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Fredholm property of partial differential operators of irregular singular type

Masatake Miyake and Masafumi Yoshino(1)

1. Introduction

In 1974 Kashiwara–Kawai–Sjöstrand showed the sufficient condition for the convergence of all formal power series solutions of the following linear partial differential equations of regular singular type

$$\mathcal{L}(x,D)u(x) \equiv \sum_{|\alpha|=|\beta|\le m} a_{\alpha\beta} D^\beta(x^\alpha u(x)) = f(x), \tag{1.1}$$

where m is a positive integer and $a_{\alpha\beta}$'s are complex constants. Here we use the standard notations of multi-indices, $D^\beta=(\partial/\partial x_1)^{\beta_1}\dots(\partial/\partial x_n)^{\beta_n}$, $|\alpha|=\alpha_1+\dots+\alpha_n$ and $x^\alpha=x_1^{\alpha_1}\dots x_n^{\alpha_n}$. They proved the following result.

Theorem 1.1. (cf. [4]) *Suppose that the following condition*

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} z^\alpha \bar{z}^\beta \ne 0, \tag{1.2}$$

is satisfied for any $z\in \mathbf{C}^n\setminus\{0\}$, where $z^\alpha=z_1^{\alpha_1}\dots z_n^{\alpha_n}$ and $\bar{z}^\beta=\bar{z}_1^{\beta_1}\dots\bar{z}_n^{\beta_n}$. Then, for any $f(x)$ analytic at the origin all formal power series solutions $u(x)$ of the equation (1.1) *converge in some neighborhood of the origin.*

They proved results for somewhat more general operators than (1.1) admitting perturbations.

Inspired from this theorem we shall study in this paper the Fredholm property of regular and irregular singular type operators including (1.1) in (formal) Gevrey spaces in a neighborhood of the origin of $\mathbf{C}^2$. We introduce a Toeplitz symbol in

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a natural way in connection with a filtration with respect to the Gevrey order. Toeplitz symbols play an important role in describing interactions of multiplications by (rational) polynomials and differentiations. We reduce our problem to the study of the Fredholm property of Toeplitz operators. Then we construct regularizers for these Toeplitz operators by use of a Riemann–Hilbert factorization condition for Toeplitz symbols associated with the differential operators (cf. (2.8), (2.9), Lemma 3.5 and Section 5). Moreover, we can show that under these conditions the index of these operators is equal to zero (cf. Theorem 4.1).

This paper is organized as follows. In Section 2 we state our main result and its applications. In Section 3 we prepare lemmas which are necessary in the proof of our main theorem. In Section 4 we reduce our problem to the Fredholm property of Toeplitz operators on the two dimensional torus $\mathbf{T}^2$. The construction of regularizers is done in Section 5.

2. Statement of the results

Let $\mathbf{N}$ be the set of non-negative integers and let $\mathbf{C}$ be the set of complex numbers. Let $\mathbf{C}[[x]]$ be the set of all formal power series

$$\mathbf{C}[[x]] := \left\{ u(x)\,;\, u(x) = \sum_{\eta\in\mathbf{N}^2} u_\eta \frac{x^\eta}{\eta!},\ u_\eta \in \mathbf{C} \right\}.$$

Let $w_j > 0$ $(j=1,2)$ and $s>0$. We set $w=(w_1, w_2)$. If we denote by

$$\mathcal{O}(\{|x_1|<w_1\}\times\{|x_2|<w_2\})$$

the set of holomorphic functions on a domain $\{|x_1|<w_1\}\times\{|x_2|<w_2\}\subset\mathbf{C}^2$, we define the class $\mathcal{G}^s_w$ by

(2.1)
$$\mathcal{G}^s_w = \left\{ u(x) = \sum u_\eta \frac{x^\eta}{\eta!} \in \mathbf{C}[[x]]\,;\, \sum_\eta u_\eta \frac{x^\eta}{|\eta|!^s} \in \mathcal{O}(\{|x_1|<w_1\}\times\{|x_2|<w_2\}) \right\}.$$

The space $\mathcal{G}^s_w$ can be regarded as a Fréchet space by the following isomorphism of Fréchet spaces

$$\mathbf{C}[[x]] \supset \mathcal{G}^s_w \xrightarrow[\sim]{\text{Borel transf.}} \mathcal{O}(\{|x_1|<w_1\}\times\{|x_2|<w_2\}), \tag{2.2}$$

where the Borel transformation is defined by

$$\mathcal{G}^s_w \ni \sum_{\eta\in\mathbf{N}^2} u_\eta \frac{x^\eta}{\eta!} \overset{\sim}{\mapsto} \sum_{\eta\in\mathbf{N}^2} u_\eta \frac{x^\eta}{|\eta|!^s} \in \mathcal{O}(\{|x_1|<w_1\}\times\{|x_2|<w_2\}). \tag{2.3}$$

We note that the space $\mathcal{G}^s_w$ coincides with a formal Gevrey space, the class of locally analytic functions and a class of entire analytic functions with finite order if $s>1$, $s=1$ and $s<1$, respectively (cf. Lemma 3.1 which follows).

We denote by D_1^{-1} integration with respect to x_1, $D_1^{-1}u(x):=\int_0^{x_1} u(y_1,x_2)\,dy_1$. The operator D_2^{-1} is defined similarly. For $\beta=(\beta_1,\beta_2)\in\mathbf{Z}^2$, we set $D^\beta=D_1^{\beta_1}D_2^{\beta_2}$, where if $\beta_j<0$ we understand that $D_j^{\beta_j}=(D_j^{-1})^{-\beta_j}$.

Let $P\equiv P(x,D_x)$ be an integro-differential operator of finite order with holomorphic coefficients in a neighborhood of the origin of $\mathbf{C}^2$,

$$P(x,D_x)=\sum_{\beta\in\mathbf{Z}^2} D^\beta a_\beta(x), \tag{2.4}$$

where $a_\beta(x)$'s are analytic functions of x in some neighborhood of the origin, and the summation with respect to β is a finite sum.

By the Taylor expansion of $a_\beta(x)$, we have

$$a_\beta(x)=\sum_\gamma a_{\gamma\beta}x^\gamma$$

with $a_{\gamma\beta}$ being complex constants. By substituting $a_\beta(x)$ in (2.4) we have the expression

$$P(x,D)=\sum_{\gamma\in\mathbf{N}^2,\beta\in\mathbf{Z}^2} a_{\gamma\beta}D^\beta x^\gamma. \tag{2.5}$$

For $D^\beta x^\gamma\cdot$ we define the s-Gevrey order $\mathrm{ord}_s D^\beta x^\gamma$ of $D^\beta x^\gamma\cdot$ by

$$\mathrm{ord}_s\, D^\beta x^\gamma\cdot:=|\beta|+(1-s)(|\gamma|-|\beta|). \tag{2.6}$$

Then the s-Gevrey order of P in (2.5) is defined by

$$\mathrm{ord}_s\, P:=\sup_{\gamma,\beta}\{|\beta|+(1-s)(|\gamma|-|\beta|)\,;a_{\gamma\beta}\neq 0\}.$$

Here and in what follows we always assume that the s-Gevrey order of $P(x,D)$ is finite. This implies that P has polynomial coefficients in case $s<1$. In case $s=1$ we further assume that for every β in (2.4) such that $|\beta|=\mathrm{ord}_1 P$, the function $a_\beta(x)$ is a *polynomial* in x.

We shall define the Toeplitz symbol associated with $P(x,D)$ by

$$L_{s,w}(z;\xi):=\sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=\mathrm{ord}_s P} a_{\alpha\beta}z^{\alpha-\beta}w^{\alpha-\beta}\xi^\alpha,\quad \xi\in\mathbf{R}^2. \tag{2.7}$$

We define the torus $\mathbf{T}^2$ by $\mathbf{T}^2=\{(z_1,z_2);z_j=e^{i\theta_j},\ 0\le\theta_j\le 2\pi,\ j=1,2\}$. Then we have the following result.

