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The quantum orbifold cohomology of weighted projective spaces

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Contents

1. Introduction

In this paper we calculate the small quantum orbifold cohomology ring of the weighted projective space $\mathbf{P}^{\mathbf{w}}=\mathbf{P}(w_0,...,w_n)$. Our approach is essentially due to Givental [19], [20], [21]. We begin with a heuristic argument relating the quantum cohomology of $\mathbf{P}^{\mathbf{w}}$ to the S^1-equivariant Floer cohomology of the loop space $L\mathbf{P}^{\mathbf{w}}$, and from this conjecture a formula for a certain generating function—the *small J-function*—for genus-zero Gromov–Witten invariants of $\mathbf{P}^{\mathbf{w}}$. The small J-function determines the small quantum orbifold cohomology of $\mathbf{P}^{\mathbf{w}}$. We then prove that our conjectural formula for the small J-function is correct by analyzing the relationship between two compactifications of the space of parameterized rational curves in $\mathbf{P}^{\mathbf{w}}$: a toric compactification (which is closely related to our heuristic model for the Floer cohomology of $L\mathbf{P}^{\mathbf{w}}$) and the space of genus-zero stable maps to $\mathbf{P}^{\mathbf{w}}\times\mathbf{P}(1,r)$ of degree $1/r$ with respect to the second factor. These compactifications carry natural $\mathbf{C}^{\times}$-actions, which one can think of as arising from rotation of loops, and there is a map between them which is $\mathbf{C}^{\times}$-equivariant. Our formula for the small J-function can be expressed in terms of integrals of $\mathbf{C}^{\times}$-equivariant cohomology classes on the toric compactification. Following Bertram [9], we use localization in equivariant cohomology to transform these into integrals of classes on the stable map compactification. This establishes our formula for the small J-function, and so allows us to determine the small quantum orbifold cohomology ring of $\mathbf{P}^{\mathbf{w}}$.

We now give precise statements of our main results. The reader unfamiliar with orbifolds or with quantum orbifold cohomology may wish first to read §2, where various basic features of the theory are outlined. Let $w_0,...,w_n$ be a sequence of positive integers and let $\mathbf{P}^{\mathbf{w}}$ be the weighted projective space $\mathbf{P}(w_0,...,w_n)$, i.e. the quotient

$$[(\mathbf{C}^{n+1}\setminus\{0\})/\mathbf{C}^{\times}],$$

where $\mathbf{C}^{\times}$ acts with weights $-w_0,...,-w_n$. Components of the inertia stack of $\mathbf{P}^{\mathbf{w}}$ correspond to elements of the set

$$F=\{k/w_i : 0\leqslant k<w_i \text{ and } 0\leqslant i\leqslant n\}$$

via

$$\mathcal{I}\mathbf{P}^{\mathbf{w}}=\coprod_{f\in F}\mathbf{P}(V^f),$$

where $\mathbf{P}(V^f)$ is the locus of points of $\mathbf{P}^{\mathbf{w}}$ with isotropy group containing $\exp(2\pi\sqrt{-1}f)\in\mathbf{C}^{\times}$. This locus is itself a weighted projective space, of dimension

$$\dim_f=|\{j : w_j f\in\mathbf{Z}\}|-1.$$

The orbifold cohomology $H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\mathbf{C})$ is equal as a vector space to

$$\bigoplus_{f\in F} H^\bullet(\mathbf{P}(V^f);\mathbf{C}).$$

It carries two ring structures and two gradings: the usual cup product on the cohomology of $\mathcal{I}\mathbf{P}^{\mathbf{w}}$, the Chen–Ruan orbifold cup product, the usual grading on the cohomology of $\mathcal{I}\mathbf{P}^{\mathbf{w}}$, and a grading where the degree of a cohomology class is shifted by a rational number (the *degree-shifting number* or *age*) depending on the component of $\mathcal{I}\mathbf{P}^{\mathbf{w}}$ on which the class is supported. In this paper, unless otherwise stated, all products should be taken with respect to the orbifold cup product; the degree of an element of $H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\mathbf{C})$ always refers to its age-shifted degree. The involution $\zeta\mapsto\zeta^{-1}$ on $\mathbf{C}^\times$ induces an involution I on $\mathcal{I}\mathbf{P}^{\mathbf{w}}$ which exchanges $\mathbf{P}(V^f)$ with $\mathbf{P}(V^{1-f})$, $f\neq 0$, and is the identity on $\mathbf{P}(V^0)$.

Since $\mathbf{P}(V^0)=\mathbf{P}^{\mathbf{w}}$, there is a canonical inclusion $H^\bullet(\mathbf{P}^{\mathbf{w}};\mathbf{C})\subset H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\mathbf{C})$. Let $P\in H^2_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\mathbf{C})$ be the image of $c_1(\mathcal{O}(1))\in H^2(\mathbf{P}^{\mathbf{w}};\mathbf{C})$ under this inclusion and let Q be the generator for $H_2(\mathbf{P}^{\mathbf{w}};\mathbf{C})$ dual to $c_1(\mathcal{O}(1))$. For each $f\in F$, write $\mathbf{1}_f$ for the image of $\mathbf{1}\in H^\bullet(\mathbf{P}(V^f);\mathbf{C})$ under the inclusion $H^\bullet(\mathbf{P}(V^f);\mathbf{C})\subset H^{\bullet+\mathrm{age}(\mathbf{P}(V^f))}_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\mathbf{C})$. We will often work with orbifold cohomology with coefficients in the ring

$$\Lambda=\mathbf{C}[[Q^{1/\mathrm{lcm}(w_0,\dots,w_n)}]].$$

This plays the role of the Novikov ring (see [37, §III 5.2.1] and [25]) in the quantum cohomology of manifolds.(1) The quantum orbifold cohomology of $\mathbf{P}^{\mathbf{w}}$ is a family of Λ-algebra structures on $H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\Lambda)$ parameterized by a neighbourhood of zero in $H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\mathbf{C})$. When the parameter is restricted to lie in $H^2(\mathbf{P}^{\mathbf{w}};\mathbf{C})\subset H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\mathbf{C})$, we refer to the resulting family of algebras as the *small quantum orbifold cohomology* of $\mathbf{P}^{\mathbf{w}}$.

Let $f_1,\dots,f_k$ be the elements of F arranged in increasing order, and set $f_{k+1}=1$. The classes

$$\begin{array}{llll}\mathbf{1}_{f_1}, & \mathbf{1}_{f_1}P, & \dots, & \mathbf{1}_{f_1}P^{\dim_{f_1}},\\ \mathbf{1}_{f_2}, & \mathbf{1}_{f_2}P, & \dots, & \mathbf{1}_{f_2}P^{\dim_{f_2}},\\ \vdots & & & \\ \mathbf{1}_{f_k}, & \mathbf{1}_{f_k}P, & \dots, & \mathbf{1}_{f_k}P^{\dim_{f_k}}\end{array} \tag{1}$$

form a Λ-basis for $H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\Lambda)$.

(1) If we were being more careful, we could take the Novikov ring to be the semigroup ring R of the semigroup of degrees of effective possibly-stacky curves in $\mathbf{P}^{\mathbf{w}}$. But the degree of such a curve is $k/\mathrm{lcm}(w_0,\dots,w_n)$ for some integer k, and so R is naturally a subring of Λ.

Theorem 1.1. (See Corollary 5.4) *The matrix, with respect to the above basis, of multiplication by the class* P *in the small quantum orbifold cohomology algebra of* $\mathbf{P}^{\mathbf{w}}=\mathbf{P}(w_0,...,w_n)$ *corresponding to the point* $tP\in H^2(\mathbf{P}^{\mathbf{w}};\mathbf{C})$ *is*

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & r_N \\ r_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & r_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & r_{N-1} & 0 \end{pmatrix},$$

where $N=\dim_{f_1}+...+\dim_{f_k}+k$,

$$r_i=\begin{cases} Q^{f_{j+1}-f_j}e^{(f_{j+1}-f_j)t}\dfrac{s_{j+1}}{s_j}, & \textit{if } i=\dim_{f_1}+...+\dim_{f_j}+j \textit{ for some } j\leqslant k, \\ 1, & \textit{otherwise}, \end{cases}$$

and

$$s_j=\begin{cases} 1, & \textit{if } j=1, \\ \prod_{i=0}^n w_i^{-\lceil f_j w_i\rceil}, & \textit{if } 2\leqslant j\leqslant k+1. \end{cases}$$

Corollary 1.2. *The small quantum orbifold cohomology algebra of* $\mathbf{P}^{\mathbf{w}}$ *is the free* Λ*-module which is generated as a* Λ*-algebra by the classes*

$$\mathbf{1}_{f_1},\quad \mathbf{1}_{f_2},\quad ...,\quad \mathbf{1}_{f_k}\quad \textit{and}\quad P$$

with identity element $\mathbf{1}_{f_1}=\mathbf{1}_0$ *and relations generated by*

$$P^{\dim_{f_j}+1}\mathbf{1}_{f_j}=Q^{f_{j+1}-f_j}e^{(f_{j+1}-f_j)t}\frac{s_{j+1}}{s_j}\mathbf{1}_{f_{j+1}},\quad 1\leqslant j\leqslant k. \tag{2}$$

In particular,

$$P^N=\frac{Qe^t}{w_0^{w_0}w_1^{w_1}...w_n^{w_n}}\mathbf{1}_0.$$

If we invert Q, *then the small quantum orbifold cohomology algebra is generated by* P.

Remark 1.3. If we set Q to zero in (2), then we obtain a presentation for the Chen–Ruan orbifold cohomology ring of $\mathbf{P}^{\mathbf{w}}$.

Remark 1.4. The combinatorial factors r_i and s_j can be simplified by rescaling the basis (1), replacing $\mathbf{1}_f$ by $s_f e^{ft}\mathbf{1}_f$. See §5 for a precise statement.

Remark 1.5. Multiplication by P preserves the $\mathbf{C}[[Q]]$-submodule of $H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\Lambda)$ with basis

$$\begin{array}{llll} Q^{f_1}\mathbf{1}_{f_1}, & Q^{f_1}\mathbf{1}_{f_1}P, & ..., & Q^{f_1}\mathbf{1}_{f_1}P^{\dim_{f_1}}, \\ Q^{f_2}\mathbf{1}_{f_2}, & Q^{f_2}\mathbf{1}_{f_2}P, & ..., & Q^{f_2}\mathbf{1}_{f_2}P^{\dim_{f_2}}, \\ \vdots & & & \\ Q^{f_k}\mathbf{1}_{f_k}, & Q^{f_k}\mathbf{1}_{f_k}P, & ..., & Q^{f_k}\mathbf{1}_{f_k}P^{\dim_{f_k}}. \end{array} \tag{3}$$

We will see in §3 that, after inverting Q, we can think of this submodule as the Floer cohomology of the loop space $L\mathbf{P}^{\mathbf{w}}$.

Remark 1.6. Theorem 1.1 and Corollary 1.2 confirm the conjectures of Mann [38]. In the case of $\mathbf{P}(w_0, w_1)$, we recover the result of [4, §9]. The Chen–Ruan orbifold cohomology ring of weighted projective space, which is obtained from the quantum cohomology ring by setting $Q=0$, has been studied by a number of authors. Weighted projective space is a toric Deligne–Mumford stack—this is spelled out in [10]—so one can compute the orbifold cohomology ring using results of Borisov–Chen–Smith [11]. One can also apply the methods of Chen–Hu [12], Goldin–Holm–Knutson [22], or Jiang [30]. The relationship between the orbifold cohomology ring of certain weighted projective spaces and the cohomology ring of their crepant resolutions has been studied by Boissiere–Mann–Perroni [10]. The relationship between the *quantum* orbifold cohomology ring of certain weighted projective spaces and that of their crepant resolutions is investigated in [16].

The small J-function of $\mathbf{P}^{\mathbf{w}}$, a function of $t\in\mathbf{C}$ taking values in

$$H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\Lambda)\otimes\mathbf{C}((z^{-1})),$$

is defined by

$$J_{\mathbf{P}^{\mathbf{w}}}(t) = ze^{Pt/z}\left(1+\sum_{\substack{d:d>0\\ \langle d\rangle\in F}} Q^d e^{dt}(I\circ \mathrm{ev}_1)_\star\left(\mathbf{1}^{\mathrm{vir}}_{0,1,d}\cap\frac{1}{z(z-\psi_1)}\right)\right).$$

Here $\mathbf{1}^{\mathrm{vir}}_{0,1,d}$ is the virtual fundamental class of the moduli space $\mathbf{P}^{\mathbf{w}}_{0,1,d}$ of genus-zero one-pointed stable maps to $\mathbf{P}^{\mathbf{w}}$ of degree d; the degree of a stable map is the integral of the pull-back of the Kähler class P over the domain curve; $\langle d\rangle = d-\lfloor d\rfloor$ denotes the fractional part of the rational number d; $\mathrm{ev}_1\colon \mathbf{P}^{\mathbf{w}}_{0,1,d}\to\mathcal{I}\mathbf{P}^{\mathbf{w}}$ is the evaluation map at the marked point;(2) ψ_1 is the first Chern class of the universal cotangent line at the marked point;

(2) This evaluation map does not in fact exist, but one can to all intents and purposes pretend that it does. See the discussion in §2.2.2.

and we expand the expression $(z-\psi_1)^{-1}$ as a power series in z^{-1}. Note that the degrees d occurring in the sum will in general be non-integral. We will see in §2 below that the small J-function determines the small quantum orbifold cohomology of $\mathbf{P}^{\mathbf{w}}$: it satisfies a system of differential equations whose coefficients are the structure constants of the small quantum orbifold cohomology algebra.

THEOREM 1.7. (See Corollary 4.6) *The small J-function $J_{\mathbf{P}^{\mathbf{w}}}(t)$ is equal to*

$$ze^{Pt/z} \sum_{\substack{d:d\geqslant 0\\ \langle d\rangle\in F}} \frac{Q^d e^{dt}}{\prod_{i=0}^{n}\prod_{b:\langle b\rangle=\langle dw_i\rangle,0<b\leqslant dw_i}(w_iP+bz)} \mathbf{1}_{\langle d\rangle}.$$

From this, we deduce the following result.

COROLLARY 1.8. *The small J-function $J_{\mathbf{P}^{\mathbf{w}}}(t)$ satisfies the differential equation*

$$\prod_{i=0}^{n}\prod_{k=0}^{w_i-1}\left(w_i z\frac{\partial}{\partial t}-kz\right)J_{\mathbf{P}^{\mathbf{w}}}(t)=Qe^t J_{\mathbf{P}^{\mathbf{w}}}(t).$$

Weighted projective complete intersections

Let $\mathcal{X}$ be a quasismooth complete intersection in $\mathbf{P}^{\mathbf{w}}$ of type $(d_0,d_1,...,d_m)$ and let $\iota:\mathcal{X}\to\mathbf{P}^{\mathbf{w}}$ be the inclusion. Define

$$I_{\mathcal{X}}(t)=ze^{Pt/z}\sum_{\substack{d:d\geqslant 0\\ \langle d\rangle\in F}} Q^d e^{dt}\frac{\prod_{j=0}^{m}\prod_{b:\langle b\rangle=\langle dd_j\rangle,0\leqslant b\leqslant dd_j}(d_jP+bz)}{\prod_{i=0}^{n}\prod_{b:\langle b\rangle=\langle dw_i\rangle,0<b\leqslant dw_i}(w_iP+bz)}\mathbf{1}_{\langle d\rangle}. \tag{4}$$

COROLLARY 1.9. (See §6) *Let*

$$k_{\mathcal{X}}=\sum_{j=0}^{m}d_j-\sum_{i=0}^{n}w_i$$

and let

$$k_f=\sum_{j=0}^{m}\lceil fd_j\rceil-\sum_{i=0}^{n}\lceil fw_i\rceil=k_{\mathcal{X}}f+\sum_{j=0}^{m}\langle -fd_j\rangle-\sum_{i=0}^{n}\langle -fw_i\rangle. \tag{5}$$

Suppose that for each non-zero $f\in F$ we have either $k_f<-1$ or

$$|\{j:d_jf\in\mathbf{Z}\}|\geqslant|\{i:w_if\in\mathbf{Z}\}|.$$

(1) *If $k_{\mathcal{X}}<-1$ then*

$$I_{\mathcal{X}}(t)=\iota_\star(z+tP+O(z^{-1}))\quad \textit{and}\quad \iota_\star J_{\mathcal{X}}(t)=I_{\mathcal{X}}(t);$$

(2) *if* $k_{\mathcal{X}}=-1$ *then*

$$I_{\mathcal{X}}(t)=\iota_\star(z+tP+s(t)\mathbf{1}_0+O(z^{-1})),$$

where $s(t)=Qe^t(\prod_{j=0}^m d_j!)/(\prod_{i=0}^n w_i!)$, *and*

$$\iota_\star(e^{s(t)/z}J_{\mathcal{X}}(t))=I_{\mathcal{X}}(t);$$

(3) *if* $k_{\mathcal{X}}=0$ *then*

$$I_{\mathcal{X}}(t)=\iota_\star(F(t)z+g(t)P+O(z^{-1}))$$

for some functions $F\colon\mathbf{C}\to\Lambda$ *and* $g\colon\mathbf{C}\to\Lambda$, *and*

$$\iota_\star J_{\mathcal{X}}(\tau(t))=\frac{I_{\mathcal{X}}(t)}{F(t)},$$

where the change of variables $\tau(t)=g(t)/F(t)$ *is invertible.*

The assumptions of Corollary 1.9 have a geometric interpretation.

PROPOSITION 1.10. (See §6) *The following conditions on* $\mathcal{X}$ *are equivalent*:

(1) $\mathcal{X}$ *is well-formed and has terminal singularities*;

(2) *for all non-zero* $f\in F$, *either* $|\{j:d_jf\in\mathbf{Z}\}|\geqslant|\{i:w_if\in\mathbf{Z}\}|$ *or*

$$\sum_{i=0}^n\langle fw_i\rangle>1+\sum_{j=0}^m\langle fd_j\rangle. \tag{6}$$

In particular, *if* $k_{\mathcal{X}}\leqslant 0$ *and* $\mathcal{X}$ *has terminal singularities*, *then the assumptions of Corollary* 1.9 *are satisfied. If* $\mathcal{X}$ *is Calabi–Yau*, *then these assumptions are equivalent to* $\mathcal{X}$ *having terminal singularities.*

Remark 1.11. We were surprised to discover the notion of terminal singularities occurring so naturally in Gromov–Witten theory.

Remark 1.12. Corollary 1.9 determines the part of the small J-function of $\mathcal{X}$ involving classes pulled back from $\mathbf{P}^{\mathbf{w}}$, and hence the part of the small quantum orbifold cohomology algebra of $\mathcal{X}$ generated by such classes.

Remark 1.13. Corollary 1.9 is an immediate consequence of a more general result, Corollary 6.2 below, which is applicable to any weighted projective complete intersection $\mathcal{X}$ with $k_{\mathcal{X}}\geqslant 0$ and which determines the part of the "big J-function" of $\mathcal{X}$ involving classes pulled back from $\mathbf{P}^{\mathbf{w}}$. The big J-function is defined in §2.3.

Remark 1.14. In dimension 3, a Calabi–Yau orbifold has terminal singularities if and only if it is smooth. Thus Corollary 1.9 applies to only 4 of the 7555 quasismooth Calabi–Yau 3-fold weighted projective hypersufaces:([3])

$$\begin{aligned} X_5 &\subset \mathbf{P}(1,1,1,1,1),\\ X_6 &\subset \mathbf{P}(1,1,1,1,2),\\ X_8 &\subset \mathbf{P}(1,1,1,1,4),\\ X_{10} &\subset \mathbf{P}(1,1,1,2,5). \end{aligned}$$

These can all be handled using methods of Givental [21] and others, by resolving the singularities of the ambient space. In dimension 4, however, there are many Gorenstein terminal quotient singularities and consequently many interesting examples. For instance,

$$X_7 \subset \mathbf{P}(1,1,1,1,1,2)$$

can be treated using Corollary 1.9 but not, to our knowledge, by existing methods.

