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An extremal problem related to Kolmogoroff's inequality for bounded functions

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ABSTRACT

Let A and B be positive numbers and m and n positive integers, $m < n$. Then there is for complex valued functions φ on R with sufficient differentiability and boundedness properties a representation

$$\varphi^{(m)} = \varphi^{(n)} * \nu_1 + \varphi * \nu_2,$$

where ν_1 and ν_2 are bounded Borel measures with ν_1 absolutely continuous, such that there exists a function φ with $|\varphi^{(n)}| \leqslant A$ and $|\varphi| \leqslant B$ on R and satisfying

$$\varphi^{(m)}(0) = A \int_R |d\nu_1| + B \int_R |d\nu_2|.$$

This result is formulated and proved in a general setting also applicable to derivatives of fractional order. Necessary and sufficient conditions are given in order that the measures and the optimal functions have the same essential properties as those which occur in the particular case stated above.

1. We denote by $M(R)$ the Banach space of bounded Borel measures on R and by $AC(R)$ the subspace of $M(R)$ consisting of all measures which are absolutely continuous with respect to the Lebesgue measure. The Fourier–Stieltjes transform $\hat{\mu}$ of a measure $\mu \in M(R)$ is defined by the relation

$$\hat{\mu}(t) = \int_R e^{-itx} \, d\mu(x),$$

for every t on the dual R. Convolution of elements in $M(R)$ is defined in the usual way such that it corresponds to pointwise multiplication of the Fourier–Stieltjes transforms.

Let μ_1 and μ_2 be given elements in $M(R)$, and μ_0 a third given element such that there exist elements $_1$ and ν_2 in $M(R)$ such that

$$\mu_0 = \mu_1 * \nu_1 + \mu_2 * \nu_2. \tag{1}$$

We assume that there exist a real number a and measures σ_0 and σ_2 in $AC(R)$ such that the three relations

$$\hat{\mu}_1(t) \neq 0, \tag{2}$$

$$\hat{\mu}_2(t) = \hat{\mu}_1(t)\hat{\sigma}_2(t), \tag{3}$$

$$\hat{\mu}_0(t) = \hat{\mu}_1(t)\hat{\sigma}_0(t) \tag{4}$$

all hold, if $|t| \geqslant a$.

H denotes the set of all pairs of bounded Borel measures $\{\nu_1, \nu_2\}$, which satisfy (1). L denotes the set of all pairs of bounded Borel measures $\{\nu_1, \nu_2\}$ such that

$$\mu_1 * \nu_1 + \mu_2 * \nu_2 = 0$$

We finally form the class K of all pairs of functions $\{\varphi_1, \varphi_2\}$ in $L^\infty(R)$, such that with the usual definition of convolution between elements in $L^\infty(R)$ and $AC(R)$,

$$\varphi_1 * \nu_1 + \varphi_2 * \nu_2 = 0,$$

for every $\nu_1 \in AC(R)$ and $\nu_2 \in AC(R)$ such that $\{\nu_1, \nu_2\} \in L$.

Theorem 1. *1°. If $\{\nu_1, \nu_2\} \in H$ then $\nu_1 \in AC(R)$.*

2°. If $\{\varphi_1, \varphi_2\} \in K$, then φ_2 is continuous, after a change in a set of Lebesgue measure 0.

3°. With this assumption on φ_2, we form for any $\{\varphi_1, \varphi_2\} \in K$ and $\{\nu_1, \nu_2\} \in H$ the functional

$$F(\varphi_1, \varphi_2) = F(\varphi_1, \varphi_2, \nu_1, \nu_2) = \int_R \varphi_1(-x)\, \nu_1'(x)\, dx + \int_R \varphi_2(-x)\, d\nu_2(x). \tag{5}$$

Its value does not depend on the choice of $\{\nu_1, \nu_2\}$.

4°. Let (A, B) be a fixed pair of positive numbers and let $K(A, B)$ denote the subset of all $\{\varphi_1, \varphi_2\} \in K$ such that $\|\varphi_1\|_\infty \leqslant A$, $\|\varphi_2\|_\infty \leqslant B$. Then there exists a $\{\Psi_1, \Psi_2\} \in K(A, B)$ such that

$$|F(\varphi_1, \varphi_2)| \leqslant F(\Psi_1, \Psi_2),$$

for every $\{\varphi_1, \varphi_2\} \in K(A, B)$.