Theorem 2.1. *The operator* $P\colon \mathcal{G}^s_w \to \mathcal{G}^s_w$ *is a Fredholm operator of index zero in the sense that the mapping has finite dimensional kernel and cokernel of the same dimension if the following conditions are satisfied*:

$$L_{s,w}(z,\xi) \neq 0 \quad \forall (z_1,z_2) \in \mathbf{T}^2,\ \forall \xi \in \mathbf{R}^2,\ |\xi|=1,\ \xi \geq 0, \tag{2.8}$$

$$\mathrm{ind}_1\, L_{s,w} = \mathrm{ind}_2\, L_{s,w} = 0. \tag{2.9}$$

Here $\mathrm{ind}_1\, L_{s,w}$ (*resp.* $\mathrm{ind}_2\, L_{s,w}$) *is defined by*

$$\mathrm{ind}_1\, L_{s,w} = \frac{1}{2\pi i} \oint_{|\zeta_1|=1} d\log L_{s,w}(\zeta_1, z_2, \xi). \tag{2.10}$$

Remark. We note that the right-hand side of (2.10) is an integer-valued continuous function of z_2 and ξ. Because the sets $\{z_2 \in \mathbf{C}; |z_2|=1\}$ and $\{\xi \in \mathbf{R}^2; |\xi|=1\}$ are connected, the integral (2.10) is constant. Hence the right-hand side is independent of z_2 and ξ. We denote this quantity by $\mathrm{ind}_1\, L_{s,w}$. We similarly define $\mathrm{ind}_2\, L_{s,w}$.

As a corollary to this theorem, we can give another characterization theorem for convergence of formal power series solutions, different from Theorem 1.1.

Corollary 2.2. *Suppose that* $n=2$ *and let* $\mathcal{L}(x,D)$ *be the partial differential operator given in Theorem* 1.1. *Let the Toeplitz symbol* $L_{1,w}(z;\xi)$ *when* $s=1$ *satisfy the conditions* (2.8) *and* (2.9). *Then every formal power series solution* $u(x)$ *of the equation* (1.1) *for any function* $f(x)$ *analytic at the origin converges at the origin.*

Proof. From Theorem 2.1, the mapping $\mathcal{L}\colon \mathcal{G}^1_w \to \mathcal{G}^1_w$ is a Fredholm operator of index zero. Let H_n be the set of homogeneous polynomials of degree $n \in \mathbf{N}$. Then $\mathcal{L}(x,D)$ maps H_n into itself. Therefore, the finite dimensional kernel of the mapping $\mathcal{L}\colon \mathcal{G}^1_w \to \mathcal{G}^1_w$ is spanned by homogeneous polynomials. This shows that $\mathcal{L}$ is invertible on H_n for sufficiently large n. It follows that the mapping $\mathcal{L}\colon \mathbf{C}[[x]] \to \mathbf{C}[[x]]$ is also Fredholm of index zero. Therefore we see that there exists N such that for every $f \in \mathcal{G}^1_w$ (resp. $f \in \mathbf{C}[[x]]$) satisfying $f = \sum_{n=N}^{\infty} f_n$, $f_n \in H_n$, the equation $\mathcal{L}(x,D)u(x) = f(x)$ has a unique solution $u = \sum_{n=N}^{\infty} u_n$, $u_n \in H_n$, $u \in \mathcal{G}^1_w$ (resp. $u \in \mathbf{C}[[x]]$). Therefore, the mapping $\mathcal{L}\colon \mathbf{C}[[x]]/\mathcal{G}^1_w \to \mathbf{C}[[x]]/\mathcal{G}^1_w$ is a bijection which implies the conclusion. □

We shall apply Theorem 2.1 to a Cauchy–Goursat–Fuchs problem in $\mathcal{G}^1_w$,

$$P(x, D_x)u = f \in \mathcal{G}^1_w, \quad u = O(x^\gamma), \tag{2.11}$$

where $\gamma = (\gamma_1, \gamma_2) \in \mathbf{N}^2$, $|\gamma| = m$ and the condition $u = O(x^\gamma)$ means that $u(x)/x^\gamma \in \mathcal{G}^1_w$. Here the operator $P(x, D_x)$ is a differential operator given by (2.4) and m is an order of P. If we set $u=D_x^{-\gamma}v$ then the problem (2.11) is equivalent to the following equation

$$P(x, D_x)D_x^{-\gamma}v = f. \tag{2.12}$$

Hence we obtain the equation (2.4) with β replaced by $\beta-\gamma$. We have the following corollary.

Corollary 2.3. *Assume that $s=1$. Suppose that there exists a β with $|\beta|=m$ in the expression (2.4) such that $a_\beta(0)\neq 0$. Define $T_w(z):=\sum_{|\beta|=m} a_\beta(0)z^{\gamma-\beta}w^{\gamma-\beta}$. Suppose that the convex hull of the image of the torus $\mathbf{T}^2$ by the map $z\mapsto T_w(z)$ does not contain the origin for some w. Then the problem (2.11) has a unique solution in $\mathcal{G}^1_{\varkappa w}$ for small $\varkappa>0$, where $\varkappa w=(\varkappa w_1, \varkappa w_2)$.*

Remark. We want to show the unique solvability of (2.11) under a so-called spectral condition. Indeed, if the spectral condition

$$|a_\gamma(0)| > \sum_{|\beta|=m, \beta\neq\gamma} |a_\beta(0)|w^{\gamma-\beta} \tag{2.13}$$

is satisfied, then the convex hull of $T_w(\mathbf{T}^2)$ does not contain the origin. In order to see this, we take a real number θ such that $e^{i\theta}a_\gamma(0)=|a_\gamma(0)|$, we have, for $z\in\mathbf{T}^2$,

$$e^{i\theta}T_w(z) = |a_\gamma(0)| + e^{i\theta}\sum_{\beta\neq\gamma} a_\beta(0)z^{\gamma-\beta}w^{\gamma-\beta}.$$

Hence by (2.13) we have $\operatorname{Re} e^{i\theta}T_w(z)>0$. By Corollary 2.3 we have the assertion.

Proof of Corollary 2.3. By making the change of variables $x_1\mapsto w_1x_1$, $x_2\mapsto w_2x_2$ if necessary one may assume that $w_1=w_2=1$ in (2.11) or (2.12). We shall show (2.8) and (2.9). The Toeplitz symbol $L_{1,\varkappa w}$ of the operator $P(x,D)D_x^{-\gamma}$ is given by $L_{1,\varkappa w}=T_w+O(\varkappa)$. Condition (2.8) is clear. To prove (2.9) let us fix $z_2, |z_2|=1$. Because the convex hull of the curve $z_1\ni\mathbf{T}\mapsto L_{1,\varkappa w}(z,\xi)$ does not contain the origin it follows that $\operatorname{ind}_1 L_{1,\varkappa w}=0$. Similarly we can prove $\operatorname{ind}_2 L_{1,\varkappa w}=0$. Hence the operator $PD_x^{-\gamma}$ given by (2.12) is a Fredholm operator.

In order to show that $PD_x^{-\gamma}$ is injective we may assume that there exists $c_0>0$ such that $\operatorname{Re} T_w(z)\geq c_0>0$ for any $z\in\mathbf{T}^2$. We write the operator $PD_x^{-\gamma}$ in the following form

$$PD_x^{-\gamma} = Q_0+Q_1+\dots+Q_j+\dots,$$

where Q_j maps homogeneous polynomials of degree ν to the ones of degree $\nu+j$. If we know that $Q_0:=\sum_{|\beta|=m} a_\beta(0)D_x^{\beta-\gamma}$ is injective, the operator $PD_x^{-\gamma}$ is injective.

Indeed, for $u=\sum_{j=0}^{\infty} u_j \in \mathcal{G}_w^s$ with u_j homogeneous of degree j such that $PD_x^{-\gamma}u=0$ we have that $0=PD_x^{-\gamma}u=Q_0u_0+(Q_0u_1+Q_1u_0)+\dots$. It follows that $Q_0u_0=0$, i.e. $u_0=0$, and inductively we have $u_1=u_2=\dots=0$.