Remark 1.15. Let $\mathcal{X}\subset\mathbf{P}^{\mathbf{w}}$ be a quasismooth hypersurface of degree $d=\sum_{i=0}^{n} w_i$. The I-function of $\mathcal{X}$ is a fundamental solution of the ordinary differential equation

$$H^{\mathrm{red}} I = 0, \quad \text{where } H = \prod_{i=0}^{n} \prod_{k=0}^{w_i-1} \left(w_i \frac{\partial}{\partial t} - k \right) - Q e^t \prod_{k=0}^{d-1} \left(d \frac{\partial}{\partial t} - k \right), \tag{7}$$

and the superscript “red” means that we are taking the main irreducible constituent: the operator obtained by removing factors that are common to both summands. It is shown in [18, Theorem 1.1] that the local system of solutions of equation (7) is $\mathrm{gr}_{n-1}^{W} R^{n-1} f_! \mathbf{R}_Y$, where $f: Y \to \mathbf{C}^\times$ is the *mirror-dual Landau–Ginzburg model*:

$$Y = \left\{ (y_0, ..., y_n, t) \in (\mathbf{C}^\times)^{n+1} \times \mathbf{C}^\times : \prod_{i=0}^{n} y_i^{w_i} = t \text{ and } \sum_{i=0}^{n} y_i = 1 \right\}.$$

This is a mirror theorem for quasismooth Calabi–Yau weighted projective hypersurfaces.

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([3]) See Gavin Brown's graded ring database `http://www.kent.ac.uk/ims/grdb/`.

with Martin Guest. In particular, Martin suggested to use Birkhoff factorization to recover quantum cohomology from the J-function; this works, but we preferred to adopt a more explicit approach here. The project owes a great deal to Alexander Givental who, directly or indirectly, taught us much of what we know about this subject.

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2. Orbifold cohomology and quantum orbifold cohomology

In this section we give an introduction to the cohomology and quantum cohomology of orbifolds following [3] and [4]. An alternative exposition can be found in [45]. We work in the algebraic category, using the term "orbifold" to mean "smooth separated Deligne–Mumford stack of finite type over **C**". Gromov–Witten theory for orbifolds was originally constructed in the symplectic setting by Chen and Ruan [13], [14]. Note that we do not require our orbifolds to be reduced (in the sense of Chen and Ruan): the stabilizer of the generic point of an orbifold is allowed to be non-trivial.

2.1. Orbifold cohomology

Let $\mathcal{X}$ be a stack. Its inertia stack $\mathcal{IX}$ is the fiber product

$$\begin{array}{ccc} \mathcal{IX} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle\Delta} \\ \mathcal{X} & \xrightarrow{\ \Delta\ } & \mathcal{X}\times\mathcal{X}, \end{array}$$

where Δ is the diagonal map. The fiber product is taken in the 2-category of stacks. One can think of a point of $\mathcal{IX}$ as a pair (x,g), where x is a point of $\mathcal{X}$ and $g\in\mathrm{Aut}_{\mathcal{X}}(x)$. There is an involution $I:\mathcal{IX}\to\mathcal{IX}$ which sends the point (x,g) to (x,g^{-1}).

The *Chen–Ruan orbifold cohomology groups* $H^{\bullet}_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$ of a Deligne–Mumford stack $\mathcal{X}$ are the cohomology groups of its inertia stack:(4)

$$H^{\bullet}_{\mathrm{orb}}(\mathcal{X};\mathbf{C})=H^{\bullet}(\mathcal{IX};\mathbf{C}).$$

(4) An introduction to the cohomology of stacks can be found in [4, §2].

If $\mathcal{X}$ is compact then there is an inner product, the *orbifold Poincaré pairing*, on orbifold cohomology defined by

$$\begin{aligned} H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})\otimes H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C}) &\longrightarrow \mathbf{C} \\ \alpha\otimes\beta &\longmapsto \int_{\mathcal{IX}} \alpha\cup I^\star(\beta). \end{aligned}$$

We denote the pairing of α and β by $(\alpha,\beta)_{\mathrm{orb}}$.

With each component $\mathcal{X}_i$ of the inertia stack $\mathcal{IX}$ we associate a rational number, the *age* of $\mathcal{X}_i$, defined as follows. Choose a geometric point (x,g) of $\mathcal{X}_i$ and write the order of $g\in\mathrm{Aut}_{\mathcal{X}}(x)$ as r. The automorphism g acts on the tangent space $T_x\mathcal{X}$, so we can write

$$T_x\mathcal{X} = \bigoplus_{0\leqslant j<r} E_j,$$

where E_j is the subspace of $T_x\mathcal{X}$ on which g acts by multiplication by $\exp(2\pi\sqrt{-1}j/r)$. The age of $\mathcal{X}_i$ is

$$\mathrm{age}(\mathcal{X}_i) = \sum_{j=0}^{r-1} \frac{j}{r}\dim E_j.$$

This is independent of the choice of the geometric point $(x,g)\in\mathcal{X}_i$.

We use these rational numbers to equip the orbifold cohomology $H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$ with a new grading: if $\alpha\in H^p(\mathcal{X}_i;\mathbf{C})\subset H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$ then the *orbifold degree* or *age-shifted degree* of α is

$$\mathrm{orbdeg}(\alpha) = p+2\,\mathrm{age}(\mathcal{X}_i).$$

Note that $(\alpha,\beta)_{\mathrm{orb}}\neq 0$ only if $\mathrm{orbdeg}\,\alpha+\mathrm{orbdeg}\,\beta=2\dim_{\mathbf{C}}\mathcal{X}$, so for a compact orbifold $\mathcal{X}$ the orbifold cohomology $H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$ is a graded inner product space.

Weighted projective space $\mathbf{P}^{\mathbf{w}}$ is the stack quotient

$$[(\mathbf{C}^{n+1}\setminus\{0\})/\mathbf{C}^\times], \tag{8}$$

where $\mathbf{C}^\times$ acts with weights $-w_0,...,-w_n$. As discussed in §1, components of the inertia stack of $\mathbf{P}^{\mathbf{w}}$ are indexed by

$$F = \{k/w_i : 0\leqslant k < w_i \text{ and } 0\leqslant i\leqslant n\}$$

via

$$\mathcal{I}\mathbf{P}^{\mathbf{w}} = \coprod_{f\in F} \mathbf{P}(V^f);$$

here

$$V^f = \{(x_0,...,x_n)\in\mathbf{C}^{n+1} : x_i = 0 \text{ unless } w_i f\in\mathbf{Z}\}$$

and $\mathbf{P}(V^f)=[(V^f\setminus\{0\})/\mathbf{C}^\times]$, so that $\mathbf{P}(V^f)$ is the locus of points of $\mathbf{P}^{\mathbf{w}}$ with isotropy group containing $\exp(2\pi\sqrt{-1}f)\in\mathbf{C}^\times$. The involution I maps the component $\mathbf{P}(V^f)$ to the component $\mathbf{P}(V^{\langle -f\rangle})$. The age of $\mathbf{P}(V^f)\subset\mathcal{I}\mathbf{P}^{\mathbf{w}}$ is $\langle -w_0f\rangle+...+\langle -w_nf\rangle$ where, as before, $\langle r\rangle$ denotes the fractional part $r-\lfloor r\rfloor$ of r.

Remark 2.1. It is logical to take the action of $\mathbf{C}^\times$ on $\mathbf{C}^{n+1}$ to have *negative* weights $-w_0, ..., -w_n$, as we now explain. One could repeat all discussions in this paper working equivariantly with respect to the (ineffective) action of the torus $\mathbf{T}^{n+1}$ on $\mathbf{P}^{\mathbf{w}}$. This action descends from an action of $\mathbf{T}^{n+1}$ on $\mathbf{C}^{n+1}$, and we should choose this action so that $H^0(\mathbf{P}^{\mathbf{w}},\mathcal{O}(1))$ is the standard representation of $\mathbf{T}^{n+1}$. This means that $\mathbf{T}^{n+1}$ acts with *negative* weights on $\mathbf{C}^{n+1}$:

$$(t_0, ..., t_n): (x_0, ..., x_n) \longmapsto (t_0^{-1}x_0, ..., t_n^{-1}x_n).$$

The action of $\mathbf{C}^\times$ in (8) is obtained from the $\mathbf{T}^{n+1}$-action on $\mathbf{C}^{n+1}$ via the map

$$\begin{aligned} \mathbf{C}^\times &\longrightarrow \mathbf{T}^{n+1}, \\ t &\longmapsto (t^{w_0}, ..., t^{w_n}), \end{aligned}$$

and so the weights of the $\mathbf{C}^\times$-action on $\mathbf{C}^{n+1}$ should be negative. To obtain the results which hold if the $\mathbf{C}^\times$-action in (8) is taken with *positive* weights $w_0, ..., w_n$, the reader should just replace the class $\mathbf{1}_f$ with the class $\mathbf{1}_{\langle -f\rangle}$ throughout §1.

Remark 2.2. One could instead define the orbifold cohomology of $\mathcal{X}$ to be the cohomology of its *cyclotomic inertia stack* constructed in [4, §3.1], or as the cohomology of its *rigidified cyclotomic inertia stack* [4, §3.4]. Geometric points of the cyclotomic inertia stack are given by representable morphisms $B\mu_r\to\mathcal{X}$. The rigidified cyclotomic inertia stack is obtained from the cyclotomic inertia stack by removing the canonical copy of μ_r from the automorphism group of each component parameterizing morphisms $B\mu_r\to\mathcal{X}$: this process is called "rigidification" [1]. From the point of view of calculation, it does not matter which definition one uses. With our definitions,

$$\mathbf{P}(V^f) = \mathbf{P}(w_{i_1}, ..., w_{i_m}),$$

where $w_{i_1}, ..., w_{i_m}$ are the weights w_j such that $w_jf\in\mathbf{Z}$. The reader who prefers the cyclotomic inertia stack—which has the advantage that its components are parameterized by representations, and one can define the age of a representation without choosing a preferred root of unity—should take

$$\mathbf{P}(V^f) = \mathbf{P}(w_{i_1}, ..., w_{i_m}),$$

but regard the index f not as the rational number j/r (in lowest terms) but as the character $\zeta\mapsto\zeta^j$ of μ_r. The reader who prefers the rigidified cyclotomic inertia stack should similarly regard f as a character of μ_r, but take

$$\mathbf{P}(V^f)=\mathbf{P}\Big(\frac{w_{i_1}}{r},\ldots,\frac{w_{i_m}}{r}\Big).$$

2.2. Ring structures on orbifold cohomology

The orbifold cup product and the quantum orbifold product are defined in terms of Gromov–Witten invariants of $\mathcal{X}$. These invariants are intersection numbers in stacks of twisted stable maps to $\mathcal{X}$.

2.2.1. Moduli stacks of twisted stable maps

Recall [4, §4] that an *n-pointed twisted curve* is a connected 1-dimensional Deligne–Mumford stack such that

- its coarse moduli space is an n-pointed pre-stable curve: a possibly-nodal curve with n distinct smooth marked points;
- it is a scheme away from marked points and nodes;
- it has cyclic quotient stack structures at marked points;
- it has *balanced* cyclic quotient stack structures at nodes: near a node, the stack is étale-locally isomorphic to

$$[(\operatorname{Spec}\mathbf{C}[x,y]/(xy))/\mu_r],$$

where $\zeta\in\mu_r$ acts as $\zeta\colon(x,y)\mapsto(\zeta x,\zeta^{-1}y)$.

A family of n-pointed twisted curves over a scheme S is a flat morphism $\pi\colon\mathcal{C}\to S$ together with a collection of n gerbes over S with disjoint embeddings into $\mathcal{C}$ such that the geometric fibers of π are n-pointed twisted curves. Note that the gerbes over S defined by the marked points need not be trivial: this will be important when we discuss evaluation maps below.

An n-pointed *twisted stable map* to $\mathcal{X}$ of genus g and degree $d\in H_2(\mathcal{X};\mathbf{Q})$ is a representable morphism $\mathcal{C}\to\mathcal{X}$ such that

- $\mathcal{C}$ is an n-pointed twisted curve;
- the coarse moduli space C of $\mathcal{C}$ has genus g;
- the induced map of coarse moduli spaces $C\to X$ is stable in the sense of Kontsevich [33];
- the push-forward $f_\star[\mathcal{C}]$ of the fundamental class of $\mathcal{C}$ is d.

A family of such objects over a scheme S is a family of twisted curves $\pi:\mathcal{C}\to S$ together with a representable morphism $\mathcal{C}\to\mathcal{X}$ such that the geometric fibers of π give n-pointed twisted stable maps to $\mathcal{X}$ of genus g and degree d. The moduli stack parameterizing such families is called the *stack of twisted stable maps to* $\mathcal{X}$. It is a proper Deligne–Mumford stack, which we denote by $\mathcal{X}_{g,n,d}$. In [3] and [4] a very similar object is denoted by $\mathcal{K}_{g,n}(\mathcal{X},\beta)$: the only difference is that Abramovich–Graber–Vistoli take the degree β to be a curve class on the coarse moduli space of $\mathcal{X}$, whereas we take d to lie in $H_2(\mathcal{X};\mathbf{Q})$. When we specialize to the case of weighted projective space we will identify degrees $d\in H_2(\mathbf{P}^{\mathbf{w}};\mathbf{Q})$ with their images under the isomorphism $H_2(\mathbf{P}^{\mathbf{w}};\mathbf{Q})\cong\mathbf{Q}$ given by cap product with $c_1(\mathcal{O}(1))$.

2.2.2. Evaluation maps

Given an n-pointed twisted stable map $f:\mathcal{C}\to\mathcal{X}$, each marked point x_i determines a geometric point $(f(x_i),g)$ of the inertia stack $\mathcal{IX}$, where g is defined as follows. Near x_i, $\mathcal{C}$ is isomorphic to $[\mathbf{C}/\mu_r]$ and since f is representable it determines an injective homomorphism $\mu_r\to\mathrm{Aut}_{\mathcal{X}}(f(x_i))$. We work over $\mathbf{C}$, so we have a preferred generator $\exp(2\pi\sqrt{-1}/r)$ for μ_r. The automorphism g is the image of this generator in $\mathrm{Aut}_{\mathcal{X}}(f(x_i))$. Thus each marked point gives an evaluation map to $\mathcal{IX}$ defined on geometric points of $\mathcal{X}_{g,n,d}$.

These maps do *not* in general assemble to give maps of stacks $\mathcal{X}_{g,n,d}\to\mathcal{IX}$. This is because things can go wrong in families: given a family

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle\pi}\downarrow & & \\ S & & \end{array}$$

of twisted stable maps, each marked point determines a μ_r-gerbe over S (for some r) and this gerbe will map to the inertia stack only if it is trivial. But, as is explained carefully in [4], there *are* evaluation maps to the rigidified cyclotomic inertia stack and one can use this to define push-forwards

$$(\mathrm{ev}_i)_\star: H^\bullet(\mathcal{X}_{g,n,d};\mathbf{C}) \longrightarrow H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$$

and pull-backs

$$(\mathrm{ev}_i)^\star: H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C}) \longrightarrow H^\bullet(\mathcal{X}_{g,n,d};\mathbf{C})$$

which behave as if evaluation maps $\mathrm{ev}_i:\mathcal{X}_{g,n,d}\to\mathcal{IX}$ existed. We will write as if the maps ev_i themselves existed, referring to "the image of ev_i", etc. This is abuse of language, but no ambiguity should result.

2.2.3. Gromov–Witten invariants

The stack $\mathcal{X}_{g,n,d}$ can be equipped [4, §4.5] with a virtual fundamental class in

$$H_\bullet(\mathcal{X}_{g,n,d};\mathbf{C}).$$

In general, $\mathcal{X}_{g,n,d}$ is disconnected and its virtual dimension—the homological degree of the virtual fundamental class—is different on different components. On the substack $\mathcal{X}_{g,n,d}^{i_1,\dots,i_n}$ of twisted stable maps such that, for each $m\in\{1,2,\dots,n\}$, the image of ev_m lands in the component $\mathcal{X}_{i_m}$ of the inertia stack, the *real* virtual dimension is

$$2n+(2-2g)(\dim_{\mathbf{C}}\mathcal{X}-3)-2K_{\mathcal{X}}(d)-2\sum_{m=1}^{n}\mathrm{age}(\mathcal{X}_{i_m}). \tag{9}$$

We will write $(\mathbf{P}^{\mathbf{w}})_{g,n,d}^{f_1,\dots,f_n}$ for the substack of $\mathbf{P}^{\mathbf{w}}_{g,n,d}$ consisting of twisted stable maps such that, for each $m\in\{1,2,\dots,n\}$, the mth marked point maps to the component $\mathbf{P}(V^{f_m})$ of $\mathcal{I}\mathbf{P}^{\mathbf{w}}$, and will denote the virtual fundamental class of $\mathbf{P}^{\mathbf{w}}_{g,n,d}$ by $\mathbf{1}^{\mathrm{vir}}_{g,n,d}$.

There are line bundles

$$L_i\longrightarrow\mathcal{X}_{g,n,d},\quad i\in\{1,2,\dots,n\},$$

called *universal cotangent lines*, such that the fiber of L_i at the stable map $f:\mathcal{C}\to\mathcal{X}$ is the cotangent line to the coarse moduli space of $\mathcal{C}$ at the ith marked point. We denote the first Chern class of L_i by ψ_i. There is a canonical map from $\mathcal{X}_{g,n,d}$ to the moduli stack $X_{g,n,d}$ of stable maps to the coarse moduli space X of $\mathcal{X}$; the bundle L_i is the pull-back to $\mathcal{X}_{g,n,d}$ of the ith universal cotangent line bundle on $X_{g,n,d}$.

Gromov–Witten invariants are intersection numbers of the form

$$\int_{\mathcal{X}^{\mathrm{vir}}_{g,n,d}}\prod_{i=1}^{n}\mathrm{ev}_i^\star(\alpha_i)\cdot\psi_i^{k_i}, \tag{10}$$

where $\alpha_1,\dots,\alpha_n\in H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$, $k_1,\dots,k_n$ are non-negative integers and the integral means cap product with the virtual fundamental class. If any of the k_i are non-zero then (10) is called a *gravitational descendant*. We will use correlator notation for Gromov–Witten invariants, writing (10) as

$$\langle\alpha_1\psi_1^{k_1},\dots,\alpha_n\psi_n^{k_n}\rangle^{\mathcal{X}}_{g,n,d}.$$

Remark 2.3. One could avoid the complications caused by the non-existence of the maps ev_i by defining orbifold cohomology in terms of the rigidified cyclotomic inertia stack: evaluation maps to this flavour of inertia stack certainly exist. Or one could replace $\mathcal{X}_{g,n,d}$ with a moduli stack of stable maps with sections to all gerbes. We will do neither of these things. In each case there is a price to pay: to get the correct Gromov–Witten invariants—the invariants which participate in the definition of an associative quantum product—one must rescale all virtual fundamental classes by rational numbers depending on the stack structures at marked points. This is described in detail in [4, §6.1.3] and [45].

2.2.4. The orbifold cohomology ring

The Chen–Ruan orbifold cup product $\underset{\mathrm{CR}}{\cup}$ is defined by

$$\big(\alpha \underset{\mathrm{CR}}{\cup} \beta, \gamma\big)_{\mathrm{orb}} = \langle \alpha, \beta, \gamma \rangle^{\mathcal{X}}_{0,3,0}.$$

It gives a super-commutative and associative ring structure on orbifold cohomology, called the *orbifold cohomology ring*. As indicated in §1, unless otherwise stated, all products of orbifold cohomology classes are taken using this ring structure.

2.2.5. Quantum orbifold cohomology

Quantum orbifold cohomology is a family of Λ-algebra structures on $H^{\bullet}_{\mathrm{orb}}(\mathcal{X};\Lambda)$, where Λ is an appropriate Novikov ring, defined by

$$(\alpha \bullet_\tau \beta, \gamma)_{\mathrm{orb}} = \sum_d \sum_{n\geqslant 0} \frac{Q^d}{n!} \langle \alpha, \beta, \gamma, \tau, \tau, ..., \tau \rangle^{\mathcal{X}}_{0,n+3,d}. \tag{11}$$

Here the first sum is over degrees d of effective possibly-stacky curves in $\mathcal{X}$, and Q^d is the element of the Novikov ring corresponding to the degree $d \in H_2(\mathcal{X};\mathbf{Q})$. In the case $\mathcal{X} = \mathbf{P}^{\mathbf{w}}$, where $H_2(\mathcal{X};\mathbf{Q})$ is 1-dimensional and

$$\Lambda = \mathbf{C}[[Q^{1/\mathrm{lcm}(w_0,\dots,w_n)}]],$$

the element of Λ corresponding to $d \in H_2(\mathcal{X};\mathbf{Q})$ is $Q^{\int_d c_1(\mathcal{O}(1))}$. To interpret (11), choose a basis $\phi_1, ..., \phi_N$ for $H^{\bullet}_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$ and set

$$\tau = \tau^1 \phi_1 + ... + \tau^N \phi_N.$$

Then the right-hand side of (11) is a formal power series in $\tau^1, ..., \tau^N$ and so (11) defines a family of product structures $\bullet_\tau$ parameterized by a formal neighbourhood of zero in $H^{\bullet}_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$. The Witten–Dijkgraaf–Verlinde–Verlinde equations [4], [13] imply that this is a family of associative products.