5°. There exists a $\{\nu_1, \nu_2\} \in H$ such that

$$F(\Psi_1, \Psi_2) = A \int_R |\nu_1'(x)|\, dx + B \int_R |d\nu_2(x)|. \tag{6}$$

Before we give the proof of Theorem 1, we shall discuss a particularly important example and make some general comments.

Let m and n be integers such that $0 < m < n$. It is easy to see that all our assumptions are fulfilled if we choose μ_1, μ_2 and μ_0 such that

$$\hat{\mu}_1(t) = e^{-t^2}(it)^n,$$

$$\hat{\mu}_2(t) = e^{-t^2},$$

$$\hat{\mu}_0(t) = e^{-t^2}(it)^m.$$

Using the definition of K we find that K consists of all pairs of bounded functions of the form $\{\varphi^{(n)}, \varphi\}$, where φ is absolutely continuous together with its $n-1$ first derivatives. Let us then form the continuous function

$$\varphi_0 = \varphi^{(n)} * \nu_1 + \varphi * \nu_2,$$

where $\{\nu_1, \nu_2\} \in H$. Convoluting this with an arbitrary measure μ with compact support and high differentiability properties, it is easy to see, by using (1) and partial integration, that we obtain

$$\varphi_0 * \mu = \varphi * \mu^{(m)},$$

where $\mu^{(m)}$ denotes the measure which, in the distribution sense, is the *m-th* derivative of the measure μ. Hence $\varphi_0 = \varphi^{(m)}$. If we substitute $x=0$ in $\varphi^{(m)}$ we obtain

$$\varphi^{(m)}(0) = F(\varphi^{(n)}, \varphi, \nu_1, \nu_2).$$

Hence our theorem shows that there is for every pair (A, B) of positive numbers a representation

$$\varphi^{(m)}(0) = \int_R \varphi^{(n)}(-x)\, \nu_1'(x)\, dx + \int_R \varphi(-x)\, d\nu_2(x)$$

such that equality can be attained in the resulting inequality

$$|\varphi^{(m)}(0)| \leqslant A \int_R |\nu_1'(x)|\, dx + B \int_R |d\nu_2(x)|,$$

if

$$\|\varphi^{(n)}\|_\infty \leqslant A, \ \|\varphi\|_\infty \leqslant B.$$

In this particular case, the value of (6) is known, since the optimal functions φ have been found by Kolmogoroff [6], see also Bang [2]. As was shown in Bang's paper, the inequality given by Kolmogoroff is very closely related to the well-known inequalities by Bernstein and Bohr. As can be seen for instance in Achiezer [1, §§ 74, 86] these two inequalities can be proved and generalized using representations which are similar to our formula (5). The starting point for our investigations was an attempt to prove and generalize Kolmogoroff's inequality using similar ideas. We shall in section 3 show that the representation obtained in Theorem 1 can give a direct information on the possibility of such generalizations.

It should finally be mentioned that the ideas in this paper have connections with questions on minimal extrapolations of Fourier–Stieltjes transforms (see for instance [3] and Herz [4, Theorem 4.1]). It is possible to put certain parts of these two theories into a common framework. Generalizations in the same direction as those in Hörmander [5] are also possible.

2. Proof of Theorem 1.

1°. If $\{\nu_1, \nu_2\} \in H$, then by (1)

$$\hat{\mu}_0 = \hat{\mu}_1 \hat{\nu}_1 + \hat{\mu}_2 \hat{\nu}_2.$$

By (2), (3) and (4) this relation implies that if $|t| \geqslant a$

$$\hat{\sigma}_0(t) = \hat{\nu}_1(t) + \hat{\sigma}_2(t)\hat{\nu}_2(t). \tag{7}$$

The difference $\hat{\nu}_3(t)$ between the left and right members of (7) is a Fourier–Stieltjes transform which vanishes for large t, hence the corresponding measure ν_3 belongs to $AC(R)$. We thus have the relation

$$\nu_1 = \sigma_0 - \nu_3 - \sigma_2 * \nu_2,$$

where σ_0, ν_3 and σ_2 belong to $AC(R)$. But the convolution of a measure in $AC(R)$ with a measure in $M(R)$ belongs to $AC(R)$. Hence $\sigma_2 * \nu_2 \in AC(R)$ and as a consequence $\nu_1 \in AC(R)$.