Suppose that a homogeneous polynomial $u=\sum u_\eta x^\eta/\eta!$ satisfies $Q_0u=0$. If we set $\widetilde{u}=\sum u_\eta e^{i\eta\theta}$, we have $Q_0u=\sum(\sum_\beta a_\beta(0)u_{\eta+\beta-\gamma})x^\eta/\eta!$ and

$$0=\langle Q_0u,u\rangle=(T_w(e^{i\theta})\widetilde{u},\widetilde{u})=\int T_w(e^{i\theta})|\widetilde{u}|^2\,d\theta,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in a finite dimensional space and $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbf{T}^2)$. Because $\operatorname{Re}T_w(z)\ge c_0>0$ it follows that $\widetilde{u}=0$. Hence Q_0 is injective. □

3. Preliminary lemmas

We define the class $G_w^s(\mu)$ ($\mu\in\mathbf{R}$) by

$$G_w^s(\mu):=\left\{u=\sum_\eta u_\eta\frac{x^\eta}{\eta!}\,;\,\|u\|:=\left(\sum_\eta\left(|u_\eta|\frac{w^\eta}{(|\eta|-\mu/s)!^s}\right)^2\right)^{1/2}<\infty\right\}, \tag{3.1}$$

where the factorial is understood as the gamma function, $r!:=\Gamma(r+1)$ for $r\ge0$ and where we set $(|\eta|-\mu/s)!=1$ if $|\eta|-\mu/s\le0$. The class $G_w^s(\mu)$ is a Hilbert space with the norm $\|\cdot\|$.

Lemma 3.1. *Let the class* $\mathcal{G}_w^s$ *be defined by* (2.1). *Then we have*

$$\mathcal{G}_w^s=\operatorname{proj}\lim_{r\uparrow w}G_r^s(\mu) \tag{3.2}$$

for every μ.

Proof. Suppose that $u(x)\in G_r^s(\mu)$ for any $r=(r_1,r_2)$ such that $r_j<w_j$ $(j=1,2)$. Then we have $|u_\eta|\le Mr^{-\eta}(|\eta|-(\mu/s))!^s$ for some $M>0$ independent of η. Therefore we have, for $|x_j|<r_j$ $(j=1,2)$,

$$\sum|u_\eta|\frac{|x|^\eta}{|\eta|!^s}\le M\sum r^{-\eta}|x^\eta|\frac{(|\eta|-\mu/s)!^s}{|\eta|!^s}.$$

Clearly, the right-hand side converges for $|x_j|<r_j$. Because $r<w$ is arbitrary we have $u\in\mathcal{G}_w^s$.

Conversely, suppose that $u=\sum u_\eta x^\eta/\eta!\in\mathcal{G}_w^s$. Then we have

$$U(x):=\sum_\eta u_\eta\frac{x^\eta}{|\eta|!^s}\in\mathcal{O}(\{|x_1|<\varrho_1\}\times\{|x_2|<\varrho_2\})$$

for any $\varrho<w$. By Cauchy's formula we have

$$v_\eta := \frac{u_\eta}{|\eta|!^s} = \frac{1}{(2\pi i)^2}\oint_{|\zeta_1|=\varrho_1}\oint_{|\zeta_2|=\varrho_2}\frac{U(\zeta)}{\zeta^{\eta+1}}\,d\zeta_1\,d\zeta_2.$$

Hence we have the estimate $|v_\eta|\leq M\varrho^{-\eta}$ for some $M>0$. Because $\varrho<w$ is arbitrary we have $u\in G_r^s(\mu)$ for any $r<w$. □

Let X_j $(j=1,2)$ be a positive number and set $X=(X_1,X_2)$. We denote by $\mathcal{O}(|x|\leq X)$ the set of holomorphic functions on $\{x\in\mathbf{C}^2; |x_j|<X_j,\ j=1,2\}$ and continuous on its closure. For $a(x)\in\mathcal{O}(|x|\leq X)$, we put $\|a\|_X:=\max_{|x_j|\leq X_j}|a(x)|$.

Lemma 3.2. *Let $s\geq 1$. Assume that $a(x)\in\mathcal{O}(|x|\leq\varrho w)$ $(\varrho>1)$. Then for any $U(x)\in G_w^s(\mu)$, we have $a(x)U(x)\in G_w^s(\mu)$ and there exists a constant C depending only on μ such that*

$$\|aU\|\leq C\left(\frac{\varrho}{\varrho-1}\right)^2\|a\|_{\varrho w}\|U\|. \tag{3.3}$$

Proof. We put $a(x)=\sum a_\gamma x^\gamma/\gamma!\in\mathcal{O}(|x|\leq\varrho w)$. Then by Cauchy's integral formula, we have $|a_\gamma|\leq\|a\|_{w\varrho}\gamma!/(\varrho w)^\gamma(\gamma\in\mathbf{N}^2)$. We put $a(x)U(x)=\sum V_\beta x^\beta/\beta!$. Then we have

$$V_\beta=\sum_{0\leq\gamma\leq\beta}a_\gamma U_{\beta-\gamma}\frac{\beta!}{(\beta-\gamma)!\gamma!}.$$

Hence we have, for $C_1>0$

$$\begin{aligned}
\sum_\beta\left(|V_\beta|\frac{w^\beta}{(|\beta|-\mu/s)!^s}\right)^2 &\leq \|a\|_{\varrho w}^2\sum_\beta\left(\sum_{0\leq\gamma\leq\beta}|U_{\beta-\gamma}|\frac{1}{(\varrho w)^\gamma}\frac{\beta!}{(\beta-\gamma)!}\frac{w^\beta}{(|\beta|-\mu/s)!^s}\right)^2\\
&\leq C_1\|a\|_{\varrho w}^2\sum_\beta\left(\sum_{0\leq\gamma\leq\beta}|U_{\beta-\gamma}|\frac{1}{\varrho^{|\gamma|}}\frac{w^{\beta-\gamma}}{(|\beta|-\mu/s-|\gamma|)!^s}\right)^2\\
&\leq C_1\|a\|_{\varrho w}^2\sum_\beta\left(\sum_\gamma\frac{1}{\varrho^{|\gamma|}}\right)\sum_\gamma\frac{1}{\varrho^{|\gamma|}}\left(|U_{\beta-\gamma}|\frac{w^{\beta-\gamma}}{(|\beta|-\mu/s-|\gamma|)!^s}\right)^2\\
&\leq C_1\left(\frac{\varrho}{\varrho-1}\right)^2\|a\|_{\varrho w}^2\sum_\gamma\frac{1}{\varrho^{|\gamma|}}\sum_{\beta\geq\gamma}\left(|U_{\beta-\gamma}|\frac{w^{\beta-\gamma}}{(|\beta-\gamma|-\mu/s)!^s}\right)^2\\
&\leq C_1\left(\frac{\varrho}{\varrho-1}\right)^4\|a\|_{\varrho w}^2\|U\|^2. \quad\square
\end{aligned}$$

Lemma 3.3. *Let* $\mu=|\beta|+(1-s)(|\alpha|-|\beta|)$ *be the s-Gevrey order of* $x^\alpha D^\beta$. *Then the map* $x^\alpha D^\beta: G^s_w(\mu)\to G^s_w(0)$ *is continuous. Moreover, for every* $\varepsilon>0$ *the map*

$$x^\alpha D^\beta: G^s_w(\mu+\varepsilon)\longrightarrow G^s_w(0)$$

is a compact operator.

Proof. We first show that for every $\varkappa<\mu$ the injection $\iota: G^s_w(\mu)\to G^s_w(\varkappa)$ is compact. Let $B\subset G^s_w(\mu)$ be a bounded set in $G^s_w(\mu)$. If we write $u=\sum_\eta u_\eta x^\eta/\eta!\in B$, then for each fixed η the set $\{u_\eta; u\in B\}$ is bounded. Hence, by the diagonal argument, we can choose a sequence $\{u^{(k)}\}\subset B$, $u^{(k)}(x)=\sum_\eta u^{(k)}_\eta x^\eta/\eta!$ such that for each η, $u^{(k)}_\eta\to u_\eta$ when $k\to\infty$. Moreover we have that

$$\begin{aligned}\sum_{|\eta|\ge N}\left(|u^{(k)}_\eta|\frac{w^\eta}{(|\eta|-\varkappa/s)!^s}\right)^2 &\le \max_{|\eta|\ge N}\frac{(|\eta|-\mu/s)!^{2s}}{(|\eta|-\varkappa/s)!^{2s}}\sum_{|\eta|\ge N}\left(|u^{(k)}_\eta|\frac{w^\eta}{(|\eta|-\mu/s)!^s}\right)^2\\ &\le K\max_{|\eta|\ge N}\frac{(|\eta|-\mu/s)!^{2s}}{(|\eta|-\varkappa/s)!^{2s}}\to 0\quad (N\to\infty),\end{aligned}$$

where $K>0$ is independent of k and N. This proves that the sequence $\{u^{(k)}\}$ converges in $G^s_w(\varkappa)$.

