces a grading

$$C(V_0^\vee) = C(V_0^\vee)_{\mathsf{even}} + C(V_0^\vee)_{\mathsf{odd}}. \tag{3.2}$$

Let ε be the automorphism of $C(V_0^\vee)$ which is $+1$ on $C(V_0^\vee)_{\mathsf{even}}$ and -1 on $C(V_0^\vee)_{\mathsf{odd}}$. Let ^{t} be the transpose antiautomorphism of $C(V_0^\vee)$ characterized by

$$\omega^t = -\omega \ \text{ for } \omega \in V^\vee \quad \text{and} \quad (ab)^t = b^t a^t \ \text{ for } a, b \in C(V_0^\vee). \tag{3.3}$$

The Pin group is

$$\mathsf{Pin}(V_0^\vee)=\{a\in C(V_0^\vee):\varepsilon(a)V_0^\vee a^{-1}\subset V_0^\vee \text{ and } a^t=a^{-1}\}. \tag{3.4}$$

It sits in a short exact sequence

$$1\longrightarrow\{\pm1\}\longrightarrow\mathsf{Pin}(V_0^\vee)\xrightarrow{\ p\ }\mathsf{O}(V_0^\vee)\longrightarrow 1, \tag{3.5}$$

where the projection p is given by $p(a)(\omega)=\varepsilon(a)\omega a^{-1}$. (Note the appearance of ε in the definition of $\mathsf{Pin}(V_0^\vee)$. This insures that p is surjective.)

We call a complex simple $C(V_0^\vee)$-module (γ,S) of dimension $2^{\lfloor\dim V/2\rfloor}$ a *spin module* for $C(V_0^\vee)$. When $\dim V$ is even, there is only one such module (up to equivalence), but if $\dim V$ is odd, there are two inequivalent spin modules. We may endow such a module with a positive definite Hermitian form $\langle\cdot,\cdot\rangle_S$ such that

$$\langle\gamma(a)s,s'\rangle_S=\langle s,\gamma(a^t)\rangle_S \quad \text{for all } a\in C(V_0^\vee) \text{ and } s,s'\in S. \tag{3.6}$$

In all cases, (γ,S) restricts to an irreducible unitary representation of $\mathsf{Pin}(V_0^\vee)$.

3.2. The Dirac operator *D*

Definition 3.1. Let $\{\omega_i\}_{i=1}^n$ and $\{\omega^i\}_{i=1}^n$ be dual bases of $V_0^\vee$, and recall the elements $\widetilde{\omega}_i\in\mathbb{H}$ from (2.13). The Dirac element is defined as

$$\mathcal{D}=\sum_{i=1}^n\widetilde{\omega}_i\otimes\omega^i\in\mathbb{H}\otimes C(V_0^\vee).$$

It is elementary to verify that $\mathcal{D}$ does not depend on the choice of dual bases.

Frequently we will work with a fixed spin module (γ,S) for $C(V_0^\vee)$ and a fixed $\mathbb{H}$-module (π,X). In this setting, it will be convenient to define the Dirac operator for X (and S) as $D=(\pi\otimes\gamma)(\mathcal{D})$. Explicitly,

$$D=\sum_{i=1}^n\pi(\widetilde{\omega}_i)\otimes\gamma(\omega^i)\in\operatorname{End}_{\mathbb{H}\otimes C(V_0^\vee)}(X\otimes S). \tag{3.7}$$

Lemma 3.2. *Let X be a Hermitian $\mathbb{H}$-module with invariant form $(\cdot,\cdot)_X$. With notation as in (3.6), endow $X\otimes S$ with the Hermitian form*

$$(x\otimes s,x'\otimes s')_{X\otimes S}=(x,x')_X\otimes(s,s')_S.$$

Then the operator D is self-adjoint with respect to $(\cdot,\cdot)_{X\otimes S}$,

$$(D(x\otimes s),x'\otimes s')_{X\otimes S}=(x\otimes s,D(x'\otimes s'))_{X\otimes S}. \tag{3.8}$$

Proof. This follows from a straightforward verification. □

We immediately deduce the following analogue of Proposition 2.8.

Proposition 3.3. *In the setting of Lemma 3.2, a Hermitian $\mathbb{H}$-module is unitary only if*

$$(D^2(x\otimes s), x\otimes s)_{X\otimes S} \geqslant 0 \quad \textit{for all } x\otimes s \in X\otimes S. \tag{3.9}$$

To be a useful criterion for unitarity, we need to establish a formula for D^2 (see Theorem 3.5 below).

3.3. The spin cover $\widetilde{W}$

The Weyl group W acts by orthogonal transformations on $V_0^\vee$, and thus is a subgroup of $\mathsf{O}(V_0^\vee)$. We define the group $\widetilde{W}$ in $\mathsf{Pin}(V_0^\vee)$ as

$$\widetilde{W} := p^{-1}(\mathsf{O}(V_0^\vee)) \subset \mathsf{Pin}(V_0^\vee), \quad \text{where } p \text{ is as in (3.5)}. \tag{3.10}$$

Therefore, $\widetilde{W}$ is a central extension of W,

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{W} \xrightarrow{\;p\;} W \longrightarrow 1. \tag{3.11}$$

We will need a few details about the structure of $\widetilde{W}$. For each $\alpha\in R$, define elements $f_\alpha \in C(V_0^\vee)$ via

$$f_\alpha = \frac{\alpha^\vee}{|\alpha^\vee|} \in V^\vee \subset C(V_0^\vee). \tag{3.12}$$

It easily follows that $p(f_\alpha)=s_\alpha$, the reflection in W through α. Thus $\{f_\alpha : \alpha\in R\}$ (or just $\{f_\alpha : \alpha \in \Pi\}$) generate $\widetilde{W}$. Obviously $f_\alpha^2 = -1$. Slightly more delicate considerations (see, e.g., [M, Theorem 3.2]) show that if $\alpha, \beta \in R$ and $\gamma = s_\alpha(\beta)$, then

$$f_\beta f_\alpha = -f_\alpha f_\gamma. \tag{3.13}$$

A representation of $\widetilde{W}$ is called *genuine* if it does not factor to W, i.e. if -1 acts non-trivially. Otherwise it is called *non-genuine*. (Similar terminology applies to $\mathbb{C}[\widetilde{W}]$-modules.) Via restriction, we can regard a spin module (γ, S) for $C(V_0^\vee)$ as a unitary $\widetilde{W}$-representation. Clearly it is genuine. Since $R^\vee$ spans $V^\vee$, it is also irreducible (see, e.g., [M, Theorem 3.3]). For notational convenience, we lift the sgn representation of W to a non-genuine representation of $\widetilde{W}$ which we also denote by sgn.

We write ϱ for the diagonal embedding of $\mathbb{C}[\widetilde{W}]$ into $\mathbb{H}\otimes C(V_0^\vee)$ defined by extending

$$\varrho(\widetilde{w}) = t_{p(\widetilde{w})} \otimes \widetilde{w} \tag{3.14}$$

linearly.

LEMMA 3.4. *Recall the notation of Definition 3.1 and (3.14). For* $\widetilde{w}\in\widetilde{W}$,

$$\varrho(\widetilde{w})\mathcal{D} = \mathsf{sgn}(\widetilde{w})\mathcal{D}\varrho(\widetilde{w})$$

as elements of $\mathbb{H}\otimes C(V^\vee)$.