2°. We choose a measure $\nu \in AC(R)$ such that $\hat{\nu}(t) = 1$ if $|t| \leqslant a$. Let $\{\varphi_1, \varphi_2\} \in K$. For any $\mu \in AC(R)$ the pair

$$\{(\sigma_2 - \sigma_2 * \nu) * \mu,\ \nu * \mu - \mu\}$$

lies in L, and both measures belong to $AC(R)$. Hence by the definition of K,

$$\varphi_1 * (\sigma_2 - \sigma_2 * \nu) * \mu + \varphi_2 * (\nu * \mu - \mu) = 0.$$

A rearrangement gives

$$\{\varphi_2 - (\varphi_2 * \nu + \varphi_1 * \sigma_2 - \varphi_1 * \sigma_2 * \nu)\} * \mu = 0.$$

This implies that

$$\varphi_2 = \varphi_2 * \nu + \varphi_1 * \sigma_2 - \varphi_1 * (\sigma_2 * \nu)$$

almost everywhere. Since ν, σ_2 and $\sigma_2 * \nu$ all belong to $AC(R)$, the right-hand member is continuous which proves the assertion 2°.

3°. We form the Banach space X of all pairs $\{\nu_1, \nu_2\}$ where $\nu_1 \in AC(R)$ and $\nu_2 \in M(R)$, with the norm

$$A \int_R |\nu_1'(x)|\, dx + B \int_R |d\nu_2(x)|,$$

and with the vector operations defined in an obvious way. L is a linear subspace in X and H a hyperplane, parallel to L. For every given $\{\varphi_1, \varphi_2\} \in K$,

$$F(\varphi_1, \varphi_2, \nu_1, \nu_2)$$

represents a linear functional on X, which vanishes for every $\{\nu_1, \nu_2\} \in L$ such that $\nu_2 \in AC(R)$. If now $\{\nu_1, \nu_2\} \in L$ is arbitrary and does not annihilate the functional, it is in view of 1° and 2° possible to find a measure $\nu \in AC(R)$ with its support concentrated to a neighborhood of $x = 0$ and such that the functional, applied to $\{\nu_1 * \nu, \nu_2 * \nu\}$, does not vanish. But this element belongs to L and $\nu_2 * \nu \in AC(R)$ which gives us a contradiction. Hence the functional vanishes on L and thus the value is constant on H.

4°. With the notions introduced above we let d denote the distance between H and L, i.e.

$$d = \inf_{\{\nu_1, \nu_2\} \in H} \left(A \int_R |\nu_1'(x)|\, dx + B \int_R |d\nu_2(x)| \right). \tag{8}$$

Obviously
$$|F(\varphi_1, \varphi_2, \nu_1, \nu_2)| \leqslant d, \tag{9}$$

if
$$\{\varphi_1, \varphi_2\} \in K(A, B), \quad \{\nu_1, \nu_2\} \in H.$$

We know from the Hahn–Banach theorem that there exists a bounded linear functional $G(\nu_1, \nu_2)$ on X, with norm 1, vanishing on L, and taking the value d on H. G is also a linear functional with norm $\leqslant 1$, on the closed subspace of X, consisting of the pairs $\{\nu_1, \nu_2\}$ where both ν_1 and ν_2 belong to $AC(R)$. The dual of this space is well known, and we obtain from this that there exist bounded measurable functions Ψ_1 and Ψ_2 such that

$$G(\nu_1 \nu_2) = \int_R \Psi_1(-x)\, \nu_1'(x)\, dx + \int_R \Psi_2(-x)\, \nu_2'(x)\, dx, \tag{10}$$

if ν_1 and $\nu_2 \in AC(R)$. Obviously

$$\|\Psi_1\|_\infty \leqslant A, \quad \|\Psi_2\|_\infty \leqslant B.$$

Since G vanishes on L, the definition of K shows that $\{\Psi_1, \Psi_2\} \in K$, in particular that we can assume Ψ_2 to be continuous. Since by (9)

