Acta Math., 203 (2009), 235–267
DOI: 10.1007/s11511-009-0042-y

Rigidity of escaping dynamics for transcendental entire functions

by

Lasse Rempe

University of Liverpool
Liverpool, U.K.

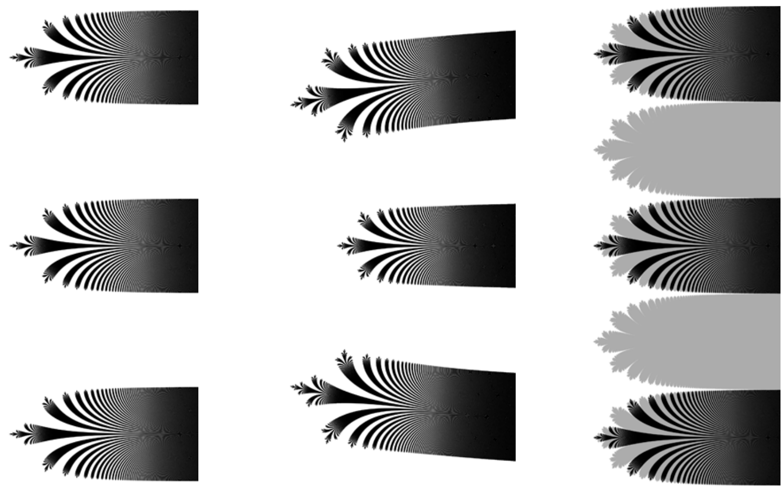
1. Introduction

The study of the dynamical behavior of transcendental functions, initiated by Fatou in 1926 [F], has enjoyed increasing interest recently. Many intriguing phenomena discovered in polynomial dynamics, relating to the behavior of high-order renormalizations of a polynomial, occur naturally for transcendental maps. Compare, for example, Shishikura's proof that the boundary of the Mandelbrot set has Hausdorff dimension 2 [S] with McMullen's treatment of the Julia set of $z \mapsto \lambda \exp(z)$ [McM1]. A more recent example is provided by work of Avila and Lyubich [AL], who proved that a constant-type Feigenbaum quadratic polynomial with positive measure Julia set would have hyperbolic dimension less than 2. Work of Urbański and Zdunik [UZ] shows that a similar phenomenon occurs for the simplest exponential maps.

In this note, we prove a structural theorem for the dynamics near a logarithmic singularity. On the one hand, this result explains the observation that many Julia sets of explicit entire transcendental functions bear striking similarities to each other, even if they are very different from a function-theoretic point of view, compare Figure 1. On the other hand, it provides a tool to better understand the Julia sets of these functions, and results in some important rigidity statements required in the study of density of hyperbolicity [RvS2].

The *Eremenko–Lyubich class* $\mathcal{B}$ is the class of transcendental entire functions for which the set $\operatorname{sing}(f^{-1})$ of critical and asymptotic values is bounded. We say that two functions $f, g \in \mathcal{B}$ are *quasiconformally equivalent near* ∞ if there exist quasiconformal

Supported in part by a postdoctoral fellowship of the German Academic Exchange Service (DAAD) and by EPSRC Advanced Research Fellowship EP/E052851/1.



(a) $f_1(z) = 2(\exp(z) - 1)$ (b) $f_2(z) = (z+1)\exp(z) - 1$ (c) $f_3(z) = \lambda \sinh z$

Figure 1. Images (a) and (b) show the Julia sets of the functions f_1 and f_2 (in black). Our results imply that these two functions are quasiconformally conjugate in a neighborhood of these sets. (Compare Theorem 3.1 and Observation 2.7.) In (c) the black set consists of points whose orbits under f_3 remain in a right half-plane. Again, restricted to this set, f_3 is quasiconformally conjugate to f_1. (The Julia set of f_3 is underlaid in gray.) Note that the three maps are function-theoretically diverse: f_1 has one asymptotic value, f_2 has both an asymptotic and a critical value, and f_3 has two critical values. (In (c), $\lambda = 0.575$.)

maps $\varphi, \psi: \mathbb{C} \to \mathbb{C}$ such that

$$\psi(f(z)) = g(\varphi(z)) \tag{1.1}$$

whenever $|f(z)|$ or $|g(\varphi(z))|$ is large enough. (When (1.1) holds on all of $\mathbb{C}$, the maps are called *quasiconformally equivalent*; compare §2. Quasiconformal equivalence classes form the natural parameter spaces of entire functions.)

THEOREM 1.1. (Conjugacy near infinity) *Let $f, g \in \mathcal{B}$ be quasiconformally equivalent near infinity. Then there exist $R > 0$ and a quasiconformal map $\vartheta: \mathbb{C} \to \mathbb{C}$ such that*

$$\vartheta \circ f = g \circ \vartheta \quad \textit{on } J_R(f) := \{z \in \mathbb{C} : |f^n(z)| \geqslant R \textit{ for all } n \geqslant 1\}.$$

Furthermore, ϑ has zero dilatation on $\{z \in J_R(f) : |f^n(z)| \to \infty\}$.

Remark 1. In fact, our methods are purely local and as such apply to any (not necessarily globally defined) function that has only logarithmic singularities over infinity. In particular, they apply to restrictions of certain entire (or meromorphic) functions that themselves do not belong to class $\mathcal{B}$. We refer the reader to §2 for the precise definition of the class of functions that is treated.

(a) $f_4(z)=\exp(z)+\varkappa$ (b) $f_5(z)=\exp(z)$

Figure 2. Two functions that are quasiconformally equivalent to the map f_1 from Figure 1, but have very different dynamics: in (a), the Julia set (in gray) is a "pinched Cantor bouquet", while in (b) it is the entire complex plane. However, on the sets $J_R(f_j)$ from Theorem 1.1 (in black), they are quasiconformally conjugate to (a suitable restriction of) f_1. (The parameter in (a) is given by $\varkappa=1.0038+2.8999i$.)

Remark 2. For functions with non-logarithmic singularities over infinity, the dynamics near infinity may vary dramatically within the same parameter space. For example, for the function $z\mapsto z-1-\exp(z)$, all points with sufficiently negative real part tend to $-\infty$ under iteration: the function has a *Baker domain* containing a left half-plane. On the other hand, the function $z\mapsto z+1-\exp(z)$ does not have any Baker domains: every orbit in the Fatou set converges to an attracting fixed point; see [W, §5.3].

Theorem 1.1 can be seen as an analog of a classical theorem of Böttcher which states that any two polynomials of the same degree $d\geqslant 2$ are conformally conjugate near ∞ [M, Theorem 18.10]. We find the generality of our theorem surprising for a number of reasons. Not only can functions that are quasiconformally equivalent near infinity have very different function-theoretic properties (recall Figure 1), but more significantly the behavior near infinity can vary widely between different functions in $\mathcal{B}$. Indeed, for the function-theoretically simplest functions in this class, such as those shown in Figure 1, and in fact all functions $f\in\mathcal{B}$ of finite order [RRRS, Theorem 1.2], the escaping set

$$I(f) := \{z\in\mathbb{C} : f^n(z)\to\infty\} \tag{1.2}$$

consists entirely of curves. On the other hand, it is is possible for the escaping set of a hyperbolic function $f\in\mathcal{B}$ to contain no non-trivial curves at all [RRRS, Theorem 8.4]. Theorem 1.1 shows that, even for such a "pathological" function, the behavior near infinity remains the same throughout its quasiconformal equivalence class.

Douady and Hubbard [DH1] used Böttcher's theorem to introduce *dynamic rays*, which have become the backbone of the successful theory of polynomial dynamics. We believe that our result will likewise be useful in the study of families of transcendental functions, even those with such wild behavior as the example mentioned above. Indeed, one corollary of Theorem 1.1 (Corollary A.1) is that any function that is quasiconformally equivalent to this example also does not contain any curves in its escaping set.

Another aspect of the theorem's generality that seems surprising is the statement about dilatation. It is worth noting that two quasiconformally equivalent functions in class $\mathcal{B}$ may have different orders of growth. (Whether this is possible for functions with *finitely many* singular values is a difficult open problem.) Hence the map φ in the definition of quasiconformal equivalence cannot, in general, be chosen to be asymptotically conformal near infinity. In such a situation, one could imagine that some of the dilatation of the quasiconformal map ϑ would be supported on the escaping set $I(f)$, but by Theorem 1.1 this is not the case.

In fact, we will show that the map ϑ is essentially unique (more precisely, it is unique up to an initial choice of isotopy class; compare Corollary 4.2); hence it follows that *no* quasiconformal conjugacy between f and g can support dilatation on the set $I(f)$.

THEOREM 1.2. (No invariant line fields) *A function $f \in \mathcal{B}$ supports no invariant line fields on its escaping set.*

Remark 1. This statement has content only in families where the set of escaping points has positive measure. As far as we know, it is new even for the family

$$z \longmapsto a\exp(z)+b\exp(-z)$$

of cosine maps, whose escaping sets have positive measure by [McM1].

Remark 2. Showing that the Julia set of a polynomial cannot support an invariant line field is a major open problem in complex dynamics. In contrast, it is known [EL2] that there are entire functions with invariant line fields on their Julia sets. In fact, the example from [EL2] has an invariant line field on $I(f) \cap J(f)$, showing that Theorem 1.2 becomes false if the assumption $f \in \mathcal{B}$ is dropped.

By the same reasoning, we also obtain further rigidity principles for the set $I(f)$, of which the following is an important special case.

THEOREM 1.3. (Quasiconformal rigidity on escaping orbits) *Suppose that f and g are entire functions with finitely many singular values, and let π be a topological conjugacy betweeen f and g. If $\mathcal{O}=\{z_0, f(z_0), f^2(z_0), ...\}$ is any escaping orbit of f, then the restriction $\pi|_{\mathcal{O}}$ extends to a quasiconformal self-map of the plane.*

While the source of the rigidity here is much softer than in the famous rigidity results for rational functions (as indicated by the absence of dynamical hypotheses), our results provide an essential step in transferring rigidity theorems from the rational to the transcendental setting. For example, in [RvS1], Theorem 1.2 is used to obtain the absence of invariant line fields on the Julia sets of a large class of "non-recurrent" transcendental functions, extending the work of Graczyk, Kotus and Świątek [GKŚ]. In [RvS2], our results are used, together with the work of Kozlovski, Shen and van Strien [KSvS1], [KSvS2] to establish density of hyperbolicity in certain families of real transcendental entire functions (including the real cosine family $a\sin x+b\cos x$, $a, b\in\mathbb{R}$).

In contrast to the polynomial case, the map ϑ from Theorem 1.1 will generally *not* extend to a conjugacy between the escaping sets of f and g [R1, Proposition 2.1]. However, in the case of *hyperbolic* functions $f\in\mathcal{B}$—i.e., those for which the postsingular set is compactly contained in the Fatou set—we can do better.

THEOREM 1.4. (Conjugacy for hyperbolic maps) *Let $f, g\in\mathcal{B}$ be quasiconformally equivalent near infinity, and suppose that f and g are hyperbolic.*

Then f and g are conjugate on their sets of escaping points.

Together with recent results of Barański [Ba], our proof of Theorem 1.4 also shows that, for hyperbolic $f\in\mathcal{B}$ of finite order, $J(f)$ can be described as a *pinched Cantor Bouquet*; i.e., as the quotient of a Cantor Bouquet (or "straight brush") by a closed equivalence relation on the endpoints. Recently, Mihaljević-Brandt [M-B] has generalized Theorem 1.4 to a large class of "subhyperbolic" entire functions. In particular, her result applies to all postcritically finite functions $f\in\mathcal{B}$ with no asymptotic values for which there is some Δ such that all critical points of f have degree at most Δ.

Structure of the article and ideas of the proofs

We begin in §2 by reviewing some basic properties of Eremenko–Lyubich functions and introducing the local class $\mathcal{B}_{\log}$. §3 is devoted to the proof of Theorem 1.1, which has two main ingredients. The first of these is the well-known fact that functions in $\mathcal{B}$ are expanding inside their logarithmic tracts. The second is that the quasiconformal maps φ and ψ do not move points near infinity more than a finite distance with respect to the hyperbolic metric in a punctured neighborhood of infinity. With these two facts, most of the theorem can be considered to be a variant of standard conjugacy results for expanding maps.

However, in order to obtain the statement on dilatation, we need to break the proof down into two cases: one where both maps f and g are dynamically simple ("disjoint-type") functions, and one where the quasiconformal maps φ and ψ are in fact affine. (In the latter case, the quasiconformality of the function ϑ, and the dilatation estimate, will be obtained via the "λ-lemma" of [MSS].) Together, these two cases combine to give the full theorem; compare also the discussion at the end of §3.

