CHARACTERISTIC VALUES ASSOCIATED WITH A CLASS OF NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS

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1. The equations to be studied in this paper are of the form

$$y'' + y\,F(y^2, x) = 0, \tag{1.1}$$

or, more generally,

$$y'' + p(x)\,y + y\,F(y^2, x) = 0, \tag{1.2}$$

where $p(x)$ is a positive and continuous function of x in a finite closed interval $[a, b]$, and the function $F(t, x)$ is subject to the following conditions:

$F(t, x)$ *is continuous in* t *and* x *for* $0 \leqslant t < \infty$ *and* $a \leqslant x \leqslant b$, *respectively;* (1.3 a)

$F(t, x) > 0$ *for* $t > 0$ *and* $x \in [a, b]$; (1.3 b)

There exists a positive number ε *such that, for any* x *in* $[a, b]$, $t^{-\varepsilon} F(t, x)$ *is a non-decreasing function of* t *for* $t \in [0, \infty]$. (1.3 c)

The statement that a function $y(x)$ is a solution of (1.1) or (1.2) in an interval $[a, b]$ will mean that $y(x)$ and $y'(x)$ are continuous in $[a, b]$ and that $y(x)$ satisfies there the equation in question.

Because of condition (1.3 c), equation (1.2) is not included in the class (1.1) and must be considered separately. Condition (1.3 b) and, in the case of equation (1.2), the fact that $p(x) > 0$ shows that a solution $y(x)$ of (1.1) or (1.2) satisfies the inequality $y y'' < 0$ for $y \neq 0$, i.e., the solution curves are concave with respect to the horizontal axis. It follows therefore from an elementary geometric argument that any solution $y(x)$ for which y and y' are finite at some point of $[a, b]$, can be continued to all points of the interval.

Our aim is to investigate the properties of those solutions $y(x)$ of (1.1) or (1.2) which satisfy the boundary conditions $y(a) = y(b) = 0$, although most of our results can be extended, by obvious modifications of the arguments employed, to more general homogeneous boundary conditions. It will be shown that there exists a countable set of such solutions, and that these solutions—which may be characterized by the number of their zeros in (a, b)—correspond to the stationary values of a certain functional. The latter will be termed "characteristic values" of the problem, and their asymptotic behavior will be studied.

We shall also consider equations (1.1) for which $F(t, x)$ is a periodic function of x, and we shall show that such an equation has a countable number of distinct periodic solutions.

We remark that no Lipschitz condition has been imposed on the function $F(t, x)$ since, with one exception, we do not need the uniqueness of the solution $y(x)$ of (1.1)—or (1.2)—corresponding to given values of $y(a)$ and $y'(a)$. The exception is the trivial solution $y(x) \equiv 0$, which has to be shown to be the only solution satisfying the initial conditions $y(a) = y'(a) = 0$. As the following argument shows, the uniqueness of this solution is a consequence of conditions (1.3).

Without losing generality, we may assume that $y(x) \not\equiv 0$ in a small intervall $[a, a+\delta]$ $(\delta > 0)$ (otherwise we may replace a by a', where a' is the largest value in $[a, b]$ such that $y(x) \equiv 0$ in $[a, a']$). We now distinguish two cases, according as there does, or does not, exist an interval $a < x < \alpha$ $(\alpha < b)$ such that $y(x) \neq 0$ in (a, α). In the first case, we replace (1.1), or (1.2), by the equivalent integral equation

$$y(x) = y(a) + y'(a)(x-a) - \int_a^x (x-s)\, y(s)\, F_1(y^2, s)\, ds,$$

where $F_1(t, x)$ stands for either $F(t, x)$ or $p(x) + F(t, x)$. Since $y(a) = y'(a) = 0$, this reduces to

$$y(x) + \int_a^x (x-s)\, y(s)\, F_1(y^2, s)\, ds = 0.$$

If x is taken to be a point of (a, α), this is seen to lead to a contradiction since $y(s)$ does not change its sign in (a, x) and $F_1(y^2, s)$ is positive.

In the second case, there will exist a sequence of points $\{x_n\}$ such that $b > x_1 > x_2 > \cdots > x_n > \cdots > a$, $\lim_{n\to\infty} x_n = a$, and $y(x_n) = 0$. In the interval $[a, x_n]$, (1.1) or (1.2) may be replaced by the integral equation

$$y(x) = \int_a^{x_n} g(x, s)\, y(s)\, F_1(y^2, s)\, ds,$$

where the Green's function $g(x,s)$ is defined by $(x_n-a)\,g(x,s)=(x-a)\,(x_n-s)$ and $(x_n-a)\,g(x,s)=(s-a)\,(x_n-x)$ in the intervals $[a,s]$ and $[s,x_n]$, respectively. Identifying x with the value at which $y(x)$ attains its maximum M_n in $[a,x_n]$, we obtain

$$M_n \leqslant M_n \int_a^{x_n} g(x,s)\,F_1(y^2,s)\,ds.$$

$F_1(t,x)$ is a positive and non-decreasing function of t for $t>0$. Since $4\,g(x,s)\leqslant x_n-a$ and $M_n\leqslant M_1$, we thus arrive at the inequality

$$4 \leqslant (x_n-a)\int_a^{x_1} F_1(M_1^2,s)\,ds.$$

Since $x_n-a\to 0$ for $n\to\infty$, this again leads to a contradiction, and the required uniqueness proof is complete.

2. Defining the function $G(t,x)$ by

$$G(t,x)=\int_0^t F(s,x)\,ds, \tag{2.1}$$

we consider the functional

$$H(y)\equiv\int_a^b [y'^2-G(y^2,x)]\,dx \tag{2.2}$$

within the class of continuous functions $y(x)$ which have a piecewise continuous derivative in $[a,b]$, and satisfy $y(a)=y(b)=0$. Although (1.1) is the Euler–Lagrange equation of the functional (2.2), it can be shown by simple examples that (2.2) has neither an upper nor a lower bound if $y(x)$ ranges over the class of functions in question. To obtain an extremum it is necessary to subject $y(x)$ to further restrictions. A restriction suitable for our purposes is given by the condition

$$\int_a^b y'^2\,dx=\int_a^b y^2 F(y^2,x)\,dx \tag{2.3}$$

which—as one easily confirms by multiplying both sides of (1.1) by $y(x)$ and integrating from a to b—is automatically satisfied by a solution of (1.1) for which $y(a)=y(b)=0$. If we add to this the condition $y(x)\not\equiv 0$, then it can be shown that within this restricted class the functional (2.2) has a positive minimum, and that this minimum is attained if $y(x)$ coincides with a solution of (1) for which $y(a)=y(b)=0$.

A proof of the existence of a solution of the extremal problem, and of the implied existence of a solution of (1.1), was carried out in a previous paper [6] which was concerned with oscillation properties of the solutions of (1.1) in an interval of infinite length (the boundary conditions considered in [6] were $y(a)=y'(b)=0$, but this causes only trivial changes in the argument). For convenient reference, we restate here the result in question.