Proof. From the definitions and Proposition 2.10 (2), we have

$$\varrho(\widetilde{w})\mathcal{D}\varrho(\widetilde{w}^{-1}) = \sum_{i=1}^{n} t_{p(\widetilde{w})}\widetilde{\omega}_i t_{p(\widetilde{w}^{-1})}\otimes\widetilde{w}\omega^i\widetilde{w}^{-1} = \sum_{i=1}^{n}\widetilde{p(\widetilde{w})\cdot\omega_i}\otimes\widetilde{w}\omega^i\widetilde{w}^{-1},$$

where we have used the $\cdot$ to emphasize the usual action of W on $S(V_0^\vee)$. We argue that in $C(V_0^\vee)$,

$$\widetilde{w}\omega^i\widetilde{w}^{-1} = \mathsf{sgn}(\widetilde{w})(p(\widetilde{w})\cdot\omega^i). \tag{3.15}$$

Then the lemma follows from the fact that the definition of $\mathcal{D}$ is independent of the choice of dual bases.

Since $\widetilde{W}$ is generated by the various f_α for α simple, it is sufficient to verify (3.15) for $\widetilde{w}=f_\alpha$. This follows from direct calculation:

$$\begin{aligned} f_\alpha\omega^i f_\alpha^{-1} &= -\frac{1}{\langle\alpha^\vee,\alpha^\vee\rangle}\alpha^\vee\omega^i\alpha^\vee = -\frac{1}{\langle\alpha^\vee,\alpha^\vee\rangle}\alpha^\vee(-\alpha^\vee\omega^i - 2\langle\omega,\alpha^\vee\rangle) \\ &= -\omega^i + (\omega,\alpha)\alpha^\vee = -s_\alpha\cdot\omega^i. \end{aligned}$$

□

3.4. A formula for $\mathcal{D}^2$

Set

$$\Omega_{\widetilde{W}} = -\frac{1}{4}\sum_{\substack{\alpha,\beta>0\\ s_\alpha(\beta)<0}} c_\alpha c_\beta|\alpha|\,|\beta| f_\alpha f_\beta. \tag{3.16}$$

This is a complex linear combination of elements of $\widetilde{W}$, i.e. an element of $\mathbb{C}[\widetilde{W}]$. Using (3.13), it is easy to see $\Omega_{\widetilde{W}}$ is invariant under the conjugation action of $\widetilde{W}$. Once again $\Omega_{\widetilde{W}}$ will play the role of a Casimir element for $\mathbb{C}[\widetilde{W}]$. The following result should be compared with [P, Proposition 3.1].

THEOREM 3.5. *With notation as in (2.4), (3.7), (3.14) and (3.16),*

$$\mathcal{D}^2 = -\Omega\otimes 1 + \varrho(\Omega_{\widetilde{W}}), \tag{3.17}$$

as elements of $\mathbb{H}\otimes C(V^\vee)$.

Proof. It will be useful below to set

$$R_\circ^2 := \{(\alpha,\beta)\in R\times R : \alpha,\beta>0, \alpha\neq\beta \text{ and } s_\alpha(\beta)<0\}. \tag{3.18}$$

To simplify notation, we fix a self-dual (orthonormal) basis $\{\omega_i\}_{i=1}^n$ of $V_0^\vee$. From Definition 3.1, we have

$$\mathcal{D}^2 = \sum_{i=1}^n \widetilde{\omega}_i^2\otimes\omega_i^2 + \sum_{i\neq j}\widetilde{\omega}_i\widetilde{\omega}_j\otimes\omega_i\omega_j.$$

Using $\omega_i^2=-1$ and $\omega_i\omega_j=-\omega_i\omega_j$ in $C(V^\vee)$ and the notation of Definition 2.7, we get

$$\mathcal{D}^2 = -\widetilde{\Omega}\otimes 1 + \sum_{i<j}[\widetilde{\omega}_i,\widetilde{\omega}_j]\otimes\omega_i\omega_j.$$

Applying Theorem 2.11 to the first term and Proposition 2.10 (3) to the second, we have

$$\mathcal{D}^2 = -\Omega\otimes 1 + \frac{1}{4}\sum_{\alpha>0}\langle\alpha,\alpha\rangle + \frac{1}{4}\sum_{(\alpha,\beta)\in R_\circ^2} c_\alpha c_\beta\langle\alpha,\beta\rangle t_{s_\alpha}t_{s_\beta}\otimes 1 - \sum_{i<j}[T_{\omega_i},T_{\omega_j}]\otimes\omega_i\omega_j.$$

Rewriting $[T_{\omega_i},T_{\omega_j}]$ using Proposition 2.10 (3), this becomes

$$\begin{aligned}\mathcal{D}^2 = -\Omega\otimes 1 &+ \frac{1}{4}\sum_{\alpha>0}\langle\alpha,\alpha\rangle + \frac{1}{4}\sum_{(\alpha,\beta)\in R_\circ^2} c_\alpha c_\beta\langle\alpha,\beta\rangle t_{s_\alpha}t_{s_\beta}\otimes 1\\ &-\frac{1}{4}\sum_{i<j}\sum_{(\alpha,\beta)\in R_\circ^2} c_\alpha c_\beta((\omega_i,\alpha)(\omega_j,\beta)-(\omega_i,\beta)(\omega_j,\alpha))t_{s_\alpha}t_{s_\beta}\otimes\omega_i\omega_j,\end{aligned}$$

and since $\omega_i\omega_j=-\omega_j\omega_i$ in $C(V^\vee)$,

$$\begin{aligned}\mathcal{D}^2 = -\Omega\otimes 1 &+ \frac{1}{4}\sum_{\alpha>0}\langle\alpha,\alpha\rangle + \frac{1}{4}\sum_{(\alpha,\beta)\in R_\circ^2} c_\alpha c_\beta\langle\alpha,\beta\rangle t_{s_\alpha}t_{s_\beta}\otimes 1\\ &-\frac{1}{4}\sum_{(\alpha,\beta)\in R_\circ^2} c_\alpha c_\beta t_{s_\alpha}t_{s_\beta}\otimes\sum_{i\neq j}((\omega_i,\alpha)(\omega_j,\beta))\omega_i\omega_j.\end{aligned}$$

Using (2.2) and the definition of f_α in (3.12), we get

$$\begin{aligned}\mathcal{D}^2 &= -\Omega\otimes 1 + \frac{1}{4}\sum_{\alpha>0}\langle\alpha,\alpha\rangle + \frac{1}{4}\sum_{(\alpha,\beta)\in R_\circ^2} c_\alpha c_\beta\langle\alpha,\beta\rangle t_{s_\alpha}t_{s_\beta}\otimes 1\\ &\quad-\frac{1}{4}\sum_{(\alpha,\beta)\in R_\circ^2} c_\alpha c_\beta t_{s_\alpha}t_{s_\beta}\otimes(|\alpha|\,|\beta| f_\alpha f_\beta+\langle\alpha,\beta\rangle)\\ &= -\Omega\otimes 1 + \frac{1}{4}\sum_{\alpha>0}\langle\alpha,\alpha\rangle - \frac{1}{4}\sum_{(\alpha,\beta)\in R_\circ^2} c_\alpha c_\beta|\alpha|\,|\beta| t_{p(f_\alpha)}t_{p(f_\beta)}\otimes f_\alpha f_\beta.\end{aligned}$$

The theorem follows. ☐

Corollary 3.6. *In the setting of Proposition 3.3, assume further that X is irreducible and unitary with central character χ_ν with $\nu \in V$ (as in Definition 2.2). Let $(\tilde{\sigma}, \widetilde{U})$ be an irreducible representation of $\widetilde{W}$ such that $\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, X \otimes S) \neq 0$. Then*

$$\langle \nu, \nu \rangle \leqslant c(\tilde{\sigma}), \tag{3.19}$$

where

$$c(\tilde{\sigma}) = \frac{1}{4} \sum_{\alpha > 0} c_\alpha^2 \langle \alpha, \alpha \rangle - \frac{1}{4} \sum_{\substack{\alpha, \beta > 0 \\ \alpha \neq \beta \\ s_\alpha(\beta) < 0}} c_\alpha c_\beta |\alpha|\,|\beta| \frac{\mathrm{tr}_{\tilde{\sigma}}(f_\alpha f_\beta)}{\mathrm{tr}_{\tilde{\sigma}}(1)} \tag{3.20}$$

is the scalar by which $\Omega_{\widetilde{W}}$ acts in $\widetilde{U}$ and $\mathrm{tr}_{\widetilde{\sigma}}$ denotes the character of $\widetilde{\sigma}$.