The proofs of Theorems 1.2 and 1.3 are given in §4. As already mentioned, they rely on the fact that the map ϑ is unique in a certain sense (Corollary 4.2). The idea of the proof can be traced back to the argument of Douady and Goldberg [DG], who proved that two topologically conjugate real exponential maps with escaping singular orbits must be conformally conjugate.

To prove Theorem 1.4 in §5, we show that hyperbolic entire functions are expanding with respect to the hyperbolic metric; the construction of a semi-conjugacy then proceeds as usual for expanding maps.

In Appendix A, we discuss the relation of our results with some well-known questions regarding escaping sets posed by Fatou [F] and Eremenko [Er].

Acknowledgements

I would like to especially thank Carsten Petersen, whose thought-provoking questions during a talk at the Institut Henri Poincaré initiated the research that led to these results. I would also like to thank Walter Bergweiler, Adam Epstein, Alex Eremenko, Jeremy Kahn, Misha Lyubich, Phil Rippon, Dierk Schleicher, Gwyneth Stallard, Sebastian van Strien and particularly Helena Mihaljević-Brandt for interesting discussions and comments, and the referee for helpful suggestions that led to marked improvement in exposition.

Background and notation

We refer the reader to [M], [Berg], [H] and [LV] for introductions to holomorphic dynamics, plane hyperbolic geometry and quasiconformal mappings.

We denote the complex plane by $\mathbb{C}$ and the Riemann sphere by $\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. All closures and boundaries will be understood to be taken in $\mathbb{C}$, unless explicitly stated otherwise. We denote the right half-plane by $\mathbb{H}:=\{z\in\mathbb{C}:\operatorname{Re} z>0\}$ ($\{\operatorname{Re} z>0\}$, in short) and the unit disk by $\mathbb{D}:=\{|z|<1\}$; more generally, we write

$$\mathbb{H}_Q:=\{\operatorname{Re} z>Q\} \quad \text{and} \quad \mathbb{D}_R(z_0):=\{|z-z_0|<R\}.$$

If $f:\mathbb{C}\to\mathbb{C}$ is an entire function, we denote its *Julia* and *Fatou* sets by $J(f)$ and $F(f)$, respectively. Recall that the *escaping set* $I(f)$ was defined in (1.2).

The set of *singular values*, $S(f)$, is the closure of the set $\operatorname{sing}(f^{-1})$ of critical and asymptotic values of f. The *Speiser class* $\mathcal{S}$ and the *Eremenko–Lyubich class* $\mathcal{B}\supset\mathcal{S}$ are defined as

$$\mathcal{S} := \{f\colon \mathbb{C}\to\mathbb{C} \text{ transcendental entire} : S(f) \text{ finite}\} \quad \text{and}$$
$$\mathcal{B} := \{f\colon \mathbb{C}\to\mathbb{C} \text{ transcendental entire} : S(f) \text{ bounded}\}.$$

2. Preliminaries

The hyperbolic metric

If $U\subset\mathbb{C}$ is open and $\mathbb{C}\setminus U$ contains at least two points, we denote the density of the hyperbolic metric in U by ϱ_U. We denote the hyperbolic distance and the length in U by dist_U and ℓ_U, respectively. The derivative of a holomorphic function f with respect to the hyperbolic metric of U (where defined) will be denoted by

$$\|Df(z)\|_U := |f'(z)|\frac{\varrho_U(f(z))}{\varrho_U(z)}.$$

Recall [M, Corollary A.8] that, if U is simply connected, then

$$\frac{1}{2\,\mathrm{dist}(z,\partial U)} \leqslant \varrho_U(z) \leqslant \frac{2}{\mathrm{dist}(z,\partial U)} \tag{2.1}$$

for all $z\in U$; we refer to this as the *standard estimate* on the hyperbolic metric. We also remind the reader that holomorphic covering maps preserve the hyperbolic metric, and that Pick's theorem [M, Theorem 2.11] implies that $\varrho_{U'}(z)>\varrho_U(z)$ for all $z\in U'$ if $U'\subsetneq U$.

In §5, we will use the following estimate on the hyperbolic metric in certain multiply-connected domains.

LEMMA 2.1. (Hyperbolic metric in countably punctured sphere) *Let $\{w_j\}_{j\in\mathbb{N}}$ be a sequence of points in $\mathbb{C}$, with $w_j\to\infty$, satisfying $|w_{j+1}|\leqslant C|w_j|$ for some constant $C>1$ and all sufficiently large $j\in\mathbb{N}$. Set $V:=\mathbb{C}\setminus\{w_j:j\in\mathbb{N}\}$. Then $1/\varrho_V(z)=O(|z|)$ as $z\to\infty$.*

Proof. By Pick's theorem, we may disregard finitely many entries in the sequence and hence suppose that $|w_{j+1}|\leqslant C|w_j|$ holds for all $j\in\mathbb{N}$. We will estimate the value $\varrho_V(z_0)$ at a given point $z_0\in\mathbb{C}$ from below by using the hyperbolic metric in a suitable doubly-punctured plane $U_{a,b}:=\mathbb{C}\setminus\{a,b\}$. Note that the map

$$\varphi_{a,b}(z) := \frac{z-a}{b-a}$$

is a conformal isomorphism between $U_{a,b}$ and $U_{0,1}$. In particular, for all $z\in U_{a,b}$ we have

$$\varrho_{U_{a,b}}(z) = \frac{\varrho_{U_{0,1}}(\varphi_{a,b}(z))}{|b-a|}.$$

Set $a:=|w_0|$ and let $z_0\in\mathbb{C}$ with $|z_0|\geqslant|a|$. Let j be minimal with $|w_j|\geqslant 2|z_0|$ and set $b:=w_j$. Then $2|z_0|\leqslant|b|\leqslant 2C|z_0|$ by our assumption on the sequence w_j, and, in particular,

$$|z_0|\leqslant|b-a|\leqslant 3C|z_0|.$$

It follows that $|\varphi_{a,b}(z_0)|\leqslant 2$. Hence, setting

$$K:=\inf_{|z|\leqslant 2}\varrho_{U_{0,1}}(z)>0,$$

we see that

$$\varrho_V(z_0)\geqslant\varrho_{U_{a,b}}(z_0)=\frac{\varrho_{U_{0,1}}(z_0)}{|b-a|}\geqslant\frac{K}{3C|z_0|}.$$

Since z_0 was arbitrary with $|z_0|\geqslant|a|$, and C and K are constants independent of z_0, this proves the claim. □

Tracts and logarithmic coordinates

A domain $U\subset\mathbb{C}$ is called an *unbounded Jordan domain* if the boundary of U on the Riemann sphere is a Jordan curve passing through ∞.

Suppose that $f\in\mathcal{B}$, and let $D\subset\mathbb{C}$ be a bounded Jordan domain chosen such that $S(f)\cup\{0,f(0)\}\subset D$ (e.g., $D=\mathbb{D}_R(0)$, where $R\geqslant 1+|f(0)|+\max_{s\in S(f)}|s|$). Set $W:=\mathbb{C}\setminus\overline{D}$ and $\mathcal{U}:=f^{-1}(W)$. Then each component T of $\mathcal{U}$ is an unbounded Jordan domain (called a *tract* of f), and $f\colon T\to W$ is a universal covering.

We can perform a *logarithmic change of coordinates* (see [EL3, §2] or [Berg, §4.8]) to obtain a $2\pi i$-periodic function $F\colon\mathcal{V}\to H$, where $H=\exp^{-1}(W)$ and $\mathcal{V}=\exp^{-1}(\mathcal{U})$, such that $\exp\circ F=f\circ\exp$. We will say that this function F is a *logarithmic transform* of f. By construction, the following properties hold:

(A) H is a $2\pi i$-periodic unbounded Jordan domain that contains a right half-plane.

(B) $\mathcal{V}\neq\varnothing$ is $2\pi i$-periodic and $\operatorname{Re} z$ is bounded from below in $\mathcal{V}$.

(C) F is $2\pi i$-periodic.

(D) Each component T of $\mathcal{V}$ is an unbounded Jordan domain that is disjoint from all its $2\pi i\mathbb{Z}$-translates. For each such T, the restriction $F\colon T\to H$ is a conformal isomorphism with $F(\infty)=\infty$. (T is called a *tract of* F; we denote the inverse of $F|_T$ by F_T^{-1}.)

(E) The components of $\mathcal{V}$ have pairwise disjoint closures and accumulate only at ∞; i.e., if $z_n\in\mathcal{V}$ is a sequence converging to some finite point $z\in\mathbb{C}$, then all but finitely many of the z_n belong to a single component of $\mathcal{V}$.

We will denote by $\mathcal{B}_{\log}$ the class of all functions

$$F: \mathcal{V} \longrightarrow H,$$

where F, $\mathcal{V}$ and H have the properties (A)–(E), regardless of whether F arises as the logarithmic transform of a function $f \in \mathcal{B}$ or not.

Remark. In [RRRS], the class $\mathcal{B}_{\log}$ is defined without requiring condition (C).

Note that any $F \in \mathcal{B}_{\log}$ extends continuously to $\overline{\mathcal{V}}$ by Carathéodory's theorem. The *Julia set* and *escaping set* of $F \in \mathcal{B}_{\log}$ are defined to be

$$J(F) := \{z \in \overline{\mathcal{V}} : F^n(z) \in \overline{\mathcal{V}} \text{ for all } n \geqslant 0\},$$
$$I(F) := \{z \in J(F) : \operatorname{Re} F^n(z) \to \infty\}.$$

If F is the logarithmic transform of a function $f \in \mathcal{B}$, then $\exp(I(F)) \subset I(f)$ and the orbit of every $z \in I(f)$ will eventually remain in $\exp(I(F))$. For $Q > 0$, we also define

$$J_Q(F) := \{z \in J(F) : \operatorname{Re} F^n(z) \geqslant Q \text{ for all } n \geqslant 1\},$$
$$I_Q(F) := I(F) \cap J_Q(F).$$

If F is the logarithmic transform of f, then clearly $\exp(J_Q(F)) = J_{e^Q}(f)$ (the latter set was defined in Theorem 1.1).

Expansion and normalization

Let us introduce two important sub-classes of $\mathcal{B}_{\log}$.

Definition 2.2. (Disjoint-type and normalized functions) Let $F: \mathcal{V} \to H$ belong to the class $\mathcal{B}_{\log}$.

(a) We say that F is of *disjoint type* if $\overline{\mathcal{V}} \subset H$.

(b) We say that F is *normalized* if $H = \mathbb{H}$ and, for all $z \in \mathcal{V}$,

$$|F'(z)| \geqslant 2. \tag{2.2}$$

Remark. If an entire function $f \in \mathcal{B}$ has a logarithmic transform F of disjoint type, then we will also say that f itself is of disjoint type. In this case, the Fatou set of f consists of a single immediate basin of attraction, and $J(f) = \exp(J(F))$. The examples from Figure 1 are of disjoint type, while those in Figure 2 are not.

Let $F: \mathcal{V} \to H$ be any element of $\mathcal{B}_{\log}$. It follows easily from (D) and the standard estimate (2.1) on the hyperbolic metric that

$$\|DF(z)\|_H \to \infty, \quad \text{as } \operatorname{Re} z \to \infty. \tag{2.3}$$

In particular, by Pick's theorem, any disjoint-type function $F \in \mathcal{B}$ is uniformly expanding with respect to the hyperbolic metric in H.

The same argument also shows, again for any function $F \in \mathcal{B}_{\log}$, that $|F'(z)| \to \infty$ as $\operatorname{Re} F(z) \to \infty$; see [EL3, Lemma 1]. In particular, there is $R>0$ such that (2.2) holds for all $z \in \mathcal{V}$ with $\operatorname{Re} F(z) \geqslant R$. By restricting F to the set $\widetilde{\mathcal{V}} := \{z \in \mathcal{V} : \operatorname{Re} F(z) > R\}$ and conjugating by $z \mapsto z - R$, we obtain the function

$$\begin{aligned} \widetilde{F}: \widetilde{\mathcal{V}} - R &\longrightarrow \mathbb{H}, \\ z &\longmapsto F(z+R) - R. \end{aligned}$$

By construction, this function $\widetilde{F}$ is a normalized element of $\mathcal{B}_{\log}$. As we are mostly concerned with the behavior of F near ∞, we usually deal only with normalized functions. However, note that a normalization of a disjoint-type map F need not be of disjoint type.