THEOREM 2.1. *Let Γ denote the class of functions $y(x)$ which are piecewise continuously differentiable in $[a,b]$, satisfy the conditions $y(a)=y(b)=0$, $y(x)\not\equiv 0$, and are subject to* (2.3). *If $H(y)$ denotes the functional* (2.2), *the problem*

$$H(y)=\min=\lambda, \quad y(x)\in\Gamma \tag{2.4}$$

is solved by a solution of (1.1) *for which $y(a)=y(b)=0$ and $y(x)>0$ in (a,b). The minimal value λ is positive.*

We remark here that (2.3) is a normalization condition. Indeed, if $u(x)$ is a function which satisfies all the other admissibility conditions, it is always possible to find a positive constant α such that $y(x)=\alpha u(x)$ satisfies (2.3). This is equivalent to finding an α such that

$$\int_a^b u'^2\,dx=\int_a^b u^2 F(\alpha^2 u^2, x)\,dx, \tag{2.5}$$

and the truth of the assertion follows from the observation that the righthand side of (2.5) is a continuous function of α which, in accordance with (1.3 c), tend to 0 for $\alpha\to 0$ and to ∞ for $\alpha\to\infty$.

We further remark that the condition $y(x)\not\equiv 0$ is essential. If this condition is omitted, the extremal problem has the trivial solution $y(x)\not\equiv 0$. The latter is a singular solution of the problem, in the sense that it cannot be approximated by other admissible functions. Indeed, if $y(x)$ is an admissible function we have

$$y^2(x)=\left(\int_a^x y'\,dx\right)^2\leqslant (x-a)\int_a^b y'^2\,dx,$$

where $\beta=\int_a^b y'^2\,dx>0$. Applying this to (2.3) and noting that $\beta>0$, we obtain

$$1\leqslant\int_a^b (x-a)\,F[\beta(x-a),x]\,dx. \tag{2.6}$$

If $\beta < 1$, it follows from (1.3 c) that

$$1 \leqslant \beta^{\varepsilon} \int_a^b (x-a) F[x-a, x] \, dx,$$

and we may conclude from (1.3 b) that there exists a positive constant β_0 such that, for every admissible function $y(x)$, we have

$$\int_a^b y'^2 \, dx \geqslant \beta_0 > 0. \tag{2.7}$$

The last inequality also shows that the functional (2.2) has a positive lower bound. By (2.1) and (1.3 c) we have

$$G(t, x) = \int_0^t F(s, x) \, ds = \int_0^t s^{\varepsilon} [s^{-\varepsilon} F(s, x)] \, ds \leqslant t^{-\varepsilon} F(t, x) \int_0^t s^{\varepsilon} \, ds = (1+\varepsilon)^{-1} t F(t, x).$$

In view of (2.2) and (2.3), we thus have

$$H(y) = \int_a^b [y^2 F(y^2, x) - G(y^2, x)] \, dx \geqslant \varepsilon (1+\varepsilon)^{-1} \int_a^b y^2 F(y^2, x) \, dx = \varepsilon (1+\varepsilon)^{-1} \int_a^b y'^2 dx, \tag{2.8}$$

and the existence of the bound follows from (2.7).

3. Theorem 2.1 establishes the existence of a solution $y(x)$ of the boundary value problem

$$y'' + y F(y^2, x) = 0, \quad y(a) = y(b) = 0. \tag{3.1}$$

In the present section it will be shown that this problem has, in addition, an infinite number of other solutions which can be obtained by solving the minimum problem (2.4) under increasingly restrictive side conditions. The minimal values of the "energy integral" (2.2) associated with these problems will be called the *characteristic values* $\lambda_1, \lambda_2, \ldots$ of the problem (3.1) where $\lambda_1 = \lambda$ is the number defined in Theorem 2.1 (formula (2.4)), and $0 < \lambda_1 < \lambda_2 < \ldots$.

To formulate the minimum problem defining the characteristic value λ_n, we choose $n+1$ distinct points a_ν such that $a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = b$. In the interval $[a_{\nu-1}, a_\nu]$ $(\nu = 1, \ldots, n)$, we consider functions $y_\nu(x)$ which are piecewise continuously differentiable, vanish for $x = a_{\nu-1}$ and $x = a_\nu$ (but not identically) and are normalized by

$$\int_{a_{\nu-1}}^{a_\nu} y'^2\,dx = \int_{a_{\nu-1}}^{a_\nu} y^2 F(y^2, x)\,dx. \tag{3.2}$$

If, for $x \in [a_{\nu-1}, a_\nu]$, we write $y(x) = y_\nu(x)$, the nth characteristic value is then defined by

$$\lambda_n = \min \int_a^b [y'^2 - G(y^2, x)]\,dx, \tag{3.3}$$

where $y(x)$ ranges over the class of all functions with the indicated properties.

Theorem 2.1 shows that it is sufficient to consider this minimum problem for functions $y(x)$ which in the intervals $[a_{\nu-1}, a_\nu]$ $(\nu = 1, \ldots, n)$ coincide, respectively, with the solutions $y_\nu(x)$ of (1.1) which vanish at $x = a_{\nu-1}$ and $x = a_\nu$, and whose existence is established by Theorem 2.1. We shall prove that the set of numbers $a_1, \ldots, a_{n-1}$ for which the right-hand side of (3.3) attains its minimum is such that the corresponding solutions $y_\nu(x)$ of (1.1) combine to a single solution $y(x)$ of (1.1) in the interval $[a, b]$. This solution $y(x)$ vanishes for $x = a$ and $x = b$, and has precisely $n-1$ zeros in (a, b).

We first show that our minimum problem has a solution. If we write $\lambda = \lambda(a, b)$ to indicate the interval to which the number λ defined in (2.4) refers, the existence of this solution will be a consequence of the following three properties of $\lambda(a, b)$.

Lemma 3.1.

(a) *If* $a \leqslant a' < b' \leqslant b$, *then* $\lambda(a, b) \leqslant \lambda(a', b')$;

(b) $\lambda(a, b) \to \infty$ *for* $b - a \to 0$;

(c) $\lambda(a, b)$ *is a continuous function of both* a *and* b.

To establish (*a*), we denote by $u(x)$ the function solving the problem (2.4) for the interval $[a', b']$, and define a function $v(x)$ as follows:

$$v(x) = u(x) \text{ for } x \in [a', b'],\ v(x) \equiv 0 \text{ for } x \in [a, a') \text{ and } x \in (b', b].$$

Since $v(x)$ is easily confirmed to be an admissible function for the problem (2.4) associated with the interval $[a, b]$, it follows from Theorem 2.1 and the definition of $v(x)$ that

$$\lambda(a, b) \leqslant H(v) = H(u) = \lambda(a', b').$$

Turning next to (b), we set $b - a = \delta$ $(\delta > 0)$ and we use the inequality (2.6). This yields

$$1 < \int_a^b (x-a) F[\beta(x-a), x] < \delta \int_a^b F(\beta\delta, x)\, dx,$$

where $\beta = \int_a^b y'^2\, dx$, and $y(x)$ is the solution of problem (2.4). If there existed a positive constant M such that $\beta \leqslant M$ for all δ such that $0 < \delta < \delta_0$, we would have

$$1 < \delta \int_a^{a+\delta_0} F(\delta_0 M, x)\, dx,$$

which is absurd. Hence, $\int_a^b y'^2\, dx \to \infty$ for $\delta \to 0$. In view of (2.8) and (2.4), this implies that $\lambda(a, b) \to \infty$.