Small quantum orbifold cohomology is the family $\circ_\tau$ of Λ-algebra structures on $H^\bullet_{\mathrm{orb}}(\mathcal{X};\Lambda)$ defined by restricting the parameter τ in $\bullet_\tau$ to lie in a formal neighbourhood of zero in $H^2(\mathcal{X};\mathbf{C})\subset H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$. This family is entirely determined by its element at $\tau=0$, as follows from the divisor equation [4, Theorem 8.3.1]:

$$\langle\alpha_1,...,\alpha_n,\gamma\rangle^{\mathcal{X}}_{0,n+1,d}=\left(\int_d\gamma\right)\langle\alpha_1,...,\alpha_n\rangle^{\mathcal{X}}_{0,n,d}$$

whenever $\gamma\in H^2(\mathcal{X};\mathbf{C})$ and either $d\neq 0$ or $n\geqslant 3$. For example in the case $\mathcal{X}=\mathbf{P}^{\mathbf{w}}$, if P is the first Chern class of $\mathcal{O}(1)$ and t lies in a formal neighbourhood of zero in $\mathbf{C}$, then

$$(\alpha\circ_{tP}\beta,\gamma)_{\mathrm{orb}}=\sum_{d\geqslant 0}Q^de^{dt}\langle\alpha,\beta,\gamma\rangle^{\mathbf{P}^{\mathbf{w}}}_{0,3,d}. \tag{12}$$

Analogous statements hold for general $\mathcal{X}$.

2.3. The *J*-function

Let us write

$$\langle\!\langle\alpha_1\psi_1^{i_1},...,\alpha_m\psi_m^{i_m}\rangle\!\rangle^{\mathcal{X}}_\tau=\sum_d\sum_{n\geqslant 0}\frac{Q^d}{n!}\langle\alpha_1\psi_1^{i_1},...,\alpha_m\psi_m^{i_m},\tau,\tau,...,\tau\rangle^{\mathcal{X}}_{0,m+n,d},$$

so that

$$(\alpha\bullet_\tau\beta,\gamma)_{\mathrm{orb}}=\langle\!\langle\alpha,\beta,\gamma\rangle\!\rangle^{\mathcal{X}}_\tau.$$

The *J-function of* $\mathcal{X}$ is

$$\mathbf{J}_{\mathcal{X}}(\tau)=z+\tau+\left\langle\!\!\left\langle\frac{\phi^\varepsilon}{z-\psi_1}\right\rangle\!\!\right\rangle^{\mathcal{X}}_\tau\phi_\varepsilon, \tag{13}$$

where $\phi^1,...,\phi^N$ is the basis for $H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$ such that $(\phi^i,\phi_j)_{\mathrm{orb}}=\delta^i{}_j$; here and henceforth we use the summation convention, summing over repeated indices, and expand $(z-\psi_1)^{-1}$ as a power series in z^{-1}. The J-function is a function of $\tau\in H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$ taking values in $H^\bullet_{\mathrm{orb}}(\mathcal{X};\Lambda)\otimes\mathbf{C}(\!(z^{-1})\!)$, defined for τ in a formal neighbourhood of zero. In other words, just as for (11), we regard the right-hand side of (13) as a formal power series in the coordinates $\tau^1,...,\tau^N$ of τ. To distinguish it from the small J-function of $\mathcal{X}$ defined below, we will sometimes refer to $\mathbf{J}_{\mathcal{X}}$ as the *big J-function* of $\mathcal{X}$.

LEMMA 2.4. *The J-function satisfies*

$$z\frac{\partial}{\partial\tau^i}\frac{\partial}{\partial\tau^j}\mathbf{J}_{\mathcal{X}}(\tau)=c(\tau)_{ij}{}^{\mu}\frac{\partial}{\partial\tau^\mu}\mathbf{J}_{\mathcal{X}}(\tau), \tag{14}$$

where

$$\phi_i\bullet_\tau\phi_j=c(\tau)_{ij}{}^{\mu}\phi_\mu.$$

Proof. This follows from the *topological recursion relations*

$$\langle\!\langle \alpha\psi_1^{a+1}, \beta\psi_2^b, \gamma\psi_3^c\rangle\!\rangle_\tau^{\mathcal{X}} = \langle\!\langle \alpha\psi_1^a, \phi_\mu\rangle\!\rangle_\tau^{\mathcal{X}} \langle\!\langle \phi^\mu, \beta\psi_2^b, \gamma\psi_3^c\rangle\!\rangle_\tau^{\mathcal{X}}, \quad a,b,c \geqslant 0,$$

exactly as in [42]. A proof of the topological recursion relations is sketched in [45]. For

$$\begin{aligned} z\frac{\partial}{\partial\tau^i}\frac{\partial}{\partial\tau^j}\mathbf{J}_{\mathcal{X}}(\tau) &= \sum_{m\geqslant 0}\frac{1}{z^m}\langle\!\langle \phi^\varepsilon\psi_1^m, \phi_i, \phi_j\rangle\!\rangle_\tau^{\mathcal{X}}\phi_\varepsilon \\ &= \langle\!\langle \phi^\varepsilon, \phi_i, \phi_j\rangle\!\rangle_\tau^{\mathcal{X}}\phi_\varepsilon + \sum_{m\geqslant 1}\frac{1}{z^m}\langle\!\langle \phi^\varepsilon\psi_1^{m-1}, \phi_\mu\rangle\!\rangle_\tau^{\mathcal{X}}\langle\!\langle \phi^\mu, \phi_i, \phi_j\rangle\!\rangle_\tau^{\mathcal{X}}\phi_\varepsilon \\ &= \langle\!\langle \phi_i, \phi_j, \phi^\mu\rangle\!\rangle_\tau^{\mathcal{X}}\frac{\partial}{\partial\tau^\mu}\mathbf{J}_{\mathcal{X}}(\tau) \end{aligned}$$

and

$$\phi_i \bullet_\tau \phi_j = \langle\!\langle \phi_i, \phi_j, \phi^\mu\rangle\!\rangle_\tau^{\mathcal{X}}\phi_\mu. \qquad \square$$

The J-function determines the quantum orbifold product, as

$$z\frac{\partial}{\partial\tau^i}\frac{\partial}{\partial\tau^j}\mathbf{J}_{\mathcal{X}}(\tau) = \phi_i \bullet_\tau \phi_j + O(z^{-1}). \tag{15}$$

2.3.1. The small J-function

The small J-function $J_{\mathcal{X}}(\tau)$ is obtained from the J-function $\mathbf{J}_{\mathcal{X}}(\tau)$ by restricting τ to lie in a formal neighbourhood of zero in $H^2(\mathcal{X};\mathbf{C}) \subset H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$. In the case of weighted projective space, we regard the small J-function as being defined on a formal neighbourhood of zero in $\mathbf{C}$, setting

$$J_{\mathbf{P}^{\mathbf{w}}}(t) = \mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(tP).$$

LEMMA 2.5. *We have*

$$J_{\mathbf{P}^{\mathbf{w}}}(t) = ze^{Pt/z}\left(1 + \sum_{d>0} Q^d e^{dt}\left\langle \frac{\phi^\varepsilon}{z(z-\psi_1)}\right\rangle_{0,1,d}^{\mathbf{P}^{\mathbf{w}}}\phi_\varepsilon\right).$$

Proof. This follows from the divisor equation [4, Theorem 8.3.1]:

$$\begin{aligned} \langle \alpha_1\psi_1^{i_1}, \dots, \alpha_n\psi_n^{i_n}, \gamma\rangle_{0,n+1,d}^{\mathcal{X}} &= \left(\int_d \gamma\right)\langle \alpha_1\psi_1^{i_1}, \dots, \alpha_n\psi_n^{i_n}\rangle_{0,n,d}^{\mathcal{X}} \\ &\quad + \sum_{j=1}^n \langle \alpha_1\psi_1^{i_1}, \dots, (\alpha_j\gamma)\psi_j^{i_j-1}, \dots, \alpha_n\psi_n^{i_n}\rangle_{0,n,d}^{\mathcal{X}} \end{aligned}$$

whenever $\gamma \in H^2(\mathcal{X};\mathbf{C})$ and either $d \neq 0$ or $n \geqslant 3$. We have

$$J_{\mathbf{P}^{\mathbf{w}}}(t) = z + tP + \sum_{\substack{d:d\geqslant 0\\ \langle d\rangle\in F}}\sum_{n\geqslant 0}\sum_{m\geqslant 0}\frac{Q^d t^n}{n!\,z^{m+1}}\langle \phi^\varepsilon\psi_1^m, P, P, \dots, P\rangle_{0,n+1,d}^{\mathbf{P}^{\mathbf{w}}}\phi_\varepsilon. \tag{16}$$

Now, using the divisor equation,

$$\begin{aligned}\sum_{\substack{d:d>0\\ \langle d\rangle\in F}}\sum_{n\geqslant 0}\sum_{m\geqslant 0}\frac{Q^d t^n}{n!z^{m+1}}\langle\phi^\varepsilon\psi_1^m,P,P,...,P\rangle_{0,n+1,d}^{\mathbf{P^w}}\phi_\varepsilon \\ &= \sum_{\substack{d:d>0\\ \langle d\rangle\in F}}\sum_{n\geqslant 0}\sum_{m\geqslant 0}\frac{Q^d t^n}{n!z^{m+1}}\left\langle\phi^\varepsilon\psi_1^m\left(\frac{P}{z}+d\right)^n\right\rangle_{0,1,d}^{\mathbf{P^w}}\phi_\varepsilon \\ &= \sum_{\substack{d:d>0\\ \langle d\rangle\in F}} Q^d\left\langle\frac{e^{Pt/z}e^{dt}\phi^\varepsilon}{z-\psi_1}\right\rangle_{0,1,d}^{\mathbf{P^w}}\phi_\varepsilon \\ &= ze^{Pt/z}\sum_{\substack{d:d>0\\ \langle d\rangle\in F}} Q^d e^{dt}\left\langle\frac{\phi^\varepsilon}{z(z-\psi_1)}\right\rangle_{0,1,d}^{\mathbf{P^w}}\phi_\varepsilon.\end{aligned} \tag{17}$$

The terms in (16) which are not in (17) are

$$z+tP+\sum_{n\geqslant 2}\sum_{m\geqslant 0}\frac{t^n}{n!z^{m+1}}\langle\phi^\varepsilon\psi_1^m,P,P,...,P\rangle_{0,n+1,0}^{\mathbf{P^w}}\phi_\varepsilon.$$

Using the divisor equation again, this is

$$z+tP+\sum_{n\geqslant 2}\sum_{m\geqslant 0}\frac{t^n}{n!z^{m+1}}\langle\phi^\varepsilon P^{n-2}\psi_1^{m-n+2},P,P\rangle_{0,3,0}^{\mathbf{P^w}}\phi_\varepsilon \tag{18}$$

and, since L_1 is trivial on $(\mathbf{P^w})_{0,3,0}$, the summand vanishes unless $m=n-2$. So (18) is

$$z+tP+\sum_{n\geqslant 2}\frac{t^n}{n!z^{n-1}}\big(\phi^\varepsilon P^{n-2}\mathbin{\underset{\mathrm{CR}}{\cup}}P,P\big)_{\mathrm{orb}}\phi_\varepsilon = z+tP+\sum_{n\geqslant 2}\frac{t^nP^n}{n!z^{n-1}} = ze^{Pt/z}.$$

Combining this with (17) gives

$$J_{\mathbf{P^w}}(t) = ze^{Pt/z}\left(1+\sum_{d>0}Q^d e^{dt}\left\langle\frac{\phi^\varepsilon}{z(z-\psi_1)}\right\rangle_{0,1,d}^{\mathbf{P^w}}\phi_\varepsilon\right). \qquad \square$$

From (15), we see that the small quantum cohomology algebra is determined by

$$\frac{\partial\mathbf{J}_{\mathcal{X}}}{\partial\tau^j}(\tau)\bigg|_{\tau\in H^2(\mathcal{X};\mathbf{C})\subset H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})}, \quad j\in\{1,2,...,N\}.$$

Let $v,w\in H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$ and let ∇_v denote the directional derivative along v, so that

$$\nabla_v\mathbf{J}(\tau) = v^\alpha\frac{\partial\mathbf{J}}{\partial\tau^\alpha}(\tau),$$

where $v=v^1\phi_1+...+v^N\phi_N$ and $\tau=\tau^1\phi_1+...+\tau^N\phi_N$. From (14),

$$z\nabla_v\nabla_w\mathbf{J}_{\mathcal{X}}(\tau)=\nabla_{v\bullet_\tau w}\mathbf{J}(\tau)=v\bullet_\tau w+O(z^{-1}).$$

Taking $\tau\in H^2(\mathcal{X};\mathbf{C})\subset H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})$ gives

$$z\nabla_v\nabla_w\mathbf{J}_{\mathcal{X}}(\tau)=\nabla_{v\circ_\tau w}\mathbf{J}(\tau)=v\circ_\tau w+O(z^{-1}),\tag{19}$$

and it follows that the small J-function determines the subalgebra of the small quantum orbifold cohomology algebra which is generated by $H^2(\mathcal{X};\mathbf{C})$. We will see below that for weighted projective spaces this subalgebra is the whole of the small quantum orbifold cohomology algebra.

3. S^1-equivariant Floer cohomology and quantum cohomology

Floer cohomology should capture information about "semi-infinite cycles" in the free loop space $L\mathbf{P}^\mathbf{w}$. Giving a rigorous definition is not easy, particularly if one wants to define a theory which applies beyond the toric setting, and we will not attempt to do so here: various approaches to the problem can be found in [5], [17], [29], [31] and [46]. Instead we will indicate roughly how one might define Floer cohomology groups $HF^\bullet(L\mathbf{P}^\mathbf{w})$ in terms of Morse theory on a covering space of $L\mathbf{P}^\mathbf{w}$, and explain how to compute them. We argue mainly by analogy with Morse theory on finite-dimensional manifolds. An excellent (and rigorous) introduction to finite-dimensional Morse theory from a compatible point of view can be found in [7]. The material in this section provides motivation and context for the rest of the paper, but most of it is not rigorous mathematics: we do not discuss the topologies on many of the spaces we consider, for example, and questions of transversality and compactness are systematically ignored. More importantly, several key steps in the argument are plausible analogies rather than rigorous proof. None of the material in this section is logically necessary, and so the reader may want to skip directly to §4.

3.1. Loops in $\mathbf{P}^\mathbf{w}$

Lupercio and Uribe have defined the loop groupoid of any topological groupoid [36]. As $\mathbf{P}^\mathbf{w}$ can be represented by a proper étale Lie groupoid [39], this defines the *loop space* $L\mathbf{P}^\mathbf{w}$. Let $\mathcal{U}=\mathbf{C}^{n+1}\setminus\{0\}$. The Lupercio–Uribe definition can be rephrased in the following equivalent ways:

(A) A loop in $\mathbf{P}^\mathbf{w}$ is a pair (γ,h) where $\gamma:[0,1]\to\mathcal{U}$ is a continuous map and $h\in\mathbf{C}^\times$ satisfies $\gamma(1)=h\gamma(0)$; two loops (γ_1,h_1) and (γ_2,h_2) are isomorphic if and only if there exists a map $k:[0,1]\to\mathbf{C}^\times$ with $\gamma_2(x)=k(x)\gamma_1(x)$ for all $x\in[0,1]$ and $h_2=k(1)h_1k(0)^{-1}$;

(B) A loop in $\mathbf{P}^{\mathbf{w}}$ is a diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & \mathcal{U} \\ \big\downarrow & & \\ S^1, & & \end{array} \tag{20}$$

where $P \to S^1$ is a principal $\mathbf{C}^\times$-bundle and f is a $\mathbf{C}^\times$-equivariant continuous map; an isomorphism between the loops

$$\begin{array}{ccc} P_1 & \xrightarrow{f_1} & \mathcal{U} \\ \big\downarrow & & \\ S^1 & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} P_2 & \xrightarrow{f_2} & \mathcal{U} \\ \big\downarrow & & \\ S^1 & & \end{array}$$

is an isomorphism $\phi: P_1 \to P_2$ of principal $\mathbf{C}^\times$-bundles such that the diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{f_1} & \mathcal{U} \\ \big\downarrow & \overset{\phi}{\searrow} & \big\uparrow f_2 \\ S^1 & \longleftarrow & P_2 \end{array}$$

commutes.

The loop space $L\mathbf{P}^{\mathbf{w}}$ can be thought of as an infinite-dimensional Kähler orbifold, as follows. A tangent vector to $L\mathbf{P}^{\mathbf{w}}$ at (γ, h) is a vector field $v: [0,1] \to T\mathcal{U}$ along γ such that $v(1) = h_\star v(0)$. Weighted projective space is a Kähler orbifold: let $\omega \in \Omega^2(\mathbf{P}^{\mathbf{w}})$ be the Kähler form on $\mathbf{P}^{\mathbf{w}}$ obtained by symplectic reduction from the standard Kähler form on $\mathcal{U}$, so that ω represents the class $P \in H^2(\mathbf{P}^{\mathbf{w}}; \mathbf{C})$, and let g be the corresponding Kähler metric on $\mathbf{P}^{\mathbf{w}}$. These structures induce a Kähler form on $L\mathbf{P}^{\mathbf{w}}$:

$$\Omega(u, v) = \int_0^1 \omega(u(t), v(t))\, dt, \quad u, v \in T_{(\gamma, h)} L\mathbf{P}^{\mathbf{w}},$$

and a Riemannian metric on $L\mathbf{P}^{\mathbf{w}}$:

$$G(u, v) = \int_0^1 g(u(t), v(t))\, dt, \quad u, v \in T_{(\gamma, h)} L\mathbf{P}^{\mathbf{w}}.$$

3.2. The symplectic action functional

There is an S^1-action on $L\mathbf{P}^{\mathbf{w}}$ given by rotation of loops (see [36]). This action is locally Hamiltonian with respect to the Kähler form Ω. The moment map $m: L\mathbf{P}^{\mathbf{w}} \to S^1$ for this action, which is called the *symplectic action functional*, is given as follows. Every loop in $\mathbf{P}^{\mathbf{w}}$ is the boundary value of a representable continuous map $f\colon D\to\mathbf{P}^{\mathbf{w}}$ from a possibly-stacky([5]) disc D. The integral $\int_D f^\star\omega$ does not depend unambiguously on the loop γ, because there are many possible choices of D and f, but the ambiguity in its value lies in the group

$$\Pi=\left\{\int_S g^\star\omega : S \text{ is a possibly-stacky sphere, } g\colon S\to\mathbf{P}^{\mathbf{w}} \text{ is representable and continuous}\right\}.$$

Since $\mathbf{R}/\Pi\cong S^1$, the map

$$m:\gamma\longmapsto\int_D\gamma$$

defines a circle-valued function on $L\mathbf{P}^{\mathbf{w}}$. This is the symplectic action functional. Pulling back the universal cover $\mathbf{R}\to S^1$ along the map $m\colon L\mathbf{P}^{\mathbf{w}}\to S^1$ defines a covering

$$p\colon\widetilde{L\mathbf{P}^{\mathbf{w}}}\to L\mathbf{P}^{\mathbf{w}}$$

and a function $\mu\colon\widetilde{L\mathbf{P}^{\mathbf{w}}}\to\mathbf{R}$. We can regard the covering $\widetilde{L\mathbf{P}^{\mathbf{w}}}$ as consisting of pairs $(\gamma,[D])$, where γ is a loop in $\mathbf{P}^{\mathbf{w}}$ and $[D]$ is a relative homology class of possibly-stacky discs D with boundary γ. The function μ gives the area of the disc D:

$$\mu\colon(\gamma,[D])\longmapsto\int_D\gamma.$$

We will study the Morse theory of μ.