In order to complete the proof we shall show that the map $x^\alpha D^\beta: G^s_w(\mu)\to G^s_w(0)$ is continuous. By simple calculations

$$x^\alpha D^\beta\sum_\eta u_\eta\frac{x^\eta}{\eta!}=\sum u_\eta\frac{x^{\eta+\alpha-\beta}}{(\eta-\beta)!}=\sum u_{\eta+\beta-\alpha}\frac{x^\eta}{(\eta-\alpha)!}. \tag{3.4}$$

Hence we have

$$\sum_\eta\left(|u_{\eta+\beta-\alpha}|\frac{w^\eta}{(|\eta|)!^s}\frac{\eta!}{(\eta-\alpha)!}\right)^2=\sum_\eta\left(|u_\eta|w^{\eta-\beta+\alpha}\frac{1}{(|\eta|-|\beta|+|\alpha|)!^s}\frac{(\eta-\beta+\alpha)!}{(\eta-\beta)!}\right)^2. \tag{3.5}$$

If η is sufficiently large the term $(\eta-\beta+\alpha)!/(\eta-\beta)!$ can be estimated from above by a constant times $|\eta|^{|\alpha|}$. Therefore we have

$$\frac{(|\eta|-\mu/s)!^s}{(|\eta|-|\beta|+|\alpha|)!^s}\frac{(\eta-\beta+\alpha)!}{(\eta-\beta)!}\le C|\eta|^{s(|\beta|-|\alpha|)-\mu}|\eta|^{|\alpha|}=C|\eta|^{s(|\beta|-|\alpha|)+|\alpha|-\mu} \tag{3.6}$$

for some constant C independent of η. Because $s(|\beta|-|\alpha|)+|\alpha|-\mu=0$ by assumption the right-hand side of (3.6) is bounded when $|\eta|$ tends to infinity. By (3.4), (3.5) and (3.6) we see that the map $x^\alpha D^\beta: G^s_w(\mu)\to G^s_w(0)$ is continuous. □

Let $p(\eta)$ be a function on $\mathbf{N}^2$ such that

$$|p(\eta)| \leq C|\eta|^m, \quad \forall \eta \in \mathbf{N}^2 \tag{3.7}$$

for some $C>0$ and $m\geq 0$ independent of η. Then we define the Euler type pseudo-differential operator $p(\partial)$ on $G_w^s(\mu)$ by

$$p(\partial)u := \sum_\eta u_\eta p(\eta)x^\eta/\eta!, \quad u = \sum_\eta u_\eta x^\eta/\eta! \in G_w^s(\mu), \tag{3.8}$$

where we set $\partial=(\partial_1, \partial_2)$, $\partial_j = x_j(\partial/\partial x_j)$, $j=1,2$. We note that if $p(\eta)=\eta_1+\eta_2$, the operator $p(\partial)=\partial_1+\partial_2$ is a so-called Euler type differential operator.

Lemma 3.4. *Let $p(\eta)$ be a function on $\mathbf{N}^2$ such that $\sup_{|\eta|\geq N}|p(\eta)|\to 0$ when $N\to\infty$. Then the map $p(\partial): G_w^s(\mu)\to G_w^s(\mu)$ is a compact operator for every $\mu\geq 0$.*

The proof of this lemma follows exactly the same arguments as the former half of the proof of Lemma 3.3. Therefore we omit the proof.

In the following we give basic properties of Fredholm operators. Let H be a Hilbert space with norm $\|\cdot\|$. We denote by $\mathcal{L}(H)$ the space of linear continuous operators on H. An operator $L\in\mathcal{L}(H)$ is said to be a Fredholm operator if the range LH of L is closed in H, the kernel and cokernel of L is of finite dimension, i.e., $\dim \operatorname{Ker} L<\infty$ and $\dim \operatorname{Coker} L<\infty$, where $\operatorname{Coker} L = H/LH$. We denote the space of Fredholm operators by $\Psi(H)$. For $L\in\Psi(H)$ we define the index of L by

$$\operatorname{ind} L := \dim \operatorname{Ker} L - \dim \operatorname{Coker} L.$$

Let $\mathcal{C}_\infty(H)$ be the space of compact operators on H, and let I denote the identity operator on H. Then the following two lemmas are well known (cf. [1]).

Lemma 3.5. *An operator $L\in\mathcal{L}(H)$ is a Fredholm operator if and only if there exist linear continuous operators $R_1\in\mathcal{L}(H)$, $R_2\in\mathcal{L}(H)$ and compact operators $K_1\in\mathcal{C}_\infty(H)$, $K_2\in\mathcal{C}_\infty(H)$ such that*

$$R_1 L = I + K_1, \quad LR_2 = I + K_2.$$

Here the operators R_1 and R_2 are called left and right regularizers, respectively.

Lemma 3.6. *The set $\Psi(H)$ is an open subset of $\mathcal{L}(H)$ and the index is constant on the connected components of $\Psi(H)$. If $L\in\Psi(H)$ and $K\in\mathcal{C}_\infty(H)$ the operator $L+K$ is a Fredholm operator and* $\operatorname{ind}(L+K)=\operatorname{ind} L$.

4. Proof of Theorem 2.1

Let m be an s-Gevrey order of P. In view of Lemma 3.1 it is sufficient to prove that for any $r<w$ the map

$$P: G_r^s(m) \to G_r^s(0) \tag{4.1}$$

is a Fredholm operator of index 0. Then Theorem 2.1 is a consequence of the following theorem.

Theorem 4.1. *For any $r<w$ the map* (4.1) *is a Fredholm operator of index zero if the conditions* (2.8) *and* (2.9) *are satisfied.*

In order to prove Theorem 4.1 we prove two propositions.

We set $\langle\eta\rangle := (1+|\eta|^2)^{1/2}$ and we denote by $\langle\partial\rangle$ the Euler type pseudodifferential operator with symbol $\langle\eta\rangle$. Let P be given by (2.5). Then we have the following result.

Proposition 4.2. *Let the operators P_0 and Q_0 be defined by*

$$Q_0 := P_0\langle\partial\rangle^{-m}, \quad P_0(x, D_x) := \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta} D^\beta x^\alpha. \tag{4.2}$$

Then Q_0 maps $G_r^s(0)$ into itself. Moreover Q_0 is a Fredholm operator of index zero if and only if $P: G_r^s(m) \to G_r^s(0)$ is a Fredholm operator of index zero.

Proof. We write P in (2.5) in the following form

$$\begin{aligned} P(x, D_x) &= \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta} D^\beta x^\alpha + \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)<m} a_{\alpha\beta} D^\beta x^\alpha \\ &=: P_0(x, D) + P_1(x, D). \end{aligned} \tag{4.3}$$

Because the s-Gevrey order of terms in P_1 is smaller than m, it follows from Lemma 3.3 that the map $P_1: G_r^s(m) \to G_r^s(0)$ is compact. Therefore by Lemma 3.6 one may assume that $P = P_0$.