To prove property (c), it is sufficient to show that $\lambda(a, b)$ is a continuous function of b, since the roles of a and b can be interchanged by an elementary transformation. To simplify the writing we set $a = 0$, and we denote by $y(x)$ the solution of problem (2.4) for the interval $[0, b]$. If $0 < b' < b$, we write $t = b b'^{-1}$ and we define a function $u(x)$ in $[0, b']$ by $u(x) = y(t x)$. As shown in section 2, there exists a positive constant α such that

$$\int_0^{b'} u'^2\, dx = \int_0^{b'} u^2 F(\alpha^2 u^2, x)\, dx. \tag{3.4}$$

With this choice of α, the function $w(x) = \alpha u(x)$ is subject to the normalization (2.3) (for the interval $[0, b']$). Since, moreover, $w(0) = w(b') = 0$, it follows from Theorem 2.1 that

$$\lambda(0, b') \leqslant H(w) = \int_0^{b'} [w'^2 - G(w^2, x)]\, dx. \tag{3.5}$$

In view of the definition of $u(x)$, (3.4) is equivalent to

$$t^2 \int_0^b y'^2\, dx = \int_0^b y^2 F(\alpha^2 y^2, x t^{-1})\, dx.$$

Since the function F is monotonic in its first argument and continuous in both arguments this shows that α is a continuous function of t for $t \geqslant 1$. The normalization condition

$$\int_0^b y'^2\, dx = \int_0^b y^2 F(y^2, x)\, dx$$

shows that $\alpha \to 1$ for $t \to 1$, and we may therefore conclude that $|\alpha - 1|$ can be made arbitrarily small by taking t close enough to 1, i.e., by taking b' close enoug to b.

Changing the integration variable in (3.5) from x to tx and observing that $w(x) = \alpha y(tx)$, we obtain

$$\lambda(0, b') \leqslant t^{-1} \int_0^b [\alpha^2 t^2 y'^2 - G(\alpha^2 y^2, x t^{-1})] \, dx, \tag{3.6}$$

where $y = y(x)$. Since the function G is continuous in both its variables and, as just shown, $t \to 1$ implies $\alpha \to 1$ we conclude that, for an arbitrarily small given positive number γ, we can make the right-hand side of (3.6) smaller than

$$\int_0^b [y'^2 - G(y^2, x)] \, dx + \gamma$$

by taking b' close enough to b. But the last expression is equal to $\lambda(0, b) + \gamma$, and we obtain $\lambda(0, b') \leqslant \lambda(0, b) + \gamma$. Since, according to property (a), $\lambda(0, b') \geqslant \lambda(0, b)$, this shows that $\lambda(0, b)$ is indeed a continuous function of b. This completes the proof of Lemma 3.1.

It is now easy to see that there exists a set of distinct points $a_1, \dots a_{n-1}$ for which the expression

$$\Lambda = \sum_{\nu=1}^{n} \lambda(a_{\nu-1}, a_\nu) \quad (a_0 = a, a_n = b)$$

attains its minimum. Indeed, according to property (c), Λ is a continuous function of the variables $a_1, \dots, a_{n-1}$, and by property (b) the values of these variables must be bounded away from each other in any sequence of sets $(a_1, \dots, a_{n-1})$ for which Λ tends to its minimum. Since the minimum of Λ coincides with the minimum of the right-hand side of (3.3) under the specified conditions, we have thus established the existence of a solution $y_n(x)$ of the minimum problem (3.3). As already mentioned, in each interval $[a_{\nu-1}, a_\nu]$ this function $y_n(x)$ coincides with a solution of (1.1) for which $y_n(a_{\nu-1}) = y_n(a_\nu) = 0$ and $y_n(x) \neq 0$ in $(a_{\nu-1}, a_\nu)$. The function $y(x)$ will thus have precisely $n-1$ zeros in (a, b), and it follows that, for different values of n, problem (3.3) will have different solutions.

Since the side conditions under which the minimum problem (3.3) is solved become more restrictive as n increases, it is clear that $\lambda_n \geqslant \lambda_{n-1}$. In order to show that equality is excluded, we denote by $y(x)$ a solution of problem (3.3), and we define a function $u(x)$ as follows: $u(x) = y(x)$ in $[a, a_{n-1}]$, where $a_1, \dots, a_{n-1}$ $(a < a_1 < \cdots < a_{n-1} < b)$ are the zeros of $y(x)$ in (a, b), and $u \equiv 0$ in $[a_{n-1}, b]$. It is easily confirmed that $u(x)$

is an admissible function for the problem (3.3) corresponding to the index $n-1$, and it follows therefore that

$$\lambda_{n-1} \leqslant \int_a^b [u'^2 - G(u^2, x)]\,dx = \int_a^{a_{n-1}} [y'^2 - G(y^2, x)]\,dx = \lambda_n - \int_{a_{n-1}}^b [y'^2 - G(y^2, x)]\,dx.$$

Since, by (2.2), (2.3), and (2.8), the last integral is positive, this proves the strict inequality $\lambda_{n-1} < \lambda_n$.

We now turn to the proof of the assertion that the minimum problem is solved by a function $y(x)$ which is a solution of equation (1.1) throughout the interval $[a, b]$. However, the following remark is in order. If, in one of the intervals $(a_{\nu-1}, a_\nu)$, the extremal function $y(x)$ is replaced by $-y(x)$, neither the admissibility conditions nor the value of the functional (3.3) are changed. In order to remove this trivial lack of uniqueness, we shall assume that the signs of $y(x)$ are chosen in such a way that $y(x)$ changes its sign at each point $a_\nu\ (\nu = 1, 2, \dots, n-1)$. As pointed out before, the function $y(x)$ must in each interval $[a_{\nu-1}, a_\nu]$ coincide with a solution of (1.1). Since $y(a_\nu) = 0\ (\nu = 0, 1, \dots, n)$, it follows therefore that the extremal function $y(x)$—if normalized in the way just indicated—will be a solution of (1.1) in $[a, b]$ if, and only if,

$$\lim_{x \to a_\nu - 0} y'(x) = \lim_{x \to a_\nu + 0} y'(x), \quad \nu = 1, \dots, n-1, \tag{3.7}$$

or, in shorter notation, $y'_-(a_\nu) = y'_+(a_\nu)$. We shall prove (3.7) by showing that $y(x)$ could not be a solution of the problem (3.3) if (3.7) fails to hold at some point a_ν.

We accordingly assume that $y'_-(a_\nu) \neq y'_+(a_\nu)$ and we set, for easier writing, $a_{\nu-1} = \alpha$, $a = c$, $a_{\nu+1} = \beta$. Without losing generality we may further assume that $y(x) > 0$ in (α, c) and therefore $y(x) < 0$ in (c, β). We now define a function $u(x)$ in the following manner. If δ denotes a small positive quantity, we set $u(x) = y(x)$ in $[\alpha, c-\delta]$ and $[c+\delta, \beta]$, and

$$u(x) = y(c-\delta) + (2\delta)^{-1}(x - c + \delta)[y(c+\delta) - y(c-\delta)], \quad c - \delta \leqslant x \leqslant c + \delta. \tag{3.8}$$

Evidently, $u(x)$ is continuous in $[\alpha, \beta]$. In $(c-\delta, c+\delta)$, the linear function (3.8) vanishes at a point $x = c'$ given by

$$2\delta\, y(c-\delta) + (c' - c + \delta)[y(c+\delta) - y(c-\delta)] = 0. \tag{3.9}$$

In order to obtain a function subject to the normalization (3.2), we multiply $u(x)$ by positive factors ϱ and σ in $[\alpha, c']$ and $[c', \beta]$, respectively, so that

$$\int_{\alpha}^{c'} u'^2\,dx = \int_{\alpha}^{c'} u^2 F(\varrho^2 u^2, x)\,dx,$$
$$\int_{c'}^{\beta} u'^2\,dx = \int_{c'}^{\beta} u^2 F(\sigma^2 u^2, x)\,dx. \tag{3.10}$$

The function $v(x)$ defined by $v(x)=\varrho\, u(x)$ and $v(x)=\sigma\, u(x)$ in $[\alpha, c']$ and $[c', \beta]$ respectively, will then be normalized in accordance with (3.2), and it is clear that the function $y_1(x)$ obtained from $y(x)$ by substituting $v(x)$ for $y(x)$ in $[\alpha, \beta]$ is an admissible function for problem (3.3).