$$|F(\varphi_1, \varphi_2)| \leqslant d = G(\nu_1, \nu_2),$$

if
$$\{\varphi_1, \varphi_2\} \in K(A, B), \quad \{\nu_1, \nu_2\} \in H,$$

4° is proved if we can show that

$$G(\nu_1, \nu_2) = \int_R \Psi_1(-x)\, \nu_1'(x)\, dx + \int_R \Psi_2(-x)\, d\nu_2(x), \tag{11}$$

for every $\{\nu_1, \nu_2\} \in X$.

We can, of course, because of (10) restrict ourselves to the case when $\nu_1 = 0$. In the case when $\hat{\nu}_2$ has a compact support, ν_2 belongs to $AC(R)$, hence by (10) the relation (11) is true, and thus we can also restrict ourselves to the case when $\hat{\nu}_2$ vanishes on the set $\{t \| t| \leqslant a\}$. Then $\{-\sigma_2 * \nu_2, \nu_2\} \in L$, hence

$$G(-\sigma_2 * \nu_2, \nu_2) = 0,$$

which gives us

$$G(0, \nu_2) = G(\sigma_2 * \nu_2, 0) = \int_R \Psi_1(-x)\, (\sigma_2 * \nu_2)'(x)\, dx. \tag{12}$$

Now let $\{\tau_n\}_1^\infty$ be a sequence of non-negative measures in $AC(R)$, all with total mass 1 and with supports contained in the set $\{x \| x| \leqslant 1/n\}$. Then ν_2 in (12) can be exchanged to $\nu_2 * \tau_n$. By a well-known property of convolutions with measures in $AC(R)$, applied to the right-hand member of (12), we see that

$$\lim_{n \to \infty} G(0, \nu_2 * \tau_n) = G(0, \nu_2),$$

Since $\nu_2 \ast \tau_n$ belong to $AC(R)$, the left-hand member of this relation can be represented using the formula (10), and this finally gives, by standard arguments,

$$G(0, \nu_2) = \int_R \Psi_2(-x)\, d\nu_2(x),$$

hence the desired result.

5°. We have to prove that there exists an element $\{\nu_1, \nu_2\} \in H$ which gives the minimum in (8). We can find a weakly convergent sequence of pairs of measures, converging to a measure $\{\nu_1^0, \nu_2^0\}$, where both ν_1^0 and ν_2^0 belong to $M(R)$. It is easy to see, using the Fourier–Stieltjes transforms, that the relation (1) which determines H, can be written

$$\lambda_0(y) = \int_{-\infty}^{\infty} \lambda_1(y-x)\, d\nu_1(x) + \int_{-\infty}^{\infty} \lambda_2(y-x)\, d\nu_2(x), \tag{13}$$

for every $y \in R$, where λ_0, λ_1 and λ_2 are the functions

$$e^{-x^2} \ast \mu_i \quad (i = 0, 1, 2).$$

But these functions belong to $C_0(R)$ and hence (13) has to be fulfilled for the limiting measures $\{\nu_1^0, \nu_2^0\}$. Hence $\{\nu_1^0, \nu_2^0\} \in H$, and it is obvious that it must realize the infimum.

3. We now return to the case when $\hat{\mu}_1 = e^{-t^2}(it)^n$, $\hat{\mu}_2 = e^{-t^2}$, $\hat{\mu}_0 = e^{-t^2}(it)^m$, where m and n are integers such that $0 < m < n$. This case was briefly discussed in the last part of § 1. Under this assumption Kolmogoroff [6] has found the optimal pairs $\{\Psi_1, \Psi_2\}$. Disregarding a constant factor with modulus 1, they are found among the functions given by the relation

$$\Psi_1 = A \operatorname{sign}(\sin(bx + c)),$$

where b and c are real, $b \neq 0$, and with the corresponding Ψ_2 determined as the primitive function of Ψ_1 of order n and with mean value 0.