We note that, for $k \ge 0$, $n \ge 0$ and $m \ge 0$

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right)^k t^m t^n &= (n+m)(n+m-1)\dots(n+m-k+1)t^{n+m-k} \\ &= t^{n+m-k}\frac{\Gamma(n+m+1)}{\Gamma(n+m-k+1)}, \end{aligned} \tag{4.4}$$

where Γ denotes the gamma function. Similarly, if $k<0$ we have

$$\left(\frac{\partial}{\partial t}\right)^k t^m t^n = \frac{t^{n+m-k}}{(n+m+1)\dots(n+m-k)} = t^{n+m-k}\frac{\Gamma(n+m+1)}{\Gamma(n+m-k+1)}.$$

Therefore if we define the Euler type operator $p_{\alpha\beta}(\partial)$ on $G^s_r(\mu)$ $(\mu\geq 0)$ by

$$p_{\alpha\beta}(\eta)=\prod_{j=1}^{2}\frac{\Gamma(\eta_j+\alpha_j+1)}{\Gamma(\eta_j+\alpha_j-\beta_j+1)}, \tag{4.5}$$

we have, for $\alpha\in\mathbf{Z}^2_+$ and $\beta\in\mathbf{Z}^2$

$$D^\beta x^\alpha u=x^{\alpha-\beta}p_{\alpha\beta}(\partial)u \quad \text{for } u\in G^s_r(\mu)\ (\mu\geq 0).$$

Therefore we have that

$$P_0(x,D)=\sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta}x^{\alpha-\beta}p_{\alpha\beta}(\partial). \tag{4.6}$$

By Lemma 3.3 the operator Q_0 maps $G^s_r(0)$ into itself. Suppose that Q_0 is a Fredholm operator. By Lemma 3.5 there exist regularizers R_j and compact operators K_j $(j=1,2)$ such that

$$R_1Q_0=I+K_1, \quad Q_0R_2=I+K_2. \tag{4.7}$$

We have $I+K_2=Q_0R_2=P_0\langle\partial\rangle^{-m}R_2$. Hence $\langle\partial\rangle^{-m}R_2$ is a right regularizer of P_0. On the other hand we have

$$I+K_1=R_1Q_0=R_1P_0\langle\partial\rangle^{-m}=R_1\langle\partial\rangle^{-m}P_0+R_1[P_0,\langle\partial\rangle^{-m}].$$

If $[P_0,\langle\partial\rangle^{-m}]$ is a compact operator, it follows that $R_1\langle\partial\rangle^{-m}$ is a left regularizer of P_0. Hence P_0 is a Fredholm operator. We can similarly prove the converse. Moreover, since $\langle\partial\rangle^{-m}$ is a bijection we have $\operatorname{ind}P_0=\operatorname{ind}Q_0$.

It remains to prove that $[P_0,\langle\partial\rangle^{-m}]$ is a compact operator. Because P_0 is a sum of operators of the form $x^\gamma p_{\alpha\beta}(\partial)$ $(\gamma=\alpha-\beta)$ it is sufficient to consider the case $P_0=x^\gamma p_{\alpha\beta}(\partial)$. We note that the operators $p_{\alpha\beta}(\partial)$ and $\langle\partial\rangle^{-m}$ commute. Because $[x^\gamma,\langle\partial\rangle^{-m}]=x^\gamma\langle\partial\rangle^{-m}-\langle\partial\rangle^{-m}x^\gamma$, we have, for $u=\sum_\eta u_\eta x^\eta/\eta!\in G^s_r(\mu)$

$$[x^\gamma,\langle\partial\rangle^{-m}]u=x^\gamma\sum_\eta u_\eta x^\eta(\langle\eta\rangle^{-m}-\langle\gamma+\eta\rangle^{-m})/\eta!\,. \tag{4.8}$$

By Taylor's formula we have

$$\langle\eta\rangle^{-m}-\langle\gamma+\eta\rangle^{-m}=m\int_0^1\gamma\cdot(\eta+s\gamma)\langle\eta+s\gamma\rangle^{-m-2}\,ds=:C_\gamma(\eta). \tag{4.9}$$

It follows that $\Lambda_\gamma(\eta):=\langle\eta\rangle^m C_\gamma(\eta)$ satisfies

$$\sup_{|\eta|\geq N}|\Lambda_\gamma(\eta)|\to 0,\quad N\to\infty. \tag{4.10}$$

Therefore we have

$$[P_0,\langle\partial\rangle^{-m}]=[x^\gamma,\langle\partial\rangle^{-m}]p_{\alpha\beta}(\partial)=x^\gamma\langle\partial\rangle^{-m}\Lambda_\gamma(\partial)p_{\alpha\beta}(\partial)=P_0\langle\partial\rangle^{-m}\Lambda_\gamma(\partial), \tag{4.11}$$

where $\Lambda_\gamma(\partial)$ is an Euler type operator with symbol given by $\Lambda_\gamma(\eta)$. It follows from (4.10), (4.11) and Lemmas 3.3 and 3.4 that $[P_0,\langle\partial\rangle^{-m}]$ is a compact operator. □

Next we shall show that the Fredholmness of the operator $Q_0: G_r^s(0)\to G_r^s(0)$ is equivalent to that of a certain Toeplitz operator. Let $\mathbf{T}^2$ be a two dimensional torus and let us take the coordinate $(e^{i\theta_1},e^{i\theta_2})\in\mathbf{T}^2$. Let $u=\sum u_\eta x^\eta/\eta!\in G_r^s(0)$. We set $v_\eta:=u_\eta r^\eta/|\eta|!^s$. Then $u\in G_r^s(0)$ if and only if the sequence $\{v_\eta\}$ is in $l_2^+:=l_2(\mathbf{Z}_+^2)$, the set of square summable sequences on $\mathbf{Z}_+^2$, where $\mathbf{Z}_+$ is the set of non-negative integers. Because the space l_2^+ and the Hardy space $H^2(\mathbf{T}^2)$ are isomorphic, it follows that $u\in G_r^s(0)$ if and only if $\sum_\eta v_\eta e^{i\theta\eta}$ is in $H^2(\mathbf{T}^2)$. Because $H^2(\mathbf{T}^2)$ is a closed subspace of $L^2(\mathbf{T}^2)$, the space of square integrable functions, there is a projection π from $L^2(\mathbf{T}^2)$ onto $H^2(\mathbf{T}^2)$. By the correspondence between the spaces $G_r^s(0)$ and $H^2(\mathbf{T}^2)$ the Euler type operator $p(\partial)$ in (3.8) on $G_r^s(0)$ also defines an Euler type pseudodifferential operator $p(D)$ ($D=i^{-1}\partial/\partial\theta$) on $H^2(\mathbf{T}^2)$. We denote by $\lambda_\alpha(D)$ the Euler type pseudodifferential operator with symbol $\lambda_\alpha(\eta):=\eta^\alpha|\eta|^{-|\alpha|}$ ($\eta\neq 0$) and $\lambda_\alpha(0)=0$. We define a Toeplitz operator on $H^2(\mathbf{T}^2)$ by

$$T=\pi\sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m}a_{\alpha\beta}r^{\alpha-\beta}e^{i(\alpha-\beta)\theta}\lambda_\alpha(D): H^2(\mathbf{T}^2)\to H^2(\mathbf{T}^2). \tag{4.12}$$

The function (2.7) with $w=r$ is called the symbol of the Toeplitz operator T.

Proposition 4.3. *The operator Q_0 is a Fredholm operator of index zero if and only if the Toeplitz operator T is a Fredholm operator of index zero.*

Proof. By the isomorphism between $G_r^s(0)$ and $H^2(\mathbf{T}^2)$ the projection π induces a projection on the formal Laurent series

$$\pi u:=\sum_{\eta\in\mathbf{Z}_+^2}u_\eta x^\eta/\eta!\quad\text{for } u=\sum_{\eta\in\mathbf{Z}^2}u_\eta x^\eta/\eta!, \tag{4.13}$$

where we use the same notation π as the projection on L^2 to H^2 for the sake of simplicity. By the definition of $p_{\alpha\beta}(\partial)$ in (4.5) we see that in the expression of $x^{\alpha-\beta}p_{\alpha\beta}(\partial)\langle\partial\rangle^{-m}u$, $u\in G_r^s(0)$ there appear no negative powers. Hence we have

$$\begin{aligned} Q_0&=\pi\sum a_{\alpha\beta}x^{\alpha-\beta}p_{\alpha\beta}(\partial)\langle\partial\rangle^{-m}\\ &=\pi\sum a_{\alpha\beta}x^{\alpha-\beta}\langle\partial\rangle^{(s-1)(|\alpha|-|\beta|)}\langle\partial\rangle^{-|\beta|}p_{\alpha\beta}(\partial), \end{aligned} \tag{4.14}$$

on $G_r^s(0)$.