By (1.3 c), $F(t, x)$ is an increasing function of t. Hence, the function $G(t, x)$ defined in (2.1) is convex in t and we have

$$G(t, x) \geqslant G(s, x) + (t-s)\,F(s, x) \tag{3.11}$$

for any non-negative s and t. Therefore,

$$\int_{\alpha}^{\beta} [v'^2 - G(v^2, x)]\,dx \leqslant \int_{\alpha}^{\beta} [v'^2 - G(y^2, x) - (v^2 - y^2)\,F(y^2, x)]\,dx$$
$$= \int_{\alpha}^{\beta} [y^2 F(y^2, x) - G(y^2, x)]\,dx + \int_{\alpha}^{\beta} [v'^2 - v^2 F(y^2, x)]\,dx.$$

In view of (3.2) and (2.2) it thus follows that

$$H(y_1) \leqslant H(y) + \int_{\alpha}^{\beta} [v'^2 - v^2 F(y^2, x)]\,dx,$$

i.e., $$H(y_1) \leqslant H(y) + \varrho^2 \int_{\alpha}^{c'} [u'^2 - u^2 F(y^2, x)]\,dx + \sigma^2 \int_{c'}^{\beta} [u'^2 - u^2 F(y^2, x)]\,dx. \tag{3.12}$$

Our aim is to show that the sum of the last two terms in (3.12) can be made negative by taking δ sufficiently small. Increasing the right-hand side of (3.12) by omitting the negative term in the integrand in the interval $(c-\delta, c+\delta)$, we have

$$H(y_1) \leqslant H(y) + \varrho^2 \int_{\alpha}^{c-\delta} [u'^2 - u^2 F(u^2, x)]\,dx + \sigma^2 \int_{c+\delta}^{\beta} [u'^2 - u^2 F(u^2, x)]\,dx$$
$$+ \varrho^2 \int_{c-\delta}^{c} u'^2\,dx + \sigma^2 \int_{c}^{c+\delta} u'^2\,dx. \tag{3.13}$$

In the intervals $[\alpha, c-\delta]$ and $[c+\delta, \beta]$ we have $u(x)=y(x)$. Observing that $y(x)$ is a solution of (1.1) in each interval, we obtain

$$\int_{\alpha}^{c-} [u'^2 - u^2 F(u^2, x)]\,dx = y(c-\delta)\,y'(c-\delta)$$

and

$$\int_{c+\delta}^{\beta} [u'^2 - u^2 F(u^2, x)]\,dx = -y(c+\delta)\,y'(c+\delta).$$

Inserting this in (3.13), and noting that

$$\varrho^2 \int_{c-\delta}^{c} u'^2\,dx = \int_{c-\delta}^{c} u'^2\,dx + (\varrho^2-1)\int_{c-\delta}^{c} u'^2\,dx$$

$$\leqslant \int_{c-\delta}^{c} u'^2\,dx + |\varrho^2-1| \int_{c-\delta}^{c+\delta} u'^2\,dx$$

and, similarly

$$\sigma^2 \int_{c}^{c+\delta} u'^2\,dx \leqslant \int_{c}^{c+\delta} u'^2\,dx + |\sigma^2-1| \int_{c-\delta}^{c+\delta} u'^2\,dx,$$

we obtain

$$H(y_1) \leqslant H(y) + y(c-\delta)\,y'(c-\delta) - y(c+\delta)\,y'(c+\delta) + \int_{c-\delta}^{c+\delta} u'^2\,dx$$

$$+ (\varrho^2-1)\,y(c-\delta)\,y'(c-\delta) - (\sigma^2-1)\,y(c+\delta)\,y'(c+\delta)$$

$$+ \{|\varrho^2-1| + |\sigma^2-1|\} \int_{c-\delta}^{c+\delta} u'^2\,dx. \tag{3.14}$$

By (1.1), $y(c)=0$ implies $y''(c)=0$. Hence, $y(c+\delta)=\delta y'_+(c)+O(\delta^3)$, $y'(c+\delta)=y'_+(c)+O(\delta^2)$, $y(c-\delta)=-\delta y'_-(c)+O(\delta^3)$, $y'(c-\delta)=y'_-(c)+O(\delta^2)$. Since $c'\to c$ for $\delta\to 0$ and the function F is continuous in both its arguments, (3.10) shows that $\varrho^2\to 1$ and $\sigma^2\to 1$ for $\delta\to 0$, and it follows that $(\varrho^2-1)\,y(c-\delta)\,y'(c-\delta)$ and $(\sigma^2-1)\,y(c+\delta)\,y'(c+\delta)$ are $o(\delta)$. By (3.8) we have

$$\int_{c-\delta}^{c+\delta} u'^2\,dx = (2\delta)^{-1}[y(c+\delta)-y(c-\delta)]^2 = \frac{\delta}{2}[y'_+(c)+y'_-(c)]^2 + O(\delta^3).$$

The last term in (3.14) is therefore also $o(\delta)$ and in view of $y(c-\delta)\,y'(c-\delta)-$ $-y(c+\delta)\,y'(c+\delta)=-\delta\,[y_-'^2(c)+y_+'^2(c)]+O(\delta^3)$, (3.14) reduces to

$$H(y_1)\leqslant H(y)-\frac{\delta}{2}[y_+'(c)-y_-'(c)]^2+o(\delta). \tag{3.15}$$

If $y_+'(c)\neq y_-'(c)$, i.e. if condition (3.7) does not hold, the expression $\delta\,[y_+'(c)-y_-'(c)]^2+$ $+o(\delta)$ can be made negative by choosing δ small enough, and the corresponding function $y_1(x)$ will satisfy the inequality $H(y_1)<H(y)$. But $y_1(x)$ is an admissible function for the extremal problem (3.3) and, in view of the definition (2.2) of the functional $H(y)$, this contradicts the assumption that $y(x)$ is a solution of problem (3.3). This contradiction can be avoided only if $y_-'(c)=y_+'(c)$. Since c may be identified with any of the numbers $a_\nu\,(\nu=1,2,\ldots,n-1)$, this establishes the relations (3.7).

The following statement summarizes the results of this section.

THEOREM 3.2. *Let Γ_n denote the class of functions $y(x)$ with the following properties: $y(x)$ is continuous and piecewise differentiable in $[a,b]$; $y(a_\nu)=0$ $(\nu=0,1,\ldots,n,$ $n\geqslant 1)$, where the a_ν are numbers such that $a=a_0<a_1<\cdots<a_{n-1}<a_n=b$; for $\nu=1,\ldots,n$,*

$$\int_{a_{\nu-1}}^{a_\nu} y'^2 d\,x=\int_{a_{\nu-1}}^{a_\nu} y^2 F(y^2,x)\,d\,x, \tag{3.16}$$

where $F(t,x)$ is subject to the conditions (1.3).