A certain lack of symmetry in the functions $\{\Psi_1, \Psi_2\}$ appears when m and n varies. This can, however, be overcome, for every fixed set $\{m, n, A, B\}$, by changing the functions $\hat{\mu}_1$ and $\hat{\mu}_2$ to

$$e^{-t^2}(it)^n e^{i\beta_1 t}$$

and

$$e^{-t^2}(it)^m e^{t\beta_2 t}.$$

respectively with the real numbers β_1 and β_2 suitably chosen. After such a change, which of course only corresponds to translations in the functions $\{\varphi_1, \varphi_2\}$ in the class considered, we see that the optimal pair $\{\psi_1, \psi_2\}$ has the following properties, for some $h > 0$:

1°. $$\Psi_1 = A \operatorname{sign}\left(\sin \frac{hx}{2}\right),$$

2°. $$\Psi_2(2n\pi/h) = B(-1)^n \quad (n \in Z),$$

while $$-B < \Psi_2(x) < B, \quad \text{if} \quad x \neq 2n\pi/h \quad (n \in Z).$$

A pair of functions $\{\varphi_1, \varphi_2\}$ with these properties is said to belong to the class $E(A, B, h)$.

A natural problem is now to investigate if pairs in the class $E(A, B, h)$ occurs as optimal pairs in other cases.

Let us then first make some assumptions on $\hat{\mu}_0$, $\hat{\mu}_1$ and $\hat{\mu}_2$. We assume that $\hat{\mu}_1 \neq 0$ almost everywhere, and that $\hat{\mu}_0/\hat{\mu}_1$ and $\hat{\mu}_2/\hat{\mu}_1 \in L^1$ except for some bounded interval. Finally we assume that

$$\sum_{-\infty}^{\infty} \hat{\mu}_2(t+hn)/\hat{\mu}_1(t+hn) \neq 0,$$

almost everywhere. Then the following theorem holds.

Theorem 2. *Suppose that* $\{\Psi_1, \Psi_2\} \in K \cap E(A, B, h)$. *Necessary and sufficient in order that* $\{\Psi_1, \Psi_2\}$ *is an optimal pair, in the sense of Theorem 1, is that there exists a pair* $\{\nu_1, \nu_2\} \in H$, *such that*

$$\nu_1'(x) \operatorname{sign}\left(\sin \frac{hx}{2}\right) = |\nu_1'(x)| \quad (x \in R),$$

while ν_2 *is a discrete measure composed of non-negative pointmasses at* $x = 4n\pi/h$, $n \in Z$, *and non-positive point masses at* $x = (4n+2)\pi/h$, $n \in Z$.

$\{\nu_1, \nu_2\}$ *is then uniquely determined by the relations*

$$\sum_{-\infty}^{\infty} \hat{\mu}_0(t+nh)/\hat{\mu}_1(t+nh) = \hat{\nu}_2(t) \sum_{-\infty}^{\infty} \hat{\mu}_2(t+nh)/\hat{\mu}_1(t+nh), \tag{14}$$

$$\hat{\nu}_1 = \hat{\mu}_0/\hat{\mu}_1 - \hat{\nu}_2\hat{\mu}_2/\hat{\mu}_1. \tag{15}$$

Proof of Theorem 2. The sufficiency is a direct consequence of the conditions on ν_1 and ν_2 which show that

$$|F(\varphi_1, \varphi_2)| = |F(\varphi_1, \varphi_2, \nu_1, \nu_2)| \leqslant F(\Psi_1, \Psi_2, \nu_1, \nu_2) = |F(\psi_1, \psi_2)|,$$

if

$$\{\varphi_1, \varphi_2\} \in K(A, B).$$

To prove the necessity we assume that $\{\Psi_1, \Psi_2\} \in K \cap E(A, B, h)$ is an optimal pair. Then by Theorem 1, 4° and 5°, there exists an optimal $\{\nu_1, \nu_2\} \in H$, which obviously has to fulfil the conditions on the signs of ν_1' and ν_2.