Remark 3.1. When applying this argument to other orbifolds $\mathcal{X}$, one should consider only the subset of $L\mathcal{X}$ consisting of loops which bound possibly-stacky discs. This condition does not arise here, as every loop in $\mathbf{P}^{\mathbf{w}}$ is the boundary value of a representable continuous map $f\colon D\to\mathbf{P}^{\mathbf{w}}$ from a disc D with one possibly-stacky point at the origin. To see this, observe that every loop in $\mathbf{P}^{\mathbf{w}}$ is homotopic to a loop which lands entirely within the image of a coordinate chart

$$\{[z_0:z_1:\ldots:z_n]\in\mathbf{P}^{\mathbf{w}}:z_i=1\}\quad\text{for some } i,$$

and consequently (because these coordinate charts are contractible) that every loop in $\mathbf{P}^{\mathbf{w}}$ is homotopic to a loop with image contained in one of the points

$$\{[z_0:z_1:\ldots:z_n]\in\mathbf{P}^{\mathbf{w}}:z_j=0 \text{ for } j\neq i \text{ and } z_i=1\}.$$

Such loops evidently bound representable continuous maps $f\colon D\to\mathbf{P}^{\mathbf{w}}$, where D is a disc with one possibly-stacky point at the origin, and the assertion follows.

([5]) Let Σ be a Riemann surface, which may have a boundary. By a *possibly-stacky* Σ we mean a reduced orbifold with coarse moduli space equal to Σ and no stacky points on the boundary.

3.3. Morse theory

As motivation, let us recall some key points from [7]. Let (X, g) be a finite-dimensional Riemannian manifold and $f: X \to \mathbf{R}$ a Morse–Bott function. Let $X_1^{\mathrm{cr}}, ..., X_r^{\mathrm{cr}}$ denote the components of the critical set of f, $X^{\mathrm{cr}} = \coprod_i X_i^{\mathrm{cr}}$, and let $\mathcal{M}$ be the set of descending gradient trajectories of f (i.e. of integral curves $\gamma: \mathbf{R} \to X$ for the vector field $-\mathrm{grad}(f)$). Under reasonable conditions on f and g, $\mathcal{M}$ is a smooth finite-dimensional manifold with a natural compactification $\overline{\mathcal{M}}$. A point of $\overline{\mathcal{M}}$ consists of a sequence of gradient trajectories $\gamma_1(t), ..., \gamma_m(t)$, where $m \geqslant 1$, such that

$$\lim_{t \to \infty} \gamma_i(t) = \lim_{t \to -\infty} \gamma_{i+1}(t), \quad 1 \leqslant i < m.$$

There is an action of $\mathbf{R}$ on $\mathcal{M}$ by "time translation":

$$\begin{aligned} \mathbf{R} \times \mathcal{M} &\longrightarrow \mathcal{M}, \\ (s, \gamma(t)) &\longmapsto \gamma(s+t), \end{aligned}$$

and this extends to give an action on $\overline{\mathcal{M}}$. Let $\gamma: \mathbf{R} \to X$ be a descending gradient trajectory. As $t \to \mp\infty$, $\gamma(t)$ approaches critical points of f; this defines upper and lower endpoint maps $u: \overline{\mathcal{M}}/\mathbf{R} \to X^{\mathrm{cr}}$ and $l: \overline{\mathcal{M}}/\mathbf{R} \to X^{\mathrm{cr}}$.

Chains in the Morse–Bott complex of f are differential forms on the critical set:

$$C_\bullet^{\mathrm{MB}} = \bigoplus_{i=1}^r \Omega^\bullet(X_i^{\mathrm{cr}}),$$

where the grading on $\Omega^\bullet(X_i^{\mathrm{cr}})$ is shifted by an integer which depends on the component X_i^{cr} (see [7]). Consider the diagram

$$X^{\mathrm{cr}} \xleftarrow{\ u\ } \overline{\mathcal{M}}/\mathbf{R} \xrightarrow{\ l\ } X^{\mathrm{cr}}.$$

The differential in the Morse–Bott complex is the sum of the de Rham differential and a contribution from the space $\overline{\mathcal{M}}/\mathbf{R}$ of gradient trajectories:

$$d_{\mathrm{MB}} \alpha = d_{\mathrm{de\,Rham}} \alpha + (-1)^j u_\star l^\star \alpha \quad \text{for } \alpha \in \Omega^j(X^{\mathrm{cr}}).$$

The homology of the complex $(C_\bullet^{\mathrm{MB}}, d_{\mathrm{MB}})$ is the cohomology of X:

$$H_\bullet(C_\bullet^{\mathrm{MB}}, d_{\mathrm{MB}}) \cong H^\bullet(X; \mathbf{R}). \tag{21}$$

Let $\alpha \in \Omega^\bullet(X_i^{\mathrm{cr}}) \subset C_\bullet^{\mathrm{MB}}$ be such that $d_{\mathrm{MB}} \alpha = 0$, and let A be a generic cycle in X_i^{cr} which is Poincaré-dual to α. Under the isomorphism (21), the class $[\alpha] \in H_\bullet(C_\bullet^{\mathrm{MB}}, d_{\mathrm{MB}})$ maps to the cohomology class on X which is Poincaré-dual to the cycle(6) swept out by gradient trajectories that end on A. So, roughly speaking, $\alpha \in \Omega^\bullet(X_i^{\mathrm{cr}})$ represents the cohomology class dual to the cycle given by upward gradient flow from $A \subset X_i^{\mathrm{cr}}$.

Furthermore if X is a finite-dimensional manifold with S^1-action, g is an S^1-invariant Riemannian metric on X, and $f: X \to \mathbf{R}$ is an S^1-invariant Morse–Bott function then, under reasonable conditions on f and g, we can compute the S^1-equivariant cohomology of X using the S^1-equivariant Morse–Bott complex of f. If we define chain groups

$$C_\bullet^{S^1,\mathrm{MB}} = \bigoplus_{i=1}^{r} \Omega^\bullet(X_i^{\mathrm{cr}}) \otimes H_{S^1}^\bullet(\mathrm{pt}; \mathbf{R}),$$

with the grading shifted as before, and use the differential d_{MB} as before, then

$$H_\bullet(C_\bullet^{S^1,\mathrm{MB}}, d_{\mathrm{MB}}) \cong H_{S^1}^\bullet(X; \mathbf{R}).$$

3.4. Floer cohomology and S^1-equivariant Floer cohomology

Recall our setup

$$\begin{array}{ccc} \widetilde{L\mathbf{P}^{\mathbf{w}}} & \xrightarrow{\mu} & \mathbf{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle t \mapsto \exp(2\pi\sqrt{-1}\,t)} \\ L\mathbf{P}^{\mathbf{w}} & \xrightarrow{m} & S^1, \end{array}$$

where m is the moment map for the S^1-action on $L\mathbf{P}^{\mathbf{w}}$ given by loop rotation. We define the *Floer cohomology* of $L\mathbf{P}^{\mathbf{w}}$ to be the homology of the Morse–Bott complex of μ. We will describe the critical set of μ in a moment. Gradient trajectories of μ, with respect to the induced Kähler metric $p^\star G$ on $\widetilde{L\mathbf{P}^{\mathbf{w}}}$, give paths of loops in $\mathbf{P}^{\mathbf{w}}$ which sweep out holomorphic cylinders. It is this—the link between Morse-theoretic gradient trajectories and holomorphic curves—which connects Floer cohomology to Gromov–Witten theory.

The critical set of μ is a covering space of the critical set of m. As m is a moment map, the critical set of m coincides with the S^1-fixed set on $L\mathbf{P}^{\mathbf{w}}$. This S^1-fixed set is canonically isomorphic to the inertia stack $\mathcal{I}\mathbf{P}^{\mathbf{w}}$ (see [36]) and so the critical set of μ is a covering space of $\mathcal{I}\mathbf{P}^{\mathbf{w}}$. The deck transformation group of this covering, and of the covering $p: \widetilde{L\mathbf{P}^{\mathbf{w}}} \to L\mathbf{P}^{\mathbf{w}}$, is $\mathbf{Z}$: let $\mathbf{C}[Q, Q^{-1}]$ denote the group ring of the group of

(6) More precisely, the Poincaré-dual cycle is the *closure* of the locus

$$\bigcup \{\gamma(t) : t \in \mathbf{R} \text{ and } \gamma \text{ is a gradient trajectory such that } l(\gamma) \in A\}.$$

deck transformations. A deck transformation changes the value of the function μ by an integer, and we have

$$(\text{critical set of } \mu)\cap\mu^{-1}(r)=\begin{cases}\text{a copy of } \mathbf{P}(V^{\langle r\rangle}), & \text{if } \langle r\rangle\in F,\\ \varnothing, & \text{otherwise.}\end{cases}$$

We will call the component of the critical set of μ which lies in $\mu^{-1}(r)$ *the component of the critical set at level* r. A point in the component of the critical set at level r is a pair $(\gamma,[D])$ where γ is an S^1-fixed loop in $\mathbf{P}^{\mathbf{w}}$ and $[D]$ is the homology class of a possibly-stacky disc bounding γ and having area r. As γ here is an S^1-fixed loop, $[D]$ is in fact the homology class of a possibly-stacky *sphere* in $\mathbf{P}^{\mathbf{w}}$ of area r.

The chain groups in the Morse–Bott complex for μ should be

$$C_\bullet^{\mathrm{MB}}=\biggl(\bigoplus_{f\in F}Q^f\Omega^\bullet(\mathbf{P}(V^f))\biggr)\otimes\mathbf{C}[Q,Q^{-1}].$$

Here we introduced fractional(7) powers Q^f so that an element $\alpha Q^r\in C_\bullet^{\mathrm{MB}}$, where $\alpha\in\Omega^\bullet\big(\mathbf{P}\big(V^f\big)\big)$ and $r\in\mathbf{Q}$, is a differential form α on the component of the critical set at level r. The grading on the chain groups is defined by

$$\deg(\alpha Q^r)=\deg\alpha+\operatorname{age}\mathbf{P}(V^f)+(w_0+\ldots+w_n)r.$$

Note that $\deg(\alpha Q^r)\in\mathbf{Z}$. As before, the differential in the Morse–Bott complex should be given by

$$d_{\mathrm{MB}}\theta=d_{\mathrm{de\,Rham}}\theta\pm u_\star l^\star\theta,$$

where u and l fit into the diagram

$$X^{\mathrm{cr}}\xleftarrow{\;u\;}\overline{\mathcal{M}}/\mathbf{R}\xrightarrow{\;l\;}X^{\mathrm{cr}}.$$

In this case the space $\mathcal{M}$ of descending gradient trajectories, each of which gives a holomorphic map $\mathcal{C}\to\mathbf{P}^{\mathbf{w}}$ from a cylinder $\mathcal{C}$, admits an S^1-action coming from the reparameterization of $\mathcal{C}$. This S^1-action extends to an S^1-action on $\overline{\mathcal{M}}$, which commutes with the $\mathbf{R}$-action on $\overline{\mathcal{M}}$:

$$\begin{array}{ccccc} X^{\mathrm{cr}} & \xleftarrow{\;u\;} & \overline{\mathcal{M}}/\mathbf{R} & \xrightarrow{\;l\;} & X^{\mathrm{cr}}\\ & \nwarrow & \downarrow & & \\ & & \overline{\mathcal{M}}/(\mathbf{R}\times S^1). & & \end{array}\tag{22}$$

(7) These fractional shifts will play an essential role later—see (26) and the discussion thereafter.

The upper and lower endpoint maps u and l are S^1-equivariant, and so for each $\theta \in \Omega^\bullet(X^{\mathrm{cr}})$ we have $u_\star l^\star \theta = 0$: we can compute the pushforward along u by first pushing forward along the vertical map in (22), and this pushforward sends the S^1-invariant differential form $l^\star\theta$ to zero. Thus in this case we should have $d_{\mathrm{MB}}\theta = d_{\mathrm{de\,Rham}}\theta$, and so

$$H_\bullet(C_\bullet^{\mathrm{MB}}, d_{\mathrm{MB}}) = \left(\bigoplus_{f\in F} Q^f H^\bullet(\mathbf{P}(V^f);\mathbf{R})\right) \otimes \mathbf{C}[Q,Q^{-1}]$$

as graded vector spaces. Here the grading on $H^\bullet(\mathbf{P}(V^f))$ is shifted by the age of $\mathbf{P}(V^f)$, and the degree of Q is $w_0+\ldots+w_n$.

It follows, as indicated in Remark 1.5, that after completing the group ring $\mathbf{C}[Q,Q^{-1}]$ we can identify $HF^\bullet(L\mathbf{P}^{\mathbf{w}}) = H_\bullet(C_\bullet^{\mathrm{MB}}, d_{\mathrm{MB}})$ with the free $\mathbf{C}(\!(Q)\!)$-submodule of

$$H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\Lambda[Q^{-1}])$$

with basis (3). The $\mathbf{C}(\!(Q)\!)$-module structure here arises from the action of deck transformations on $\widetilde{L\mathbf{P}^{\mathbf{w}}}$. Let z be the first Chern class of the tautological line bundle over BS^1, so that $H^\bullet_{S^1}(\mathrm{pt}) = \mathbf{C}[z]$. Identical arguments and conventions suggest that the S^1-equivariant Floer cohomology $HF^\bullet_{S^1}(L\mathbf{P}^{\mathbf{w}})$ should be the free $\mathbf{C}[z](\!(Q)\!)$-submodule of

$$H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\mathbf{C}) \otimes \mathbf{C}[z][[Q^{1/\mathrm{lcm}(w_0,\ldots,w_n)}]][Q^{-1}]$$

with basis (3).

3.5. Floer cohomology and small quantum cohomology

We think of elements of $HF^\bullet(L\mathbf{P}^{\mathbf{w}})$ as representing semi-infinite cycles in $\widetilde{L\mathbf{P}^{\mathbf{w}}}$, as follows. Recall that gradient trajectories of $\mu: \widetilde{L\mathbf{P}^{\mathbf{w}}} \to \mathbf{R}$ sweep out holomorphic cylinders in $\mathbf{P}^{\mathbf{w}}$. Recall further that we are using bases $\phi_1, \ldots, \phi_N$ and $\phi^1, \ldots, \phi^N$ for $H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\mathbf{C})$ such that $(\phi^i, \phi_j)_{\mathrm{orb}} = \delta^i{}_j$. Suppose that $\phi_\beta \in H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\mathbf{C})$ is supported on $\mathbf{P}(V^f) \subset \mathcal{I}\mathbf{P}^{\mathbf{w}}$, and let B be a generic cycle in $\mathbf{P}(V^f)$ which is Poincaré-dual to ϕ_β. The Floer cohomology class $\phi_\beta Q^r \in HF^\bullet(L\mathbf{P}^{\mathbf{w}})$ represents the semi-infinite cycle in $\widetilde{L\mathbf{P}^{\mathbf{w}}}$ swept out by upward gradient flow from the copy of B in the component of the critical set at level r.(8) The projection of this semi-infinite cycle to $L\mathbf{P}^{\mathbf{w}}$ consists of loops(9) in $\mathbf{P}^{\mathbf{w}}$ which bound a holomorphic disc $\{z: |z| \leqslant 1\} \to \mathbf{P}^{\mathbf{w}}$ with a possibly-stacky point at the origin such that the S^1-fixed loop defined by the origin of the disc lies in $B \subset \mathcal{I}\mathbf{P}^{\mathbf{w}}$.

(8) Note that $\langle r\rangle = f$, and so the component of the critical set at level r is a copy of $\mathbf{P}(V^f)$.

(9) More precisely, the projection consists of the closure of the set of such loops. In the rest of this section, we will ignore such distinctions.

From this point of view, it is not obvious that $HF^\bullet(L\mathbf{P}^\mathbf{w})$ should carry a ring structure: the transverse intersection of two semi-infinite cycles need not be semi-infinite, so we should not expect an intersection product here. But the transverse intersection of a finite-codimension cycle with a semi-infinite cycle will be semi-infinite, and this should give a map

$$H^\bullet(\widetilde{L\mathbf{P}^\mathbf{w}})\otimes HF^\bullet(L\mathbf{P}^\mathbf{w}) \longrightarrow HF^\bullet(L\mathbf{P}^\mathbf{w}).$$

Evaluation at $1\in S^1$ gives a map $\widetilde{L\mathbf{P}^\mathbf{w}}\to\mathbf{P}^\mathbf{w}$, and via pull-back we get a map

$$\begin{aligned} H^\bullet(\mathbf{P}^\mathbf{w};\mathbf{C})\otimes HF^\bullet(L\mathbf{P}^\mathbf{w}) &\longrightarrow HF^\bullet(L\mathbf{P}^\mathbf{w}), \\ \phi_\alpha\otimes\phi_\beta Q^r &\longmapsto \sum_{d\in\mathbf{Q}}\sum_\gamma n(d)_{\alpha\beta}{}^\gamma\phi_\gamma Q^{d+r}, \end{aligned} \tag{23}$$

which([10]) commutes with the action of $\mathbf{C}(\!(Q)\!)$. The structure constants of this map have a geometric interpretation, as follows. If everything intersects transversely, the structure constant $n(d)_{\alpha\beta}{}^\gamma$ should count the number of isolated points in the intersection of three cycles in $\widetilde{L\mathbf{P}^\mathbf{w}}$:

(a) the finite-codimension cycle corresponding to ϕ_α;

(b) the semi-infinite cycle corresponding to $\phi_\beta Q^r$;

(c) a semi-infinite cycle representing the element of Floer *homology* corresponding to $\phi_\gamma Q^{d+r}$.

Cycle (a) is the pre-image in $\widetilde{L\mathbf{P}^\mathbf{w}}$ of the cycle in $L\mathbf{P}^\mathbf{w}$ consisting of loops such that the point $1\in S^1$ maps to a generic cycle in $\mathbf{P}^\mathbf{w}$ which is Poincaré-dual to ϕ_α. Cycle (b) was described above. Cycle (c) is swept out by *downward* gradient flow from an appropriate cycle in the component of the critical set at level $d+r$. Its projection to $L\mathbf{P}^\mathbf{w}$ consists of loops which bound a holomorphic disc $\{z:|z|\geqslant 1\}\to\mathbf{P}^\mathbf{w}$ with a possibly-stacky point at ∞ such that the S^1-fixed loop defined by the point ∞ lies in a generic cycle in $\mathcal{I}\mathbf{P}^\mathbf{w}$ which is Poincaré-dual to ϕ^γ. So $n(d)_{\alpha\beta}{}^\gamma$ counts—or, in the non-transverse situation, gives a virtual count of—the number of isolated holomorphic spheres in $\mathbf{P}^\mathbf{w}$ of degree $d\in\mathbf{Q}$ carrying exactly two possibly-stacky points $\{0,\infty\}$ and incident at the points $\{0,1,\infty\}$ to generic cycles in $\mathcal{I}\mathbf{P}^\mathbf{w}$ which are Poincaré-dual to ϕ_β, ϕ_α and ϕ^γ, respectively. In other words, the structure constants $n(d)_{\alpha\beta}{}^\gamma$ of the map (23) coincide with the structure constants (12) of the small orbifold quantum cohomology algebra.

Remark 3.2. This shows that small quantum orbifold multiplication by a class in the untwisted sector $H^\bullet(\mathbf{P}^\mathbf{w};\mathbf{C})\subset H^\bullet_{\mathrm{orb}}(\mathbf{P}^\mathbf{w};\mathbf{C})$ can be thought of as an operation on Floer cohomology. It would be interesting to find an interpretation of multiplication by other orbifold cohomology classes in these terms.

([10]) Note that (23) involves the subspace $H^\bullet(\mathbf{P}^\mathbf{w};\mathbf{C})\subset H^\bullet_{\mathrm{orb}}(\mathbf{P}^\mathbf{w};\mathbf{C})$ and not the full orbifold cohomology group $H^\bullet_{\mathrm{orb}}(\mathbf{P}^\mathbf{w};\mathbf{C})$.