If $G(t,x)$ denotes the function defined by (2.1), *the extremal problem*

$$\int_a^b [y'^2-G(y^2,x)]\,d\,x=\min=\lambda_n,\quad y(x)\in\Gamma_n, \tag{3.17}$$

has a solution $y_n(x)$ whose derivative is continuous throughout $[a,b]$, and the characteristic values λ_n are strictly increasing with n. The function $y_n(x)$ has precisely $n-1$ zeros in (a,b), and it is a solution of the differential system

$$y''+y\,F(y^2,x)=0,\quad y(a)=y(b)=0. \tag{3.18}$$

The question whether the problem (3.17) has (up to the factor -1) only one solution $y_n(x)$ with a continuous derivative, remains open. Another question which remains unanswered is whether the system (3.18) may have additional solutions with $n-1$ zeros in (a,b) which are not, at the same time, solutions of (3.17). For the sake of convenient formulation we shall, nevertheless, occasionally refer to the λ_n as the *characteristic values of the system* (3.18).

We add here a few remarks which will be used later. In deriving inequality (3.15) no use was made of the assumption that the number ε in (1.3 c) is positive, and (3.15) will therefore remain valid if (1.3 c) is replaced by the weaker condition that $F(t, x)$ be a non-decreasing function of t.

The second remark concerns the behavior of $\lambda_n(a, b)$—where a and b are the ends of the interval to which the minimum problem (3.17) refers—as a function of a and b. Since $\lambda_n(a, b) = \min \sum_{\nu=1}^{n} \lambda(a_{\nu-1}, a_\nu)$ $(a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b)$, it follows from Lemma 3.1 and an elementary argument that $\lambda_n(a, b)$, too, has the three properties stated in Lemma 3.1.

Finally, if the argument resulting in (3.15) is carried through under the assumption that $y(x) \equiv 0$ in $[c, \beta]$, the inequality (3.15) will be replaced by

$$H(y_1) \leqslant H(y) - \frac{\delta}{2} y'^2(c) + o(\delta), \tag{3.19}$$

where $y_1(\alpha) = y_1(c + \delta) = 0$. In view of Theorem 2.1, we thus have the estimate

$$\lambda(\alpha, c + \delta) \leqslant \lambda(a, c) - \frac{\delta}{2} y'^2(c) + o(\delta). \tag{3.20}$$

4. According to Theorem 3.2, the characteristic values λ_n of the problem (3.16) are strictly increasing with n. The following result gives additional information regarding the growth of λ_n for large n.

THEOREM 4.1. *If λ_n is the n-th characteristic value of the problem* (3.17), *then*

$$\lim_{n \to \infty} \frac{\lambda_n}{n^2} = \infty. \tag{4.1}$$

The exponent 2 cannot be replaced by a larger number.

Using the same notation as in Theorem 3.2, we have, for $x \in (a_{\nu-1}, a_\nu)$,

$$y^2(x) = \left(\int_{a_{\nu-1}}^{x} y' \, dx \right)^2 \leqslant (a_\nu - a_{\nu-1}) \int_{a_{\nu-1}}^{a_\nu} y'^2 \, dx.$$

Applying this inequality to (3.16), we obtain

$$1 \leqslant (a_\nu - a_{\nu-1}) \int_{a_{\nu-1}}^{a_\nu} F(y^2, x)\, dx,$$

whence
$$\int_a^b F(y^2, x)\, dx \geqslant \sum_{\nu=1}^{n} (a_\nu - a_{\nu-1})^{-1}.$$

For fixed $a_0 = a$ and $a_n = b$, the right-hand side of this inequality attains its minimum for $a_1 - a_0 = a_2 - a_1 = \cdots = a_n - a_{n-1} = n^{-1}(b-a)$, and we have therefore

$$\int_a^b F(y^2, x)\, dx \geqslant n^2 (b-a)^{-1}. \tag{4.2}$$

If α is a positive constant, if follows from (3.11) that

$$\int_a^b G(\alpha^2, x)\, dx \geqslant \int_a^b G(y^2, x)\, dx + \int_a^b (\alpha^2 - y^2) F(y^2, x),$$

or, in view of (3.16),

$$\int_a^b [y'^2 - G(y^2, x)]\, dx \geqslant \alpha^2 \int_a^b F(y^2, x)\, dx - \int_a^b G(\alpha^2, x)\, dx.$$

We now identify $y(x)$ with the solution of (3.17). In view of (3.17) and (4.2), the last inequality will then lead to

$$\lambda_n \geqslant \alpha^2 (b-a)^{-1} n^2 - \int_a^b G(\alpha^2, x)\, dx,$$

and we conclude that
$$\liminf_{n\to\infty} n^{-2} \lambda_n \geqslant \alpha^2 (b-a)^{-1}.$$

Since the constant α may be taken arbitrarily large, this proves (4.1).

In order to show that, in (4.1), n^2 cannot be replaced by a higher power of n, we compute the characteristic values λ_n associated with the differential system

$$y'' + y^{2m+1} = 0, \quad y(0) = y(b) = 0, \tag{4.3}$$

where m is a positive integer. Our result will be that

$$\lambda_n = \frac{m(m+1)^{1/m}}{m+2} [4\beta^2 b^{-1} n^2]^{1+1/m}, \tag{4.4}$$

where $$\beta = \int_0^1 (1 - t^{2m+2})^{-\frac{1}{2}} \, dt. \tag{4.5}$$

Since m may be taken arbitrarily large, this shows that (4.1) is indeed the best possible result of its kind.

If $y(x)$ is a solution of equation (4.3) for which $y(0) = 0$, $y'(0) = \alpha > 0$, we have

$$y'^2 + (m+1)^{-1} y^{2m+2} = \alpha^2. \tag{4.6}$$

Elementary considerations show that $y(x)$ is periodic and that its value oscillates between the limits $\pm M$, where M is the positive number determined by

$$M^{2m+2} = (m+1)\,\alpha^2. \tag{4.7}$$

If $x = T$ is the lowest positive value for which $y^2(x) = M^2$, the zeros of $y(x)$ in $(0, \infty)$ are at $x = 2T, x = 4T, \ldots$, and it is easily seen that all these zeros move to the left as α^2 grows. As a consequence, there exists precisely one solution of the system (4.3) with $n-1$ zeros in $(0, b)$. By Theorem 3.2, this solution is necessarily identical with the solution of the extremal problem (3.17) (with $F(y^2, x) \equiv y^{2m}$), and we have

$$\lambda_n = \int_0^b [y'^2 - (m+1)^{-1} y^{2m+2}] \, dx,$$

if $y(x)$ is the solution in question. Since $y(x)$ is subject to the identities $y(T+x) = y(T-x)$ and $y(x + 2T) = -y(T)$—as one confirms by substituting these functions in the differential equation and using a few trivial transformations—this is equivalent to

$$\lambda_n = 2n \int_0^T [y'^2 - (m+1)^{-1} y^{2m+2}] \, dx, \quad T = b\,(2n)^{-1}.$$

Multiplying equation (4.3) by $y(x)$ and integrating from 0 to T, we obtain

$$\int_0^T y'^2 \, dx = \int_0^T y^{2m+2} \, dx. \tag{4.8}$$

Hence, $$\lambda_n = 2nm\,(m+1)^{-1} \int_0^T y'^2 \, dx, \quad T = b\,(2n)^{-1}. \tag{4.9}$$

Integrating (4.6) from 0 to T, we have

$$\int_0^T y'^2\,dx + (m+1)^{-1}\int_0^T y^{2m+2}\,dx = \alpha^2 T,$$

and thus, in view of (4.8),

$$\int_0^T y'^2\,dx = (m+1)\,\alpha^2\,T\,(m+2)^{-1}. \tag{4.10}$$

To compute α, we observe that $y(x)$ is increasing in $(0, T)$, and we may therefore conclude from (4.6) that

$$T = \int_0^M [\alpha^2 - (m+1)^{-1}\,y^{2m+2}]^{-\frac{1}{2}}\,dy.$$

In view of (4.7) and (4.5), this leads to

$$T = (m+1)^{\frac{1}{2}}\,M^{-m}\int_0^1 [1 - t^{2m+2}]^{-\frac{1}{2}}\,dt = (m+1)^{\frac{1}{2}}\,M^{-m}\,\beta.$$

Using (4.7) again, we thus obtain

$$\alpha^2 = (m+1)^{-1}\,[(m+1)^{\frac{1}{2}}\,\beta\,T^{-1})^{2(1+1/m)}.$$

Combining this with (4.9) and (4.10), and observing that $T = b\,(2n)^{-1}$, we arrive at (4.4).