We shall study the operators $\pi x^\gamma \langle\partial\rangle^{(s-1)|\gamma|}$ $(\gamma=\alpha-\beta)$ and $\langle\partial\rangle^{-|\beta|}p_{\alpha\beta}(\partial)$. Let $u=\sum u_\eta x^\eta/\eta!\in G_r^s(0)$. We set $v_\eta:=u_\eta r^\eta/|\eta|!^s$. Then we have

$$\begin{aligned}\pi x^\gamma \langle\partial\rangle^{(s-1)|\gamma|}\sum u_\eta \frac{x^\eta}{\eta!} &= \pi x^\gamma \langle\partial\rangle^{(s-1)|\gamma|}\sum v_\eta r^{-\eta}|\eta|!^s \frac{x^\eta}{\eta!}\\ &= \sum_{\eta+\gamma\geq 0,\eta\geq 0} v_\eta r^{-\eta}|\eta|!^s\langle\eta\rangle^{(s-1)|\gamma|}\frac{x^{\eta+\gamma}}{\eta!}\\ &= \sum v_{\eta-\gamma} r^{\gamma-\eta}(|\eta|-|\gamma|)!^s\langle\eta-\gamma\rangle^{(s-1)|\gamma|}\frac{\eta!}{(\eta-\gamma)!}\frac{x^\eta}{\eta!}.\end{aligned}$$

Therefore the map $\pi x^\gamma \langle\partial\rangle^{(s-1)|\gamma|}: l_2^+\to l_2^+$ is given by

$$\pi x^\gamma \langle\partial\rangle^{(s-1)|\gamma|}\{v_\eta\} = \left\{v_{\eta-\gamma} r^\gamma \frac{(|\eta|-|\gamma|)!^s}{|\eta|!^s}\langle\eta-\gamma\rangle^{(s-1)|\gamma|}\frac{\eta!}{(\eta-\gamma)!}\right\}\in l_2^+ \tag{4.15}$$

for $\{v_\eta\}\in l_2^+$.

We define the pseudodifferential operator $A_\gamma(D)$ with symbol $A_\gamma(\eta)$ by

$$A_\gamma(\eta) := \frac{|\eta|!^s\langle\eta\rangle^{(s-1)|\gamma|}}{(|\eta|+|\gamma|)!^s}\frac{(\eta+\gamma)!}{\eta!}. \tag{4.16}$$

Let S_γ be a multiplication operator by a function $e^{i\gamma\theta}$, $S_\gamma\sum u_\eta e^{i\eta\theta}=\sum u_\eta e^{i(\eta+\gamma)\theta}$. Then it follows from (4.15) that the operator $\pi x^\gamma\langle\partial\rangle^{(s-1)|\gamma|}$ corresponds to

$$\pi S_\gamma A_\gamma(\partial) r^\gamma. \tag{4.17}$$

We have, assuming $\eta+\gamma\geq 0$,

$$\frac{|\eta|!^s\langle\eta\rangle^{(s-1)|\gamma|}}{(|\eta|+|\gamma|)!^s}\frac{(\eta+\gamma)!}{\eta!} = \tilde\lambda_\gamma(\eta)+r_\gamma(\eta), \quad \tilde\lambda_\gamma(\eta) = \frac{(\eta_1+1)^{\gamma_1}(\eta_2+1)^{\gamma_2}}{\langle\eta\rangle^{|\gamma|}}, \tag{4.18}$$

where $r_\gamma(\eta)$ consists of terms such that $r_\gamma(\eta)\to 0$ when $|\eta|\to\infty$.

Indeed the quantity $|\eta|!^s\langle\eta\rangle^{s|\gamma|}(|\eta|+|\gamma|)!^{-s}$ tends to 1 when $|\eta|\to\infty$ and γ is fixed. On the other hand, we get, assuming $\eta_j+\gamma_j\geq 0$,

$$\frac{(\eta_j+\gamma_j)!}{\eta_j!}\langle\eta\rangle^{-\gamma_j} = \left(\frac{\eta_j+1}{\langle\eta\rangle}\right)^{\gamma_j}+\Psi_j(\eta), \tag{4.19}$$

with $\Psi_j(\eta)$ satisfying $\Psi_j(\eta)\to 0$ when $|\eta|\to\infty$. By these estimates we have (4.18).

It follows from (4.17), (4.18) and the definition of π that

$$\pi S_\gamma A_\gamma(\partial)r^\gamma = \pi S_\gamma \tilde{\lambda}_\gamma(\partial)r^\gamma + \pi R_\gamma(\partial) \tag{4.20}$$

where $R_\gamma(\partial) := S_\gamma r_\gamma(\partial) r^\gamma$, with $r_\gamma(\partial)$ being the Euler type psedodifferential operator with the symbol $r_\gamma(\eta)$. We note that by Lemma 3.4 $\pi R_\gamma(\partial)$ is a compact operator.

Next we consider the operator $\langle\partial\rangle^{-|\beta|}p_{\alpha\beta}(\partial)$. Because we have

$$\langle\partial\rangle^{-|\beta|}p_{\alpha\beta}(\partial)\sum u_\eta \frac{x^\eta}{\eta!} = \sum v_\eta r^{-\eta}|\eta|!^s \langle\eta\rangle^{-|\beta|}p_{\alpha\beta}(\eta)\frac{x^\eta}{\eta!},$$

the operator $\langle\partial\rangle^{-|\beta|}p_{\alpha\beta}(\partial)$ defines the pseudodifferential operator on H^2 with the symbol $\langle\eta\rangle^{-|\beta|}p_{\alpha\beta}(\eta)$. If $\eta+\gamma\geq 0$ we have

$$\langle\eta\rangle^{-|\beta|}p_{\alpha\beta}(\eta) = \tilde{\lambda}_\beta(\eta) + \tilde{r}_{\alpha\beta}(\eta),$$

where $\tilde{r}_{\alpha\beta}(\eta)$ satisfies that $\sup_{|\eta|\geq n}|\tilde{r}_{\alpha\beta}(\eta)| \to 0$ when n tends to infinity. Therefore we can replace $\langle\partial\rangle^{-|\beta|}p_{\alpha\beta}(\partial)$ in (4.14) with $\tilde{\lambda}_\beta(\partial)+\tilde{r}_{\alpha\beta}(\partial)$.

By (4.20) with $\gamma=\alpha-\beta$ we get from (4.14) that Q_0 corresponds to the operator

$$\begin{aligned} &\pi\sum a_{\alpha\beta}(S_{\alpha-\beta}\tilde{\lambda}_{\alpha-\beta}r^{\alpha-\beta} + R_{\alpha-\beta})(\tilde{\lambda}_\beta + \tilde{r}_{\alpha\beta}) \\ &\qquad = \pi\sum a_{\alpha\beta}S_{\alpha-\beta}r^{\alpha-\beta}\tilde{\lambda}_{\alpha-\beta}\tilde{\lambda}_\beta + \pi\sum K_{\alpha\beta} \\ &\qquad = \pi\sum a_{\alpha\beta}S_{\alpha-\beta}\tilde{\lambda}_\alpha r^{\alpha-\beta} + \pi\sum K_{\alpha\beta}, \end{aligned} \tag{4.21}$$

where

$$K_{\alpha\beta} = S_{\alpha-\beta}\tilde{\lambda}_{\alpha-\beta}r^{\alpha-\beta}\tilde{r}_{\alpha\beta} + R_{\alpha-\beta}\tilde{\lambda}_\beta + R_{\alpha-\beta}\tilde{r}_{\alpha\beta}. \tag{4.22}$$

For each α and β, $K_{\alpha\beta}$ is a compact operator by the definition of symbols $R_{\alpha-\beta}$ and $\tilde{r}_{\alpha\beta}$ and Lemma 3.4. Because the sum $\sum a_{\alpha\beta}K_{\alpha\beta}$ is a finite sum, the second term in the right-hand side of (4.21) is a compact operator. Because $\tilde{\lambda}_\alpha - \eta^\alpha/|\eta|^{|\alpha|}$ defines a compact operator the right-hand side of (4.21) is equal to T modulo compact operators. Moreover we have $\operatorname{ind} Q_0 = \operatorname{ind} T$. □

5. Proof of Theorem 4.1

In order to prove Theorem 4.1 we recall the following lemma.

Lemma 5.1. *Let $A(\theta, D)$ and $B(\theta, D)$ be classical pseudodifferential operators of order zero on $\mathbf{T}^2$ with smooth symbols. Then the commutator $[A,B]:=AB-BA$ is a classical pseudodifferential operator of order -1. Especially, the commutator $[A,B]$ is a compact operator on $L^2(\mathbf{T}^2)$.*

This lemma is elementary in the theory of pseudodifferential operators and the proof is a routine work. So we omit the proof.