3.6. The $\mathcal{D}$-module structure on S^1-equivariant Floer cohomology

In this section we explain why the S^1-equivariant Floer cohomology $HF^{\bullet}_{S^1}(L\mathbf{P}^{\mathbf{w}})$ should carry a natural $\mathcal{D}$-module structure. Recall that Ω is the Kähler form on $L\mathbf{P}^{\mathbf{w}}$ induced by the Kähler structure on $\mathbf{P}^{\mathbf{w}}$, and that we consider the covering space $p: \widetilde{L\mathbf{P}^{\mathbf{w}}} \to L\mathbf{P}^{\mathbf{w}}$. We have $[\Omega] = \mathrm{ev}_1^\star P$. The form $p^\star\Omega$ is not equivariantly closed, so it does not define an S^1-equivariant cohomology class on $\widetilde{L\mathbf{P}^{\mathbf{w}}}$, but $p^\star\Omega + z\mu$ is equivariantly closed—this follows from the fact that m is a moment map. Let $\wp$ be the class of $p^\star\Omega + z\mu$ in $H^2_{S^1}(\widetilde{L\mathbf{P}^{\mathbf{w}}})$. Consider the map $\mathbf{P}: HF^{\bullet}_{S^1}(L\mathbf{P}^{\mathbf{w}}) \to HF^{\bullet}_{S^1}(L\mathbf{P}^{\mathbf{w}})$ given by multiplication by $\wp$, and the map $\mathbf{Q}: HF^{\bullet}_{S^1}(L\mathbf{P}^{\mathbf{w}}) \to HF^{\bullet}_{S^1}(L\mathbf{P}^{\mathbf{w}})$ given by pull-back by the deck transformation Q^{-1}. Since

$$(Q^{-1})^\star \wp = \wp - z$$

we have $[\mathbf{P}, \mathbf{Q}] = z\mathbf{Q}$. In other words, if we define $\mathcal{D}$ to be the Heisenberg algebra

$$\mathcal{D} = \mathbf{C}[z][[\mathbf{Q}]][\mathbf{Q}^{-1}]\langle \mathbf{P} \rangle \quad \text{such that} \quad [\mathbf{P}, \mathbf{Q}] = z\mathbf{Q},$$

then $HF^{\bullet}_{S^1}(L\mathbf{P}^{\mathbf{w}})$ should carry the structure of a $\mathcal{D}$-module where $\mathbf{Q}$ acts by pull-back by Q^{-1} and $\mathbf{P}$ acts by multiplication by $\wp$.

In the non-equivariant limit ($z \to 0$) this structure degenerates to a $\mathbf{C}(\!(Q)\!)[P]$-module structure on $HF^{\bullet}(L\mathbf{P}^{\mathbf{w}})$, where P acts via (23). Thus we can recover the part of the small orbifold quantum cohomology algebra generated by P—which, as we will see below, is the whole thing—from the $\mathcal{D}$-module structure on $HF^{\bullet}_{S^1}(L\mathbf{P}^{\mathbf{w}})$. It is clear that $HF^{\bullet}(L\mathbf{P}^{\mathbf{w}})$ should be generated as a $\mathbf{C}(\!(Q)\!)[P]$-module by $\{Q^f \mathbf{1}_f\}$, so we expect $HF^{\bullet}_{S^1}(L\mathbf{P}^{\mathbf{w}})$ to be finitely generated as a $\mathcal{D}$-module. Our analysis below will show that $HF^{\bullet}_{S^1}(L\mathbf{P}^{\mathbf{w}})$ is of rank 1, generated by $\mathbf{1}_0 Q^0$. This generator is Givental's "fundamental Floer cycle"—it represents the semi-infinite cycle in $\widetilde{L\mathbf{P}^{\mathbf{w}}}$ swept out by upward gradient flow from the component of the critical set at level 0. The projection to $L\mathbf{P}^{\mathbf{w}}$ of the fundamental Floer cycle consists of all loops which bound holomorphic discs with a possibly-stacky point at the origin.

The link between Floer cohomology and Gromov–Witten theory appears here as a conjectural $\mathcal{D}$-module isomorphism between $HF^{\bullet}_{S^1}(L\mathbf{P}^{\mathbf{w}})$ and the $\mathcal{D}$-module generated by the small J-function. We have seen how $\mathcal{D}$ acts on $HF^{\bullet}_{S^1}(L\mathbf{P}^{\mathbf{w}})$. Another realization of $\mathcal{D}$ is by differential operators

$$\mathbf{P}: f \longmapsto z\frac{\partial f}{\partial t} \quad \text{and} \quad \mathbf{Q}: f \longmapsto Qe^t f$$

acting on the space of analytic functions $f: \mathbf{C} \to H^{\bullet}_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}}; \Lambda[Q^{-1}]) \otimes \mathbf{C}(\!(z^{-1})\!)$. The small J-function is such a function (see §2.3.1) and so it generates a $\mathcal{D}$-module; relations in this

$\mathcal{D}$-module are differential equations satisfied by $J_{\mathbf{P^w}}(t)$ (see equations (14), (19) and the discussions thereafter). We will make use of this conjectural $\mathcal{D}$-module isomorphism in the next section, where we write down a concrete model for $HF^{\bullet}_{S^1}(L\mathbf{P^w})$ as a $\mathcal{D}$-module and then identify the fundamental Floer cycle in this model with the small J-function $J_{\mathbf{P^w}}(t)$. This will give a conjectural formula for the small J-function.

3.7. Computing the $\mathcal{D}$-module structure

As we lack a concrete model for $\widetilde{L\mathbf{P^w}}$, we consider instead the space of *polynomial loops*

$$L_{\mathrm{poly}} = \{(f^0, ..., f^n) : f^i \in \mathbf{C}[t, t^{-1}] \text{ and not all the } f^i \text{ are zero}\}/\mathbf{C}^{\times},$$

where $\alpha \in \mathbf{C}^{\times}$ acts on a vector-valued Laurent polynomial as

$$(f^0, ..., f^n) \longmapsto (\alpha^{-w_0} f^0, ..., \alpha^{-w_n} f^n).$$

The space L_{poly} is quite different from $\widetilde{L\mathbf{P^w}}$—it is, for example, certainly not a covering space([11]) of $L\mathbf{P^w}$. But L_{poly} is in some ways a good analog for $\widetilde{L\mathbf{P^w}}$. We will see below that there is an S^1-action on L_{poly} such that the S^1-fixed subset is a covering space of the inertia stack $\mathcal{I}\mathbf{P^w}$ with deck transformation group $\mathbf{Z}$. So for computations involving quantities which localize to the S^1-fixed set—such as S^1-equivariant semi-infinite cohomology—L_{poly} is a good substitute for $\widetilde{L\mathbf{P^w}}$. Working by analogy with the discussion in the previous section, we now construct an action of $\mathcal{D}$ on the "S^1-equivariant semi-infinite cohomology" of L_{poly}. This will be our concrete model for $HF^{\bullet}_{S^1}(L\mathbf{P^w})$.

The space L_{poly} is an infinite-dimensional weighted projective space. It carries an S^1-action coming from loop rotation, which is Hamiltonian with respect to the Fubini–Study form $\Omega' \in \Omega^2(L_{\mathrm{poly}})$. The moment map for this action is

$$\mu' : \left[\left(\sum_{k\in\mathbf{Z}} a^0_k t^k, ..., \sum_{k\in\mathbf{Z}} a^n_k t^k\right)\right] \longmapsto -\frac{\sum_{l=0}^n \sum_{k\in\mathbf{Z}} k|a^l_k|^2}{\sum_{l=0}^n \sum_{k\in\mathbf{Z}} w_l|a^l_k|^2}.$$

A polynomial loop

$$[(f^0(t), ..., f^n(t))] \in L_{\mathrm{poly}}$$

is fixed by loop rotation if and only if

$$(f^0(\lambda t), ..., f^n(\lambda t)) = (\alpha(\lambda)^{-w_0} f^0(t), ..., \alpha(\lambda)^{-w_n} f^n(t))$$

([11]) The "obvious map" $L_{\mathrm{poly}} \to L\mathbf{P^w}$, given by restricting a polynomial map $f(t)$ to the circle $\{t \in \mathbf{C} : |t| = 1\}$ and filling in where necessary using continuity, is not even continuous.

for all $\lambda \in S^1$ and some possibly multi-valued function $\alpha(\lambda)$. We need $\alpha(\lambda) = \lambda^{-k/w_i}$ for some integer k, so components of the S^1-fixed set are indexed by

$$\widetilde{F} = \{k/w_i : k \in \mathbf{Z} \text{ and } 0 \leqslant i \leqslant n\}.$$

For $r \in \widetilde{F}$, the corresponding S^1-fixed component

$$\mathrm{Fix}_r = \{[(b_0 t^{w_0 r}, ..., b_n t^{w_n r})] \in L_{\mathrm{poly}} : b_i = 0 \text{ unless } w_i r \in \mathbf{Z}\}$$

is a copy of the component $\mathbf{P}(V^{\langle r \rangle})$ of the inertia stack, and the value of μ' on this fixed component is $-r$. The normal bundle to Fix_r is

$$\bigoplus_{i=0}^{n} \bigoplus_{\substack{j \in \mathbf{Z} \\ j \neq w_i r}} \mathcal{O}(w_i P + (j - w_i r) z),$$

where $\mathcal{O}(aP + bz)$ denotes the bundle $\mathcal{O}(a)$ on $\mathrm{Fix}_r = \mathbf{P}(V^{\langle r \rangle})$ which has weight b with respect to loop rotation.

Let $\wp'$ be the class of $\Omega' + z\mu'$ in $H^2_{S^1}(L_{\mathrm{poly}})$, so that

$$H^\bullet_{S^1}(L_{\mathrm{poly}}) = \mathbf{C}[z, \wp'],$$

and introduce an action of $\mathbf{Z}$ on L_{poly} by "deck transformations":

$$Q^m : \left[\left(\sum_{k \in \mathbf{Z}} a_k^0 t^k, ..., \sum_{k \in \mathbf{Z}} a_k^n t^k\right)\right] \longmapsto \left[\left(\sum_{k \in \mathbf{Z}} a_k^0 t^{k - m w_0}, ..., \sum_{k \in \mathbf{Z}} a_k^n t^{k - m w_n}\right)\right], \quad m \in \mathbf{Z}.$$

The deck transformation Q^m changes the value of μ' by m, and sends Fix_r to Fix_{r-m}. We let $\mathbf{Q}$ act on $H^\bullet_{S^1}(L_{\mathrm{poly}})$ by pull-back by Q^{-1}, and $\mathbf{P}$ act on $H^\bullet_{S^1}(L_{\mathrm{poly}})$ by cup product with $\wp'$. As

$$(Q^{-1})^\star \wp' = \wp' - z,$$

so that $[\mathbf{P}, \mathbf{Q}] = z\mathbf{Q}$, this gives an action of $\mathcal{D}$ on $H^\bullet_{S^1}(L_{\mathrm{poly}})$.

We now consider the "S^1-equivariant semi-infinite cohomology" of L_{poly}. We will work formally, representing semi-infinite cohomology classes by infinite products in

$$H^\bullet_{S^1}(L_{\mathrm{poly}}).$$

These products, interpreted naïvely, definitely diverge, but one can make rigorous sense of them by considering them as the limits of finite products and at the same time considering L_{poly} as the limit of spaces of Laurent polynomials of bounded degree. This is explained in [19] and [29]. Recall that the fundamental Floer cycle in $\widetilde{L\mathbf{P}^{\mathbf{w}}}$ consists (roughly speaking) of loops which bound holomorphic discs. The analog of the fundamental Floer cycle in L_{poly} is the cycle of Laurent polynomials which are regular at $t=\infty$. We represent this by the infinite product

$$\Delta=\prod_{i=0}^{n}\prod_{k>0}(w_i\wp'+kz).$$

To interpret this, observe that the Fourier coefficient a_k^i of the loop

$$\left[\left(\sum_{k\in\mathbf{Z}}a_k^0t^k,\dots,\sum_{k\in\mathbf{Z}}a_k^nt^k\right)\right]\in L_{\mathrm{poly}}$$

gives a section of the bundle $\mathcal{O}(w_i)$ over

$$L_{\mathrm{poly}}\cong\mathbf{P}(\dots,w_n,w_0,w_1,\dots,w_n,w_0,w_1,\dots,w_n,w_0,\dots),$$

which has weight k with respect to loop rotation. Our candidate for the Floer fundamental cycle is cut out by the vanishing of the a_k^i, $k>0$, and so Δ is a candidate for the S^1-equivariant Thom class of its normal bundle—that is, for its S^1-equivariant Poincaré-dual. We have

$$\prod_{i=0}^{n}\prod_{j=0}^{w_i-1}(w_i\mathbf{P}-jz)\Delta=\mathbf{Q}\Delta. \tag{24}$$

This is an equation in the S^1-equivariant semi-infinite cohomology of L_{poly}, regarded as a $\mathcal{D}$-module via the actions of $\mathbf{P}$ and $\mathbf{Q}$ defined above. As a $\mathcal{D}$-module, the S^1-equivariant semi-infinite cohomology of L_{poly} is generated by Δ.

We cannot directly identify Δ with the small J-function, as the $\mathcal{D}$-module generated by Δ involves shift operators

$$\mathbf{P}\colon g(\wp')\longmapsto\wp' g(\wp')\quad\text{and}\quad\mathbf{Q}\colon g(\wp')\longmapsto g(\wp'-z),$$

whereas that generated by the small J-function involves differential operators

$$\mathbf{P}\colon f(t)\longmapsto z\frac{\partial f}{\partial t}\quad\text{and}\quad\mathbf{Q}\colon f(t)\longmapsto Qe^tf(t).$$

We move between the two via a sort of Fourier transform. We expect, by analogy with the Atiyah–Bott localization theorem [6], that there should be a localization map Loc from localized S^1-equivariant semi-infinite cohomology of L_{poly} to the cohomology $H^\bullet_{S_1}(L_{\mathrm{poly}}^{S^1})\otimes\mathbf{C}(z)$ of the S^1-fixed set. We consider

$$\mathrm{Loc}(e^{\wp' t/z}\Delta) \tag{25}$$

as this should satisfy

$$\mathbf{P}\operatorname{Loc}(e^{\wp' t/z}\Delta) = z\frac{\partial}{\partial t}\operatorname{Loc}(e^{\wp' t/z}\Delta) = \operatorname{Loc}(e^{\wp' t/z}\wp'\Delta) = \operatorname{Loc}(e^{\wp' t/z}\mathbf{P}\Delta),$$
$$\mathbf{Q}\operatorname{Loc}(e^{\wp' t/z}\Delta) = Qe^t\operatorname{Loc}(e^{\wp' t/z}\Delta) = e^t\operatorname{Loc}((Q^{-1})^\star(e^{\wp' t/z}\Delta)) = \operatorname{Loc}(e^{\wp' t/z}\mathbf{Q}\Delta).$$

The class $\wp'\in H^2_{S^1}(L_{\mathrm{poly}})$ restricts to the class $c_1(\mathcal{O}(1))-zr\in H^2(\mathrm{Fix}_r)\otimes\mathbf{C}(z)$, and we can write this as the Chen–Ruan orbifold cup product

$$(P-zr)\mathbf{1}_{\langle -r\rangle}.$$

Thus $\operatorname{Loc}(e^{\wp' t/z}\Delta)$ should be something like

$$\sum_{r\in\widetilde{F}} Q^{-r}e^{Pt/z}e^{-rt}\frac{\prod_{i=0}^n\prod_{k>0}(w_iP+(k-w_ir)z)}{\prod_{i=0}^n\prod_{j\in\mathbf{Z},j\neq w_ir}(w_iP+(j-w_ir)z)}\mathbf{1}_{\langle -r\rangle}, \tag{26}$$

where the numerator records the restriction of Δ to Fix_r and the denominator stands for the S^1-equivariant Euler class of the normal bundle to Fix_r. We need to make sense of this expression.

Note first that if $r>0$, the numerator in (26) is divisible by $P^{\dim_{\langle r\rangle}+1}$ and hence vanishes for dimensional reasons. So our expression is

$$\sum_{\substack{r\in\widetilde{F}\\ r\geqslant 0}} Q^r e^{Pt/z}e^{rt}\prod_{i=0}^n\frac{1}{\prod_{b:\langle b\rangle=\langle rw_i\rangle,0<b\leqslant w_ir}(w_iP+bz)}\frac{1}{\prod_{b:\langle b\rangle=\langle rw_i\rangle,b<0}(w_iP+bz)}\mathbf{1}_{\langle r\rangle}.$$

This expression still does not make sense due to the divergent infinite product on the right. We "regularize" it by simply dropping these factors—which depend on r only through $\langle r\rangle$—and multiplying by z, obtaining the *I-function*

$$I(t) = ze^{Pt/z}\sum_{\substack{r\in\widetilde{F}\\ r\geqslant 0}} Q^r e^{rt}\frac{1}{\prod_{i=0}^n\prod_{b:\langle b\rangle=\langle rw_i\rangle,0<b\leqslant w_ir}(w_iP+bz)}\mathbf{1}_{\langle r\rangle}.$$

This is a formal function of t taking values in $H^\bullet_{\mathrm{orb}}(\mathbf{P}^\mathbf{w};\Lambda)$. It satisfies

$$\prod_{i=0}^n\prod_{j=0}^{w_i-1}\left(w_iz\frac{\partial}{\partial t}-jz\right)I = Qe^tI,$$

so the $\mathcal{D}$-modules generated by Δ and by I are isomorphic (see (24)). We conjecture that this $\mathcal{D}$-module is isomorphic to the $\mathcal{D}$-module generated by the small J-function, and that

$$J_{\mathbf{P}^\mathbf{w}}(t) = I(t).$$

4. Calculation of the small J-function

4.1. Summary: the basic diagram

In this section we describe a certain commutative diagram of stacks with $\mathbf{C}^\times$-action which lies at the heart of our proof of Theorem 1.7. We begin by showing that for each genus-zero one-pointed twisted stable map to $\mathbf{P}^{\mathbf{w}}$, the component of $\mathcal{I}\mathbf{P}^{\mathbf{w}}$ to which the marked point maps is determined by the degree of the map.

LEMMA 4.1. *Fix a positive rational number $d>0$.*

(1) *If the moduli stack $\mathbf{P}^{\mathbf{w}}_{0,0,d}$ is non-empty, then d is an integer.*

(2) *If the moduli stack $(\mathbf{P}^{\mathbf{w}})^f_{0,1,d}$ is non-empty, then $f=\langle -d\rangle$.*

Proof. Let $\mathcal{C}$ be a balanced twisted curve, and assume that there is a stable representable morphism $\varphi:\mathcal{C}\to\mathbf{P}^{\mathbf{w}}$ of degree d:

$$\int_{\mathcal{C}} \varphi^\star\mathcal{O}(1)=d.$$

Applying Riemann–Roch for twisted curves [4, Theorem 7.2.1], we find that

$$\chi(\mathcal{C},\varphi^\star\mathcal{O}(1))=\begin{cases} 1+d, & \text{in case (1),}\\ 1+d-\langle -f\rangle, & \text{in case (2).}\end{cases}$$

As $\chi(\mathcal{C},\varphi^\star\mathcal{O}(1))$ is an integer, the result follows. □

Notation 4.2. The lemma says that in $(\mathbf{P}^{\mathbf{w}})^f_{0,1,d}$ we always have $f=\langle -d\rangle$. It is therefore safe to drop f from the notation, and we do so in what follows. Fix now $d=m/r$ in lowest terms and write $f=\langle -d\rangle\in F$. We introduce the following notation:

(1) $M_d=\overline{\mathcal{M}}_{0,1}(\mathbf{P}^{\mathbf{w}},d)$ is, using the notation of [4], the moduli stack of genus-zero one-pointed balanced twisted stable morphisms of degree d to $\mathbf{P}^{\mathbf{w}}$ *with section to the gerbe marking.* There are maps

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\ \varphi\ } & \mathbf{P}^{\mathbf{w}} \\ {\scriptstyle\sigma}\uparrow\downarrow{\scriptstyle\pi} & & \uparrow \\ M_d & \xrightarrow[\ \mathrm{ev}_1\ ]{} & \mathbf{P}(V^f), \end{array}$$

where $\pi:\mathcal{U}\to M_d$ denotes the universal family, $\sigma: M_d\to\mathcal{U}$ the section and $\mathrm{ev}_1: M_d\to\mathbf{P}(V^f)$ the evaluation map. As usual, we write $\psi_1=c_1(L_1)$, where L_1 is the universal cotangent line at the marked point.