5. The expression (4.4) for the characteristic values of the problem (4.3) may be used in order to determine the asymptotic behavior of the characteristic values λ_n associated with the system

$$y'' + p(x)\,y^{2m+1} = 0, \quad y(a) = y(b) = 0, \tag{5.1}$$

where $p(x)$ is positive and continuous in $[a, b]$ (for other properties of equation (5.1), cf. [1, 5]). Our method of proof will present certain analogies to the classical procedure by which the asymptotic behavior of the eigenvalues of the Sturm–Liouville problem $y'' + \mu\,p(x)\,y = 0$, $y(a) = y(b) = 0$ is obtained from the known eigenvalues of the problem $y'' + \mu\,y = 0$, $y(a) = y(b) = 0$ [3]. We shall establish the following result.

THEOREM 5.1. *If λ_n is the n-th characteristic value associated with the differential system* (5.1), *then, for large n,*

$$\lambda_n = A\,n^{2(1+1/m)}\,[1 + O(n^{-2})], \tag{5.2}$$

where
$$A = \frac{m(m+1)^{1/m}}{m+2}(2\beta)^{2(1+1/m)}\left[\int_a^b [p(x)]^{1/(m+2)}\,dx\right]^{-1-1/m} \tag{5.3}$$

and β *is given by* (4.5).

In the course of the proof it will be necessary to assume that $p(x)$ has two continuous derivatives. However, in the final result no derivatives of $p(x)$ appear and it is therefore possible to extend the result to an arbitrary continuous and positive $p(x)$ by means of an approximation argument. The necessary steps are elementary but tedious, and will be omitted.

The proof will be based on a transformation of equation (5.1) which may be regarded as a generalization of the classical Liouville transformation of second-order linear differential equation [3]. We introduce a new independent variable t and a new dependent variable $u = u(t)$ by means of the relations

$$t = \int_a^x [p(\xi)]^{1/(m+2)}\,d\xi,\quad y(x) = [p(x)]^{-1/2(m+2)}u(t). \tag{5.4}$$

A formal computation shows that the system (5.1) transforms into

$$\ddot{u} - g(t)\,u + u^{2m+1} = 0,\quad u(0) = u(T) = 0, \tag{5.5}$$

where

$$g(t) = -\sigma(\sigma^{-1})^{\cdot\cdot},\quad \sigma = \sigma(t) = [p(x)]^{-1/2(m+2)},\quad T = \int_a^b [p(x)]^{1/(m+2)}(dx), \tag{5.6}$$

and the dot denotes differentiation with respect to t.

The extremal problem (3.17) (for $F(y^2, x) = p(x)\,y^{2m}$) transforms into

$$\lambda_n = \min \int_0^T (\dot{u}^2 - g\,u^2 - (m+1)^{-1}u^{2m+2})\,dt,$$

where $u(t_\nu) = 0$, $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, and $u(t)$ satisfies the conditions

$$\int_{t_{\nu-1}}^{t_\nu} [\dot{u}^2 - g\,u^2]\,dt = \int_{t_{\nu-1}}^{t_\nu} u^{2m+2}\,dt. \tag{5.7}$$

Because of (5.7), the definition of λ_n may also be written in the form

$$\lambda_n = m(m+1)^{-1}\min\int_0^T u^{2m+2}\,dt. \tag{5.8}$$

If $u(t)$ is the nth characteristic function of this problem, we define a set of positive numbers $\alpha_1, \ldots, \alpha_n$ by the conditions

$$\int_{t_{\nu-1}}^{t_\nu} \dot{u}^2\,dt = \alpha_\nu^{2n} \int_{t_{\nu-1}}^{t_\nu} u^{2m+2}\,dt, \quad \nu = 1, \ldots, n. \tag{5.9}$$

The function $v(t)$ defined by $v(t) = \alpha_\nu u(t)$ in $[t_{\nu-1}, t_\nu]$ will then have the normalization

$$\int_{t_{\nu-1}}^{t_\nu} \dot{v}^2\,dt = \int_{t_{\nu-1}}^{t_\nu} v^{2m+2}\,dt,$$

and it also satisfies all the other admissibility conditions for the nth extremal problem associated with the system

$$\ddot{v} + v^{2m+1} = 0, \quad v(0) = v(T) = 0. \tag{5.10}$$

If we denote the nth characteristic value of this problem by μ_n, we thus have

$$\mu_n \leqslant m(m+1)^{-1} \int_0^T v^{2m+2}\,dt = m(m+1)^{-1} \left\{ \sum_{\nu=1}^{n} \alpha^{2m+2} \int_{t_{\nu-1}}^{t_\nu} u^{2m+2}\,dt \right\}.$$

Using the inequality

$$m(m+1)^{-1}\,\alpha_\nu^{2m+2} \leqslant m(m+1)^{-1} + \alpha_\nu^2(\alpha_\nu^{2m} - 1), \tag{5.11}$$

we obtain $$\mu_n \leqslant m(m+1)^{-1} \int_0^T u^{2m+2}\,dt + \sum_{\nu=1}^{n} \alpha_\nu^2(\alpha_\nu^{2m} - 1) \int_{t_{\nu-1}}^{t_\nu} u^{2m+2}\,dt,$$

whence, in view of (5.8),

$$\mu_n \leqslant \lambda_n + \sum_{\nu=1}^{n} \alpha_\nu^2(\alpha_\nu^{2m} - 1) \int_{t_{\nu-1}}^{t_\nu} u^{2m+2}\,dt.$$

By (5.7) and (5.9), we have

$$(\alpha_\nu^{2m} - 1) \int_{t_{\nu-1}}^{t_\nu} u^{2m+2}\,dt = \int_{t_{\nu-1}}^{t_\nu} g u^2\,dt \tag{5.12}$$

and the last inequality simplifies to

$$\mu_n \leqslant \lambda_n + \sum_{\nu=1}^{n} \alpha_\nu^2 \int_{t_{\nu-1}}^{t_\nu} g u^2 \, dt. \tag{5.13}$$

We next show that there exists a positive constant c, independent of n, such that $\alpha_\nu^2 \leqslant c$. By (5.12) and the Hölder inequality,

$$\alpha_\nu^{2m} \int_{t_{\nu-1}}^{t_\nu} u^{2m+2} \, dt \leqslant \int_{t_{\nu-1}}^{t_\nu} u^{2m+2} \, dt + c_1 \left[\int_{t_{\nu-1}}^{t_\nu} u^{2m+2} \, dt \right]^{1/(m+1)},$$

where

$$c_1 = \left[\int_0^T |g|^{(m+1)/m} \, dt \right]^{m/(m+1)}. \tag{5.14}$$

Hence,

$$\alpha_\nu^{2m} \leqslant 1 + c_1 \left[\int_{t_{\nu-1}}^{t_\nu} u^{2m+2} \, dt \right]^{-m/(m+1)}. \tag{5.15}$$

Except for the factor $m(m+1)^{-1}$, the integral appearing in (5.15) is the first characteristic value of the problem (5.8) under the side condition (5.7) (for a specific ν) and $u(t_{\nu-1}) = u(t_\nu) = 0$. By Lemma 3.1, this characteristic value decreases if the interval to which it refers is increased. Since $[t_{\nu-1}, t_\nu] \subset [0, T]$, we may therefore conclude that the integral in (5.15) is larger than $(m+1) m^{-1} \lambda_1$. This shows that, indeed, $\alpha_\nu^2 \leqslant c$, where the constant c does not depend on n.