Lemma 2.3. ($J(F)$ has empty interior) *If $F \in \mathcal{B}_{\log}$ is normalized or of disjoint type, then $J(F)$ has empty interior.*

Sketch of proof. This is the same argument as in [EL3, Theorem 1], using the uniform expansion of the function F in the Euclidean metric (in the normalized case), resp. the hyperbolic metric (for disjoint-type maps). □

Remark. It follows that for *any* $F \in \mathcal{B}_{\log}$, $J_Q(F)$ has empty interior for sufficiently large Q; if F is the logarithmic transform of a function $f \in \mathcal{B}$, then similarly $\exp(J_Q(F)) \subset J(f)$ for sufficiently large Q.

It is easy to see that $J_Q(F) \neq \varnothing$ for all Q; in fact, the following is true.

Proposition 2.4. (Unbounded sets of escaping points [R3, Theorem 2.4]) *Let F be any element of $\mathcal{B}_{\log}$ and let T be a tract of F. Then there is an unbounded, closed, connected set $A \subset T \cap I(F)$ such that $\operatorname{Re} F^j(z) \to +\infty$, as $j \to \infty$, uniformly on A.*

Remark. In [R3], the theorem is stated for entire functions in the Eremenko–Lyubich class, but the proof applies also to functions in $\mathcal{B}_{\log}$. It follows from the results of [R2] that the set A can be chosen to be forward-invariant, but we do not require this. Compare [BRS] for the existence of unbounded connected sets of escaping points in more general situations.

COROLLARY 2.5. (Density of escaping sets) *Let* $F\in\mathcal{B}_{\log}$ *and* $Q\geqslant 0$. *Then* $I_Q(F)$ *is non-empty, and* $\operatorname{Re} z$ *is unbounded from above in* $I_Q(F)$.

Furthermore, if $Q'\geqslant Q$ *is sufficiently large, then*

$$J_{Q'}(F)\subset\overline{I_Q(F)}.$$

Sketch of proof. We may assume that F is normalized. The previous proposition implies that there is $Q'\geqslant Q+\frac{1}{2}\pi$ such that, for every $M\geqslant Q'$, there is a point $z\in I_Q(F)$ with $\operatorname{Re} z=M$.

Let $z\in J_{Q'}(F)$, and note that $I_Q(F)$ is $2\pi i$-invariant. Therefore, for every $n\geqslant 1$ we can find $w^n\in I_Q(F)$ with $\operatorname{Re} w^n=\operatorname{Re} F^n(z)$ and $|\operatorname{Im} w^n-\operatorname{Im} F^n(z)|\leqslant\pi$.

Pulling w^n back along the orbit of z, and using the expansion property (2.2), we obtain a sequence of points $\omega^n\in I_Q(F)$ with $|\omega^n-z|\leqslant\pi/2^n$. Hence $z\in\overline{I_Q(F)}$, as required. $\square$

Quasiconformal equivalence

Following [EL3, §3], two entire functions $f,g\in\mathcal{B}$ are called *quasiconformally equivalent* if there exist quasiconformal maps $\varphi,\psi\colon\mathbb{C}\to\mathbb{C}$ with

$$g\circ\varphi=\psi\circ f. \tag{2.4}$$

The set of all functions g that are quasiconformally equivalent to f can be considered the natural parameter space of f. (If $S(f)$ is finite, then this set forms a finite-dimensional complex manifold [EL3, §3].)

Similarly, let us say that two functions $F,G\in\mathcal{B}_{\log}$ (with domains $\mathcal{V}$ and $\mathcal{W}$) are *quasiconformally equivalent* if there are quasiconformal maps $\Phi,\Psi\colon\mathbb{C}\to\mathbb{C}$ such that

(a) Φ and Ψ commute with $z\mapsto z+2\pi i$;

(b) $\operatorname{Re}\Phi(z)\to\pm\infty$ as $\operatorname{Re} z\to\pm\infty$ (and similarly for Ψ);

(c) for sufficiently large R, $\Phi(F^{-1}(\mathbb{H}_R))\subset\mathcal{W}$ and $\Phi^{-1}(G^{-1}(\mathbb{H}_R))\subset\mathcal{V}$;

(d) $\Psi\circ F=G\circ\Phi$ wherever both compositions are defined.

Let $\varphi\colon\mathbb{C}\to\mathbb{C}$ be a quasiconformal map. Since φ is an order-preserving homeomorphism fixing ∞, we can define a branch of $\arg\varphi(z)-\arg z$ in a punctured neighborhood of ∞. It is well known [EL3, Lemma 4] that there is some $C>1$ such that

$$|z|^{1/C}\leqslant|\varphi(z)|\leqslant|z|^C, \tag{2.5}$$

$$|\arg\varphi(z)-\arg z|\leqslant C\log|z| \tag{2.6}$$

when z is sufficiently large.(1) Translating this statement into logarithmic coordinates, we obtain the following fact.

Lemma 2.6. (Hyperbolic distance of pullbacks) *Suppose that* $F,G\in\mathcal{B}_{\log}$ *are normalized and quasiconformally equivalent. Then there are constants* $C>0$ *and* $M>0$ *such that*

$$\mathrm{dist}_{\mathbb{H}}(F_T^{-1}(z), G_{\widetilde{T}}^{-1}(w)) \leqslant C+\tfrac{1}{2}\,\mathrm{dist}_{\mathbb{H}}(z,w)$$

for all tracts T *of* F *and* $z,w\in\mathbb{H}_M$, *where* $\widetilde{T}$ *is the tract of* G *containing* $\Phi(F_T^{-1}(\mathbb{H}_M))$.

Sketch of proof. Let Φ and Ψ be the maps from the definition of quasiconformal equivalence. There are quasiconformal maps $\varphi,\psi\colon\mathbb{C}\to\mathbb{C}$ such that $\varphi\circ\exp=\exp\circ\Phi$ and $\psi\circ\exp=\exp\circ\Psi$. Applying (2.5) and (2.6) to φ and ψ^{-1}, we easily see that there is some $M_0>0$ such that $\mathrm{dist}_{\mathbb{H}}(z,\Phi(z))$ and $\mathrm{dist}_{\mathbb{H}}(z,\Psi^{-1}(z))$ are bounded, say by ϱ, when $z\in\mathbb{H}_{M_0}$.

By (2.3), we may also choose M_1 sufficiently large so that $\|DF(z)\|_H\geqslant 2$ when $\operatorname{Re}F(z)>M_1$. Finally, let $M>\max\{M_0,M_1,R\}$, where R is as in part (c) of the definition of quasiconformal equivalence, be sufficiently large so that $\operatorname{Re}z>M_0$ whenever $\Psi(F(z))\in\mathbb{H}_M$.

If $w\in\mathbb{H}_M$, we have $G_{\widetilde{T}}^{-1}(w)=\Phi(F_T^{-1}(\Psi^{-1}(w)))$, and hence

$$\begin{aligned}\mathrm{dist}_{\mathbb{H}}(F_T^{-1}(z), G_{\widetilde{T}}^{-1}(w)) &\leqslant \varrho+\mathrm{dist}_{\mathbb{H}}(F_T^{-1}(z), F_T^{-1}(\Psi^{-1}(w))) \leqslant \varrho+\tfrac{1}{2}\,\mathrm{dist}_{\mathbb{H}}(z,\Psi^{-1}(w))\\ &\leqslant \varrho+\tfrac{1}{2}(\varrho+\mathrm{dist}_{\mathbb{H}}(z,w)) = \tfrac{3}{2}\varrho+\tfrac{1}{2}\,\mathrm{dist}_{\mathbb{H}}(z,w)\end{aligned}$$

when $z,w\in\mathbb{H}_M$. □

Observation 2.7. (Functions with quasidisk tracts) It is not always easy to check whether two given functions are quasiconformally equivalent. However, suppose that U and $\widetilde{U}$ are quasidisks whose boundaries contain ∞. Let $f\colon U\to W$ and $g\colon\widetilde{U}\to W$ be universal covering maps (where again $W=\mathbb{C}\setminus\overline{D}$ for a bounded Jordan domain D) that extend continuously to the boundary of U (resp. $\widetilde{U}$) in $\mathbb{C}$.

Then we can pick a conformal isomorphism $\varphi\colon U\to\widetilde{U}$ such that $g\circ\varphi=f$. Because U and $\widetilde{U}$ are quasidisks, φ extends to a quasiconformal map $\varphi\colon\mathbb{C}\to\mathbb{C}$ (see [LV, Satz 8.3] or [H, §4.9]).

Hence, if $f,g\in\mathcal{B}$ are such that $f^{-1}(\{|z|>R\})$ and $g^{-1}(\{|z|>R|\})$ are quasidisks for large R, then f and g are quasiconformally equivalent near infinity. More generally, if $F\colon\mathcal{V}\to\mathbb{H}$ is a function in $\mathcal{B}_{\log}$ such that $\exp(\mathcal{V})$ is a quasidisk, then F is quasiconformally equivalent to any function $G\in\mathcal{B}_{\log}$ with the same property. The tracts of the functions in Figures 1 and 2 are all quasidisks.

(1) While there surely is a classical reference for (2.6), we were unable to locate one; Eremenko and Lyubich refer to [LV], but we did not find it there. A short proof can be found in [vS, Lemma 4.1].

External addresses

Let $F\in\mathcal{B}_{\log}$. We say that $z,w\in J(F)$ *have the same external address* (under F) if, for every $n\geqslant 0$, the points $F^n(z)$ and $F^n(w)$ belong to the closure of the same tract T_n of F.

The sequence $\underline{s}=T_0T_1T_2\ldots$ is called the *external address* of z (and w) under F; compare [RRRS] for a more detailed discussion.

LEMMA 2.8. (Expansion along orbits) *Suppose that $F\in\mathcal{B}_{\log}$ is normalized. If z and w have the same external address under F, then*

$$|F^n(z)-F^n(w)|\geqslant 2^n|z-w|$$

for all $n\geqslant 0$.

Proof. This is a direct consequence of the expansion property (2.2). □

Further properties of quasiconformal maps

Throughout the article, we require a number of well-known properties of quasiconformal maps. We collect a few of these here for the reader's convenience. By convention, the "dilatation" of a quasiconformal map ψ will always mean the *complex dilatation*; that is,

$$\operatorname{dil}(\psi)=\frac{\bar\partial\psi}{\partial\psi}.$$

PROPOSITION 2.9. (Compactness of quasiconformal mappings [LV, §II.5 and §IV.5]) *Consider a sequence $\Psi_n\colon\mathbb{C}\to\mathbb{C}$ of quasiconformal maps, and suppose that there is a dense set $E\subset\mathbb{C}$ such that $\{\Psi_n\}_{n\in\mathbb{N}}$ stabilizes on E; i.e., for all $z\in E$ there is n_0 such that $\Psi_n(z)=\Psi_{n_0}(z)$ for all $n\geqslant n_0$.*

If the maximal dilatation of the maps Ψ_n is bounded independently of n, then the sequence Ψ_n converges locally uniformly to a quasiconformal map $\Theta\colon\mathbb{C}\to\mathbb{C}$.

If furthermore the complex dilatations $\operatorname{dil}(\Psi_n)$ *converge pointwise almost everywhere, then their limit agrees with* $\operatorname{dil}(\Theta)$ *almost everywhere.*

PROPOSITION 2.10. (Royden's glueing lemma [Bers, Lemma 2], [DH2, Lemma 2]) *Suppose that $U\subset\mathbb{C}$ is open and that $\varphi\colon U\to\varphi(U)\subset\mathbb{C}$ is quasiconformal. Suppose furthermore that $\psi\colon\mathbb{C}\to\mathbb{C}$ is a quasiconformal map such that the function*

$$\begin{aligned}\vartheta\colon\mathbb{C}&\longrightarrow\mathbb{C},\\ z&\longmapsto\begin{cases}\varphi(z), & \text{if } z\in U,\\ \psi(z), & \text{otherwise.}\end{cases}\end{aligned}$$

is a homeomorphism. Then ϑ is quasiconformal.

Proposition 2.11. (Quasiconformal maps of an annulus [L]) *Let $A,B\subset\mathbb{C}$ be two bounded annuli, each bounded by two Jordan curves. Suppose that $\psi,\varphi:\mathbb{C}\to\mathbb{C}$ are quasiconformal maps such that ψ maps the inner boundary α^- of A to the inner boundary β^- of B, and φ takes the outer boundary α^+ of A to the outer boundary β^+ of B.*

Let $z\in\alpha^-$ and $w\in\alpha^+$, let γ be a curve in A connecting z and w, and let $\widetilde{\gamma}$ be a curve connecting $\psi(z)$ and $\varphi(w)$ in B.