(2) G_d is the *graph space* of degree d; its definition depends on whether or not d has a non-trivial fractional part:

$$G_d=\begin{cases} \overline{\mathcal{M}}_{0,1}(\mathbf{P}^{\mathbf{w}}\times\mathbf{P}^{1,r},d\times 1/r), & \text{if } \langle d\rangle>0,\\ \overline{\mathcal{M}}_{0,0}(\mathbf{P}^{\mathbf{w}}\times\mathbf{P}^{1},d\times 1), & \text{if } \langle d\rangle=0.\end{cases}$$

More precisely, if $\langle d\rangle>0$ then G_d denotes the moduli stack of graphs with the following specified *character* at the marked point: a point of G_d is a pair of morphisms

$$(f_1, f_2)\colon C \longrightarrow \mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r},$$

where $f_1\colon C\to\mathbf{P}^{\mathbf{w}}$, $f_2\colon C\to\mathbf{P}^{1,r}$ and we require that, at the marked point, f_1 evaluates in $\mathbf{P}(V^{\langle -d\rangle})\subset\mathcal{I}\mathbf{P}^{\mathbf{w}}$ and f_2 evaluates in $\mathbf{P}(V^{(r-1)/r})\subset\mathcal{I}\mathbf{P}^{1,r}$. In other words, denoting by $x\in C$ the marked point,

$$\begin{aligned} \operatorname{Aut}_C(x) &\longrightarrow \operatorname{Aut}_{\mathbf{P}^{\mathbf{w}}}(f_1(c)) \times \operatorname{Aut}_{\mathbf{P}^{1,r}}(0), \\ e^{2\pi\sqrt{-1}/r} &\longmapsto (e^{2\pi\sqrt{-1}f}, e^{-2\pi\sqrt{-1}/r}). \end{aligned} \tag{27}$$

As a result of this choice, the marked point $x\in C$ is constrained to lie above the orbifold point $0\in\mathbf{P}^{1,r}$. Note again that, if $\langle d\rangle>0$, our graphs have a gerbe marking and G_d is a moduli stack of morphisms with section to the gerbe marking.

(3) L_d is the stack of polynomial morphisms $\mathbf{P}^{1,r}\to\mathbf{P}^{\mathbf{w}}$ of degree d. This is described in detail in §4.2.

Notation 4.3. In what follows

(1) all group actions are strict (see, e.g., [44]);

(2) all stacks that we consider are Deligne–Mumford stacks, except where we explicitly say "Artin stack";

(3) we write "stable morphism" instead of "balanced twisted stable morphism";

(4) by "part" we mean "union of connected components".

The action of the group $\mathbf{C}^\times$ on $\mathbf{C}^2$,

$$\lambda\colon (s_0, s_1) \longmapsto (s_0, \lambda^{-1}s_1), \quad \lambda\in\mathbf{C}^\times, \tag{28}$$

descends to give an action of $\mathbf{C}^\times$ on $\mathbf{P}^{1,r}=\mathbf{P}(1,r)$. This action induces actions on the stacks G_d and L_d; see below for additional details and discussion.

Notation 4.4. Let z be the first Chern class of the tautological line bundle on $B\mathbf{C}^\times$, so that $H^\bullet_{\mathbf{C}^\times}(\{\mathrm{pt}\})=\mathbf{C}[z]$.

THEOREM 4.5. *There is a commutative diagram of stacks with* $\mathbf{C}^\times$*-action*:

$$\begin{array}{ccc} G_d & \xrightarrow{\;u\;} & L_d \\ {\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle j} \\ M_d & \xrightarrow[\mathrm{ev}_1]{} & \mathbf{P}(V^f) \end{array}$$

such that the following properties hold:

(1) *The inclusion* $j: \mathbf{P}(V^f) \hookrightarrow L_d$ *is a connected component of the* $\mathbf{C}^\times$*-fixed substack, and the* $\mathbf{C}^\times$*-equivariant Euler class of its normal bundle is*

$$e(N_j) = \prod_{i=0}^{n} \prod_{\substack{b:\langle b\rangle = \langle dw_i\rangle \\ 0<b\leqslant dw_i}} (w_i P + bz).$$

(2) *The inclusion* $\iota: M_d \hookrightarrow G_d$ *is the part of the* $\mathbf{C}^\times$*-fixed substack of* G_d *which maps to* $\mathbf{P}(V^f)$*. The canonical perfect obstruction theory on* M_d *coincides with the perfect obstruction theory inherited from* G_d*, and the* $\mathbf{C}^\times$*-equivariant Euler class of the virtual normal bundle to* M_d *in* G_d *is*

$$e(N_\iota^{\mathrm{vir}}) = z(z-\psi_1).$$

(3) *The morphism* u *is "virtually birational"; in other words, when* G_d *is endowed with its canonical perfect obstruction theory and* L_d *with its intrinsic perfect obstruction theory, then*

$$u_\star \mathbf{1}_{G_d}^{\mathrm{vir}} = \mathbf{1}_{L_d}.$$

More details on obstruction theory can be found below.

Corollary 4.6. *Theorem* 1.7 *of the introduction holds. That is, we have the following formula for the small* J*-function of* $\mathbf{P}^{\mathbf{w}}$:

$$J_{\mathbf{P}^{\mathbf{w}}}(t) = ze^{Pt/z} \sum_{\substack{d:d\geqslant 0 \\ \langle d\rangle \in F}} \frac{Q^d e^{dt} \mathbf{1}_{\langle d\rangle}}{\prod_{i=0}^{n} \prod_{b:\langle b\rangle = \langle dw_i\rangle, 0<b\leqslant dw_i} (w_i P + bz)}.$$

Proof. We calculate, using the basic diagram and properties stated in Theorem 4.5,

$$\begin{aligned}
\mathbf{1}_f &= j^\star \mathbf{1}_{L_d} \\
&= j^\star u_\star \mathbf{1}_{G_d}^{\mathrm{vir}} && (29)\\
&= j^\star u_\star \iota_\star \left(\mathbf{1}_{M_d}^{\mathrm{vir}} \cap \frac{1}{e(N_\iota^{\mathrm{vir}})} \right) && (30)\\
&= j^\star u_\star \iota_\star \left(\mathbf{1}_{M_d}^{\mathrm{vir}} \cap \frac{1}{z(z-\psi_1)} \right) \\
&= j^\star j_\star (\mathrm{ev}_1)_\star \left(\mathbf{1}_{M_d}^{\mathrm{vir}} \cap \frac{1}{z(z-\psi_1)} \right) \\
&= e(N_j)(\mathrm{ev}_1)_\star \left(\mathbf{1}_{M_d}^{\mathrm{vir}} \cap \frac{1}{z(z-\psi_1)} \right). && (31)
\end{aligned}$$

Equality (29) here holds because u is virtually birational. Equality (30) follows from the virtual localization formula of Graber and Pandharipande [23] and the fact that M_d is the part of the $\mathbf{C}^\times$-fixed substack of G_d which maps to $\mathbf{P}(V^f)$. The Graber–Pandharipande formula requires all stacks to admit a global equivariant embedding in a smooth stack; the main result of [2] shows that this is true here. From equation (31), we conclude that

$$(I\circ \mathrm{ev}_1)_\star\left(\mathbf{1}_{M_d}^{\mathrm{vir}}\cap\frac{1}{z(z-\psi_1)}\right)=\frac{\mathbf{1}_{\langle d\rangle}}{\prod_{i=0}^{n}\prod_{b:\langle b\rangle=\langle dw_i\rangle,0<b\leqslant dw_i}(w_iP+bz)}. \qquad \square$$

Note that M_d can consist of several connected components, some of which can be singular or of excess dimension; this does not affect the calculation. Similarly the graph space G_d also, in general, has several irreducible or connected components. The fact that u is virtually birational implies that only the component which generically consists of morphisms from $\mathbf{P}^{1,r}$ contributes non-trivially to the calculation.

4.2. The stack L_d of polynomial maps and the morphism $j:\mathbf{P}(V^f)\hookrightarrow L_d$

Definition 4.7. Let $\mathbf{C}(w)$ denote the 1-dimensional vector space $\mathbf{C}$ equipped with a weight-w action of $\mathbf{C}^\times$.

By $\mathbf{C}^\times$ here we mean the $\mathbf{C}^\times$ which occurs in the quotient (8), not the $\mathbf{C}^\times$ which acts by "rotation of loops" (28). Recall that $d=m/r$ in lowest terms, and that $f=\langle -d\rangle\in F$.

Definition 4.8.

$$L_d=\left[\left(\bigoplus_{i=0}^{n}\mathbf{C}(-w_i)^{\oplus(1+\lfloor dw_i\rfloor)}\setminus\{0\}\right)/\mathbf{C}^\times\right].$$

We regard L_d as the stack of polynomial maps $\mathbf{P}^{1,r}\to\mathbf{P}^{\mathbf{w}}$ of degree d, as follows. Such a map is given([12]) by polynomials $P_0,P_1,...,P_n$, not all zero, where $P_i=P_i(s_0,s_1)$ is homogeneous of degree mw_i in the variables s_0 and s_1, with $\deg s_0=1$ and $\deg s_1=r$. Each P_i can be written as

$$P_i(s_0,s_1)=A_0s_0^{mw_i}+A_1s_0^{mw_i-r}s_1+...+A_{\lfloor dw_i\rfloor}s_0^{r\langle dw_i\rangle}s_1^{\lfloor dw_i\rfloor}, \tag{32}$$

and hence

$$L_d=\left[\left(\bigoplus_{i=0}^{n}\mathbf{C}(-w_i)^{\oplus(1+\lfloor dw_i\rfloor)}\setminus\{0\}\right)/\mathbf{C}^\times\right].$$

([12]) See also Claim 4.17 below.

The stack L_d is itself a weighted projective space. Recall that

$$V^f = \bigoplus_{i:fw_i \in \mathbf{Z}} \mathbf{C}(-w_i),$$

and note that fw_i is an integer if and only if dw_i is an integer. We define the map $j: \mathbf{P}(V^f) \hookrightarrow L_d$ by

$$j: \mathbf{C}(-w_i) \ni A_i \mathbf{e}_i \longmapsto A_i s_1^{dw_i} \in \mathbf{C}(-w_i).$$

The action (28) of $\mathbf{C}^\times$ on $\mathbf{P}^{1,r}$ induces an action on L_d in the obvious way.

Remark 4.9. It is clear that $j: \mathbf{P}(V^f) \hookrightarrow L_d$ is a component of the $\mathbf{C}^\times$-fixed substack.

Notation 4.10. Given a stack $\mathcal{X}$ and a scheme S, we write $\mathcal{X}(S)$ for the category of morphisms from S to $\mathcal{X}$.

Remark 4.11. Consider an action $\Psi: G \times \mathcal{X} \to \mathcal{X}$ of a group scheme G on a stack $\mathcal{X}$. A substack $\iota: \mathcal{Y} \hookrightarrow \mathcal{X}$ is *fixed* by the action if for all schemes S we have a diagram

$$\begin{array}{ccc} G(S) \times \mathcal{Y}(S) & \xrightarrow{\mathrm{pr}_2(S)} & \mathcal{Y}(S) \\ \Big\downarrow {\scriptstyle \mathrm{id}_G(S) \times \iota(S)} & \nearrow\!\!\!\nearrow & \Big\downarrow {\scriptstyle \iota(S)} \\ G(S) \times \mathcal{X}(S) & \xrightarrow[\Psi(S)]{} & \mathcal{X}(S), \end{array} \tag{33}$$

where the $\Rightarrow$ means that there is an *isomorphism of functors*

$$\Psi(S) \circ (\mathrm{id}_G(S) \times \iota(S)) \Longrightarrow \iota(S) \circ \mathrm{pr}_2(S). \tag{34}$$

By definition, a fixed substack $\iota: \mathcal{Y} \hookrightarrow \mathcal{X}$ is *the G-fixed substack* if it satisfies the obvious universal property: if $j: \mathcal{Z} \hookrightarrow \mathcal{X}$ is any other fixed substack, than it factors uniquely through $\iota: \mathcal{Y} \hookrightarrow \mathcal{X}$.

LEMMA 4.12. *Let N_j be the normal bundle of the inclusion $j: \mathbf{P}(V^f) \hookrightarrow L_d$. The $\mathbf{C}^\times$-equivariant Euler class of N_j is*

$$e(N_j) = \prod_{i=0}^{n} \prod_{\substack{b: \langle b \rangle = \langle dw_i \rangle \\ 0 < b \leqslant dw_i}} (w_i P + bz).$$

Proof. Contemplate the following diagram on $\mathbf{P}(V^f)$. The bottom two rows are the Euler sequence for weighted projective space.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \displaystyle\bigoplus_{i=0}^{n} \bigoplus_{\substack{b:\langle b\rangle=\langle dw_i\rangle \\ 0<b\leqslant dw_i}} \mathcal{O}(w_iP+bz) & = & N_j & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbf{C} & \longrightarrow & \displaystyle\bigoplus_{i=0}^{n} \bigoplus_{\substack{b:\langle b\rangle=\langle dw_i\rangle \\ 0\leqslant b\leqslant dw_i}} \mathcal{O}(w_iP+bz) & \longrightarrow & T_{L_d}|_{\mathbf{P}(V^f)} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbf{C} & \longrightarrow & \displaystyle\bigoplus_{i:dw_i\in\mathbf{Z}} \mathcal{O}(w_iP) & \longrightarrow & T_{\mathbf{P}(V^f)} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

□

4.3. Deformations and obstructions

We review the canonical obstruction theories on M_d and G_d and prove that the obstruction theory on M_d is inherited from the obstruction theory on G_d.

4.3.1. The $\mathbf{C}^{\times}$-action on G_d

The (left) action of $\mathbf{C}^{\times}$ on $\mathbf{P}^{\mathbf{w}}\times\mathbf{P}^{1,r}$, where $\mathbf{C}^{\times}$ acts on the second factor only via (28), induces an action on the stack G_d by "dragging" the image of the morphism. More precisely, given a scheme S, an object of $G_d(S)$ is a stable morphism over S:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ f\ } & \mathbf{P}^{\mathbf{w}}\times\mathbf{P}^{1,r} \\
{\scriptstyle\sigma}\uparrow\downarrow{\scriptstyle p} & & \\
S & &
\end{array}
$$

and the group action is described as

$$\lambda\colon f \longmapsto {}^{\lambda}f=\ell_{\lambda^{-1}}\circ f, \quad \lambda\in\mathbf{C}^{\times},$$

where $\ell_{\lambda}\colon \mathbf{P}^{\mathbf{w}}\times\mathbf{P}^{1,r}\to\mathbf{P}^{\mathbf{w}}\times\mathbf{P}^{1,r}$ is left translation by λ.

4.3.2. The stack $\iota\colon M_d \hookrightarrow G_d$ is part of the fixed substack

We now construct the morphism of stacks $\iota\colon M_d \to G_d$ used in Theorem 4.5. Throughout this subsection we assume that $\langle d \rangle \neq 0$. The results remain true if $\langle d \rangle = 0$; the proofs in this case are slightly different but similar and easier.

For all schemes S, we need functors $\iota(S)\colon M_d(S) \to G_d(S)$ satisfying various compatibilities. An object of $M_d(S)$ is a stable morphism

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{f'} & \mathbf{P}^{\mathbf{w}} \\ {\scriptstyle \sigma'} \uparrow \downarrow {\scriptstyle p'} & & \\ S, & & \end{array} \tag{35}$$

where σ' is a section of the gerbe marking. Denote by $C_{r,r}$ the twisted curve with coarse moduli space $\mathbf{P}^1$ and stack structure with stabilizer μ_r at 0 and ∞ determined by charts([13])

$$\begin{aligned} &[\mathbf{C}/\mu_r], \quad \text{where } \mu_r \text{ acts in the standard way at } 0, \text{ and} \\ &[\mathbf{C}/\mu_r], \quad \text{where } \mu_r \text{ acts as } \zeta\colon z \mapsto \zeta^{-1} z \text{ at } \infty. \end{aligned}$$

There is a natural morphism of stacks $C_{r,r} \to \mathbf{P}^{1,r}$ of degree $1/r$; this morphism is representable at 0 and non-representable at ∞. We denote by

$$\begin{array}{ccc} \mathcal{C}'' & \xrightarrow{f''} & \mathbf{P}^{1,r} \\ {\scriptstyle \sigma''_0, \sigma''_\infty} \uparrow \downarrow {\scriptstyle p''} & & \\ S & & \end{array}$$

the trivial family $\mathcal{C}'' = S \times C_{r,r}$ over S with (non-representable) morphism to $\mathbf{P}^{1,r}$. By definition, the functor $\iota(S)\colon M_d(S) \to G_d(S)$ maps the family (35) to the family

$$\begin{array}{ccc} \mathcal{C}/S \longrightarrow \mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r} := \mathcal{C}' \cup_{\sigma', \sigma''_\infty} \mathcal{C}'' & \xrightarrow{(f', \infty) \cup (\mathrm{ev}'_1\, p'', f'')} & \mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r} \\ {\scriptstyle \sigma''_0} \uparrow \downarrow {\scriptstyle p' \cup p''} & & \\ S. & & \end{array} \tag{36}$$

The glued family $\mathcal{C}' \cup_{\sigma', \sigma''_\infty} \mathcal{C}''$ here is constructed using [4, Proposition A.1.1]. It is easy to see that the functors $\iota(S)\colon M_d(S) \to G_d(S)$ combine to give a closed substack $\iota\colon M_d \hookrightarrow G_d$.

([13]) We have $C_{r,r} = [\mathbf{P}^1/\mu_r]$, where μ_r acts via $\xi\colon [a_0 : a_1] \mapsto [\xi a_0 : a_1]$.

LEMMA 4.13. *The substack* $\iota: M_d \hookrightarrow G_d$ *is a* $\mathbf{C}^{\times}$*-fixed substack.*

Sketch of proof. This is an extended exercise in unravelling the definition of fixed substack, which was given in Remark 4.11. We give a sketch since we could find no adequate reference in the literature. A well-written and careful treatment of group actions on stacks can be found in [44].

Consider an object $\xi'_S = (f': \mathcal{C}'/S \to \mathbf{P}^{\mathbf{w}})$ of $M_d(S)$ as in (35), and let

$$\xi_S = \iota(S)(\xi'_S) = (f: \mathcal{C}/S \to \mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r})$$

be the family of diagram (36). We must exhibit, for every S-point $\lambda \in \mathrm{Mor}(S, \mathbf{C}^{\times})$, an isomorphism from ${}^{\lambda}\xi_S$ to ξ_S which is sufficiently natural that it satisfies all the necessary compatibilities and produces the isomorphism of functors $\Rightarrow$. This all follows from the claim below.

CLAIM 4.14. *In the notation of the preceding paragraph, there is a natural* $\mathbf{C}^{\times}$*-action on* $\mathcal{C}$ *which covers the trivial action on* S *such that the morphism* $f: \mathcal{C} \to \mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}$ *is* $\mathbf{C}^{\times}$*-equivariant.*

This is obvious: the family $\mathcal{C}$ is obtained by gluing the families $\mathcal{C}'$ and $\mathcal{C}'' = S \times C_{r,r}$. $\mathbf{C}^{\times}$ acts on $\mathcal{C}''$ by acting on the second factor alone, and this action glues with the trivial action on $\mathcal{C}'$ to give an action on $\mathcal{C}$.

Now the claim precisely says that, for all $\lambda \in \mathbf{C}^{\times}(S)$, the left translation $\ell_{\lambda^{-1}}: \mathcal{C} \to \mathcal{C}$ sits in a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \ell_{\lambda^{-1}}\ } & \mathcal{C} \\ & {}^{\lambda}f \searrow \quad \swarrow f & \\ & \mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}. & \end{array}$$

That is, exactly as desired, $\ell_{\lambda^{-1}}$ defines an isomorphism from ${}^{\lambda}\xi_S$ to ξ_S. This shows that $\iota: M_d \hookrightarrow G_d$ is a $\mathbf{C}^{\times}$-fixed substack. □

We show in Lemma 4.21 below that $\iota: M_d \hookrightarrow G_d$ is a *part* of the $\mathbf{C}^{\times}$-fixed substack.

4.3.3. Perfect obstruction theory

We recall some facts about perfect obstruction theories from [8] and [34]. For a morphism $q: \mathcal{X} \to \mathcal{S}$ of stacks, we denote by $L_q^{\bullet}$ the first-two-term cutoff of the cotangent complex of q. The official references for the cotangent complex are [27] and [28], but an accessible introduction to the first-two-term cutoff can be found in [24]. Recall that a relative perfect obstruction theory is a q-perfect 2-term complex $E^\bullet$ on $\mathcal{X}$ together with a morphism $\varphi: E^\bullet \to L_q^\bullet$ which is an isomorphism on H^0 and surjective on H^{-1}; a relative perfect obstruction theory produces a virtual fundamental class $\mathbf{1}_q^{\mathrm{vir}} \in CH_\bullet(\mathcal{X})$.