Applying this to (5.13), we obtain

$$\mu_n \leqslant \lambda_n + c \int_0^T |g| \, u^2 \, dt \leqslant \lambda_n + c\, c_1 \left[\int_0^T u^{2m+2} \, dt \right]^{1/(m+1)},$$

where c_1 is the constant (5.14). Since $u(t)$ is the solution of the extremal problem (5.8), this is equivalent to

$$\mu_n \leqslant \lambda_n \, (1 + c_2 \, \lambda_n^{-m/(m+1)}), \tag{5.16}$$

where the constant c_2 is again independent of n.

Since, by Theorem 4.1, $\lambda_n \to \infty$ for $n \to \infty$, (5.16) shows that $\mu_n \leqslant c_3 \lambda_n$, provided n is large enough. Using this to estimate λ_n in the square bracket on the right-hand side of (5.16) and remembering that μ_n is gixen by the right-hand side of (4.4), we arrive at the inequality

$$\mu_n \leqslant \lambda_n (1 + c_4 n^{-2}), \tag{5.17}$$

where c_4 is a suitable constant.

In order to obtain an upper bound for λ_n, we use essentially the same argument but interchange the roles of the systems (5.1) and (5.10). If $v(t)$ is the nth characteristic function of (5.10) and $t_0, t_1, \ldots t_n$ $(0 = t_0 < t_1 < \cdots t_{n-1} < t_n = T)$ its zeros in $[0, T]$, we define positive constants $\beta_1, \ldots, \beta_n$ by the conditions

$$\int_{t_{\nu-1}}^{t_\nu} [\dot{v}^2 - g v^2]\, d t = \beta_\nu^{2m} \int_{t_{\nu-1}}^{t_\nu} v^{2m+2}\, d t. \tag{5.18}$$

(This is always possible since transformation (5.4) brings the left-hand side of (5.18) into the form $\int_a^b \tau'^2 (x)\, d x$, where $\tau(x)$ is a suitable function.) The function $w(t)$ defined by $w(t) = \beta_\nu v(t)$ in $[t_{\nu-1}, t_\nu]$ has the normalization (5.7), and it follows from (5.8) that

$$\lambda_n \leqslant m (m+1)^{-1} \int_0^T w^{2m+2}\, d t = m (m+1)^{-1} \sum_{\nu=1}^{n} \beta_\nu^{2m+2} \int_{t_{\nu-1}}^{t_\nu} v^{2m+2}\, d t.$$

Using (5.11), we obtain

$$\lambda_n \leqslant m (m+1)^{-1} \int_0^T v^{2m+2}\, d t + \sum_{\nu=1}^{n} \beta_\nu^2 (\beta_\nu^{2m} - 1) \int_{t_{\nu-1}}^{t_\nu} v^{2m+2}\, d t,$$

or, since $v(t)$ is the function minimizing (5.8),

$$\lambda_n \leqslant \mu_n + \sum_{\nu=1}^{n} \beta_\nu^2 (\beta_\nu^{2m} - 1) \int_{t_{\nu-1}}^{t_\nu} v^{2m+2}\, d t.$$

In view of (5.18) and the normalization conditions

$$\int_{t_{\nu-1}}^{t_\nu} \dot{v}^2\, d t = \int_{t_{\nu-1}}^{t_\nu} v^{2m+2}\, d t,$$

this is equivalent to

$$\lambda_n \leqslant \mu_n - \sum_{\nu=1}^{n} \beta_\nu^2 \int_{t_{\nu-1}}^{t_\nu} g v^2\, d t.$$

The existence of a constant c such that $\beta_\nu^2 \leqslant c$ (for all n) follows in the same way as the corresponding fact for constants α_ν. Applying Hölder's inequality, we thus obtain

$$\lambda_n \leqslant \mu_n + cc_1 \left[\int_0^T v^{2m+2}\, dt \right]^{1/(m+1)},$$

where c_1 is the constant (5.14). According to (5.7) (for $g(t) \equiv 0$),

$$\mu_n = m(m+1)^{-1} \int_0^T v^{2m+2}\, dt,$$

and the last inequality may therefore be written in the form

$$\lambda_n \leqslant \mu_n [1 + c_5 \mu^{-m/(m+1)}],$$

where the constant c_5 does not depend on n. Since μ_n is given by the right-hand side of (4.4), this is equivalent to

$$\lambda_n \leqslant \mu_n [1 + c_6 n^{-2}].$$

In view of (5.17) and the value of μ_n, this proves Theorem 5.1.

6. In the present section we show that, in the case of the general differential equation $y'' + yF(y^2, x) = 0$, the asymptotic behavior of the characteristic values is essentially determined by the behavior of the function $F(t, x)$ for large values of t. We shall establish the following result.

THEOREM 6.1. *Let $F(t, x)$ and $F_1(t, x)$ be functions subject to the conditions* (1.3), *and let*

$$\lim_{t\to\infty} \frac{F(t, x)}{F_1(t, x)} = 1 \tag{6.1}$$

uniformly in x. If λ_n and λ_n' denote, respectively, the n-th characteristic values associated with the systems

$$u'' + uF(u^2, x) = 0, \quad u(a) = u(b) = 0, \tag{6.2}$$

and

$$v'' + vF_1(v^2, x) = 0, \quad v(a) = v(b) = 0, \tag{6.3}$$

respectively, then

$$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_n'} = 1. \tag{6.4}$$

We choose an arbitrary small positive number δ and we consider, in addition to (6.2) and (6.3), the system

$$w'' + (1+\delta)\, w F_1(w^2, x) = 0, \quad w(a) = w(b) = 0, \tag{6.5}$$

with the characteristic values λ_n''. If $a = a_0, a_1, \ldots, a_{n-1}, a_n = b$ are the zeros of the nth characteristic function of (6.2), and if the constants α_ν are determined by the conditions

$$\int_{a_{\nu-1}}^{a_\nu} u'^2\, dx = (1+\delta) \int_{a_{\nu-1}}^{a_\nu} u^2 F_1(\alpha_\nu^2 u^2, x)\, dx, \tag{6.6}$$

then, by Theorem 3.2,

$$\lambda_n'' \leqslant \sum_{\nu=1}^{n} \int_{a_{\nu-1}}^{a_\nu} [\alpha_\nu^2 u'^2 - (1+\delta)\, G_1(\alpha_\nu^2 u^2, x)]\, dx$$

$$= \sum_{\nu=1}^{n} \int_{a_{\nu-1}}^{a_\nu} [\alpha_\nu^2 u'^2 - G(\alpha_\nu^2 u^2, x)]\, dx + \sum_{\nu=1}^{n} \int_{a_{\nu-1}}^{a_\nu} [G(\alpha_\nu^2 u^2, x) - (1+\delta)\, G_1(\alpha_\nu^2 u^2, x)]\, dx,$$

where $G_1(t, x)$ is defined by (2.1) (with F replaced by F_1). In view of (3.11) and the conditions

$$\int_{a_{\nu-1}}^{a_\nu} u'^2\, dx = \int_{a_{\nu-1}}^{a_\nu} u^2 F(u^2, x)\, dx,$$

we have

$$\sum_{\nu=1}^{n} \int_{a_{\nu-1}}^{a_\nu} [\alpha_\nu^2 u'^2 - G(\alpha_\nu^2 u^2, x)]\, dx \leqslant \sum_{\nu=1}^{n} \int_{a_{\nu-1}}^{a_\nu} [\alpha_\nu^2 u'^2 - G(u^2, x) - (\alpha_\nu^2 - 1)\, u^2 F(u^2, x)]\, dx$$

$$= \int_a^b [u^2 F(u^2, x) - G(u^2, x)]\, dx = \lambda_n,$$

and thus

$$\lambda_n'' \leqslant \lambda_n + \sum_{\nu=1}^{n} \int_{a_{\nu-1}}^{a_\nu} [G(\alpha_\nu^2 u^2, x) - (1+\delta)\, G_1(\alpha_\nu^2 u^2, x)]\, dx. \tag{6.7}$$