Then there is a quasiconformal map $\widetilde{\varphi}:\mathbb{C}\to\mathbb{C}$ that agrees with ψ on the bounded component of $\mathbb{C}\setminus A$ and with φ on the unbounded component of $\mathbb{C}\setminus A$, and such that $\widetilde{\varphi}(\gamma)$ is homotopic to $\widetilde{\gamma}$ relative to ∂B.

Remark. The statement about the homotopy class is not made in [L], but follows directly from the proof. (Alternatively, $\widetilde{\varphi}$ can be obtained from *any* quasiconformal map that interpolates ψ and φ by postcomposition with a suitable Dehn twist.)

Let us also formulate the translation of the preceding result to logarithmic coordinates, since we frequently use it in this setting.

Corollary 2.12. (Interpolation of quasiconformal maps) *Suppose that H and H' are $2\pi i$-periodic, unbounded Jordan domains, both containing some right half-plane, with $\overline{H}'\subset H$.*

Suppose that $\Psi,\Phi:\mathbb{C}\to\mathbb{C}$ are quasiconformal maps, commuting with translation by $2\pi i$, such that $\overline{\Phi(H')}\subset\Psi(H)$. Then there is a quasiconformal map $\tilde{\Phi}:\mathbb{C}\to\mathbb{C}$ that agrees with Ψ on $\mathbb{C}\setminus H$, agrees with Φ on H' and commutes with translation by $2\pi i$.

Finally, we will use the "λ-lemma" for holomorphic motions, as developed in [MSS] and improved in [BR]; compare [H, §5.2].

Proposition 2.13. (λ-lemma [BR, Theorem 1]) *Let $E\subset\mathbb{C}$ and $R>0$, and suppose that the functions*

$$h_\lambda: E\longrightarrow\mathbb{C},\quad \lambda\in\mathbb{D}_R(0),$$

are injective, with $h_0=\mathrm{id}$, and furthermore depend holomorphically on λ for fixed $z\in E$. (Under these assumptions, we say that the h_λ form a holomorphic motion *of the set E.)*

Then each h_λ extends to a quasiconformal self-map of the plane. The complex dilatation of this map is bounded by $|\lambda|/R$.

Remark 1. [H, §5.2] even establishes the stronger fact, due to Słodkowski, that the extensions of the h_λ can themselves be chosen to depend holomorphically on λ.

Remark 2. If each h_λ commutes with translation by $2\pi i$, then the extension can also be chosen with this property. (Apply the above theorem to the holomorphic motion g_λ of $\exp(E)\cup\{0\}$ defined by $g_\lambda(0)=0$ and $g_\lambda(\exp(z)):=\exp(h_\lambda(z))$.)

3. Conjugacy near infinity

In this section, we prove Theorem 1.1. We begin by treating the special case where both maps are of disjoint type.

THEOREM 3.1. (Conjugacy between disjoint-type maps) *Suppose that two functions in* $\mathcal{B}_{\log}$,

$$F\colon \mathcal{V}\to H \quad \textit{and} \quad G\colon \Phi(\mathcal{V})\to\Psi(H)$$

are quasiconformally equivalent, $\Psi\circ F=G\circ\Phi$. *Suppose furthermore that* F *and* G *are of disjoint type*; *i.e.*, $\overline{\mathcal{V}}\subset H$ *and* $\overline{\Phi(\mathcal{V})}\subset\Psi(H)$.

Then there is a quasiconformal map $\Theta\colon\mathbb{C}\to\mathbb{C}$ *with the following properties*:

(a) $\Theta|_{\mathcal{V}}$ *is isotopic to* $\Phi|_{\mathcal{V}}$ *relative to* $\partial\mathcal{V}$;

(b) Θ *is a conjugacy between* F *and* G; *i.e.* $\Theta\circ F=G\circ\Theta$ *on* $\mathcal{V}$;

(c) $\operatorname{dil}(\Theta)=0$ *almost everywhere on* $J(F)$;

(d) $\Theta(z+2\pi i)=\Theta(z)+2\pi i$.

Proof. By Corollary 2.12 (picking a $2\pi i$-invariant unbounded Jordan domain H' with $\mathcal{V}\subset H'$ and $\overline{\Phi(H')}\subset\Psi(H)$), we can find a quasiconformal map $\tilde{\Phi}\colon\mathbb{C}\to\mathbb{C}$ such that $\tilde{\Phi}$ agrees with Φ on $\mathcal{V}$ and with Ψ on $\mathbb{C}\setminus H$ (and such that $\tilde{\Phi}$ still commutes with addition by $2\pi i$). Since Φ and $\tilde{\Phi}$ agree on the domain of definition of F, we clearly have $\Psi\circ F=G\circ\tilde{\Phi}$.

In analogous manner, we can modify Ψ to a quasiconformal map $\Psi_0\colon\mathbb{C}\to\mathbb{C}$ that is conformal on a neighborhood of $\overline{\mathcal{V}}$, agrees with Ψ on $\mathbb{C}\setminus H$, and commutes with addition by $2\pi i$. (Compare also the main result of [L].) Note that we are not claiming that this modified map Ψ_0 will satisfy the same functional equation as Ψ.

By the Alexander trick, the isotopy class of a homeomorphism between two Jordan domains is determined by its boundary values (compare also [H, Proposition 6.4.9]). Hence the maps $\Psi|_H$, $\tilde{\Phi}|_H$ and $\Psi_0|_H$ all belong to a single isotopy class relative to ∂H.

We now define a sequence of maps $\Psi_n\colon\mathbb{C}\to\mathbb{C}$ inductively, starting with Ψ_0, by setting

$$\Psi_{n+1}|_T := G^{-1}_{\Phi(T)}\circ\Psi_n\circ F|_T$$

for every tract T of F, and

$$\Psi_{n+1}|_{\mathbb{C}\setminus\mathcal{V}} := \tilde{\Phi}|_{\mathbb{C}\setminus\mathcal{V}}.$$

Clearly each Ψ_n is a homeomorphism (recall that the components of $\mathcal{V}$ accumulate only at infinity by definition). By the glueing lemma (Proposition 2.10), it follows that each Ψ_n is quasiconformal. Since F and G are holomorphic, the maximal dilatation of Ψ_n depends only on that of Ψ_0 and $\tilde{\Phi}$, and is hence bounded independent of n.

Furthermore, $\Psi_n|_H$ is isotopic to $\Psi|_H$ relative to ∂H for all n. This implies that the maps $\Psi_{n+1}|_{\mathcal{V}}$ and $\Phi|_{\mathcal{V}}$ are isotopic relative to $\partial\mathcal{V}$.

By construction, $\Psi_n \circ F = G \circ \Psi_{n+1}$, and Ψ_n and Ψ_{n+1} agree outside the set $F^{-n}(H)$, so the sequence Ψ_n stabilizes on the set

$$\mathbb{C} \setminus \bigcap_{n=0}^{\infty} F^{-n}(H) = \mathbb{C} \setminus J(F).$$

By Lemma 2.3, $\mathbb{C}\setminus J(f)$ is a dense subset of $\mathbb{C}$, and it follows from Proposition 2.9 that the Ψ_n converge to some quasiconformal map $\Theta: \mathbb{C} \to \mathbb{C}$ with $\Theta \circ F = G \circ \Theta$.

The dilatations of the maps Ψ_n stabilize on the set $\mathbb{C}\setminus J(F)$, but on the other hand each Ψ_n is conformal on a neighborhood of $J(F)$, so that its complex dilatation is zero there. In particular, the dilatations converge pointwise, and it follows from the second part of Proposition 2.9 that $\operatorname{dil}(\Theta)=0$ almost everywhere on $J(F)$.

Furthermore, $\Theta|_{\mathcal{V}}$ belongs to the isotopy class of $\Phi|_{\mathcal{V}}$ relative to $\partial\mathcal{V}$. Since each Ψ_n has $\Psi_n(z+2\pi i) = \Psi_n(z) + 2\pi i$, the same is true of Θ. $\square$

Now let

$$F_0: \mathcal{V} \longrightarrow \mathbb{H}$$

be an arbitrary normalized function in $\mathcal{B}_{\log}$. We consider the one-dimensional family

$$F_\varkappa: (\mathcal{V} - \varkappa) \longrightarrow \mathbb{H}, \qquad z \longmapsto F_0(z+\varkappa),$$

where $\varkappa \in \mathbb{C}$. Note that all maps $F_\varkappa$ are normalized. We will now prove Theorem 1.1 for this family, which implies the general statement when combined with Theorem 3.1; see Corollary 3.5 below.

For given $\varkappa \in \mathbb{C}$, we define maps $\Theta_n = \Theta_n^\varkappa$ by $\Theta_0(z) := z$ and

$$\Theta_{n+1}(z) := (F_0)_T^{-1}(\Theta_n(F_0(z))) - \varkappa$$

(whenever defined), where T is the tract of F_0 containing z. In other words, Θ_n is obtained by iterating forward n times under F_0, and then taking the corresponding pullbacks under $F_\varkappa$.

THEOREM 3.2. (Convergence to a conjugacy) *Let $\varkappa \in \mathbb{C}$ and let $Q > 2|\varkappa| + 1$. Then the functions Θ_n are defined and continuous on $J_Q(F_0)$, where they converge uniformly to a map*

$$\Theta = \Theta^\varkappa : J_Q(F_0) \longrightarrow J(F_\varkappa)$$

that satisfies $\Theta \circ F_0 = F_\varkappa \circ \Theta$,

$$|\Theta(z) - z| \leqslant 2|\varkappa| \tag{3.1}$$

and is a homeomorphism onto its image.

For fixed $Q > 1$ and $z \in J_Q(F_0)$ the function $\varkappa \mapsto \Theta^\varkappa(z)$ is holomorphic on $\mathbb{D}_{(Q-1)/2}$.

Proof. The functions Θ_n are clearly continuous where defined. Let us show inductively that $\Theta_n(z)$ is defined and

$$|\Theta_n(z)-z|\leqslant 2|\varkappa| \tag{3.2}$$

whenever $z\in J_Q(F_0)$. Indeed, for such z we have $\operatorname{Re} F_0(z)\geqslant Q>2|\varkappa|+1$, so the induction hypothesis implies that $\Theta_n(F_0(z))\in\mathbb{H}$, and thus $\Theta_{n+1}(z)$ is defined. Furthermore, by the expansion property (2.2) of F_0 and the induction hypothesis, we see that

$$\begin{aligned}|\Theta_{n+1}(z)-z| &= |(F_0)_T^{-1}(\Theta_n(F_0(z)))-\varkappa-(F_0)_T^{-1}(F_0(z))|\\ &\leqslant \tfrac{1}{2}|\Theta_n(F_0(z))-F_0(z)|+|\varkappa|\leqslant|\varkappa|+|\varkappa|=2|\varkappa|,\end{aligned}$$

as required.

Using (3.2), we see that

$$\begin{aligned}|\Theta_{n+k}(z)-\Theta_n(z)| &= |(F_0)_T^{-1}(\Theta_{n-1+k}(F_0(z)))-(F_0)_T^{-1}(\Theta_{n-1}(F_0(z)))|\\ &\leqslant \frac{1}{2}|\Theta_{n-1+k}(F_0(z))-\Theta_{n-1}(F_0(z))|\leqslant\ldots\leqslant\frac{1}{2^n}|\Theta_k(F_0^n(z))-\Theta_0(F_0^n(z))|\leqslant\frac{2|\varkappa|}{2^n}.\end{aligned}$$

Hence the functions Θ_n form a Cauchy sequence, and thus converge to some function

$$\Theta=\Theta^{\varkappa}\colon J_Q(F_0)\longrightarrow J_1(F_\varkappa)$$

satisfying (3.1) and $\Theta\circ F_0=F_\varkappa\circ\Theta$. Since the convergence is locally uniform in $\varkappa$ and each Θ_n is holomorphic in $\varkappa$, the map Θ likewise depends holomorphically on $\varkappa$.

It remains to verify that Θ has the stated properties. Note that, by definition of Θ, the external address $\underline{\tilde{s}}$ of $\Theta(z)$ under $F_\varkappa$ is determined uniquely by the external address $\underline{s}$ of z under F_0. Indeed, if $\underline{s}=T_1T_2\ldots$, then $\underline{\tilde{s}}=\widetilde{T}_1\widetilde{T}_2\ldots$, where $\widetilde{T}_j=T_j-\varkappa$.

To see that Θ is injective, suppose that $\Theta(z)=\Theta(w)$. Then z and w have the same external address under F_0, and by (3.1) their orbits are never separated by more than $4|\varkappa|$. By Lemma 2.8, this is impossible unless $z=w$; so Θ is indeed injective.