Proof of Theorem 4.1. The proof below is done by arranging the argument in [1, Sect. 8.23]. In view of Propositions 4.2 and 4.3 we shall show that the Toeplitz operator (4.12) is a Fredholm operator of index zero if the conditions (2.8) and (2.9) are satisfied. We define the closed subspaces H_1, H_2 of $L^2(\mathbf{T}^2)$ by

$$H_1 := \left\{u \in L^2\,; u = \sum_{\zeta_1 \geq 0} u_\zeta e^{i\zeta\theta}\right\}, \quad H_2 := \left\{u \in L^2\,; u = \sum_{\zeta_2 \geq 0} u_\zeta e^{i\zeta\theta}\right\}. \tag{5.1}$$

By definition we see that $H^2(\mathbf{T}^2) := H_1 \cap H_2$. We define the projections π_1 and π_2 by

$$\pi_1 : L^2(\mathbf{T}^2) \to H_1, \quad \pi_2 : L^2(\mathbf{T}^2) \to H_2. \tag{5.2}$$

We note that the projection $\pi: L^2(\mathbf{T}^2) \to H^2(\mathbf{T}^2)$ is equal to $\pi_1\pi_2$, by definition. We define the Toeplitz operators $T_{+\cdot}$ and $T_{\cdot+}$ by

$$T_{+\cdot} := \pi_1 \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta} r^{\alpha-\beta} e^{i(\beta-\alpha)\theta} \lambda_\alpha(D)\pi_1 : H_1 \to H_1, \tag{5.3}$$

$$T_{\cdot+} := \pi_2 \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta} r^{\alpha-\beta} e^{i(\beta-\alpha)\theta} \lambda_\alpha(D)\pi_2 : H_2 \to H_2. \tag{5.4}$$

If we denote by $\mathcal{L}_{s,r}$ the pseudodifferential operator with symbol $L_{s,r}$, our Toeplitz operator T is given by $T = \pi_1\pi_2\mathcal{L}_{s,r}\pi_1\pi_2$.

If we fix the branch of $\log L_{s,r}$ appropriately, the function $b(z,\xi) := \log L_{s,r}(z,\xi)$ is a smooth function of $z = (e^{i\theta_1}, e^{i\theta_2}) \in \mathbf{T}^2$ and $\xi \in \mathbf{R}^2$, $|\xi| = 1$, $\xi \geq 0$. We expand $b = b(\theta)$ into Fourier series, $b = b_1 + b_2 + b_3 + b_4$, where b_1, b_2, b_3 and b_4 are functions such that the supports of their Fourier coefficients are contained in the regions $I := \{\eta_1 \geq 0,\ \eta_2 \geq 0\}$, $II := \{\eta_1 \leq 0,\ \eta_2 \geq 0\}$, $III := \{\eta_1 \leq 0,\ \eta_2 \leq 0\}$ and $IV := \{\eta_1 \geq 0,\ \eta_2 \leq 0\}$, respectively. We have a Riemann–Hilbert factorization of $L_{s,r}$

$$L_{s,r} = e^{b_1} e^{b_2} e^{b_3} e^{b_4} =: \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 \tilde{a}_4. \tag{5.5}$$

Because these regions are convex, the supports of the Fourier coefficients of $\tilde{a}_j$ $(j = 1,2,3,4)$ together with their inverses are contained in I, II, III and IV, in this order.

We want to show that $T_{+\cdot}$ and $T_{\cdot+}$ are invertible modulo compact operators and the inverses (modulo compact operators) are given respectively by

$$T_{+\cdot}^{-1}=\pi_1 a_1^{-1}a_4^{-1}\pi_1 a_2^{-1}a_3^{-1}\pi_1, \tag{5.6}$$

$$T_{\cdot+}^{-1}=\pi_2 a_1^{-1}a_2^{-1}\pi_2 a_4^{-1}a_3^{-1}\pi_2, \tag{5.7}$$

where we understand the equality sign modulo compact operators, and where a_j and a_j^{-1} denote pseudodifferential operators on $\mathbf{T}^2$ with symbols $\tilde{a}_j(\theta,\xi)$ and $\tilde{a}_j^{-1}(\theta,\xi)$ with ξ being covariable of θ. In the following we write $A\equiv B$ if two operators $A, B\in\mathcal{L}$ are equal modulo compact operators.

By Lemma 5.1 the commutators of pseudodifferential operators a_j and a_j^{-1} are compact. We note that $\mathcal{L}_{s,r}\equiv a_1a_2a_3a_4$ since the principal symbols of both sides coincide. It follows that

$$\begin{aligned}T_{+\cdot}\pi_1 a_1^{-1}a_4^{-1}\pi_1 a_2^{-1}a_3^{-1}\pi_1 &\equiv \pi_1 a_1a_2a_3a_4\pi_1 a_1^{-1}a_4^{-1}\pi_1 a_2^{-1}a_3^{-1}\pi_1\\ &\equiv \pi_1 a_2a_3a_1a_4a_1^{-1}a_4^{-1}\pi_1 a_2^{-1}a_3^{-1}\pi_1\\ &\quad +\pi_1 a_2a_3a_1a_4(I-\pi_1)a_1^{-1}a_4^{-1}\pi_1 a_2^{-1}a_3^{-1}\pi_1\\ &\equiv \pi_1 a_2a_3\pi_1 a_2^{-1}a_3^{-1}\pi_1,\end{aligned} \tag{5.8}$$

where we have used the relation $(I-\pi_1)a_1^{-1}a_4^{-1}\pi_1=0$. Therefore the right-hand side of (5.8) is equal to

$$\pi_1 a_2a_3a_2^{-1}a_3^{-1}\pi_1+\pi_1 a_2a_3(I-\pi_1)a_2^{-1}a_3^{-1}\pi_1\equiv\pi_1.$$

Here we have used the relation $\pi_1a_2a_3(I-\pi_1)=0$. Similarly we can show that $\pi_1 a_1^{-1}a_4^{-1}\pi_1 a_2^{-1}a_3^{-1}\pi_1 T_{+\cdot}\equiv\pi_1$. Hence we have proved (5.6). By the same arguments we can show (5.7).

We shall show that the left and right regularizers R of $T:=\pi\mathcal{L}_{s,r}\pi$ is given by

$$R=\pi(T_{+\cdot}^{-1}+T_{\cdot+}^{-1}-\mathcal{L}_{s,r}^{-1})\pi, \tag{5.9}$$

where $\mathcal{L}_{s,r}^{-1}$ is a pseudodifferential operator with symbol $L_{s,r}^{-1}$. First we recall that $\pi=\pi_1\pi_2$. By (5.6) we have

(5.10)

$$\begin{aligned}\pi T_{+\cdot}^{-1}\pi\mathcal{L}_{s,r}\pi &= \pi_1\pi_2T_{+\cdot}^{-1}\pi_1\pi_2\mathcal{L}_{s,r}\pi_1\pi_2\\ &=\pi_1\pi_2T_{+\cdot}^{-1}\pi_1\mathcal{L}_{s,r}\pi_1\pi_2-\pi_1\pi_2T_{+\cdot}^{-1}\pi_1(I-\pi_2)\mathcal{L}_{s,r}\pi_1\pi_2\\ &\equiv\pi_1\pi_2-\pi_1\pi_2a_1^{-1}a_4^{-1}\pi_1a_2^{-1}a_3^{-1}\pi_1(I-\pi_2)\mathcal{L}_{s,r}\pi_1\pi_2\\ &=\pi_1\pi_2-\pi_1\pi_2a_1^{-1}a_4^{-1}(\pi_1\pi_2+\pi_1(I-\pi_2))a_2^{-1}a_3^{-1}\pi_1(I-\pi_2)\mathcal{L}_{s,r}\pi_1\pi_2.\end{aligned}$$