Proof. The corollary follows by applying Proposition 3.3 to a vector $x \otimes s$ in the $\widetilde{\sigma}$-isotypic component of $X \otimes S$, and then using the formula for $D^2 = (\pi \otimes \gamma)(\mathcal{D}^2)$ from Theorem 3.5 and the formula for $\pi(\Omega)$ from Lemma 2.5. □

4. Dirac cohomology and Vogan's conjecture

Let (π, X) be an irreducible $\mathbb{H}$-module with central character χ_ν. By Lemma 3.4, the kernel of the Dirac operator on $X \otimes S$ is invariant under $\widetilde{W}$. Suppose that $\ker(D)$ is non-zero and that $\widetilde{\sigma}$ is an irreducible representation of $\widetilde{W}$ appearing in $\ker(D)$. Then, in the notation of Corollary 3.6, Theorem 3.5 and Lemma 2.5 imply that

$$\langle \nu, \nu \rangle = c(\tilde{\sigma}).$$

In particular, $\langle \nu, \nu \rangle$ is determined by the $\widetilde{W}$-structure of $\ker(D)$. Theorem 4.4 below says that χ_ν itself is determined by this information.

4.1. Dirac cohomology

As discussed above, Vogan's conjecture suggests that for an irreducible unitary representation X, the $\widetilde{W}$-structure of the kernel of the Dirac operator D should determine the central character of X. This is certainly false for non-unitary representations. But since it is difficult to imagine a proof of an algebraic statement which applies only to unitary representations, we use an idea of Vogan and enlarge the class of irreducible unitary representations for which $\ker(D)$ is non-zero to the class of representations with non-zero Dirac cohomology in the following sense.

Definition 4.1. In the setting of Definition 3.1, define

$$H^D(X) := \frac{\ker(D)}{\ker(D)\cap \operatorname{im}(D)} \tag{4.1}$$

and call it the *Dirac cohomology* of X. (For example, if X is unitary, Lemma 3.2 implies that $\ker(D)\cap\operatorname{im}(D)=0$, and so $H^D(X)=\ker(D)$.)

We roughly follow the outline proposed for real groups by Vogan in [V, Lecture 3] and completed in [HP]. The main technical result of this section is the following algebraic statement. (See [HP, Theorem 2.5 and Corollary 3.5] for the analogous result for real groups.)

THEOREM 4.2. *Let $\mathbb{H}$ be the graded affine Hecke algebra attached to a root system Φ and parameter function c (Definition 2.1). Let $z\in Z(\mathbb{H})$ be given. Then there exists $a\in\mathbb{H}\otimes C(V_0^\vee)$ and a unique element $\zeta(z)$ in the center of $\mathbb{C}[\widetilde{W}]$ such that*

$$z\otimes 1 = \varrho(\zeta(z)) + \mathcal{D}a + a\mathcal{D}$$

as elements in $\mathbb{H}\otimes C(V_0^\vee)$. Moreover, the map $\zeta\colon Z(\mathbb{H})\to\mathbb{C}[\widetilde{W}]^{\widetilde{W}}$ is a homomorphism of algebras.

The proof of Theorem 4.2 is given after Remark 4.6.

Definition 4.3. Theorem 4.2 allows one to attach canonically, to every irreducible $\widetilde{W}$-representation $(\tilde\sigma,\widetilde{U})$, a homomorphism $\chi^{\tilde\sigma}\colon Z(\mathbb{H})\to\mathbb{C}$, i.e. a central character of $\mathbb{H}$. More precisely, for every $z\in Z(\mathbb{H})$, $\tilde\sigma(\zeta(z))$ acts by a scalar, and we denote this scalar by $\chi^{\tilde\sigma}(z)$. Since ζ is an algebra homomorphism by Theorem 4.2, the map $\chi^{\tilde\sigma}$ is in fact a homomorphism.

With this definition, Vogan's conjecture takes the following form.

THEOREM 4.4. *Let $\mathbb{H}$ be the graded affine Hecke algebra attached to a root system Φ and parameter function c (Definition 2.1). Suppose that (π, X) is an $\mathbb{H}$-module with central character χ_ν with $\nu\in V$ (as in Definition 2.2). In the setting of Definition 4.1, suppose that $H^D(X)\neq 0$. Let $(\tilde\sigma,\widetilde{U})$ be an irreducible representation of $\widetilde{W}$ such that*

$$\operatorname{Hom}_{\widetilde{W}}(\widetilde{U}, H^D(X))\neq 0.$$

Then

$$\chi_\nu = \chi^{\tilde\sigma},$$

where $\chi^{\tilde\sigma}$ is as in Definition 4.3.

THEOREM 4.5. *Theorem 4.2 implies Theorem 4.4.*

Proof. In the setting of Theorem 4.4, suppose that $\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, H^D(X))\neq 0$. Then there exists $\tilde{x}=x\otimes s\neq 0$ in the $\widetilde{\sigma}$-isotypic component of $X\otimes S$ such that $\tilde{x}\in\ker(D)\setminus\mathrm{im}(D)$. Thus, for every $z\in Z(\mathbb{H})$, we have

$$(\pi(z)\otimes 1)\tilde{x}=\chi_\nu(z)\tilde{x}$$

and

$$(\pi\otimes\gamma)(\varrho(\zeta(z)))\tilde{x}=\tilde{\sigma}(\zeta(z))\tilde{x}.$$

Note that the right-hand sides of the previous two equations are scalar multiples of $\tilde{x}$. Assuming Theorem 4.2, we have

$$(\pi(z)\otimes 1-(\pi\otimes\gamma)\varrho(\zeta(z)))\tilde{x}=(Da+aD)\tilde{x}=Da\tilde{x}, \tag{4.2}$$

which would imply that $\tilde{x}\in\mathrm{im}(D)$, unless $Da\tilde{x}=0$. So we must have $Da\tilde{x}=0$, and therefore

$$\tilde{\sigma}(\zeta(z)) \text{ acts by the scalar } \chi_\nu(z) \quad \text{for all } z\in Z(\mathbb{H}). \tag{4.3}$$

Now the claim follows by comparison with Definition 4.3. □

Remark 4.6. In §5, we shall describe explicitly the central characters $\chi^{\tilde{\sigma}}$, when the graded Hecke algebra $\mathbb{H}$ has a geometric realization ([L1], [L3]).

4.2. Proof of Theorem 4.2

Motivated by [HP, §3], we define

$$d\colon \mathbb{H}\otimes C(V_0^\vee)\longrightarrow \mathbb{H}\otimes C(V_0^\vee) \tag{4.4}$$

on a simple tensor of the form $a=h\otimes v_1 \dots v_k$ (with $h\in H$ and $v_i\in V^\vee$) via

$$d(a)=\mathcal{D}a-(-1)^k a\mathcal{D},$$

and extend linearly to all of $\mathbb{H}\otimes C(V_0^\vee)$.