Furthermore, $\lim_{z\to\infty}\Theta(z)=\infty$, again by (3.1), so Θ extends to a continuous injective map on the compact space $J_Q(F_0)\cup\{\infty\}$, and thus is a homeomorphism onto its image. □

LEMMA 3.3. (Image of Θ) *Let $\varkappa\in\mathbb{C}$, and let Q and Θ be as in the preceding theorem. Then $\Theta(J_Q(F_0))\supset J_{2Q}(F_\varkappa)$.*

Proof. Set $G_0:=F_\varkappa$ and consider the family $G_\lambda(z):=G_0(z+\lambda)$. Then $F_0=G_{-\varkappa}$. Applying Theorem 3.2 to this family, we obtain a map $\Theta'\colon J_Q(F_\varkappa)\to J(F_0)$ satisfying $\Theta'\circ F_\varkappa=F_0\circ\Theta'$ and (3.1). Now, if $w\in J_{2Q}(F_\varkappa)$, then $z:=\Theta'(w)$ satisfies

$$\operatorname{Re} F^k(z)\geqslant\operatorname{Re} F^k(w)-2|\varkappa|\geqslant 2Q-2|\varkappa|>Q.$$

So $z\in J_Q(F_0)$. The points w and $w':=\Theta(z)$ have the same external address under $F_\varkappa$. Furthermore, $F_\varkappa^k(w')=\Theta(F_0^k(z))$ and $F_0^k(z)=\Theta'(F_\varkappa^k(w))$ for all k, and hence

$$|F_\varkappa^k(w)-F_\varkappa^k(w')|\leqslant|F_\varkappa^k(w)-\Theta'(F_\varkappa^k(w))|+|F_0^k(z)-\Theta(F_0^k(z))|\leqslant 4K.$$

So, by Lemma 2.8, we have $w=w'=\Theta(z)\in\Theta(J_Q(F_0))$, as required. □

THEOREM 3.4. (Quasiconformal extension and dilatation of Θ) *Let $\varkappa\in\mathbb{C}$, and let Q and Θ be as in Theorem 3.2. Then Θ extends to a quasiconformal map $\Theta:\mathbb{C}\to\mathbb{C}$. This extension can be chosen such that $\Theta(z+2\pi i)=\Theta(z)+2\pi i$, and such that $\Theta|_\mathcal{V}$ is isotopic to $\Phi(z):=z-\varkappa$ relative to $\partial\mathcal{V}$.*

Furthermore, the maximal dilatation of Θ on $J_{Q'}(F_0)$ tends to zero as $Q'\to\infty$. In particular, the dilatation of Θ is zero almost everywhere on $I(F_0)\cap J_Q(F_0)$.

Proof. For abbreviation, let us set $J_Q^\varkappa:=J_Q(F_\varkappa)$, and also write $J_Q:=J_Q^0$. Then the functions $\Theta=\Theta^\varkappa$ define a holomorphic motion of the set J_Q. By the λ-lemma (Proposition 2.13), each of these functions extends to a quasiconformal self-map $\Theta^\varkappa$ of the plane.

As pointed out in Remark 2 after Proposition 2.13, Θ can be chosen to commute with translation by $2\pi i$. Also, by (3.1),

$$\Theta(F_0(J_Q))\subset\overline{\mathbb{H}}_1,$$

so we can use Corollary 2.12 to obtain a quasiconformal map $\Theta':\mathbb{C}\to\mathbb{C}$ that agrees with Θ on $F_0(J_Q)$, but is the identity on $\mathbb{C}\setminus\mathbb{H}$ (and is hence isotopic to the identity relative to $\partial\mathbb{H}$). Consider the map Θ'', defined by

$$\Theta''(z):=(F_0)_T^{-1}(\Theta'(F_0(z)))-\varkappa$$

when z belongs to a tract T of F, and $\Theta''(z)=\Phi(z)$ otherwise. This map is quasiconformal, isotopic to Φ relative to $\partial\mathcal{V}$, and agrees with Θ', and hence with Θ, on $J_Q(F_0)$.

To discuss dilatation, recall from Theorem 3.2 that the maps $\Theta^\varkappa|_{J_{Q'}}$, for $Q'>Q$, define a holomorphic motion over the disk $\mathbb{D}_{(Q'-1)/2}(0)$ in $\varkappa$-space. It follows from the dilatation statement in the λ-lemma that $\Theta|_{J_{Q'}}$ extends to a quasiconformal map with dilatation bounded by $2|\varkappa|/(Q'-1)$. In particular,

$$\operatorname{dil}(\Theta)\leqslant\frac{2|\varkappa|}{Q'-1}\quad\text{a.e. on } J_{Q'}(F_0);$$

clearly this bound tends to 0 as $Q'\to\infty$, as claimed.

Finally, recall that we have

$$\Theta\circ F_0^n=F_\varkappa^n\circ\Theta$$

on J_Q. Since F_0 and $F_\varkappa$ are holomorphic, we see that (for $Q' \geqslant Q$) the maximal dilatation of Θ on

$$X^n_{Q'} := \{z \in J_Q : F_0^n(z) \in J_{Q'}\}$$

is the same as the maximal dilatation of Θ on $J_{Q'}$, which tends to 0 as $Q' \to \infty$. Since the bound is independent of n, the same is true for

$$X_{Q'} := \bigcup_{n=0}^{\infty} X^n_{Q'}.$$

But $I_Q(F_0) = \bigcap_{Q' \geqslant Q} X_{Q'}$, so the dilatation of Θ on $I_Q(F_0)$ is zero, as required. □

We are now ready to prove Theorem 1.1, which we restate (with some additional details) in logarithmic coordinates.

Corollary 3.5. (Conjugacy between quasiconformal equivalent maps) *Suppose that* $F, G \in \mathcal{B}_{\log}$ *are quasiconformally equivalent*, $\Psi \circ F = G \circ \Phi$. *For sufficiently large* $Q > 0$, *there exists a quasiconformal map* Θ *such that*

(a) $\Theta|_{\mathcal{V}}$ *is isotopic to* $\Phi|_{\mathcal{V}}$ *relative to* $\partial\mathcal{V}$ (*where* $\mathcal{V}$ *is the domain of* F);

(b) $\Theta \circ F = G \circ \Theta$ *on* $J_Q(F)$;

(c) $\Theta(J_Q(F)) \supset J_{Q'}(G)$ *for some* $Q' > Q$;

(d) *the dilatation of* Θ *is zero on* $J_Q(F) \cap I(F)$;

(e) $\Theta(z + 2\pi i) = \Theta(z) + 2\pi i$.

Proof. Let $\mathcal{W}$ be the domain of G. By restriction and conjugation, as discussed in §2, we may suppose without loss of generality that F and G are normalized, and that $\Phi(\mathcal{V}) \subset \mathcal{W}$.

Choose $K, L > 0$ sufficiently large so that

$$F_0 \colon \overbrace{\mathcal{V}+K}^{=:\mathcal{V}_0} \longrightarrow \mathbb{H}, \quad z \longmapsto F(z-K) \qquad \text{and} \qquad G_0 \colon \overbrace{\mathcal{W}+L}^{=:\mathcal{W}_0} \longrightarrow \mathbb{H}, \quad z \longmapsto G(z-L)$$

are of disjoint type, and that furthermore $\overline{\Phi(\mathcal{V})} + L \subset \Psi(\mathbb{H})$.

Now we can apply Theorem 3.1 to obtain a quasiconformal conjugacy Θ_2 between F_0 and G_0.

Furthermore, we can apply Theorems 3.2 and 3.4 to F and F_0, as well as to G and G_0, obtaining quasiconformal maps Θ_1 and Θ_3. It is easy to check that the function

$$\Theta := \Theta_3^{-1} \circ \Theta_2 \circ \Theta_1$$

has the required properties. □

Proof of Theorem 1.1. Suppose that $f,g\in\mathcal{B}$ are quasiconformally equivalent near infinity, i.e.

$$\psi(f(z))=g(\varphi(z)) \tag{3.3}$$

whenever $|f(z)|$ or $|g(\varphi(z))|$ is large enough, with $\varphi,\psi:\mathbb{C}\to\mathbb{C}$ quasiconformal. Without loss of generality, we may assume that $\varphi(0)=0$ and $\psi(0)=0$ (otherwise we modify these maps inside some bounded disk, using Proposition 2.11).

Pick a logarithmic transform $F:\mathcal{V}\to H$, where the disk $D=\mathbb{C}\setminus\overline{\exp(H)}$ is chosen sufficiently large to ensure that (3.3) holds for $z\in\exp(\mathcal{V})$. Let $\Phi:\mathbb{C}\to\mathbb{C}$ and $\Psi:\mathbb{C}\to\mathbb{C}$ be lifts of φ and ψ, respectively, under the exponential map. Then

$$G:=\Psi\circ F\circ\Phi^{-1}$$

is a logarithmic transform of g, and F and G are quasiconformally equivalent by definition. (Note that we are not claiming that *all* logarithmic transforms of f and g are quasiconformally equivalent.) We define ϑ by $\vartheta(\exp(z)):=\exp(\Theta(z))$, where Θ is the map from the previous theorem, and are done. □

We subdivided the proof of Theorem 1.1 into two steps, using Theorem 3.1 to reduce the problem to the simpler family $F_{\varkappa}$. We remark that this would not be necessary if we were willing to forgo the statement that the dilatation of ϑ on the escaping set is zero.

Indeed, we can adapt the proof of Theorem 3.2 to construct a suitable map Θ for any two quasiconformally equivalent functions $F,G\in\mathcal{B}_{\log}$, as follows.

Letting Ψ and Φ denote the maps from the definition of quasiconformal equivalence, we set $\Theta_0(z):=z$ and define Θ_n inductively as follows. If T is a tract of F and $\widetilde{T}$ is the tract of G that contains $\Phi(F_T^{-1}(\mathbb{H}_M))$ for sufficiently large M, we define, for $z\in T$,

$$\Theta_{n+1}(z):=G_{\widetilde{T}}^{-1}(\Theta_n(F(z)))$$

(where defined).

By virtue of Lemma 2.6, the proof of Theorem 3.2 goes through as before if we replace uniform convergence in the Euclidean metric by uniform convergence in the hyperbolic metric. That is, for sufficiently large Q, the maps Θ_n are all defined on $J_Q(F)$ and converge uniformly to a map $\Theta: J_Q(F)\to J(G)$ that is a homeomorphism onto its image.

It is important to observe that, for fixed F, the convergence is uniform not only in z but also in G if Φ and Ψ range over a compact set of quasiconformal mappings. Hence it follows that the conjugacy Θ still depends holomorphically on G (which was not immediately clear from our original proof of Corollary 3.5). We state this result formally for future reference.

PROPOSITION 3.6. (Analytic dependence on parameters) *Let $f\in\mathcal{B}$ and let M be a finite-dimensional complex manifold, with a base point $\lambda_0\in M$. Suppose that $\{f_\lambda\}_{\lambda\in M}$ is a family of entire functions quasiconformally equivalent to f, with the equivalences given by $\psi_\lambda\circ f=f_\lambda\circ\varphi_\lambda$, where $\psi_{\lambda_0}=\varphi_{\lambda_0}=\mathrm{id}$, and φ_λ and ψ_λ depend analytically on λ.*

Let $N\ni\lambda_0$ be a compact subset of M. Then there exists a constant $R>0$ such that, for every $\lambda\in N$, there is an injective function $\vartheta=\vartheta^\lambda: J_R(f)\to J(f_\lambda)$ with the following properties:

(a) $\vartheta^{\lambda_0}=\mathrm{id}$;

(b) $\vartheta^\lambda\circ f=f_\lambda\circ\vartheta^\lambda$;

(c) *for fixed $z\in J_R(f)$, the function $\lambda\mapsto\vartheta^\lambda(z)$ is analytic in λ (on the interior of N).*

In particular, we can again use the λ-lemma to show that ϑ^λ has a quasiconformal extension, as in Theorem 3.4. If one was able to furnish a direct proof of the statement that the dilatation on the escaping set is zero—our argument used the fact that the parameter space of the family $F_\varkappa$ is a parabolic surface, and hence does not generalize—then Theorem 3.1 would no longer be required for the proof of Theorem 1.1.

It is not difficult to show directly that the map Θ constructed above agrees with the map from Corollary 3.5. (In particular, it *does* have zero dilatation on the escaping set.) This also follows from the results proved in the next section (see Corollary 4.2).

4. Rigidity

Let us now show that a (not necessarily quasiconformal) conjugacy between two quasiconformally equivalent maps $F,G\in\mathcal{B}_{\log}$ only moves escaping orbits by a bounded hyperbolic distance, provided that it "preserves combinatorics" (condition (d) below). This, together with the existence results from the previous section, will allow us to deduce a number of rigidity statements (Corollaries 4.2 and 4.3 and Theorems 1.2 and 1.3).