It remains to prove the formulas (14) and (15). (15) is a direct consequence of (1). It shows, together with the conditions on $\hat{\mu}_0$, $\hat{\mu}_1$ and $\hat{\mu}_2$, that $\hat{\nu}_1 \in L^1(R)$. Hence we can assume that ν_1' is continuous. The condition on the sign variation of ν_1' then implies that

$$\nu_1'(2n\pi/h) = 0 \quad (n \in Z).$$

But ν_1' is the inverse Fourier transform of the function $\hat{\nu}_1$ in the L^1 sense. Hence the periodic function

$$\sum_{-\infty}^{\infty} \hat{\nu}_1(t+nh),$$

which locally belongs to L^1, has its Fourier coefficients determined by the values $\nu_1'(2n\pi/h)$ in such a way that they, too, must vanish. Hence

$$\sum_{-\infty}^{\infty} \hat{\nu}_1(t+nh)=0,$$

almost everywhere.

A direct summation of (15) then gives (14).

Theorem 2 is applicable in the case when μ_1, μ_2 and μ_0 are given by the relations

$$\hat{\mu}_1(t)=e^{-t^2}(it)^{\alpha_1}e^{i\beta_1 t},$$

$$\hat{\mu}_2(t)=e^{-t^2},$$

$$\hat{\mu}_0(t)=e^{-t^2}\,(it)^{\alpha_2}e^{i\beta_2 t},$$

where α_1, β_1, α_2 and β_2 are real and

$$0<\alpha_2<\alpha_1-1.$$

$(\text{it})^{\alpha_j}$, $j=1, 2$, is then to be interpreted as

$$\exp\left(\frac{\pi}{2}\, i \text{ sign } t+\log|t|\right)\alpha_j.$$

If α_1, α_2, A and B are given, it is possible to show that we can choose h, β_1 and β_2 such that there is a pair of functions $\{\varphi_1, \varphi_2\}\in K\cap E(A, B, h)$. The problem is to decide whether this pair is an optimal one.

If both α_1 and α_2 are integers this is the case, which follows from Kolmogoroff's result. The only new in this case is the existence and the explicit form of the optimal pair $\{\nu_1, \nu_2\}$.

The case when some α_j is not an integer corresponds to the problem for fractional derivatives (see Bang [2]). In this case it is easy to see from (14) that $\hat{\nu}_2(t)$ is analytic within the period $(-h/2, h/2)$ except at $t=0$, where it has a singularity of a type which excludes the possibility of the demanded sign-variation in $d\,\nu_2$. Hence $\{\Psi_1, \Psi_2\}$ is not optimal in this case.

This completes a discussion of Bang [2] where he came to the same conclusion in the case $0<\alpha_2<\alpha_1=1$. In that case, assuming $\beta_1=\beta_2=0$, the optimal pairs $\{\Psi_1, \Psi_2\}$ are explicitly known. They consist essentially of all $\{\Psi', \Psi\}$ with $\|\Psi'\|_\infty\leqslant A$, $\|\Psi\|_\infty\leqslant B$, where Ψ is absolutely continuous and where

$$\Psi(x)=-B, \quad \text{if} \quad x\leqslant -\frac{2B}{A},$$

$$\Psi'(x)=A, \qquad \text{if} \quad -\frac{2B}{A}<x\leqslant 0.$$

By means of Theorem 1 we can easily give the necessary and sufficient condition on $\hat{\mu}_0$, fulfilling (4) with given

$$\hat{\mu}_1=(it)e^{-t^2},$$

$$\hat{\mu}_2=e^{-t^2}$$

in order that all these functions $\{\psi', \psi\}$ give optimal pairs. Easy arguments show that the condition is that

$$\mu_0 = \mu_1 * \nu,$$

where ν is an absolutely continuous not necessarily bounded measure, such that for some c

$$\begin{cases} \nu' = \text{constant} \leqslant c, \quad \text{if} \quad x < 0, \\ \nu' \geqslant c, \quad \text{if} \quad 0 < x < \dfrac{2B}{A}, \\ \nu' \text{ is } \leqslant c, \quad \text{bounded and decreasing}, \quad \text{if} \quad x > \dfrac{2B}{A}. \end{cases}$$

In particular we obtain Bang's case when for some $d > 0$, and for $0 < \alpha < 1$,

$$\begin{cases} \nu' = 0, \quad x < 0, \\ \nu' = d\,|x|^{-\alpha}, \quad x > 0. \end{cases}$$

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