Then Lemma 3.4 implies that d interchanges the spaces

$$(\mathbb{H}\otimes C(V_0^\vee))^{\mathsf{triv}}=\{a\in\mathbb{H}\otimes C(V_0^\vee):\varrho(\widetilde{w})a=a\varrho(\widetilde{w})\} \tag{4.5}$$

and

$$(\mathbb{H}\otimes C(V_0^\vee))^{\mathsf{sgn}}=\{a\in\mathbb{H}\otimes C(V_0^\vee):\varrho(\widetilde{w})a=\mathsf{sgn}(\widetilde{w})a\varrho(\widetilde{w})\}. \tag{4.6}$$

(Such complications are not encountered in [HP] since the underlying real group is assumed to be connected.) Let d^{triv} (resp. d^{sgn}) denote the restriction of d to the space in (4.5) (resp. (4.6)). We will deduce Theorem 4.2 from the following result.

Theorem 4.7. *With notation as in the previous paragraph,*

$$\ker(d^{\mathsf{triv}})=\operatorname{im}(d^{\mathsf{sgn}})\oplus\varrho(\mathbb{C}[\widetilde{W}]^{\widetilde{W}}).$$

To see that Theorem 4.7 implies the first assertion of Theorem 4.2, take $z\in Z(\mathbb{H})$. Since $z\otimes 1$ is in $(\mathbb{H}\otimes C_{\mathsf{even}}(V_0^\vee))^{\mathsf{triv}}$ and clearly commutes with $\mathcal{D}$, $z\otimes 1$ is in the kernel of d^{triv}. So the conclusion of Theorem 4.7 implies that $z\otimes 1=d^{\mathsf{sgn}}(a)+\varrho(\zeta(z))$ for a unique $\zeta(z)\in\mathbb{C}[\widetilde{W}]^{\widetilde{W}}$ and an element a of $(\mathbb{H}\otimes C_{\mathsf{odd}}(V_0^\vee))^{\mathsf{sgn}}$. In particular $d^{\mathsf{sgn}}(a)=\mathcal{D}a+a\mathcal{D}$. Thus

$$z\otimes 1=\varrho(\zeta(z))+\mathcal{D}a+a\mathcal{D},$$

which is the main conclusion of Theorem 4.2. (The assertion that ζ is an algebra homomorphism is treated in Lemma 4.16 below.)

Thus everything comes down to proving Theorem 4.7. We begin with some preliminaries.

Lemma 4.8. *We have*

$$\varrho(\mathbb{C}[\widetilde{W}]^{\widetilde{W}})\subset\ker(d^{\mathsf{triv}}).$$

Proof. Fix $\widetilde{w}\in\widetilde{W}$ and let $s_{\alpha_1}\dots s_{\alpha_k}$ be a reduced expression of $p(w)$ with α_i simple. Then (after possibly replacing α_1 with $-\alpha_1$), $\widetilde{w}=f_{\alpha_1}\dots f_{\alpha_k}$. Set $a=\varrho(\widetilde{w})$. Then the definition of d and Lemma 3.4 imply that

$$d(a)=\mathcal{D}a-(-1)^k a\mathcal{D}=(1-(-1)^k\mathsf{sgn}(\widetilde{w}))\mathcal{D}a=(1-(-1)^k(-1)^k)\mathcal{D}a=0,$$

as claimed. □

Lemma 4.9. *We have*

$$(d^{\mathsf{triv}})^2=(d^{\mathsf{sgn}})^2=0.$$

Proof. For any $a\in\mathbb{H}\otimes C(V_0^\vee)$, one computes directly from the definition of d to find that

$$d^2(a)=\mathcal{D}^2a-a\mathcal{D}^2.$$

By Theorem 3.5, $\mathcal{D}^2=-\Omega\otimes 1+\varrho(\Omega_{\widetilde{W}})$. By Lemma 2.4, $-\Omega\otimes 1$ automatically commutes with a. If we further assume that a is in $(\mathbb{H}\otimes C(V_0^\vee))^{\mathsf{triv}}$, then a commutes with $\varrho(\Omega_{\widetilde{W}})$ as well. Since each term in the definition of $\Omega_{\widetilde{W}}$ is in the kernel of sgn, the same conclusion holds if a is in $(\mathbb{H}\otimes C(V_0^\vee))^{\mathsf{sgn}}$. The lemma follows. □

We next introduce certain graded objects. Let $S^j(V^\vee)$ denote the subspace of elements of degree j in $S(V^\vee)$. Let $\mathbb{H}^j$ denote the subspace of $\mathbb{H}$ consisting of products of elements in the image of $\mathbb{C}[W]$ and $S^j(V^\vee)$ under the maps described in (1) and (2) in Definition 2.1. Then it is easy to check (using (2.3)) that $\mathbb{H}^0 \subset \mathbb{H}^1 \subset \ldots$ is an algebra filtration. Set $\overline{\mathbb{H}}^j = \mathbb{H}^j / \mathbb{H}^{j-1}$ and let $\overline{\mathbb{H}} = \bigoplus_{j=1}^{\infty} \overline{\mathbb{H}}^j$ denote the associated graded algebra. Then $\overline{\mathbb{H}}$ identifies with $\mathbb{C}[W] \rtimes S(V^\vee)$, where $\mathbb{C}[W]$ acts in a natural way:

$$t_w \omega t_{w^{-1}} = w(\omega).$$

We will invoke this identification often without comment. Note that $\overline{\mathbb{H}}$ does not depend on the parameter function c used to define $\mathbb{H}$.

The map d of (4.4) induces a map

$$\bar{d}\colon \overline{\mathbb{H}} \otimes C(V_0^\vee) \longrightarrow \overline{\mathbb{H}} \otimes C(V_0^\vee). \tag{4.7}$$

Explicitly, if we fix a self-dual basis $\{\omega_i\}_{i=1}^n$ of $V^\vee$, then the value of $\bar{d}$ on a simple tensor of the form $a = t_w f \otimes v_1 \ldots v_k$ (with $t_w f \in \mathbb{C}[W] \ltimes S(V^\vee)$ and $v_i \in V^\vee$) is given by

$$\begin{aligned} \bar{d}(a) &= \sum_{i=1}^n \omega_i t_w f \otimes \omega_i v_1 \ldots v_k - (-1)^k \sum_{i=1}^n t_w f \omega_i \otimes v_1 \ldots v_k \omega_i \\ &= \sum_{i=1}^n t_w w^{-1}(\omega_i) f \otimes \omega_i v_1 \ldots v_k - (-1)^k \sum_{i=1}^n t_w f \omega_i \otimes v_1 \ldots v_k \omega_i. \end{aligned} \tag{4.8}$$

We will deduce Theorem 4.7 from the computation of the cohomology of $\bar{d}$. We need some final preliminaries.

LEMMA 4.10. *The map $\bar{d}$ of (4.8) is an odd derivation in the sense that if*

$$a = t_w f \otimes v_1 \ldots v_k \in \overline{\mathbb{H}}$$

and $b \in \overline{\mathbb{H}}$ is arbitrary, then

$$\bar{d}(ab) = \bar{d}(a) b + (-1)^k a \bar{d}(b).$$

Proof. Fix a as in the statement of the lemma. A simple induction reduces the general case of the lemma to the following three special cases:

(i) $b = \omega \otimes 1$ for $\omega \in V^\vee$;

(ii) $b = 1 \otimes \omega$ for $\omega \in V^\vee$;

(iii) $b = t_s \otimes 1$ for a simple reflection $s = s_\alpha$ in W.