THEOREM 4.1. (Restriction on conjugacies) *Let $F,G\in\mathcal{B}_{\log}$ be normalized and quasiconformally equivalent, say $\Psi\circ F=G\circ\Phi$. Suppose that $Q>0$ and that $\Pi: J_Q(F)\to J(G)$ is continuous such that*

(a) $\Pi\circ F=G\circ\Pi$;

(b) $\Pi(z)\to\infty$ *as* $z\to\infty$;

(c) $\Pi(z+2\pi i)=\Pi(z)+2\pi i$;

(d) *for every $z\in J_Q(F)$, $\Pi(z)$ and $\Phi(z)$ belong to the same tract of G.*

If Q' is sufficiently large, then the hyperbolic distance $\mathrm{dist}_{\mathbb{H}}(z,\Pi(z))$ is uniformly bounded on $J_{Q'}(F)$.

Remark. The hypothesis that Π is defined on $J_Q(F)$ can be considerably weakened (with the same proof). For example, it would be sufficient to assume that Π is defined and continuous on a forward invariant set $A \subset J_Q(F)$ with the property that A contains the grand orbit (in $J_Q(F)$) of at least one sufficiently large point z_0.

Proof. Let $C, M > 0$ be the constants from Lemma 2.6; by enlarging M if necessary, we may assume that $M \geqslant Q$. By Corollary 2.5, we can choose some point $z_0 \in J_Q(F)$ such that $\operatorname{Re} z_0 \geqslant M$ and $\operatorname{Re} \Pi(z_0) \geqslant M$; we set

$$\varrho := \max\{2C, \operatorname{dist}_{\mathbb{H}}(z_0, \Pi(z_0))\}.$$

Set $Q' := e^{\varrho} \operatorname{Re}(z_0) + 2\pi > Q + 2\pi$. We will show that

$$\operatorname{dist}_{\mathbb{H}}(z, \Pi(z)) \leqslant \varrho \quad \text{for all } z \in J_{Q'}(F).$$

Claim. For every $z \in J_{Q'}(F)$, there is a point $\zeta \in J_Q(F)$, belonging to the same tract of F as z, with $|z - \zeta| < 2\pi$ and $F(\zeta) \in \{z_0 + 2\pi i k : k \in \mathbb{Z}\}$.

Proof of the claim. F maps the boundary of the tract T containing z to the imaginary axis, and the distance of z to ∂T is at most π. Since $\operatorname{Re} F(z) \geqslant Q' \geqslant \operatorname{Re} z_0$, we can hence find a point $\zeta_1 \in T$ with $|z - \zeta_1| < \pi$ and $\operatorname{Re} F(\zeta_1) = \operatorname{Re} z_0$. There is a point $\zeta_2 \in \{z_0 + 2\pi i k : k \in \mathbb{Z}\}$ with $|F(\zeta_1) - \zeta_2| \leqslant \pi$. We set $\zeta := F_T^{-1}(\zeta_2)$. By the expansion property (2.2) of F, we have $|\zeta - \zeta_1| \leqslant \frac{1}{2}\pi$, and are done. $\square$

Now let $z \in J_{Q'}(F)$. For each $n \geqslant 0$, we can apply the claim to $F^n(z)$ to obtain a point $\zeta^n \in J_Q(F)$ with $|F^n(z) - \zeta^n| < 2\pi$ and $F(\zeta^n) \in \{z_0 + 2\pi i k : k \in \mathbb{Z}\}$. We now pull back ζ^n along the orbit of z to obtain a point z^n; i.e.,

$$z^n = F_{T_0}^{-1}(F_{T_1}^{-1}(\dots F_{T_{n-1}}^{-1}(\zeta^n) \dots)),$$

where $T_0 T_1 \dots$ is the external address of z. By induction and by the expansion property (2.2), we have

$$|F^j(z) - F^j(z^n)| < \frac{2\pi}{2^{n-j}} \tag{4.1}$$

for $j = 0, \dots, n$. In particular, $z^n \in J_Q(F)$ and $z^n \to z$.

We set $z_j^n := F^j(z^n)$ and $w_j^n := \Pi(z_j^n) = G^j(\Pi(z^n))$. Let us prove inductively that

$$\operatorname{dist}_{\mathbb{H}}(z_j^n, w_j^n) \leqslant \varrho \tag{4.2}$$

for $j = n+1, n, \dots, 0$. Indeed, we have $z_{n+1}^n = z_0 + 2\pi i k$ for some $k \in \mathbb{Z}$, and hence

$$\operatorname{dist}_{\mathbb{H}}(z_{n+1}^n, w_{n+1}^n) = \operatorname{dist}_{\mathbb{H}}(z_0, \Pi(z_0)) \leqslant \varrho,$$

by property (c) and the definition of ϱ.

Furthermore, for $j \leqslant n$, we have

$$w_j^n = (G|_{\widetilde{T}})^{-1}(w_{j+1}^n),$$

where $\widetilde{T}$ is the tract of G containing w_j^n. By the assumption (d), $\widetilde{T}$ is also the tract of G containing $\Phi(z_j^n)$.

We observe that $z_{j+1}^n, w_{j+1}^n \in \mathbb{H}_M$. Indeed, if $j=n$, this is true by the choice of z_0. If $j<n$, recall that $\operatorname{Re} z_{j+1}^n \geqslant Q' - 2\pi$ by (4.1), and $\operatorname{dist}_{\mathbb{H}}(z_{j+1}^n, w_{j+1}^n) \leqslant \varrho$ by the induction hypothesis. Our choice of Q' implies that $\operatorname{Re} w_{j+1} \geqslant \operatorname{Re} z_0 \geqslant M$.

By Lemma 2.6 and the induction hypothesis, it follows that

$$\operatorname{dist}_{\mathbb{H}}(z_j^n, w_j^n) \leqslant C + \tfrac{1}{2} \operatorname{dist}_{\mathbb{H}}(z_{j+1}^n, w_{j+1}^n) \leqslant C + \tfrac{1}{2}\varrho \leqslant \varrho,$$

as claimed.

We have $z_0^n = z^n \to z$, and hence, by continuity of Π, also $w_0^n = \Pi(z_0^n) \to \Pi(z)$. Therefore (4.2) implies that $\operatorname{dist}_{\mathbb{H}}(z, \Pi(z)) \leqslant \varrho$, as desired. □

Corollary 4.2. (Uniqueness of conjugacies) *Let F and G be quasiconformally equivalent. Then for every $Q>0$, there is $Q' \geqslant Q$ with the following property. If*

$$\Pi_1, \Pi_2 \colon J_Q(F) \longrightarrow J(G)$$

are continuous functions satisfying the hypotheses (a)–(d) *of the previous theorem, then $\Pi_1(z) = \Pi_2(z)$ for all $z \in J_{Q'}(F)$.*

Proof. We may assume without loss of generality that F and G are both normalized. Let $Q' \geqslant Q$ be chosen such that $J_{Q'}(F) \subset \overline{I_Q(F)}$ (recall Corollary 2.5).

It follows from Theorem 4.1 that there is $Q'' \geqslant Q$ such that, for all $z \in J_{Q''}(F)$, the points $\Pi_1(z)$ and $\Pi_2(z)$ have the same external address, and stay within bounded hyperbolic distance of each other. By the expansion property (2.3) of G, this implies that $\Pi_1(z) = \Pi_2(z)$ (provided Q'' was chosen large enough).

So we have proved that $\Pi_1 = \Pi_2$ on $J_{Q''}(F)$. Using (d), we see that $\Pi_1 = \Pi_2$ on $I_Q(F)$. But $I_Q(F)$ is dense in $J_{Q'}(F)$, so we are done. □

Corollary 4.3. (No invariant line fields) *Let $F \in \mathcal{B}_{\log}$. Then F has no invariant line fields on its escaping set $I(F)$.*

Proof. Recall that the existence of an invariant line field is equivalent to the existence of a non-zero F-invariant Beltrami form whose support is contained in $I(F)$ [McM2, §3.5].

So suppose that μ was such a Beltrami form. Recall that

$$I(F)=\bigcap_{Q>0}\bigcup_{n\geqslant 0}F^{-n}(J_Q(F)).$$

Since F is holomorphic, this implies that there is no $Q>0$ such that $\mu|_{J_Q(F)}$ is zero almost everywhere. Also observe that $2\pi i$-periodicity of F implies that μ is $2\pi i$-periodic.

By the measurable Riemann mapping theorem [H, Theorem 4.6.1], μ gives rise to a quasiconformal homeomorphism $\Phi:\mathbb{C}\to\mathbb{C}$, which we may choose to commute with translation by $2\pi i$. The map

$$G:=\Phi\circ F\circ\Phi^{-1}$$

is holomorphic, and clearly quasiconformally equivalent to F.

By Corollary 3.5, there is a quasiconformal map Θ, isotopic to Φ relative to the boundary of the domain of definition $\mathcal{V}$ of F, which conjugates F and G on $J_Q(F)$, where $Q>0$ is sufficiently large.

By Corollary 4.2, we then have

$$\Theta|_{J_{Q'}(F)}=\Phi|_{J_{Q'}(F)}$$

for sufficiently large Q'. Hence the dilatation of Θ and Φ agree almost everywhere on $I_{Q'}(F)$. This is a contradiction: the dilatation of Θ on $I_{Q'}(F)$ is zero almost everywhere, but this is false for the dilatation μ of the map Φ. □

Proof of Theorem 1.2. Let $f\in\mathcal{B}$, and let F be a logarithmic transform of f. If f supported an invariant line field on its escaping set, then the same would be true for F. (As in the proof of Corollary 4.3, the support of the line field has non-trivial intersection with every set of the form $\{z\in I(f):|f^n(z)|\geqslant R\}$, $R>0$.) Hence the theorem follows from Corollary 4.3. □

Proof of Theorem 1.3. Suppose that f and g are entire functions with finitely many singular values, let $\pi:\mathbb{C}\to\mathbb{C}$ be a topological conjugacy betweeen f and g, and let $\mathcal{O}$ be the orbit of some escaping point $z_0\in I(f)$.

For simplicity, let us assume that $f(0)=0$, and that $\pi(0)=0$. This is no loss of generality, since any $f\in\mathcal{B}$ has infinitely many fixed points (see [Ep, Lemma 69] or [EL1]; compare also [LZ] for a more general result). However, we would like to point out that this assumption is not essential for the proof, and is made purely for convenience.

Let $S:=S(f)\cup\{0\}$. We can pick a quasiconformal homeomorphism (in fact, a diffeomorphism) $\psi:\mathbb{C}\to\mathbb{C}$ that is isotopic to π relative to S. Using the functional relation $\pi\circ f=g\circ\pi$, the isotopy between π and ψ lifts to an isotopy between π and a quasiconformal map $\varphi:\mathbb{C}\to\mathbb{C}$ with

$$\psi\circ f=g\circ\varphi.$$

(Compare also [EL3, §3].) In particular, f and g are quasiconformally equivalent.

Now, as usual, we change to logarithmic coordinates: we let $F\colon\mathcal{V}\to H$ be a logarithmic transform of F, and Π be a lift of π; i.e., $\pi\circ\exp=\exp\circ\Pi$. Then $G:=\Pi\circ F\circ\Pi^{-1}$ is a logarithmic transform of g.

The isotopies between π and ψ, resp. φ, lift to isotopies between Π and maps Ψ, resp. Φ, satisfying $\Psi\circ F=G\circ\Phi$, so F and G are quasiconformally equivalent as elements of $\mathcal{B}_{\log}$.

Furthermore, if $M>0$ is sufficiently large, then no point $z\in\mathbb{H}_M$ leaves the domain H under the isotopy between Π and Ψ. It follows that, if T is a tract of F and $z\in T$ with $F(z)\in\mathbb{H}_M$, then $\Phi(z)\in\Pi(T)$.

Let Θ be the map from Corollary 3.5. Then, by Corollary 4.2, we have

$$\Theta|_{J_{Q'}(F)}=\Pi|_{J_{Q'}(F)}$$

for some $Q'\geqslant 0$. If ϑ is the quasiconformal map defined by $\vartheta\circ\exp=\exp\circ\Theta$, then ϑ and π agree on the set

$$J_{e^{Q'}}(f)=\{z\in\mathbb{C}:|f^n(z)|\geqslant e^{Q'} \text{ for all } n\geqslant 1\}.$$

Pick $k_0\in\mathbb{N}$ such that $f^{k_0}(z_0)\in J_{e^{Q'}}(f)$. Then π agrees with the quasiconformal map ϑ on the tail $\mathcal{O}_{k_0}:=\{f^k(z_0):k\geqslant k_0\}$ of the orbit $\mathcal{O}$.