Similarly it follows from (5.7) that

(5.11)

$$\begin{aligned}\pi T_{\cdot+}^{-1}\pi\mathcal{L}_{s,r}\pi &= \pi_1\pi_2 T_{\cdot+}^{-1}\pi_1\pi_2\mathcal{L}_{s,r}\pi_1\pi_2\\ &= \pi_1\pi_2 T_{\cdot+}^{-1}\pi_2\mathcal{L}_{s,r}\pi_2\pi_1 - \pi_1\pi_2 T_{\cdot+}^{-1}\pi_2(I-\pi_1)\mathcal{L}_{s,r}\pi_1\pi_2\\ &\equiv \pi_1\pi_2 - \pi_1\pi_2 a_1^{-1}a_2^{-1}\pi_2 a_4^{-1}a_3^{-1}\pi_2(I-\pi_1)\mathcal{L}_{s,r}\pi_1\pi_2\\ &= \pi_1\pi_2 - \pi_1\pi_2 a_1^{-1}a_2^{-1}(\pi_1\pi_2+\pi_2(I-\pi_1))a_4^{-1}a_3^{-1}\pi_2(I-\pi_1)\mathcal{L}_{s,r}\pi_1\pi_2.\end{aligned}$$

On the other hand, by using $\mathcal{L}_{s,r}^{-1}\mathcal{L}_{s,r}\equiv I$ we have

$$-\pi\mathcal{L}_{s,r}^{-1}\pi\mathcal{L}_{s,r}\pi = -\pi_1\pi_2\mathcal{L}_{s,r}^{-1}\pi_1\pi_2\mathcal{L}_{s,r}\pi_1\pi_2 \equiv -\pi_1\pi_2 - \pi_1\pi_2\mathcal{L}_{s,r}^{-1}(\pi_1\pi_2 - I)\mathcal{L}_{s,r}\pi_1\pi_2.$$

By using that

$$\pi_1\pi_2 - I = \pi_1(\pi_2 - I) + (\pi_1 - I)\pi_2 - (\pi_1 - I)(\pi_2 - I)$$

we have

$$\begin{aligned}-\pi\mathcal{L}_{s,r}^{-1}\pi\mathcal{L}_{s,r}\pi \equiv& -\pi_1\pi_2 - \pi_1\pi_2\mathcal{L}_{s,r}^{-1}\pi_1(\pi_2 - I)\mathcal{L}_{s,r}\pi_1\pi_2\\ & -\pi_1\pi_2\mathcal{L}_{s,r}^{-1}(\pi_1 - I)\pi_2\mathcal{L}_{s,w}\pi_1\pi_2\\ & +\pi_1\pi_2\mathcal{L}_{s,r}^{-1}(\pi_1 - I)(\pi_2 - I)\mathcal{L}_{s,r}\pi_1\pi_2.\end{aligned} \tag{5.12}$$

By adding (5.10), (5.11) and (5.12) we have

$$\begin{aligned}RT \equiv& \pi_1\pi_2 - \pi_1\pi_2 a_1^{-1}a_4^{-1}(\pi_1\pi_2+\pi_1(I-\pi_2))a_2^{-1}a_3^{-1}\pi_1(I-\pi_2)\mathcal{L}_{s,r}\pi_1\pi_2\\ & -\pi_1\pi_2 a_1^{-1}a_2^{-1}(\pi_1\pi_2+\pi_2(I-\pi_1))a_4^{-1}a_3^{-1}\pi_2(I-\pi_1)\mathcal{L}_{s,r}\pi_1\pi_2\\ & -\pi_1\pi_2\mathcal{L}_{s,r}^{-1}\pi_1(\pi_2 - I)\mathcal{L}_{s,r}\pi_1\pi_2 - \pi_1\pi_2\mathcal{L}_{s,r}^{-1}(\pi_1 - I)\pi_2\mathcal{L}_{s,r}\pi_1\pi_2\\ & +\pi_1\pi_2\mathcal{L}_{s,r}^{-1}(\pi_1 - I)(\pi_2 - I)\mathcal{L}_{s,r}\pi_1\pi_2.\end{aligned}$$

We note that

$$\begin{aligned}\pi_1\pi_2+\pi_1(I-\pi_2) &= I-(\pi_1-I)(\pi_2-I)-\pi_2(I-\pi_1),\\ \pi_1\pi_2+\pi_2(I-\pi_1) &= I-(\pi_1-I)(\pi_2-I)-\pi_1(I-\pi_2).\end{aligned}$$

Therefore we have

(5.13)

$$\begin{aligned}RT - \pi_1\pi_2 \equiv& \pi_1\pi_2 a_1^{-1}a_4^{-1}((\pi_1-I)(\pi_2-I)+\pi_2(I-\pi_1))a_2^{-1}a_3^{-1}\pi_1(I-\pi_2)\mathcal{L}_{s,r}\pi_1\pi_2\\ & +\pi_1\pi_2\mathcal{L}_{s,r}^{-1}(\pi_1 - I)(\pi_2 - I)\mathcal{L}_{s,r}\pi_1\pi_2\\ & +\pi_1\pi_2 a_1^{-1}a_2^{-1}((\pi_1-I)(\pi_2-I)+\pi_1(I-\pi_2))a_4^{-1}a_3^{-1}\pi_2(I-\pi_1)\mathcal{L}_{s,r}\pi_1\pi_2.\end{aligned}$$

We shall show that the operators

$$\pi_1\pi_2\varphi(\pi_1-I)(\pi_2-I), \quad \pi_2(I-\pi_1)\varphi\pi_1(I-\pi_2), \quad \pi_1(I-\pi_2)\varphi\pi_2(I-\pi_1) \tag{5.14}$$

are compact, where φ is an appropriately chosen smooth function. To this end, let $u=\sum_\alpha u_\alpha e^{i\alpha\theta}\in L^2$ and $\varphi(\xi)=\sum_\beta \varphi_\beta(\xi)e^{i\beta\theta}$ be Fourier expansions of $u\in L^2$ and $\varphi\in C^\infty$. Because $\varphi(\theta, D)$ is a pseudodifferential operator of order zero with smooth coefficients it follows that the Fourier coefficients $\varphi_\beta(\xi)$ are rapidly decreasing in β when $|\beta|\to\infty$ uniformly in ξ. We have

$$\pi_1\pi_2\varphi(\pi_1-I)(\pi_2-I)u=\sum_{\mu\in I}\Bigg(\sum_{\alpha+\beta=\mu,\alpha\in III}\varphi_\beta(\mu)u_\alpha\Bigg)e^{i\mu\theta}. \tag{5.15}$$

In view of the definitions of I and III we see that, in (5.15), β satisfies that $|\beta|=|\mu-\alpha|\geq|\mu|$ because $\mu\in I$ and $-\alpha\in I$. It follows that, for $n\geq1$

$$|\mu|^n\sum_{\alpha+\beta=\mu,\alpha\in III}|\varphi_\beta(\mu)||u_\alpha|\leq\sum|\beta|^n|\varphi_\beta(\mu)||u_\alpha|<\infty$$

for any μ because $|\varphi_\beta(\mu)||\beta|^n$ is bounded in μ and β. Hence the Fourier series (5.15) in μ converges uniformly with respect to $u\in L^2$. In view of the proof of Lemma 3.4 this shows that $\pi_1\pi_2\varphi(\pi_1-I)(\pi_2-I)$ is compact. The compactness of other operators will be proved similarly. Therefore we see that R is a left regularizer of T. We can similarly show that R is a right regularizer of T. Hence we see that T is a Fredholm operator.

It remains to prove that $\operatorname{ind} T=0$ if (2.8) and (2.9) are fulfilled. By the factorization (5.5) we set $\phi_t=e^{tb_1}e^{tb_2}e^{tb_3}e^{tb_4}$ $(0\leq t\leq1)$. Clearly ϕ_t $(0\leq t\leq1)$ is a one parameter family of symbols satisfying (2.8), (2.9), $\phi_0=1$ and $\phi_1=L_{s,r}$, which is continuous in t in $L^\infty(\mathbf{T}^2)$. Because the operator norms of Toeplitz operators S_t with symbol ϕ_t are continuous with respect to L^∞ norm of ϕ_t it follows from Lemma 3.6 that the index is constant. Hence it is equal to zero. In view of Proposition 4.2 and 4.3 we have proved Theorem 4.1. □

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Masatake Miyake
Department of Mathematics
Faculty of Science
Nagoya University
Nagoya 464-01
Japan

Masafumi Yoshino
Faculty of Economics
Chuo University
Higashinakano Hachioji
Tokyo 192-03
Japan