(The point is that these three types of elements generate $\overline{\mathbb{H}}$.) Each of these cases follows from a straightforward verification. For example, consider the first case, $b = \omega \otimes 1$. Then, from the definition of $\bar{d}$, we have

$$\bar{d}(ab) = \sum_{i=1}^n \omega_i t_w f \omega \otimes \omega_1 v_1 \ldots v_k - (-1)^k \sum_{i=1}^n t_w f \omega \omega_i \otimes v_1 \ldots v_k \omega_i. \tag{4.9}$$

On the other hand, since it is easy to see that $d(b)=0$ in this case, we have

$$\bar{d}(a)b+(-1)^k a\bar{d}(b)=\bar{d}(a)b=\sum_{i=1}^{n}\omega_i t_w f\omega\otimes\omega_1 v_1 \dots v_k-(-1)^k\sum_i t_w f\omega_i\omega\otimes v_1 \dots v_k\omega_i. \tag{4.10}$$

Since $S(V^\vee)$ is commutative, (4.9) and (4.10) coincide, and the lemma holds in this case. The other two remaining cases hold by similar direct calculations. We omit the details. □

LEMMA 4.11. *The map $\bar{d}$ satisfies $\bar{d}^2=0$.*

Proof. Fix $a=t_w f\otimes v_1 \dots v_k\in\overline{\mathbb{H}}$ and let b be arbitrary. Using Lemma 4.10, one computes directly from the definitions to find that

$$\bar{d}^2(ab)=\bar{d}^2(a)b+a\bar{d}^2(b).$$

It follows that to establish the current lemma in general, it suffices to check that $d^2(b)=0$ for each of the three kinds of generators b appearing in the proof of Lemma 4.10. Once again this is a straightforward verification whose details we omit. (Only case (iii) is non-trivial.) □

LEMMA 4.12. *Let $\bar{\varrho}$ denote the diagonal embedding of $\mathbb{C}[\widetilde{W}]$ in $\overline{\mathbb{H}}\otimes C(V_0^\vee)$ defined by linearly extending*

$$\bar{\varrho}(\widetilde{w})=t_{p(f_\alpha)}\otimes\widetilde{w}$$

for $\widetilde{w}\in\widetilde{W}$. Then

$$\bar{\varrho}(\mathbb{C}[\widetilde{W}])\subset\ker(\bar{d}).$$

Proof. As noted in §3.3, the various $f_\alpha=\alpha^\vee/|\alpha^\vee|$ (for α simple) generate $\widetilde{W}$. Furthermore $p(f_\alpha)=s_\alpha$. So Lemma 4.10 implies that the current lemma will follow if we can prove that

$$\bar{d}(t_{s_\alpha}\otimes\alpha^\vee)=0$$

for each simple α. For this, we compute directly

$$\begin{aligned}\bar{d}(t_{s_\alpha}\otimes\alpha^\vee)&=\sum_{i=1}^{n}\omega_i t_{s_\alpha}\otimes\omega_i\alpha^\vee+\sum_{i=1}^{n}t_{s_\alpha}\omega_i\otimes\alpha^\vee\omega_i\\&=\sum_{i=1}^{n}t_{s_\alpha}s_\alpha(\omega_i)\otimes\omega_i\alpha^\vee+\sum_{i=1}^{n}t_{s_\alpha}\omega_i\otimes\alpha^\vee\omega_i\\&=\sum_{i=1}^{n}t_{s_\alpha}(\omega_i-(\alpha,\omega_i))\alpha^\vee\otimes\omega_i\alpha^\vee+\sum_{i=1}^{n}t_{s_\alpha}\omega_i\otimes\alpha^\vee\omega_i\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} t_{s_\alpha}\omega_i \otimes (\omega_i\alpha^\vee + \alpha^\vee\omega_i) - \sum_{i=1}^{n} t_{s_\alpha}(\alpha,\omega_i)\alpha^\vee \otimes \omega_i\alpha^\vee \\
&= -2\sum_{i=1}^{n} t_{s_\alpha}\omega_i \otimes \langle\alpha^\vee,\omega_i\rangle - \sum_{i=1}^{n} t_{s_\alpha}\alpha^\vee \otimes (\alpha,\omega_i)\omega_i\alpha^\vee \\
&= -2\sum_{i=1}^{n} t_{s_\alpha}\langle\alpha^\vee,\omega_i\rangle\omega_i \otimes 1 - \sum_{i=1}^{n} t_{s_\alpha}\alpha^\vee \otimes \frac{\langle\omega_i,\alpha\rangle}{\langle\alpha^\vee,\alpha^\vee\rangle}\omega_i\alpha^\vee \\
&= -2t_{s_\alpha}\alpha^\vee \otimes 1 - t_{s_\alpha}\alpha^\vee \otimes \frac{(\alpha^\vee)^2}{\langle\alpha^\vee,\alpha^\vee\rangle} \\
&= -2t_{s_\alpha}\alpha^\vee \otimes 1 + 2t_{s_\alpha}\alpha^\vee \otimes 1 \\
&= 0.
\end{aligned}
$$
□

From (4.8) it follows that $\bar{d}$ preserves the subspace $S(V^\vee)\otimes C(V_0^\vee)\subset\overline{\mathbb{H}}\otimes C(V_0^\vee)$. Write $\bar{d}'$ for the restriction of $\bar{d}$ to $S(V^\vee)\otimes C(V_0^\vee)$.

LEMMA 4.13. *With notation as in the previous paragraph,*

$$\ker(\bar{d}') = \operatorname{im}(\bar{d}') \oplus \mathbb{C}(1\otimes 1).$$

Proof. An elementary calculation shows that $\bar{d}'$ is a multiple of the differential in the Koszul complex whose cohomology is well known. (See, e.g., [HP, Lemma 4.1].) □

We can now assemble these lemmas into the computation of the cohomology of $\bar{d}$.

PROPOSITION 4.14. *We have*

$$\ker(\bar{d}) = \operatorname{im}(\bar{d}) \oplus \bar{\varrho}(\mathbb{C}[\widetilde{W}]).$$

Proof. By Lemmas 4.11 and 4.12, $\operatorname{im}(\bar{d})+\bar{\varrho}(\mathbb{C}[\widetilde{W}])\subset\ker(\bar{d})$. Since it follows from the definition of $\bar{d}$ that $\operatorname{im}(\bar{d})$ and $\bar{\varrho}(\mathbb{C}[\widetilde{W}])$ intersect trivially, we need only establish the reverse inclusion. Fix $a\in\ker(\bar{d})$ and write it as a sum of simple tensors of the form $t_w f\otimes v_1 \dots v_k$. For each $w_j\in W$, let a_j denote the sum of the simple tensors appearing in this expression for a which have t_{w_j} in them. Thus $a=a_1+\dots+a_l$, and we can arrange the indexing so that each a_i is non-zero. Since $\bar{d}(a)=0$,

$$\bar{d}(a_1)+\dots+\bar{d}(a_l) = 0. \tag{4.11}$$

Each term $\bar{d}(a_i)$ is a sum of simple tensors of the form $t_{w_i} f\otimes v_1 \dots v_k$. Since the w_i are distinct, the only way (4.11) can hold is if each $\bar{d}(a_i)=0$. Choose $\widetilde{w}_i\in\widetilde{W}$ such that $p(\widetilde{w}_i)=w_i$. Set

$$a_i' = \bar{\varrho}(\widetilde{w}_1^{-1})a_i \in S(V)\otimes C(V_0^\vee).$$

Using Lemmas 4.10 and 4.12, we have that

$$\bar{\varrho}(\widetilde{w}_i)\bar{d}(a_i') = \bar{d}(\bar{\varrho}(\widetilde{w}_i)a_i) = \bar{d}(a_i) = 0.$$