We can modify the map ϑ (e.g. using Proposition 2.11) on a compact subset of $\mathbb{C}\setminus\mathcal{O}_{k_0}$ to a quasiconformal function that maps $f^k(z_0)$ to $\pi(f^k(z_0))$ for $0\leqslant k<k_0$. This is the desired quasiconformal extension of $\pi|_{\mathcal{O}}$. □

Remark. Note that the assumption that $S(f)$ is finite was used only to find a quasiconformal map ψ isotopic to π. Hence we can weaken the assumptions of Theorem 1.3 to require only that $f,g\in\mathcal{B}$ and that the conjugacy π is isotopic, relative to $S(f)$, to a quasiconformal self-map of the plane.

5. Hyperbolic maps

Recall that $f\in\mathcal{B}$ is *hyperbolic* if $S(f)$ is contained in the union of attracting basins of f. Since $S(f)$ is compact by definition, there are then only finitely many such basins, which together make up the Fatou set. In particular, f is hyperbolic if and only if the postsingular set

$$\mathcal{P}(f)=\overline{\bigcup_{j\geqslant 0} f^j(S(f))}$$

is a compact subset of the Fatou set.

In the following, we assume without loss of generality that 0 is one of the attracting periodic points of f.

We will show that such an f is semi-conjugate on its Julia set to a disjoint-type map quasiconformally equivalent to f, and this semi-conjugacy is a conjugacy when restricted to the escaping set. In view of Theorem 3.1, this implies that any two hyperbolic maps that are quasiconformally equivalent near infinity are in fact topologically conjugate on their sets of escaping points, and hence proves Theorem 1.4.

It is easy to see that there is a bounded open neighborhood U of the postsingular set $\mathcal{P}(f)$ such that $\overline{f(\overline{U})}\subset U$. We set $W:=\mathbb{C}\setminus\overline{U}$ and $V:=f^{-1}(W)\subset W$. Then

$$f\colon V\longrightarrow W$$

is a covering map, and hence expands the hyperbolic metric of W. We claim that this map is in fact *uniformly* expanding. (Compare also [RS, Theorem C].)

LEMMA 5.1. (Uniform expansion) *There is $C>1$ such that $\|Df(z)\|_W\geqslant C$ for all $z\in V$.*

Proof. Since f is a covering map, we just need to show that the inclusion $\iota\colon V\to W$ is uniformly contracting. Since the density of the hyperbolic metric of V tends to ∞ near ∂V, and V and W have no common finite boundary points, it is sufficient to prove that $\varrho_W(z)/\varrho_V(z)\to 0$ as $z\to\infty$.

The hyperbolic density of W satisfies $\varrho_W(z)=O(1/|z|\log|z|)$ as $z\to\infty$. We now estimate the hyperbolic metric of V, using Lemma 2.1. Fix some point $w\in\mathbb{C}\setminus W=\overline{U}$ such that w belongs to the unbounded component of $\mathbb{C}\setminus S(f)$.

Claim. There is a constant C and a sequence $\{w_j\}_{j\in\mathbb{N}}$ of (pairwise distinct) preimages of w under f such that $|w_{j+1}|\leqslant C|w_j|$ for all j.

Proof of the claim. Pick a Jordan curve γ surrounding $S(f)$, but not surrounding w, and let G be the unbounded component of $\mathbb{C}\setminus\gamma$. If $\widetilde{G}$ is a component of $f^{-1}(G)$, then $f\colon\widetilde{G}\to G$ is a universal covering. Hence we can find a sequence $\{w_j\}_{j\in\mathbb{N}}$ of preimages of w in $\widetilde{G}$ such that the hyperbolic distance (in $\widetilde{G}$) between w_j and w_{j+1} is constant. By the standard estimate (2.1) on the hyperbolic distance in the simply connected domain $\widetilde{G}$, this implies that $|w_{j+1}|\leqslant C|w_j|$ for some C and sufficiently large j, as desired. □

By Lemma 2.1, the hyperbolic metric of the domain $V':=\mathbb{C}\setminus\{w_n:n\in\mathbb{N}\}$ satisfies $1/\varrho_{V'}(z)=O(|z|)$. Since $\varrho_V\geqslant\varrho_{V'}$ by Pick's theorem, this means that $\varrho_W(z)/\varrho_V(z)\to 0$ as $z\to\infty$, as claimed. □

Let $K\geqslant 1$; if K is chosen sufficiently large, then $\overline{U}\subset\mathbb{D}_{K/2}(0)$. Furthermore, choose $R\geqslant K$ such that

$$f^{-1}(\{|z|>R\})\subset\{|z|>K+1\}.$$

We define $M:=R/K$ and $g(z):=f(z/M)$. Then g is of disjoint type. Indeed, we have $\mathcal{U}:=g^{-1}(\{|z|>R\})\subset\{|z|>R+M\}$. We define

$$\mathcal{V}:=f^{-1}(\{|z|>R\}) \quad \text{and} \quad \widetilde{\mathcal{V}}:=f^{-1}(\{|z|>K\}),$$

and, for all j,

$$\mathcal{U}_j:=g^{-j}(\{|z|>R\}), \quad \mathcal{V}_j:=f^{-j}(\{|z|>R\}) \quad \text{and} \quad \widetilde{\mathcal{V}}_j:=f^{-j}(\{|z|>K\}).$$

Note that $\mathcal{V}_j\subset\widetilde{\mathcal{V}}_j\subset W$ for all j.

We now define a sequence ϑ_k, where $\vartheta_0=\mathrm{id}$ and

$$\vartheta_k:\mathcal{U}_{k-1}\longrightarrow\widetilde{\mathcal{V}}_{k-1}$$

is a conformal isomorphism for $k\geqslant 1$, such that

$$f(\vartheta_{k+1}(z))=\vartheta_k(g(z)).$$

Begin by setting $\vartheta_1(z):=z/M$. Furthermore, for $z\in\mathcal{U}_0$, let $\gamma_1(z)\subset\widetilde{\mathcal{V}}_0$ be the straight line segment connecting $z=\vartheta_0(z)$ and $z/M=\vartheta_1(z)$.

To define ϑ_2 let $z\in\mathcal{U}_1$. Since

$$f(\vartheta_1(z))=\vartheta_0(g(z)),$$

the curve $\gamma_1(g(z))$ has a preimage component $\gamma_2(z)\subset\widetilde{\mathcal{V}}_1$ under f with one endpoint at $\vartheta_1(z)$; we define $\vartheta_2(z)$ to be the other endpoint. Then $f(\vartheta_2(z))=\vartheta_1(g(z))$.

We continue inductively: the curve $\gamma_{j+1}(z)\subset\widetilde{\mathcal{V}}_j$ is the pullback of $\gamma_j(g(z))$ with one endpoint at $\vartheta_j(z)$, and $\vartheta_{j+1}(z)$ is defined as the other endpoint of this curve.

It follows from the definition that each ϑ_{k+1} is continuous. Hence, for every component G of $\mathcal{U}_k$, $\vartheta_{k+1}|_G$ is a branch of $f^{-1}\circ\vartheta_k\circ g$, and hence a conformal isomorphism onto some component of $\widetilde{\mathcal{V}}_k$. It is likewise easy to check that ϑ_{k+1} is surjective, so these maps are indeed conformal isomorphisms between $\mathcal{U}_k$ and $\widetilde{\mathcal{V}}_k$.

We furthermore note that $\vartheta_k(\mathcal{U}_k)=\mathcal{V}_k$ for $k\geqslant 0$ by the inductive construction.

Theorem 5.2. (Convergence to a semiconjugacy) *In the hyperbolic metric of W, the maps $\vartheta_k|_{J(g)}$ converge uniformly to a continuous surjection*

$$\vartheta\colon J(g)\longrightarrow J(f)$$

with $f\circ\vartheta=\vartheta\circ g$ and $\vartheta(I(g))=\vartheta(I(f))$. Furthermore, $\vartheta\colon I(g)\to I(f)$ is a homeomorphism.

Proof. Let $z\in\mathcal{U}_k$. By definition,

$$\mathrm{dist}_W(\vartheta_{k+1}(z),\vartheta_k(z))\leqslant \ell_W(\gamma_{k+1}(z)).$$

We have

$$\ell_W(\gamma_1(z))\leqslant \ell_{\{|w|>K/2\}}(\gamma_1(z))=\log\left(1+\frac{\log M}{\log(2|z|/MK)}\right)\leqslant \log\left(1+\frac{\log M}{\log 2}\right)=:\mu$$

for all $z\in\mathcal{U}_0$. Since $\gamma_{k+1}(z)$ is obtained from $\gamma_1(g^k(z))$ by a branch of f^{-k}, and f is uniformly expanding on W by Lemma 5.1, we see that

$$\mathrm{dist}_W(\vartheta_{k+1}(z),\vartheta_k(z))\leqslant \frac{\mu}{C^k} \tag{5.1}$$

for all $z\in\mathcal{U}_k$, where C is the constant from Lemma 5.1.

In particular, the maps $\vartheta_k|_{J(g)}$ form a Cauchy sequence and, by the completeness of the hyperbolic metric, have a (continuous) limit

$$\vartheta\colon J(g)\longrightarrow W.$$

By (5.1), ϑ satisfies

$$\mathrm{dist}_W(\vartheta(z),z)\leqslant \mu\frac{C}{C-1}. \tag{5.2}$$

By definition, if $z\in J(g)$, then $f^k(\vartheta(z))=\vartheta(g^k(z))\in W$ for all $k\in\mathbb{N}$. Thus $\vartheta(z)\in J(f)$. Also note that, by (5.2), $\vartheta(z_n)\to\infty$ if and only if $z_n\to\infty$, so ϑ maps escaping points to escaping points.

The map $\vartheta\colon I(g)\to I(f)$ is clearly surjective. Indeed, if $w\in I(f)$, then $w\in\widetilde{V}_k$ for all sufficiently large k. Any limit point z of the sequence $z_k=\vartheta_k^{-1}(w)$ will have $\vartheta(z)=w$. (Note that $\{z_k\}_{k\in\mathbb{N}}$ cannot diverge to infinity by (5.1)).

To prove injectivity on $I(g)$, suppose by contradiction that $\vartheta(z_1)=\vartheta(z_2)$, where $z_1,z_2\in I(g)$, $z_1\neq z_2$. It follows from the construction of ϑ that also $g^j(z_1)\neq g^j(z_2)$ for all $j\geqslant 0$. However, ϑ is injective on a set of the form

$$J_{R'}(g)\cap I(g)=\{z\in I(g):|g^j(z)|\geqslant R' \text{ for all } j\geqslant 1\}.$$

(This follows from Corollary 4.2, or alternatively from an argument analogous to the proof of injectivity in Theorem 3.2.) Since $g^j(z_1)$ and $g^j(z_2)$ belong to $J_{R'}(g)\cap I(g)$ for sufficiently large j, we obtain the desired contradiction. The details are left to the reader.

Finally, $\vartheta(J(g))\cup\{\infty\}$ is the continuous image of a compact set, and thus itself compact. Since $I(f)\subset\vartheta(J(g))\subset J(f)$ and $J(f)\subset\overline{I(f)}$ by [Er], we see that ϑ is surjective. The compactness of $J(g)\cup\{\infty\}$ and the fact that $\vartheta^{-1}(I(f))=I(g)$ imply that the image of any relatively closed subset of $I(g)$ under ϑ is relatively closed in $I(f)$. Hence $(\vartheta|_{I(g)})^{-1}$ is continuous. $\square$

Recall that, by a "pinched Cantor bouquet" we mean a metric space that is the quotient of a *straight brush* in the sense of [AO] by a closed equivalence relation on its endpoints. As a corollary of Theorem 1.4, we obtain the following.

COROLLARY 5.3. (Pinched Cantor bouquets) *Let $f\in\mathcal{B}$ be hyperbolic and of finite order. Then every dynamic ray of f lands, and the Julia set is a pinched Cantor bouquet.*

Proof. Barański [Ba] proved that, in the disjoint-type case, the Julia set is a straight brush, where all points except (some of) the endpoints of the brush belong to $I(f)$. The corollary then follows immediately from our Theorem 5.2. □

Remark 1. More generally, the Julia set of a disjoint-type function f that can be written as the composition of finitely many finite-order functions in $\mathcal{B}$ is homeomorphic to a straight brush. This follows from [RRRS, Theorem 5.10] (which is independent of [Ba]). Hence Corollary 5.3 also holds for all hyperbolic functions that can be written as such a composition.