Let $\mathcal{X}$ be a stack and $d \in H_2(\mathcal{X}; \mathbf{Q})$. Denote, as usual, by $\mathcal{X}_{g,n,d}$ the moduli stack of genus-zero n-pointed stable morphisms to $\mathcal{X}$ of degree d; analogous remarks apply to the stacks $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ of n-pointed stable morphisms with sections to all gerbes. There are, as we now recall, two natural obstruction theories on $\mathcal{X}_{g,n,d}$ and they produce the same virtual fundamental class. Consider the universal family

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle\pi}\downarrow & & \\ \mathcal{X}_{g,n,d}. & & \end{array}$$

(1) The *relative obstruction theory* $E_{\mathrm{rel}}^{\bullet\vee} = R\pi_\star f^\star T_{\mathcal{X}}$ is an obstruction theory relative to the canonical morphism $q: \mathcal{X}_{g,n,d} \to \mathfrak{M}_{g,n}^{\mathrm{tw}}$ to the Artin stack of pre-stable twisted curves. The relative obstruction theory is used in [3] and [4], because it is well-suited to checking the axioms of Gromov–Witten theory.

(2) The *absolute obstruction theory* is

$$E^{\bullet\vee} = R\pi_\star R\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{U}}}(L_f^\bullet, \mathcal{O}_{\mathcal{U}}), \quad \text{where } L_f^\bullet = [f^\star \Omega_{\mathcal{X}}^1 \to \Omega_\pi^1(\log)]$$

is the cotangent complex of f; here $\Omega_\pi^1(\log)$ denotes the sheaf of Kähler differentials with logarithmic poles along the markings.

It is well-known that the absolute and relative obstruction theories produce the same fundamental class (see [23, Appendix B], [32, Proposition 3] and [40, §5.3.5]). In what follows, we use the absolute theory.

4.3.4. Obstructions and virtual normal bundle

In this section we compare obstruction theories and calculate the virtual normal bundle of $\iota: M_d \hookrightarrow G_d$.

We recall a few general notions from [23]. Let G be a group scheme acting on a stack $\mathcal{X}$ and let $E^\bullet \to L^\bullet$ be a G-linearized perfect obstruction theory. Let $\iota: \mathcal{Y} \hookrightarrow \mathcal{X}$ be the G-fixed substack. Then G acts on $E^\bullet|_{\mathcal{Y}}$, and it is a fact that the complex of G-invariants $E^{-1}|_{\mathcal{Y}}^G \to E^0|_{\mathcal{Y}}^G$ is an obstruction theory for $\mathcal{Y}$. We call this the *inherited obstruction theory*. Writing $E^i|_{\mathcal{Y}} = E^i|_{\mathcal{Y}}^G + E^i|_{\mathcal{Y}}^{\mathrm{mov}}$, the moving part $E^0|_{\mathcal{Y}}^{\mathrm{mov}\vee} \to E^{-1}|_{\mathcal{Y}}^{\mathrm{mov}\vee}$ is the *virtual normal bundle*.

LEMMA 4.15. (1) *The obstruction theory on M_d inherited from $\iota: M_d \hookrightarrow G_d$ is the natural absolute obstruction theory on M_d.*

(2) *Denoting the virtual normal bundle of ι by N_ι^{vir}, we have*

$$e(N_\iota^{\mathrm{vir}}) = z(z-\psi_1).$$

Sketch of proof. The statement is well known in a similar context, so we just give a sketch of the proof here. We start with an object $f': \mathcal{C}'/S \to \mathbf{P}^{\mathbf{w}}$ of $M_d(S)$ as in (35) and apply the functor $\iota(S)$ to make $f: \mathcal{C}/S \to \mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}$ as in diagram (36). The first statement means that the natural homomorphism

$$Rp_\star R\underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{C}}(L_f^\bullet, \mathcal{O}_\mathcal{C}) \longrightarrow Rp'_\star R\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{C}'}}(L_{f'}^\bullet, \mathcal{O}_{\mathcal{C}'}) \tag{37}$$

induces an isomorphism from the direct summand

$$Rp_\star^{\mathbf{C}^\times} R\underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{C}}(L_f^\bullet, \mathcal{O}_\mathcal{C}) \longrightarrow Rp'_\star R\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{C}'}}(L_{f'}^\bullet, \mathcal{O}_{\mathcal{C}'}).$$

Since both complexes are *perfect*, we can check this after base change to all geometric points; in effect we can and do from now on assume that $S = \mathrm{Spec}\,\mathbf{C}$, that $\mathcal{C} = C$ is a pre-stable curve over $\mathrm{Spec}\,\mathbf{C}$, etc.

Applying the cohomological functor $R\,\mathrm{Hom}_{\mathcal{O}_C}(-, \mathcal{O}_C)$ to the exact triangle

$$\Omega_p^1(\log) \longrightarrow L_f^\bullet \longrightarrow f^\star \Omega^1_{\mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}}[1] \xrightarrow{+1}$$

we calculate $E^{0\vee} = \mathbf{T}_f^1$ and $E^{-1\vee} = \mathbf{T}_f^2$ from the well-known exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(C, \Theta_C(-\log)) \longrightarrow H^0(C, f^\star T_{\mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}}) \longrightarrow \mathbf{T}_f^1 \longrightarrow \\ \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C^1(\log), \mathcal{O}_C) \longrightarrow H^1(C, f^\star T_{\mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}}) \longrightarrow \mathbf{T}_f^2 \longrightarrow 0. \end{aligned} \tag{38}$$

Our goal is to determine each piece in the exact sequence (38) as a representation of $\mathbf{C}^\times$; we make the following simple observations.

(1) $\Theta_C(-\log) = \Theta_{C'}(-\log) \oplus \Theta_{\mathbf{P}^1}(-0-\infty)$, and hence

$$H^0(C, \Theta_C(-\log)) = H^0(C', \Theta_{C'}(-\log)) \oplus \mathbf{C}(z)$$

with the first summand being a trivial representation.

(2) $f^\star T_{\mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}} = f_1^\star T_{\mathbf{P}^{\mathbf{w}}} \oplus f_2^\star T_{\mathbf{P}^{1,r}}$, where $f_1: C \to \mathbf{P}^{\mathbf{w}}$ and $f_2: C \to \mathbf{P}^{1,r}$ are the natural morphisms. Thus

$$H^0(C, f^\star T_{\mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}}) = H^0(C', (f')^\star T_{\mathbf{P}^{\mathbf{w}}}) \oplus H^0(\mathbf{P}^{1,r}, T_{\mathbf{P}^{1,r}}),$$

where the first summand is $\mathbf{C}^\times$-fixed and the second summand is moving (and easy to calculate as a representation using the equivariant Euler sequence on $\mathbf{P}^{1,r}$).

(3) We calculate $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C(\log), \mathcal{O}_C)$ with the standard local-to-global spectral sequence:

$$0 \longrightarrow H^1(C, \Theta_C(-\log)) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C(\log), \mathcal{O}_C) \longrightarrow H^0(C, \underline{\mathrm{Ext}}^1_{\mathcal{O}_C}(\Omega^1_C(\log), \mathcal{O}_C)) \longrightarrow 0.$$

Now, $H^1(C, \Theta_C(-\log)) = H^1(C', \Theta'(-\log))$ is a trivial representation, whereas

$$H^0(C, \underline{\mathrm{Ext}}^1_{\mathcal{O}_C}(\Omega^1_C(\log), \mathcal{O}_C)) = H^0(C', \underline{\mathrm{Ext}}^1_{\mathcal{O}_{C'}}(\Omega^1_{C'}(\log), \mathcal{O}_{C'})) \oplus (T_{C',\sigma'} \otimes \mathbf{C}(z)),$$

where the first summand is the trivial representation and the second summand is isomorphic to $\mathbf{C}(z)$. From this and the Five Lemma, we conclude that

$$\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C(\log), \mathcal{O}_C) = \mathrm{Ext}^1_{\mathcal{O}_{C'}}(\Omega^1_{C'}(\log), \mathcal{O}_{C'}) \oplus (T_{C',\sigma'} \otimes \mathbf{C}(z))$$

as the sum of fixed and moving parts.

(4) As before,

$$H^1(C, f^\star T_{\mathbf{P}^\mathbf{w} \times \mathbf{P}^{1,r}}) = H^1(C', (f')^\star T_{\mathbf{P}^\mathbf{w}}).$$

Using the above facts and the Five Lemma, it is easy to finish the proof. □

4.4. Construction and properties of the morphism u

We give a precise construction of the morphism u following closely the argument of Jun Li [35, Lemma 2.6]. Finally, we show that the morphism $u: G_d \to L_d$ is virtually birational.

LEMMA 4.16. *There is a natural morphism $u: G_d \to L_d$.*

Proof. We sketch the proof, which follows closely [35, Lemma 2.6]. For all schemes S, we construct functors $G_d(S) \to L_d(S)$. This is not difficult to do, since L_d is itself a weighted projective space. It therefore satisfies a universal property which makes it easy to construct elements of $L_d(S)$. Let us spell this out more precisely. We set

$$W = \mathbf{C}(-1) \oplus \mathbf{C}(-r), \quad \text{so that} \quad \mathbf{P}^{1,r} = [(W \setminus \{0\})/\mathbf{T}^1].$$

Note that the free polynomial algebra $S^*W^\vee$ generated by $W^\vee$ is a representation of $\mathbf{C}^\times$. We denote by $S^mW^\vee$ the isotypic component on which $\mathbf{C}^\times$ acts with weight $m \in \mathbf{Z}$; $S^*W^\vee$ is generated by a basis element $s_0 \in W^\vee$ of degree 1 and a basis element $s_1 \in W^\vee \cap S^rW^\vee$ of degree r. A polynomial map $\mathbf{P}^{1,r} \to \mathbf{P}^\mathbf{w}$ of degree $d = m/r$ is given by polynomials $P_0, ..., P_n \in S^{mw_i}W^\vee$, not all identically zero:

$$L_d = \left[\left(\bigoplus_{i=0}^n S^{mw_i}W^\vee \setminus \{0\}\right)/\mathbf{C}^\times\right].$$

From this we conclude the claim below.

Claim 4.17. *Let S be a scheme. An object of $L_d(S)$ consists of a line bundle $\mathcal{L}$ on S and a nowhere-vanishing sheaf homomorphism*

$$(P_0, ..., P_n): \mathcal{O}_S^n \longrightarrow \bigoplus_{i=0}^{n} \mathcal{L}^{\otimes w_i} \otimes S^{m w_i} W^{\vee}.$$

Let us now proceed to the proof of Lemma 4.16. An object of $G_d(S)$ is a stable morphism

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{(p_2,p_3)} & \mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r} \\ {\scriptstyle p_1}\downarrow & & \\ S. & & \end{array} \tag{39}$$

(Depending on whether or not d is an integer, there may be a section $\sigma: S \to \mathcal{C}$; the section plays no role in what follows.) Let us rearrange the diagram as

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{p_2} & \mathbf{P}^{\mathbf{w}} \\ {\scriptstyle q:=(p_1,p_3)}\downarrow & & \\ S \times \mathbf{P}^{1,r}. & & \end{array} \tag{40}$$

Claim 4.18. (1) *The sheaves*

$$\mathcal{F}_k = q_\star p_2^\star \mathcal{O}_{\mathbf{P}^{\mathbf{w}}}(k)$$

are flat over S and generically of rank 1.

(2) *There is a line bundle $\mathcal{L}$ on S such that*

$$\det \mathcal{F}_k = \mathcal{L}^{\otimes k} \boxtimes \mathcal{O}_{\mathbf{P}^{1,r}}(mk).$$

This is proved in [35], and it easily implies the result. The canonical sections $x_i \in H^0(\mathbf{P}^{\mathbf{w}}, \mathcal{O}(w_i))$ give elements $p_2^\star x_i \in H^0(S \times \mathbf{P}^{1,r}, \mathcal{F}_{w_i})$, and using the canonical sheaf homomorphism $\mathcal{F}_k \to \det \mathcal{F}_k$ ($\mathcal{F}_k$ has rank 1!), these map to elements P_i of

$$H^0(S \times \mathbf{P}^{1,r}, \mathcal{L}^{\otimes w_i} \boxtimes \mathcal{O}_{\mathbf{P}^{1,r}}(m w_i)) = H^0(S, \mathcal{L}^{\otimes w_i} \otimes S^{m w_i} W^{\vee}).$$

Thus we have constructed a sequence $(P_0, ..., P_n)$ of elements of $H^0(S, \mathcal{L}^{\otimes w_i} \otimes S^{m w_i} W^{\vee})$ and this, by virtue of Claim 4.17, gives an object of $L_d(S)$. □

It is useful to know the morphism u explicitly at geometric points. Consider an element $\varphi\colon\mathcal{C}\to\mathbf{P}^{\mathbf{w}}\times\mathbf{P}^{1,r}$ of $\overline{\mathcal{M}}_{0,1}(\mathbf{P}^{\mathbf{w}}\times\mathbf{P}^{1,r},d\times 1/r)$. Write

$$\mathcal{C}=\bigcup_{j=0}^{N}\mathcal{C}_j, \tag{41}$$

where

(1) $\mathcal{C}_0$ is the distinguished component mapping one-to-one to $\mathbf{P}^{1,r}$;

(2) the curves $\mathcal{C}_j$ for $j\geqslant 1$ are "vertical": they map to points $y_j\in\mathbf{P}^{1,r}$ given by equations $s_1-a_js_0^r=0$.

Assume for simplicity that the marked point $x_0\in\mathcal{C}$ *lies on* $\mathcal{C}_0$, and note that

(1) the marked point x_0 lies above $0\in\mathbf{P}^{1,r}$;

(2) for each $j\geqslant 1$ the curve $\mathcal{C}_j$ meets $\mathcal{C}_0$ in a unique point x_j, which lies above $y_j\in\mathbf{P}^{1,r}$, and the induced morphism $\varrho_j\colon(\mathcal{C}_j,x_j)\to\mathbf{P}^{\mathbf{w}}$ is representable and stable;

(3) the morphism $\varrho_0\colon(\mathcal{C}_0,\{x_0,x_1,...,x_N\})\to\mathbf{P}^{\mathbf{w}}$ is representable and pre-stable.

Write $d_j=\deg\varrho_j$ and $\bar f_j=\langle -d_j\rangle$, so that $\varrho_j\in\overline{\mathcal{M}}_{0,1}(\mathbf{P}^{\mathbf{w}},d_j)$. Clearly, $d=\sum_{j=0}^N d_j$. The gerbe at x_j in $\mathcal{C}_0$ evaluates to $\mathbf{P}(V^{f_j})$, where $f_0=f=\langle -d\rangle$ and, for $j\geqslant 1$,

$$f_j=\begin{cases}1-\bar f_j, & \text{if } \bar f_j\neq 0,\\ 0, & \text{if } \bar f_j=0.\end{cases}$$

Lemma 4.19. *In these circumstances, the polynomial map* $u(\varphi)\in L_d$ *constructed in* Lemma 4.16 *is given by homogeneous polynomials*

$$\begin{pmatrix}P_0(s_0,s_1)\\ \vdots\\ P_i(s_0,s_1)\\ \vdots\\ P_n(s_0,s_1)\end{pmatrix}=\begin{pmatrix}Q_0(s_0,s_1)\prod_{j=1}^N(s_1-a_js_0^r)^{\lfloor d_j\rfloor w_0}\\ \vdots\\ Q_i(s_0,s_1)\prod_{j=1}^N(s_1-a_js_0^r)^{\lfloor d_j\rfloor w_i}\\ \vdots\\ Q_n(s_0,s_1)\prod_{j=1}^N(s_1-a_js_0^r)^{\lfloor d_j\rfloor w_n}\end{pmatrix},$$

where $\deg Q_i=r\big(d_0+\sum_{j=1}^N f_j\big)w_i$.

Proof. This follows closely the classical case [35, Lemma 2.6]. □

We have

$$\deg P_i=r\left(d_0+\sum_{j=1}^N f_j\right)w_i+r\left(\sum_{j=1}^N\lfloor d_j\rfloor\right)w_i=r\left(d_0+\sum_{j=1}^N(\lfloor d_j\rfloor+f_j)\right)w_i=rdw_i=mw_i.$$

In addition, one should note that the polynomials Q_i themselves usually must contain common factors which account for the "stacky behaviour" of the morphism ϱ_0 above the points $y_j\in\mathbf{P}^{1,r}$. More precisely, for all i,

$$(s_1-a_js_0^r)^{\langle\bar f_jw_i\rangle+f_jw_i}\ \text{is a factor of } Q_i(s_0,s_1),$$

and it is an exact factor for at least one i such that w_if_j is an integer.

Corollary 4.20. *The basic diagram of Theorem* 4.5, *where all stacks and morphisms have by now been constructed*, *is a commutative diagram of stacks with* $\mathbf{C}^\times$-*action.*

Lemma 4.21. *The substack* $\iota\colon M_d \hookrightarrow G_d$ *is the part of the* $\mathbf{C}^\times$-*fixed substack that lies above* $j\colon \mathbf{P}(V^f) \hookrightarrow L_d$.

Proof. The basic diagram of Theorem 4.5 is a commutative diagram of stacks with $\mathbf{C}^\times$-action. The $\mathbf{C}^\times$-fixed substack of G_d is therefore a disjoint union of parts lying above the connected components of the $\mathbf{C}^\times$-fixed substack of L_d. The image of $j\colon \mathbf{P}(V^f) \hookrightarrow L_d$ is one of these components, and we show that $\iota\colon M_d \hookrightarrow G_d$ is the part of the $\mathbf{C}^\times$-fixed stack lying above $\mathbf{P}(V^f)$ by showing that it has the required universal property.

First, we show that this holds over geometric points. Let $\varphi\colon \mathcal{C} \to \mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}$ be a $\mathbf{C}^\times$-fixed point of G_d. Write $\mathcal{C} = \bigcup_{j=0}^{N} \mathcal{C}_j$ as in (41), so that $\mathcal{C}_0$ is the distinguished component mapping one-to-one to $\mathbf{P}^{1,r}$ and the $\mathcal{C}_j$ are vertical for $j \geqslant 1$. Since φ is $\mathbf{C}^\times$-fixed, by the very way the $\mathbf{C}^\times$-action is defined, the image $\varphi(\mathcal{C}) \subset \mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}$ is invariant under the action of $\mathbf{C}^\times$ on $\mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}$ acting on the second factor only. This implies that $\varphi(\mathcal{C}_0)$ is a *horizontal curve*; it then follows from Lemma 4.19 and Corollary 4.20 that there is only one vertical curve $\mathcal{C}_j$ and that it is joined to $\mathcal{C}_0$ over $\infty \in \mathbf{P}^{1,r}$. In other words, φ is *isomorphic* to a point in the image of ι.

We are now ready to finish the proof of the lemma. Consider a base scheme S and a $\mathbf{C}^\times$-fixed object of $G_d(S)$:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\;f\;} & \mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r} \\ {\scriptstyle \sigma}\big\uparrow\big\downarrow{\scriptstyle p} & & \\ S. & & \end{array} \tag{42}$$

All we need to show is that $\mathcal{C} = \mathcal{C}' \cup_{\sigma', \sigma''_\infty} \mathcal{C}''$ as in diagram (36). First of all, by what we said on geometric points, family (42), considered as a family of pre-stable curves, is the pull-back from a unique morphism to the "boundary" substack

$$\mathfrak{M}^{\mathrm{tw}}_{0,2} \times_{B\mu_r} \mathfrak{M}^{\mathrm{tw}}_{0,1} \longrightarrow \mathfrak{M}^{\mathrm{tw}}_{0,1},$$

where $\mathfrak{M}^{\mathrm{tw}}_{g,n}$ is the *smooth* Artin stack of pre-stable n-pointed twisted curves of genus g constructed in [41]. That is, $\mathcal{C} = \mathcal{C}' \cup_{\sigma', \sigma''_\infty} \mathcal{C}''$ *as a family of pre-stable curves.* Now [4, Proposition 5.2.2] implies that $\mathcal{C} = \mathcal{C}' \cup_{\sigma', \sigma''_\infty} \mathcal{C}''$ as families of stable morphisms. □

Lemma 4.22. *The morphism* u *is virtually birational*:

$$u_\star \mathbf{1}^{\mathrm{vir}}_{G_d} = \mathbf{1}_{L_d}.$$

Before proving this, it is useful to calculate the virtual dimension of the two stacks.