Thus for each i,

$$\bar{d}(a_i') = 0.$$

Since each $a_i' \in S(V^\vee)\otimes C(V_0^\vee)$, Lemma 4.13 implies that $a_i' = \bar{d}(b_i') \oplus c_i(1\otimes 1)$ with $b_i' \in S(V^\vee)\otimes C(V_0^\vee)$ and $c_i \in \mathbb{C}$. Using Lemmas 4.10 and 4.12 once again, we have that

$$a_i = \bar{\varrho}(\widetilde{w}_i)a_i' = \bar{\varrho}(\widetilde{w}_i)(\bar{d}'(b_i') + c_i(1\otimes 1)) = \bar{d}(\varrho(\widetilde{w}_i)b_i') + c_i\bar{\varrho}(\widetilde{w}_i) \in \operatorname{im}(\bar{d}) + \varrho(\mathbb{C}[\widetilde{W}]).$$

Hence $a = a_1 + \ldots + a_l \in \operatorname{im}(\bar{d}) + \varrho(\mathbb{C}[\widetilde{W}])$ and the proof is complete. □

The considerations around (4.5) also apply in the graded setting. In particular, using an argument as in the proof of Lemma 3.4, we conclude that $\bar{d}$ interchanges the spaces

$$(\overline{\mathbb{H}}\otimes C(V_0^\vee))^{\mathsf{triv}} = \{a \in \overline{\mathbb{H}}\otimes C(V_0^\vee) : \varrho(\widetilde{w})a = a\varrho(\widetilde{w})\} \tag{4.12}$$

and

$$(\overline{\mathbb{H}}\otimes C(V_0^\vee))^{\mathsf{sgn}} = \{a \in \overline{\mathbb{H}}\otimes C(V_0^\vee) : \varrho(\widetilde{w})a = \mathsf{sgn}(\widetilde{w})a\varrho(\widetilde{w})\}. \tag{4.13}$$

As before, we let $\bar{d}^{\mathsf{triv}}$ (resp. $\bar{d}^{\mathsf{sgn}}$) denote the restriction of d to the space in (4.5) (resp. (4.6)). Passing to the subspace

$$(\overline{\mathbb{H}}\otimes C(V_0^\vee))^{\mathsf{triv}} \oplus (\overline{\mathbb{H}}\otimes C(V_0^\vee))^{\mathsf{sgn}}$$

in Proposition 4.14, we obtain the following corollary.

Corollary 4.15. *With notation as in the previous paragraph,*

$$\ker(\bar{d}^{\mathsf{triv}}) = \operatorname{im}(\bar{d}^{\mathsf{sgn}}) \oplus \bar{\varrho}(\mathbb{C}[\widetilde{W}]^{\widetilde{W}}).$$

Theorem 4.7, and hence the first part of Theorem 4.2, now follow from Corollary 4.15 by an easy induction based on the degree of the filtration. All that remains is to prove that the map $\zeta: Z(\mathbb{H}) \to \mathbb{C}[\widetilde{W}]^{\widetilde{W}}$ is an algebra homomorphism.

Lemma 4.16. *The map $\zeta: Z(\mathbb{H}) \to \mathbb{C}[\widetilde{W}]^{\widetilde{W}}$ defined by Theorem 4.2 is a homomorphism of algebras.*

Proof. The non-trivial part of the claim is the fact that $\zeta(z_1z_2) = \zeta(z_1)\zeta(z_2)$ for all $z_1, z_2 \in Z(\mathbb{H})$. From Theorem 4.7, there exist elements $a_1, a_2 \in (\mathbb{H}\otimes C_{\mathsf{odd}}(V_0^\vee))^{\mathsf{sgn}}$ such that $z_i \otimes 1 = \varrho(\zeta(z_i)) + d^{\mathsf{sgn}}(a_i)$, $i = 1, 2$. Therefore,

$$\begin{aligned} z_1z_2\otimes 1 &= \varrho(\zeta(z_1)\zeta(z_2)) + \varrho(\zeta(z_1))d^{\mathsf{sgn}}(a_2) + d^{\mathsf{sgn}}(a_1)\varrho(\zeta(z_2)) + d^{\mathsf{sgn}}(a_1)d^{\mathsf{sgn}}(a_2) \\ &= \varrho(\zeta(z_1)\zeta(z_2)) + d^{\mathsf{sgn}}(\varrho(\zeta(z_1))a_2 + a_1\varrho(\zeta(z_2)) + a_1d^{\mathsf{sgn}}(a_2)), \end{aligned}$$

by Lemmas 4.8 and 4.10. The claim now follows by applying Theorem 4.7 to $z_1z_2\otimes 1$. □

This completes the proof of Theorem 4.2 and hence (by Theorem 4.5) Theorem 4.4 as well.

5. Vogan's conjecture and geometry of nilpotent orbits

In this section (for the reasons mentioned in the introduction), we fix a crystallographic root system Φ and set the parameter function c in Definition 2.1 to be identically 1, i.e. $c_\alpha = 1$ for all $\alpha \in R$.

5.1. Geometry of irreducible representations of $\widetilde{W}$

Let $\mathfrak{g}$ denote the complex semisimple Lie algebra corresponding to Φ. In particular, $\mathfrak{g}$ has a Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{h} \simeq V$ canonically. Write $\mathcal{N}$ for the nilpotent cone in $\mathfrak{g}$. Let G denote the adjoint group $\mathrm{Ad}(\mathfrak{g})$ acting by the adjoint action on $\mathcal{N}$.

Given $e \in \mathcal{N}$, let $\{e, h, f\} \subset \mathfrak{g}$ denote an $\mathfrak{sl}_2$-triple with $h \in \mathfrak{h}$ semisimple. Set

$$\nu_e = \tfrac{1}{2} h \in \mathfrak{h} \simeq V. \tag{5.1}$$

The element ν_e depends on the choices involved. But its W-orbit (and in particular $\langle \nu_e, \nu_e \rangle$ and the central character χ_{ν_e} of Definition 2.2) are well defined independent of the G-orbit of e.

Let

$$\mathcal{N}_{\mathsf{sol}} = \{ e \in \mathcal{N} : \text{the centralizer of } e \text{ in } \mathfrak{g} \text{ is a solvable Lie algebra}\}. \tag{5.2}$$

Then G also acts on $\mathcal{N}_{\mathsf{sol}}$.

Next, let $A(e)$ denote the component group of the centralizer of $e \in \mathcal{N}$ in G. To each $e \in \mathcal{N}$, Springer has defined a graded representation of $W \times A(e)$ (depending only on the G-orbit of e) on the total cohomology $H^\bullet(\mathcal{B}^e)$ of the Springer fiber over e. Set $d(e) = 2\dim(\mathcal{B}^e)$ and define

$$\sigma_{e,\phi} = (H^{d(e)}(\mathcal{B}^e))^\phi \in \mathsf{Irr}(W) \cup \{0\}, \tag{5.3}$$

the ϕ invariants in the top degree. (In general, given a finite group H, we write $\mathsf{Irr}(H)$ for the set of equivalence classes of its irreducible representations.) Let $\mathsf{Irr}_0(A(e)) \subset \mathsf{Irr}(A(e))$ denote the subset of representations of "Springer type", i.e. those ϕ such that $\sigma_{e,\phi} \neq 0$.

Finally, let $\mathsf{Irr}_{\mathsf{gen}}(\widetilde{W}) \subset \mathsf{Irr}(\widetilde{W})$ denote the subset of genuine representations.