Remark 2. We should note that the fact that the Julia set is homeomorphic to a straight brush is not explicitly proved either in [Ba] or in [RRRS]. However, it is not difficult to deduce this from the results proved there using the topological characterization given in [AO].

Appendix A. Structure of escaping sets

In this section, we discuss the bearing that our results have on some intriguing questions about escaping sets of entire functions that go back to Fatou's original article of 1926 [F], and Eremenko's study of the escaping set [Er]. Fatou observed that the Julia sets of certain explicit entire functions contain curves on which the iterates tend to ∞, and asked whether this property holds for much more general functions. Eremenko showed that (for an arbitrary entire function f), every component of the closure $\overline{I(f)}$ is unbounded. He then asked whether, in fact, every component of $I(f)$ is unbounded, and also whether every point of $I(f)$ can be connected to ∞ by a curve in $I(f)$. (For a more detailed discussion of these questions and their history, compare [RRRS].)

This suggests the study of the following properties for an entire function f.

(F) *Fatou property*: There is a curve to ∞ in $I(f)$.

(E) *Eremenko property*: Every connected component of $I(f)$ is unbounded.

(S) *Strong Eremenko property*: Every point $z\in I(f)$ can be connected to ∞ by a curve in $I(f)$.

It is shown in [RRRS] that there exist hyperbolic functions $f\in\mathcal{B}$ for which the Julia set does not contain any curves to ∞. Thus, property (F) (and, in particular, (S)) can fail for functions in class $\mathcal{B}$. In fact, there are even hyperbolic functions whose Julia set does not contain any non-trivial curves at all. Together with Theorem 1.1, this implies the following result.

Corollary A.1. (No curves in the escaping set) *There exists an entire function $f\in\mathcal{B}$ with the following property: if $g\in\mathcal{B}$ is quasiconformally equivalent to f near infinity, then the escaping set $I(g)$ does not contain any non-trivial curves.*

Proof. Let f be the example constructed in [RRRS, Theorem 8.4], whose Julia set does not contain any non-trivial curves. If g is quasiconformally equivalent to f near infinity, then, by Theorem 1.1, for sufficiently large R the set $J_R(f)$ of points whose forward orbits are contained in $\mathbb{C}\setminus\mathbb{D}_R(0)$ is homeomorphic to a subset of the Julia set of f. Therefore $J_R(f)$ does not contain any non-trivial curves either. Since the image of a non-trivial curve under f is again a non-trivial curve, the same holds for all sets $f^{-n}(J_R(f))$, $n\geqslant 0$.

Suppose, by contradiction, that $I(g)$ does contain a non-trivial curve $\gamma: [0,1]\to I(g)$; we may assume that γ is not constant on any interval. For every n, $\gamma^{-1}(f^{-n}(J_R(f)))$ is a closed subset of $[0,1]$ that does not contain any intervals, and hence is nowhere dense. However, we have

$$[0,1]=\gamma^{-1}(I(f))\subset\bigcup_{n\geqslant 0}\gamma^{-1}(f^{-n}(J_R(f))),$$

which contradicts the Baire category theorem. □

In [R2], we establish Eremenko's property for every hyperbolic function $f\in\mathcal{B}$, and more generally any function $f\in\mathcal{B}$ with bounded postsingular set. This shows that a situation as in Corollary A.1 cannot occur for property (E). Whether there is any entire function for which property (E) fails remains an open problem. We remark that even in the exponential family, the study of connected components of $I(f)$ is far from trivial: while in the hyperbolic case, each such component consists of a single dynamic ray [BDDJM], for many non-hyperbolic exponential maps, including $z\mapsto\exp(z)$ and $z\mapsto 2\pi i\exp(z)$, the escaping set is a connected subset of the complex plane [R4], [J], [R5].

Note that our Theorem 1.4 also shows that

> *for any quasiconformal equivalence class in class $\mathcal{B}$, each of the properties* (F), (E) *and* (S) *either holds for all hyperbolic maps or fails for all hyperbolic maps.*

Now consider the following uniform variants of the above properties.

(UE) For every $z \in I(f)$, there exists some unbounded connected set $A \ni z$ such that $f^n|_A \to \infty$ *uniformly*.

(US) Every $z \in I(f)$ can be connected to ∞ by a curve γ such that $f^n|_\gamma \to \infty$ *uniformly*.

In many proofs of the Eremenko property, or the strong Eremenko property, they are in fact established in this uniform sense. It is possible, following the construction in [RRRS], to construct an entire function for which property (UE) fails.

Theorem 1.1 shows that

> *for any quasiconformal equivalence class of Eremenko–Lyubich functions*, *each of the properties* (UE) *and* (US) *either holds for all maps or fails for all maps.*

In [RRRS], property (US) is established for a large subset of $\mathcal{B}$, in particular for those of finite order (as well as finite compositions of such functions). The above-mentioned recent results of Barański [Ba] also imply this property for *disjoint-type* functions $f \in \mathcal{B}$ of finite order (i.e., hyperbolic maps with a single fixed attractor). Hence Theorem 1.1, together with [Ba], provides an alternative proof of property (US)—and thus a positive answer to Fatou's and Eremenko's questions—for functions $f \in \mathcal{B}$ of finite order.

References

[AO] Aarts, J. M. & Oversteegen, L. G., The geometry of Julia sets. *Trans. Amer. Math. Soc.*, 338 (1993), 897–918.

[AL] Avila, A. & Lyubich, M., Hausdorff dimension and conformal measures of Feigenbaum Julia sets. *J. Amer. Math. Soc.*, 21 (2008), 305–363.

[Ba] Barański, K., Trees and hairs for some hyperbolic entire maps of finite order. *Math. Z.*, 257 (2007), 33–59.

[Berg] Bergweiler, W., Iteration of meromorphic functions. *Bull. Amer. Math. Soc.*, 29 (1993), 151–188.

[BRS] Bergweiler, W., Rippon, P. J. & Stallard, G. M., Dynamics of meromorphic functions with direct or logarithmic singularities. *Proc. Lond. Math. Soc.*, 97 (2008), 368–400.

[Bers] Bers, L., The moduli of Kleinian groups. *Uspeki Mat. Nauk*, 29 (1974), 86–102 (Russian); English translation in *Russian Math. Surveys*, 29 (1974), 88–102.

[BR] Bers, L. & Royden, H. L., Holomorphic families of injections. *Acta Math.*, 157 (1986), 259–286.

[BDDJM] Bhattacharjee, R., Devaney, R. L., Deville, R. E. L., Josić, K. & Moreno-Rocha, M., Accessible points in the Julia sets of stable exponentials. *Discrete Contin. Dyn. Syst. Ser. B*, 1 (2001), 299–318.

[DG] Douady, A. & Goldberg, L. R., The nonconjugacy of certain exponential functions, in *Holomorphic Functions and Moduli*, Vol. I (Berkeley, CA, 1986), Math. Sci. Res. Inst. Publ., 10, pp. 1–7. Springer, New York, 1988.

[DH1] Douady, A. & Hubbard, J. H., *Étude dynamique des polynômes complexes.* I, II. Publications Mathématiques d'Orsay, 84, 85. Université de Paris-Sud, Département de Mathématiques, Orsay, 1984, 1985.

[DH2] — On the dynamics of polynomial-like mappings. *Ann. Sci. École Norm. Sup.*, 18 (1985), 287–343.

[Ep] Epstein, A. L., *Towers of Finite Type Complex Analytic Maps.* Ph.D. Thesis, City University of New York, 1995. `http://pcwww.liv.ac.uk/~lrempe/adam/thesis.pdf`.

[Er] Eremenko, A. È., On the iteration of entire functions, in *Dynamical Systems and Ergodic Theory* (Warsaw, 1986), Banach Center Publ., 23, pp. 339–345. PWN, Warsaw, 1989.

[EL1] Eremenko, A. È. & Lyubich, M. Y. Structural stability in some families of entire functions. Preprint, Ukrainian Physico-Technical Institute, Kharkov, 1984 (Russian).

[EL2] — Examples of entire functions with pathological dynamics. *J. London Math. Soc.*, 36 (1987), 458–468.

[EL3] — Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier* (*Grenoble*), 42 (1992), 989–1020.

[F] Fatou, P., Sur l'itération des fonctions transcendantes entières. *Acta Math.*, 47 (1926), 337–370.

[GKŚ] Graczyk, J., Kotus, J. & Świątek, G., Non-recurrent meromorphic functions. *Fund. Math.*, 182 (2004), 269–281.

[H] Hubbard, J. H., *Teichmüller Theory and Applications to Geometry*, *Topology*, *and Dynamics.* Vol. 1. Matrix Editions, Ithaca, NY, 2006.

[J] Jarque, X., On the connectivity of the escaping set for complex exponential Misiurewicz parameters. Preprint, 2009.

[KSvS1] Kozlovski, O., Shen, W. & van Strien, S., Rigidity for real polynomials. *Ann. of Math.*, 165 (2007), 749–841.

[KSvS2] — Density of hyperbolicity in dimension one. *Ann. of Math.*, 166 (2007), 145–182.

[LZ] Langley, J. K. & Zheng, J. H., On the fixpoints, multipliers and value distribution of certain classes of meromorphic functions. *Ann. Acad. Sci. Fenn. Math.*, 23 (1998), 133–150.

[L] Lehto, O., An extension theorem for quasiconformal mappings. *Proc. London Math. Soc.*, 14a (1965), 187–190.

[LV] Lehto, O. & Virtanen, K. I., *Quasikonforme Abbildungen.* Die Grundlehren der mathematischen Wissenschaften, 126. Springer, Berlin, 1965 (German); English translation: *Quasiconformal Mappings in the Plane*, Springer, New York, 1973.

[MSS] Mañé, R., Sad, P. & Sullivan, D., On the dynamics of rational maps. *Ann. Sci. École Norm. Sup.*, 16 (1983), 193–217.

[McM1] McMullen, C., Area and Hausdorff dimension of Julia sets of entire functions. *Trans. Amer. Math. Soc.*, 300 (1987), 329–342.

[McM2] — *Complex Dynamics and Renormalization.* Annals of Mathematics Studies, 135. Princeton University Press, Princeton, NJ, 1994.

[M-B] Mihaljević-Brandt, H., Semiconjugacies, pinched Cantor bouquets and hyperbolic orbifolds. Preprint, 2009. `arXiv:0907.5398 [math.DS]`.

[M] Milnor, J., *Dynamics in One Complex Variable.* Annals of Mathematics Studies, 160. Princeton University Press, Princeton, NJ, 2006.

[R1] Rempe, L., Topological dynamics of exponential maps on their escaping sets. *Ergodic Theory Dynam. Systems*, 26:6 (2006), 1939–1975.
[R2] — On a question of Eremenko concerning escaping components of entire functions. *Bull. Lond. Math. Soc.*, 39 (2007), 661–666.
[R3] — Siegel disks and periodic rays of entire functions. *J. Reine Angew. Math.*, 624 (2008), 81–102.
[R4] — The escaping set of the exponential. To appear in *Ergodic Theory Dynam. Systems*.
[R5] — Connected escaping sets of exponential maps. Preprint, 2009. `arXiv:0910.4680 [math.DS]`.
[RvS1] Rempe, L. & van Strien, S., Absence of line fields and Mañé's theorem for non-recurrent transcendental functions. To appear in *Trans. Amer. Mat. Soc.*
[RvS2] — Density of hyperbolicity for certain families of real transcendental entire functions. In preparation.
[RS] Rippon, P. J. & Stallard, G. M., Iteration of a class of hyperbolic meromorphic functions. *Proc. Amer. Math. Soc.*, 127:11 (1999), 3251–3258.
[RRRS] Rottenfusser, G., Rückert, J., Rempe, L. & Schleicher, D., Dynamic rays of bounded-type entire functions. To appear in *Ann. of Math.*
[S] Shishikura, M., The boundary of the Mandelbrot set has Hausdorff dimension two, in *Complex Analytic Methods in Dynamical Systems* (Rio de Janeiro, 1992). *Astérisque*, 222 (1994), 7, 389–405.
[vS] van Strien, S., Misiurewicz maps unfold generically (even if they are critically non-finite). *Fund. Math.*, 163 (2000), 39–54.
[UZ] Urbański, M. & Zdunik, A., The finer geometry and dynamics of the hyperbolic exponential family. *Michigan Math. J.*, 51 (2003), 227–250.
[W] Weinreich, J. M., *Boundaries which Arise in the Iteration of Transcendental Entire Functions*. Ph.D. Thesis, Imperial College, London, 1991.

Lasse Rempe
Department of Mathematical Sciences
University of Liverpool
Liverpool, L69 7ZL
U.K.
l.rempe@liverpool.ac.uk

Received March 11, 2008