LEMMA 4.23. *We have*

$$\dim \mathbf{1}_{G_d}^{\mathrm{vir}} = \dim L_d = n + \sum_{i=0}^{n} \lfloor d w_i \rfloor.$$

Proof. We calculate using the dimension formula of equation (9),

$$\begin{aligned}\dim \mathbf{1}_{G_d}^{\mathrm{vir}} &= 1 + \dim(\mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}) - 3 - K_{\mathbf{P}^{\mathbf{w}} \times \mathbf{P}^{1,r}} \cdot (d, 1/r) - \mathrm{age} \\ &= 1 + n + 1 - 3 + d\left(\sum_{i=0}^{n} w_i\right) + \frac{r+1}{r} - \sum_{i=0}^{n} \langle -f w_i \rangle - \frac{1}{r} \\ &= n + \sum_{i=0}^{n} \lfloor d w_i \rfloor.\end{aligned}$$

□

Proof of Lemma 4.22. There is a unique component of G_d generically parameterizing morphisms from irreducible curves and it maps generically one-to-one to L_d. This component of G_d is generically smooth and of the expected dimension; the virtual fundamental class of this component thus coincides with the usual fundamental class and pushes forward to give the fundamental class of L_d. If a component of G_d generically parameterizes morphisms from reducible curves, it maps to a proper subvariety of L_d. □

4.5. Proof of Theorem 4.5

Putting together all the pieces, we have a proof of Theorem 4.5. The existence of the commutative diagram was shown in Corollary 4.20; the first statement is Remark 4.9 and Lemma 4.12; the second statement is Lemma 4.21 and Lemma 4.15; the third statement is Lemma 4.22.

5. The small quantum cohomology of weighted projective spaces

In this section we prove Theorem 1.1. As was discussed in §2.3.1, and as we will see rather explicitly below, to determine the small quantum orbifold cohomology algebra of $\mathbf{P}^{\mathbf{w}}$, it suffices to compute the directional derivatives

$$\nabla_{\phi_i} \mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)|_{\tau \in H^2(\mathcal{X};\mathbf{C}) \subset H^{\bullet}_{\mathrm{orb}}(\mathcal{X};\mathbf{C})}, \quad i \in \{1, 2, ..., N\}, \tag{43}$$

where $\phi_1, ..., \phi_N$ is a basis for $H^{\bullet}_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}}; \mathbf{C})$. We have computed the small J-function $J_{\mathbf{P}^{\mathbf{w}}}(t)$, which is the restriction of $\mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)$ to $H^2(\mathbf{P}^{\mathbf{w}}; \mathbf{C}) \subset H^{\bullet}_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}}; \mathbf{C})$:

$$J_{\mathbf{P}^{\mathbf{w}}}(t) = \mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(tP).$$

This does not, a priori, determine the directional derivatives

$$\nabla_y \mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)|_{\tau\in H^2(\mathcal{X};\mathbf{C})\subset H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbf{C})}$$

along directions y not in $H^2(\mathbf{P}^{\mathbf{w}};\mathbf{C})$, but it does allow us to calculate multiple derivatives

$$\nabla_P \dots \nabla_P \mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)|_{\tau=tP} = \frac{\partial}{\partial t} \dots \frac{\partial}{\partial t} J_{\mathbf{P}^{\mathbf{w}}}(t).$$

We will combine these calculations with the differential equations (19) to determine the directional derivatives (43).

Let $N=w_0+\dots+w_n$ and let $c_1,\dots,c_N$ be the sequence obtained by arranging the terms

$$\frac{0}{w_0},\frac{1}{w_0},\dots,\frac{w_0-1}{w_0},\quad \frac{0}{w_1},\frac{1}{w_1},\dots,\frac{w_1-1}{w_1},\quad \dots,\quad \frac{0}{w_n},\frac{1}{w_n},\dots,\frac{w_n-1}{w_n}$$

in increasing order. Define differential operators

$$D_j = \begin{cases} \mathrm{id}, & \text{if } j=1,\\ Q^{-c_j}e^{-c_jt}\prod_{m=1}^{j-1}(z\partial/\partial t - zc_m), & \text{if } 1<j\leqslant N.\end{cases}$$

LEMMA 5.1. *There exist* $v_1,\dots,v_N\in H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\Lambda)$ *such that*

$$z^{-1}D_jJ_{\mathbf{P}^{\mathbf{w}}}(t) = \nabla_{v_j}\mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)|_{\tau=tP},\quad j\in\{1,2,\dots,N\}.$$

Furthermore,

(a) $v_1=\mathbf{1}_0$;

(b) $v_{j+1}=Q^{c_j-c_{j+1}}e^{(c_j-c_{j+1})t}P\circ_{tP}v_j$, $1\leqslant j<N$;

(c) $v_j=\sigma_jP^{r_j}\mathbf{1}_{c_j}$, $1\leqslant j\leqslant N$, *where*

$$\sigma_j = \frac{\prod_{m:c_m<c_j}(c_j-c_m)}{\prod_{i=0}^n\prod_{b:\langle b\rangle=\langle c_jw_i\rangle,0<b\leqslant c_jw_i} b}$$

and

$$r_j = |\{i : i<j \text{ and } c_i=c_j\}|.$$

In particular, $v_1,\dots,v_N$ *is a basis for* $H^\bullet_{\mathrm{orb}}(\mathbf{P}^{\mathbf{w}};\mathbf{C})$.

Remark 5.2. Note that the sequence $c_1,\dots,c_N$ is

$$\underbrace{f_1,\dots,f_1}_{\dim_{f_1}+1},\quad \underbrace{f_2,\dots,f_2}_{\dim_{f_2}+1},\quad \dots,\quad \underbrace{f_k,\dots,f_k}_{\dim_{f_k}+1}$$

and that the sequence $\sigma_1,\dots,\sigma_N$ is

$$\underbrace{s_1,\dots,s_1}_{\dim_{f_1}+1},\quad \underbrace{s_2,\dots,s_2}_{\dim_{f_2}+1},\quad \dots,\quad \underbrace{s_k,\dots,s_k}_{\dim_{f_k}+1},$$

where $f_1,\dots,f_k$ are defined above equation (1) and $s_1,\dots,s_k$ are defined in Theorem 1.1.

Proof of Lemma 5.1. The string equation [4, Theorem 8.3.1] implies that

$$z\nabla_{\mathbf{1}_0}\mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)=\mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau),$$

so we can take $v_1=\mathbf{1}_0$. Assume that

$$z^{-1}D_jJ_{\mathbf{P}^{\mathbf{w}}}(t)=\nabla_{v_j}\mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)|_{\tau=tP}$$

for some j with $1\leqslant j\leqslant N-1$. Since

$$z\frac{\partial}{\partial t}D_j=Q^{c_{j+1}-c_j}e^{(c_{j+1}-c_j)t}D_{j+1},$$

we have

$$\begin{aligned}z^{-1}D_{j+1}J_{\mathbf{P}^{\mathbf{w}}}(t)&=Q^{c_j-c_{j+1}}e^{(c_j-c_{j+1})t}\frac{\partial}{\partial t}D_jJ_{\mathbf{P}^{\mathbf{w}}}(t)\\&=Q^{c_j-c_{j+1}}e^{(c_j-c_{j+1})t}z\frac{\partial}{\partial t}(\nabla_{v_j}\mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)|_{\tau=tP})\\&=Q^{c_j-c_{j+1}}e^{(c_j-c_{j+1})t}\nabla_{P\circ_\tau v_j}\mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)|_{\tau=tP}\qquad\text{(cf. (19) and Lemma 2.4).}\end{aligned}$$

Thus, we can take

$$v_{j+1}=Q^{c_j-c_{j+1}}e^{(c_j-c_{j+1})t}P\circ_{tP}v_j.$$

By induction, this proves the existence of $v_1,...,v_N$. It also proves (a) and (b).

We know that

$$\nabla_{v_j}\mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)=v_j+O(z^{-1})$$

and that

$$\nabla_{v_j}\mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)|_{\tau=tP}=\frac{1}{z}D_jJ_{\mathbf{P}^{\mathbf{w}}}(t),$$

so to establish (c) we need to compute the coefficient of z in

$$D_jJ_{\mathbf{P}^{\mathbf{w}}}(t)=ze^{Pt/z}\sum_{\substack{d:d\geqslant0\\\langle d\rangle\in F}}Q^{d-c_j}e^{(d-c_j)t}\mathbf{1}_{\langle d\rangle}\frac{\prod_{m=1}^{j-1}(P+(d-c_m)z)}{\prod_{i=0}^n\prod_{b:\langle b\rangle=\langle dw_i\rangle,0<b\leqslant dw_i}(w_iP+bz)}.$$

The degree in z of the denominator of the dth summand here is

$$\lceil w_0d\rceil+\lceil w_1d\rceil+...+\lceil w_nd\rceil,$$

which is the number of fractions k/w_i, $k\geqslant0$ and $0\leqslant i\leqslant n$, which are strictly less than d. If $d>c_j$ then this exceeds the degree in z of the numerator and so the dth summand, when expanded as a Laurent series in z^{-1}, is $O(z^{-1})$. Recall that $\mathbf{P}(V^f)$ is a weighted projective space of dimension $\dim_f$. If $d<c_j$ then, by Remark 5.2, there are $\dim_d+1$ values of l such that $l\in\{1,2,...,j\}$ and $c_l=d$. This implies that the dth summand above contains a factor of

$$P^{\dim_d+1}\mathbf{1}_d,$$

which vanishes for dimensional reasons. Thus only the summand where $d=c_j$ contributes to the coefficient of z:

$$D_j J_{\mathbf{P}^{\mathbf{w}}}(t)=ze^{Pt/z}\mathbf{1}_{c_j}\frac{P^{r_j}\prod_{m:c_m<c_j}(P+(c_j-c_m)z)}{\prod_{i=0}^{n}\prod_{b:\langle b\rangle=\langle c_jw_i\rangle,0<b\leqslant c_jw_i}(w_iP+bz)}+o(z).$$

The degree in z of the numerator and denominator here are equal, so

$$D_j J_{\mathbf{P}^{\mathbf{w}}}(t)=z\mathbf{1}_{c_j}\frac{P^{r_j}\prod_{m:c_m<c_j}(c_j-c_m)}{\prod_{i=0}^{n}\prod_{b:\langle b\rangle=\langle c_jw_i\rangle,0<b\leqslant c_jw_i}b}+o(z)$$

and therefore $v_j=\sigma_j P^{r_j}\mathbf{1}_{c_j}$, as claimed. □

Lemma 5.3. *We have*

$$P\circ_{tP}v_N=\frac{1}{w_0^{w_0}w_1^{w_1}\dots w_n^{w_n}}Q^{1-c_N}e^{(1-c_N)t}\mathbf{1}_0.$$

Proof. On the one hand

$$\begin{aligned}\nabla_{P\circ_{tP}v_N}\mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)|_{\tau=tP}&=z\nabla_P\nabla_{v_N}\mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)|_{\tau=tP}\qquad\text{(cf. (19))}\\&=\frac{\partial}{\partial t}D_N J_{\mathbf{P}^{\mathbf{w}}}(t),\end{aligned}$$

and on the other hand

$$\nabla_{P\circ_{tP}v_N}\mathbf{J}_{\mathbf{P}^{\mathbf{w}}}(\tau)|_{\tau=tP}=P\circ_{tP}v_N+O(z^{-1}),$$

so we need to compute the coefficient of z^0 in

$$\frac{\partial}{\partial t}D_N J_{\mathbf{P}^{\mathbf{w}}}(t)=e^{Pt/z}\sum_{\substack{d:d\geqslant 0\\ \langle d\rangle\in F}}Q^{d-c_N}e^{(d-c_N)t}\mathbf{1}_{\langle d\rangle}\frac{\prod_{m=1}^{N}(P+(d-c_m)z)}{\prod_{i=0}^{n}\prod_{b:\langle b\rangle=\langle dw_i\rangle,0<b\leqslant dw_i}(w_iP+bz)}.$$

Arguing exactly as in the proof of Lemma 5.1 (c), we see that only the summand with $d=1$ contributes and that

$$\frac{\partial}{\partial t}D_N J_{\mathbf{P}^{\mathbf{w}}}(t)=Q^{1-c_N}e^{(1-c_N)t}\mathbf{1}_0\frac{\prod_{m=1}^{N}(1-c_m)}{\prod_{i=0}^{n}w_i!}+O(z^{-1}).$$

Thus,

$$P\circ_{tP}v_N=Q^{1-c_N}e^{(1-c_N)t}\mathbf{1}_0\frac{\prod_{m=1}^{N}(1-c_m)}{\prod_{i=0}^{n}w_i!}=\frac{1}{w_0^{w_0}w_1^{w_1}\dots w_n^{w_n}}Q^{1-c_N}e^{(1-c_N)t}\mathbf{1}_0.\qquad □$$

Lemmas 5.1 and 5.3 together show that the matrix of small orbifold quantum multiplication $P\circ_{tP}$ with respect to the basis

$$Q^{c_1}e^{c_1t}v_1, \quad Q^{c_2}e^{c_2t}v_2, \quad \ldots, \quad Q^{c_N}e^{c_Nt}v_N \tag{44}$$

is

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & Qe^t/w_0^{w_0}w_1^{w_1}\ldots w_n^{w_n} \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

Corollary 5.4. *Theorem* 1.1 *holds.*

Proof. The basis (44) differs from the basis (1) by factors of σ_j, Q^j and e^{c_jt}. Taking account of these differences yields Theorem 1.1. □

6. Weighted projective complete intersections

Let $\mathcal{X}$ be a quasismooth complete intersection of type $(d_0, d_1, \ldots, d_m)$ in $\mathbf{P}^{\mathbf{w}}$ and let $\iota: \mathcal{X} \to \mathbf{P}^{\mathbf{w}}$ be the inclusion. Let $k_{\mathcal{X}} = \sum_{j=0}^m d_j - \sum_{i=0}^n w_i$. The main result of this section, Corollary 6.2, determines part of the big J-function of $\mathcal{X}$; it applies to quasismooth complete intersections with $k_{\mathcal{X}} \leqslant 0$.

We begin with a combinatorial lemma.

Lemma 6.1. (1) *If* $k_{\mathcal{X}} \leqslant 0$ *then, for all* $f \in F$,

$$\sum_{j=0}^m \lceil fd_j \rceil - \sum_{i=0}^n \lceil fw_i \rceil \leqslant fk_{\mathcal{X}}.$$

(2) *If* $k_{\mathcal{X}} = 0$ *then, for all non-zero* $f \in F$,

$$\sum_{j=0}^m \lceil fd_j \rceil - \sum_{i=0}^n \lceil fw_i \rceil < 0.$$

Proof. The proof is elementary; see [26, §8] for some useful facts about quasismooth complete intersections. Fix $f \in F$ and let $I = \{i : w_i f \in \mathbf{Z}\}$. Since $\mathcal{X}$ is quasismooth along $\mathbf{P}(V^f) \subset \mathbf{P}^{\mathbf{w}}$, we can reorder the d_j's and the w_i's such that

(1) for $j \leqslant l$, fd_j is not an integer and there is a monomial $x_I^{M_I}$ in the variables $\{x_i : i \in I\}$ such that $x_j x_I^{M_I}$ has degree d_j; in particular, this implies that $fd_j \equiv fw_j \bmod \mathbf{Z}$;

(2) for $l < j$, there is a monomial $x_I^{M_I}$ of degree d_j in the variables $\{x_i : i \in I\}$; in particular, this implies that fd_j is an integer.

Then

$$\sum_{j=0}^{m}\lceil fd_j\rceil = fk_{\mathcal{X}}+\sum_{i=0}^{l}\lceil fw_i\rceil+\sum_{i\in I} fw_i+\sum_{i\notin\{0,...,l\}\cup I} fw_i \leqslant fk_{\mathcal{X}}+\sum_{i=0}^{n}\lceil fw_i\rceil, \qquad (45)$$

and this is part (1) of the statement. If $k_{\mathcal{X}}=0$ then part (2) also follows unless we have equality in equation (45), that is, unless $\{0,...,l\}\cup I=\{0,...,n\}$. We show that this leads to a contradiction. Let $G_0,...,G_m$ be the equations of $\mathcal{X}$ of degrees $\deg G_j=d_j$. For $j=0,...,l$, we have that $fd_j\notin\mathbf{Z}$; this implies that $\mathbf{P}(V^f)=\{(x_0,...,x_n):x_0=...=x_l=0\}$ is an irreducible component of $\{x:G_0(x)=...=G_l(x)=0\}$. This in turn implies that $\mathcal{X}$ itself is reducible, a contradiction. □

COROLLARY 6.2. (1) *If* $k_{\mathcal{X}}<0$, *then*

$$I_{\mathcal{X}}(t)=\iota_\star(z+\tau(t)+O(z^{-1}))$$

for some function $\tau:\mathbf{C}\to H^\bullet_{\mathrm{orb}}(\mathcal{X};\Lambda)$, *and*

$$\iota_\star\mathbf{J}_{\mathcal{X}}(\tau(t))=I_{\mathcal{X}}(t).$$

(2) *If* $k_{\mathcal{X}}=0$, *then*

$$I_{\mathcal{X}}(t)=\iota_\star(F(t)z+G(t)+O(z^{-1}))$$

for some functions $F:\mathbf{C}\to\Lambda$, $G:\mathbf{C}\to H^\bullet_{\mathrm{orb}}(\mathcal{X};\Lambda)$ *and*

$$\iota_\star\mathbf{J}_{\mathcal{X}}(\tau(t))=\frac{I_{\mathcal{X}}(t)}{F(t)}, \quad \textit{where } \tau(t)=\frac{G(t)}{F(t)}.$$

Proof. The assertions $I_{\mathcal{X}}(t)=\iota_\star(...)$ follow by expanding $I_{\mathcal{X}}(t)$ as a Laurent series in z^{-1} and applying Lemma 6.1. The rest follows by combining Theorem 1.7 with the "Quantum Lefschetz" theorem [15, Corollary 5.1]. □

Corollary 1.9 follows immediately from Corollary 6.2, by computing the functions $\tau(t)$ in (1) and $G(t)$ in (2) using Lemma 6.1.

Proof of Proposition 1.10. We recall the Reid–Tai criterion for terminal singularities [43]. Fix a positive integer r and a set of integer weights $a_1,...,a_n$ and consider the space

$$\frac{1}{r}(a_1,...,a_n):=\mathbf{C}^n/\mu_r, \quad \text{where } \mu_r \text{ acts with weights } a_1,...,a_n.$$

We say that the set of weights is *well-formed* if $\gcd(r,a_1,...,a_{i-1},a_{i+1},...,a_n)=1$ for all i, that is, if the action of μ_r is faithful and there are no quasi-reflections. This means that the orbifold is "non-singular" in codimensions 0 and 1. The Reid–Tai criterion states that X is well-formed with terminal singularities if and only if

$$\sum_{i=1}^{n}\left\langle\frac{ka_i}{r}\right\rangle>1 \quad \text{for } k=1,2,...,r-1. \tag{46}$$

Terminal singularities are defined in [43]; for the purpose of this proof, the reader can take the Reid–Tai criterion as a definition.

We now proceed to the proof of the proposition. Let us assume that $\mathcal{X}=\mathcal{X}_{d_0,...,d_m}\subset \mathbf{P}^{\mathbf{w}}$ is quasismooth and well-formed with terminal singularities. Choose a non-zero $f\in F$. Assuming that

$$c=|\{i:fw_i\in\mathbf{Z}\}|-|\{j:fd_j\in\mathbf{Z}\}|>0,$$

we want to show that

$$\sum_{i=0}^{n}\langle fw_i\rangle>1+\sum_{j=0}^{m}\langle d_jf\rangle. \tag{47}$$

As in the proof of Lemma 6.1, we can reorder the d_j's and the w_i's so that

(1) $fd_j\equiv fw_j \bmod \mathbf{Z}$ for $j\leqslant l$, and none of these numbers is an integer;

(2) $fd_j\in\mathbf{Z}$ for $l<j$, and $fw_i\in\mathbf{Z}$ for $l<i\leqslant m+c$.

The singularities of $\mathcal{X}$ along $\mathbf{P}(V^f)$ are locally of the form

$$\frac{1}{r}(\underbrace{0,0,...,0}_{c},w_{m+c+1},...,w_n) \tag{48}$$

Inequality (47) is equivalent to

$$\sum_{i=m+c+1}^{n}\langle fw_i\rangle>1$$

and it holds by the Reid–Tai criterion for the singularity (48). The above argument can be read in reverse to show the converse: if the condition of Proposition 1.10 holds, then $\mathcal{X}$ has terminal singularities. □

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