Theorem 5.1. ([C]) *Recall the notation of* (5.1)–(5.3). *Then there is a surjective map*

$$\Psi \colon \mathsf{Irr}_{\mathsf{gen}}(\widetilde{W}) \longrightarrow G \backslash \mathcal{N}_{\mathsf{sol}} \tag{5.4}$$

with the following properties:

(1) *If* $\Psi(\tilde{\sigma}) = G \cdot e$, *then*

$$c(\tilde{\sigma}) = \langle \nu_e, \nu_e \rangle, \tag{5.5}$$

where $c(\widetilde{\sigma})$ *is defined in* (3.20);

(2a) *For a fixed spin module* (γ, S) *for* $C(V_0^\vee)$, *if* $e\in\mathcal{N}_{\mathsf{sol}}$ *and* $\phi\in\mathsf{Irr}_0(A(e))$, *then there exists* $\widetilde{\sigma}\in\Psi^{-1}(G\cdot e)$ *such that*

$$\mathrm{Hom}_W\,(\sigma_{e,\phi}, \widetilde{\sigma}\otimes S)\neq 0;$$

(2b) *If* $\Psi(\widetilde{\sigma})=G\cdot e$, *then there exists* $\phi\in\mathsf{Irr}_0(A(e))$ *and a spin module* (γ, S) *for* $C(V_0^\vee)$ *such that*

$$\mathrm{Hom}_W(\sigma_{e,\phi}, \widetilde{\sigma}\otimes S)\neq 0.$$

Together with Corollary 3.6, we immediately obtain the following result.

COROLLARY 5.2. *Let* (π, X) *be an irreducible unitary* $\mathbb{H}$*-module with central character* χ_ν *with* $\nu\in V$ *(as in Definition 2.2). Fix a spin module* (γ, S) *for* $C(V_0^\vee)$.

(a) *Let* $(\tilde{\sigma}, \widetilde{U})$ *be a representation of* $\widetilde{W}$ *such that* $\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, X\otimes S)\neq 0$. *In the notation of Theorem* 5.1, *write* $\Psi(\widetilde{\sigma})=G\cdot e$. *Then*

$$\langle\nu,\nu\rangle\leqslant\langle\nu_e,\nu_e\rangle. \tag{5.6}$$

(b) *Suppose that* $e\in\mathcal{N}_{\mathsf{sol}}$ *and* $\phi\in\mathsf{Irr}_0(A(e))$ *are such that* $\mathrm{Hom}_W(\sigma_{(e,\phi)}, X)\neq 0$. *Then*

$$\langle\nu,\nu\rangle\leqslant\langle\nu_e,\nu_e\rangle. \tag{5.7}$$

Remark 5.3. The bounds in Corollary 5.2 represent the best possible in the sense that there exist X such that the inequalities are actually equalities. For example, consider part (b) of the corollary, fix $\phi\in\mathsf{Irr}_0(A(e))$ and let $X_t(e,\phi)$ be the unique tempered representation of $\mathbb{H}$ parameterized by (e,ϕ) in the Kazhdan–Lusztig classification ([KL]). Thus $X_t(e,\phi)$ is an irreducible unitary representation with central character χ_{ν_e} and, as a representation of W,

$$X_t(e,\phi)\simeq H^\bullet(\mathcal{B}_e)^\phi.$$

In particular, $\sigma_{e,\phi}$ occurs with multiplicity 1 in $X_t(e,\phi)$ (in the top degree).

Thus the inequality in Corollary 5.2 (b) applied to $X_t(e,\phi)$ is an equality.

The representations $X_t(e,\phi)$ will play an important role in our proof of Theorem 5.8.

5.2. Applications to unitary representations

Recall that there exists a unique open dense G-orbit in $\mathcal{N}$, the *regular orbit*; let $\{e_{\mathsf{r}}, h_{\mathsf{r}}, f_{\mathsf{r}}\}$ be a corresponding $\mathfrak{sl}_2$ with $h_{\mathsf{r}}\in\mathfrak{h}$, and set $\nu_{\mathsf{r}}=\frac{1}{2}h_{\mathsf{r}}$. If $\mathfrak{g}$ is simple, then there exists a unique open dense G-orbit in the complement of $G{\cdot}e_{\mathsf{r}}$ in $\mathcal{N}$ called the *subregular orbit*. Let $\{e_{\mathsf{sr}}, h_{\mathsf{sr}}, f_{\mathsf{sr}}\}$ be an $\mathfrak{sl}_2$-triple for the subregular orbit with $h_{\mathsf{sr}}{\in}\mathfrak{h}$, and set $\nu_{\mathsf{sr}}{=}\frac{1}{2}h_{\mathsf{sr}}$.

The tempered module $X_t(e_{\mathsf{r}},\mathsf{triv})$ is the Steinberg discrete series, and we have $X_t(e_{\mathsf{r}},\mathsf{triv})|_W{=}\mathsf{sgn}$. When $\mathfrak{g}$ is simple, the tempered module $X_t(e_{\mathsf{sr}},\mathsf{triv})$ has dimension $\dim V+1$, and $X_t(e_{\mathsf{sr}},\mathsf{triv})|_W{=}\mathsf{sgn}{\oplus}\mathsf{refl}$, where refl is the reflection W-type.

Now we can state certain bounds for unitary $\mathbb{H}$-modules.

Corollary 5.4. *Let (π, X) be an irreducible unitary $\mathbb{H}$-module with central character χ_ν with $\nu{\in}V$ (as in Definition 2.2). Then, we have*

(1) $\langle\nu,\nu\rangle\leqslant\langle\nu_{\mathsf{r}},\nu_{\mathsf{r}}\rangle$;

(2) *if $\mathfrak{g}$ is simple of rank at least 2, and X is* not *the trivial or the Steinberg $\mathbb{H}$-module, then $\langle\nu,\nu\rangle\leqslant\langle\nu_{\mathsf{sr}},\nu_{\mathsf{sr}}\rangle$.*

Proof. The first claim follows from Corollary 5.2 (1), since $\langle\nu_e,\nu_e\rangle\leqslant\langle\nu_{\mathsf{r}},\nu_{\mathsf{r}}\rangle$ for every $e{\in}\mathcal{N}$.

For the second claim, assume that X is not the trivial nor the Steinberg $\mathbb{H}$-module. Then X contains a W-type σ such that $\sigma{\neq}\mathsf{triv},\mathsf{sgn}$. We claim that $\sigma{\otimes}S$, where S is a fixed irreducible spin module, contains a $\widetilde{W}$-type $\tilde\sigma$ which is not a spin module. If this were not the case, assuming for simplicity that $\dim V$ is even, we would find that $\sigma{\otimes}S{=}S{\oplus}...{\oplus}S$, where there are $\mathrm{tr}_\sigma(1)$ copies of S in the right-hand side. In particular, we would get $\mathrm{tr}_\sigma(s_\alpha s_\beta)\,\mathrm{tr}_S(f_\alpha f_\beta){=}\mathrm{tr}_\sigma(1)\,\mathrm{tr}_S(f_\alpha f_\beta)$. Notice that this formula is true when $\dim V$ is odd too, since the two inequivalent spin modules in this case have characters which have the same value on $f_\alpha f_\beta$. If $\langle\alpha,\beta\rangle{\neq}0$, then we know that $\mathrm{tr}_S(f_\alpha f_\beta){\neq}0$ ([M]). This means that $\mathrm{tr}_\sigma(s_\alpha s_\beta){=}\mathrm{tr}_\sigma(1)$, for all non-orthogonal roots α and β. One verifies directly that, when Φ is simple of rank 2, this relation does not hold. Thus we obtain a contradiction.

Returning to the second claim in the corollary, let $\tilde\sigma$ be a $\widetilde{W}$-type appearing in $X{\otimes}S$ which is not a spin module. Let e be a non-regular nilpotent element such that $\Psi(\tilde\sigma){=}G{\cdot}e$. Corollary 5.2 says that $\langle\nu,\nu\rangle\leqslant\langle\nu_e,\nu_e\rangle$. To complete the proof, recall that if $\mathfrak{g}$ is simple, the largest value for $\langle\nu_e,\nu_e\rangle$, when e is not a regular element, is obtained when e is a subregular nilpotent element. □

Remark 5.5. Standard considerations for reducibility of principal series allow one to deduce a strengthened version of Corollary 5.4 (1) assuming $\nu{\in}V_0$, namely that ν is contained in the convex hull of the Weyl group orbit of ν_{r}. This corresponds to a classical results of Howe and Moore [HM]. We also note that Corollary 5.4 (2) implies, in particular, that, when the root system is simple, the only unitary irreducible $\mathbb{H}$-modules with central character ν_{r} are the trivial and the Steinberg modules. This is a version of a well-known result of Casselman [Ca]. Moreover, Corollary 5.4 (2) shows that the trivial $\mathbb{H}$-module is isolated in the unitary dual of $\mathbb{H}$ for all simple root systems of rank at least 2, and it gives the best possible spectral gap for the trivial module.

Remark 5.6. There is another, subtler application of Corollary 5.2 (2). Assume that $X(s,e,\psi)$ is an irreducible $\mathbb{H}$-module parameterized in the Kazhdan–Lusztig classification by the G-conjugacy class of $\{s,e,\psi\}$, where $s\in V_0$, $[s,e]=e$ and $\psi\in\mathsf{Irr}_0 A(s,e)$. The group $A(s,e)$ embeds canonically in $A(e)$. Let $\mathsf{Irr}_0 A(s,e)$ denote the subset of elements in $\mathsf{Irr} A(s,e)$ which appear in the restriction of an element of $\mathsf{Irr}_0 A(e)$. The module $X(s,e,\psi)$ is characterized by the property that it contains every W-type $\sigma_{(e,\phi)}$, $\phi\in\mathsf{Irr}_0 A(e)$ such that $\mathrm{Hom}_{A(s,e)}(\psi,\phi)\neq 0$.

Let $\{e,h,f\}$ be an $\mathfrak{sl}_2$-triple containing e. One may choose s such that $s=\frac{1}{2}h+s_z$, where $s_z\in V_0$ centralizes $\{e,h,f\}$ and s_z is orthogonal to h with respect to $\langle\cdot,\cdot\rangle$. When $s_z=0$, we have $A(s,e)=A(h,e)=A(e)$, and $X(\frac{1}{2}h,e,\psi)$ is the tempered module $X_t(e,\phi)$ ($\phi=\psi$) from before.

Corollary 5.2 (2) implies that if $e\in\mathcal{N}_{\mathsf{sol}}$, then $X(s,e,\psi)$ is unitary *if and only if* $X(s,e,\psi)$ is tempered.

5.3. Dirac cohomology and nilpotent orbits

In this setting, we can sharpen the results from §4, in particular Theorem 4.4, by making use of Theorem 5.1. (For comments related to dropping the equal-parameter crystallographic condition, see Remark 5.9.)

PROPOSITION 5.7. *Let (π,X) be an irreducible unitary $\mathbb{H}$-module with central character χ_ν with $\nu\in V$ (as in Definition 2.2). In the setting of Definition 3.1, suppose that $(\widetilde{\sigma},\widetilde{U})$ is an irreducible representation of $\widetilde{W}$ such that $\mathrm{Hom}_{\widetilde{W}}(\widetilde{U},X\otimes S)\neq 0$. Write $\Psi(\widetilde{\sigma})=G\cdot e$ as in Theorem 5.1. Assume further that $\langle\nu,\nu\rangle=\langle\nu_e,\nu_e\rangle$. Then*

$$\mathrm{Hom}_{\widetilde{W}}(\widetilde{U},H^D(X))\neq 0.$$

Proof. Let $x\otimes s$ be an element of the $\tilde{\sigma}$-isotypic component of $X\otimes S$. By Theorems 3.5 and 5.1, we have that

$$D^2(x\otimes s)=(-\langle\nu,\nu\rangle+\langle\nu_e,\nu_e\rangle)(x\otimes s)=0. \tag{5.8}$$

Since X is unitary, $\ker(D)\cap\mathrm{im}(D)=0$, and so (5.8) implies that

$$x\otimes s\in\ker(D)=H^D(X). \qquad \square$$

THEOREM 5.8. *Let $\mathbb{H}$ be the graded affine Hecke algebra attached to a crystallographic root system Φ and constant parameter function $c\equiv 1$ (Definition 2.1). Suppose that (π, X) is an $\mathbb{H}$-module with central character χ_ν with $\nu\in V$ (as in Definition 2.2). In the setting of Definition 4.1, suppose that $H^D(X)\neq 0$. Let $(\tilde{\sigma}, \widetilde{U})$ be a representation of $\widetilde{W}$ such that $\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, H^D(X))\neq 0$. Using Theorem 5.1, write $\Psi(\widetilde{\sigma})=G\cdot e$. Then*

$$\chi_\nu = \chi_{\nu_e}.$$

Proof. The statement of Theorem 5.8 will follow from Theorem 4.4, if we can show that $\chi^{\tilde{\sigma}}(z)=\chi_{\nu_e}(z)$ for all $z\in Z(\mathbb{H})$, where $\Psi(\widetilde{\sigma})=G\cdot e$ as in Theorem 5.1.

Using Theorem 5.1 (2b), choose $\phi\in \mathsf{Irr}_0(A(e))$ such that

$$\mathrm{Hom}_{\widetilde{W}}(\widetilde{\sigma}, \sigma_{e,\phi}\otimes S)\neq 0, \tag{5.9}$$

and consider the unitary $\mathbb{H}$-module $X(e,\phi)$ of Remark 5.3 with central character χ_{ν_e}. Then, since $X_t(e,\phi)$ contains the W-type $\sigma_{e,\phi}$, (5.9) implies that

$$\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, X\otimes S)\neq 0.$$

So Proposition 5.7 implies that

$$\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, H^D(X))\neq 0.$$

Since $X(e,\phi)$ has central character χ_{ν_e}, (4.3) applies to give that $\tilde{\sigma}(\zeta(z))$ acts by the scalar $\chi_{\nu_e}(z)$ for all $z\in\mathbb{H}$. This completes the proof. □

Remark 5.9. Note that the proof of Theorem 5.8 depended on two key ingredients beyond Theorems 4.2 and 4.4: Theorem 5.1 and the classification (and W-structure) of tempered modules. Both results are available for the algebras considered by Lusztig in [L1], the former by [C, Theorem 3.10.1] and the latter by [L3]. Thus our proof establishes a version of Theorem 5.8 for cases of the unequal parameters as in [L1].

It would be interesting to consider the problem of identifying the central characters $\chi^{\tilde{\sigma}}$ (Definition 4.3), for every irreducible $\widetilde{W}$-representation $\tilde{\sigma}$, in the setting of a graded affine Hecke algebra $\mathbb{H}$ attached to an arbitrary root system Φ and an arbitrary parameter function c. We expect that the set $\{\chi^{\tilde{\sigma}}:\tilde{\sigma}\in\mathsf{Irr}\widetilde{W}\}$ is always a subset of the set of central characters of irreducible tempered $\mathbb{H}$-modules, and at least when c is non-constant, we expect it is, in fact, precisely the set of central characters for elliptic tempered $\mathbb{H}$-modules (in the sense of [OS]).

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