To estimate the sum on the right-hand side of (6.7) we observe that, by (2.1),

$$G(\alpha_\nu^2 u^2, x) - (1+\delta)\, G_1(\alpha_\nu^2 u^2, x) = \int_0^{\alpha_\nu^2 u^2} [F(t, x) - (1+\delta)\, F_1(t, x)]\, dt.$$

By (6.1), the integrand becomes negative for $t > M^2$, where M depends on δ. Hence,

$$G(\alpha_\nu^2 u^2, x) - (1+\delta) G_1(\alpha_\nu^2 u^2, x) \leqslant \int_0^{M^2} F(t, x)\, dx = G(M^2, x)$$

and we conclude from (6.7) that

$$\lambda_n'' \leqslant \lambda_n + \int_a^b G(M^2, x)\, dx. \tag{6.8}$$

To compare λ_n' and λ_n'', we denote by $a = a_0', a_1', \ldots, a_{n-1}', a_n' = b$ the zeros of the nth characteristic function of (6.5), and determine constants $\beta_1, \ldots, \beta_n$ by the conditions

$$\int_{a_{\nu-1}'}^{a_\nu'} w'^2\, dx = \int_{a_{\nu-1}'}^{a_\nu'} w^2 F_1(\beta_\nu^2 w^2, x)\, dx. \tag{6.9}$$

Since, by (6.5),
$$\int_{a_{\nu-1}'}^{a_\nu'} w'^2\, dx = (1+\delta) \int_{a_{\nu-1}'}^{a_\nu'} w^2 F_1(w^2, x)\, dx, \tag{6.10}$$

we have $\beta_\nu > 1$ and therefore, by (1.3 c), $F_1(\beta_\nu^2 w^2, x) \geqslant \beta_\nu^{2\varepsilon} F_1(w^2, x)$, where ε is a fixed positive number. Hence, by (6.9) and (6.10),

$$(1+\delta) \int_{a_{\nu-1}}^{a_\nu} w^2 F_1(w^2, x)\, dx \geqslant \beta_\nu^{2\varepsilon} \int_{a_{\nu-1}}^{a_\nu} w^2 F_1(w^2, x)\, dx,$$

i.e., $\beta_\nu^{2\varepsilon} \leqslant 1 + \delta$, or
$$\beta_\nu^2 \leqslant c, \tag{6.11}$$

where the constant c does not depend on n.

The function $w_1(x)$ defined by $w_1(x) = \beta_\nu w(x)$ in $[a_{\nu-1}', a_\nu']$ has the normalization

$$\int_{a_{\nu-1}'}^{a_\nu'} w_1'^2\, dx = \int_{a_{\nu-1}'}^{a_\nu'} w_1^2 F_1(w_1^2, x)\, dx,$$

and it follows therefore from Theorem 3.2 that the nth characteristic value λ_n' associated with the system (6.3) can be estimated by

$$\lambda_n' \leqslant \int_a^b [w_1'^2 - G_1(w_1^2, x)]\,dx = \sum_{\nu=1}^{n} \int_{a_{\nu-1}'}^{a_\nu'} [\beta_\nu'^2 w^2 - G_1(\beta_\nu^2 w^2, x)]\,dx$$

$$= \sum_{\nu=1}^{n} \int_{a_{\nu-1}'}^{a_\nu'} [\beta_\nu'^2 w^2 - (1+\delta)\,G_1(\beta_\nu^2 w^2, x)]\,dx + \delta \sum_{\nu=1}^{n} \int_{a_{\nu-1}'}^{a_\nu'} G_1(\beta_\nu^2 w^2, x)\,dx.$$

Since, in view of (3.11) and (6.10),

$$\sum_{\nu=1}^{n} \int_{a_{\nu-1}'}^{a_\nu'} [\beta_\nu^2 w'^2 - (1+\delta)\,G_1(\beta_\nu^2 w^2, x)]\,dx$$

$$\leqslant \sum_{\nu=1}^{n} \int_{a_{\nu-1}'}^{a_\nu'} [\beta^2 w'^2 - (1+\delta)\,G_1(w^2, x) - (1+\delta)(\beta_\nu^2 - 1)\,w^2 F(w^2, x)]\,dx$$

$$= \int_a^b [w'^2 - (1+\delta)\,G_1(w^2, x)]\,dx = \lambda_n'',$$

it follows that

$$\lambda_n' \leqslant \lambda_n'' + \delta \sum_{\nu=1}^{n} \int_{a_{\nu-1}'}^{a_\nu'} G_1(\beta_\nu^2 w^2, x)\,dx. \tag{6.12}$$

By (3.11) (for $t=0$, $s=\beta_\nu^2 w^2$),

$$G_1(\beta_\nu^2 w^2, x) \leqslant \beta_\nu^2 w^2 F_1(\beta_\nu^2 w^2, x),$$

and thus, in view of (6.9) and (6.11),

$$\int_{a_{\nu-1}'}^{a_\nu'} G_1(\beta_\nu^2 w^2, x)\,dx \leqslant \beta_\nu^2 \int_{a_{\nu-1}'}^{a_\nu'} w'^2\,dx \leqslant c \int_{a_{\nu-1}'}^{a_\nu'} w'^2\,dx.$$

The inequality (6.12) may therefore be replaced by

$$\lambda_n' \leqslant \lambda_n'' + \delta c \int_a^b w'^2\,dx.$$

Since, by (2.8) (if this inequality is applied to the corresponding quantities associated with the system (6.5)) and (6.10),

$$\int_a^b w'^2\,dx \leqslant (1+\varepsilon)\,\varepsilon^{-1}\int_a^b [w'^2-(1+\delta)\,G_1(w^2,\,x)]\,dx = (1+\varepsilon)\,\varepsilon^{-1}\,\lambda'',$$

we finally obtain
$$\lambda_n' \leqslant \lambda_n''\,(1+\delta B),$$

where B is a constant depending on n. Combining this with (6.8) and observing that the constant M in (6.8) is likewise independent of n, we find that

$$\limsup_{n\to\infty} \frac{\lambda_n'}{\lambda_n} \leqslant 1+\delta B,$$

or, since δ can be chosen arbitrarily small and B does not depend on δ,

$$\limsup_{n\to\infty} \frac{\lambda_n'}{\lambda_n} \leqslant 1.$$

If the roles of $F(t,\,x)$ and $F_1(t,\,x)$ are reversed, the same procedure yields

$$\limsup_{n\to\infty} \frac{\lambda_n}{\lambda_n'} \leqslant 1,$$

and the proof of Theorem 6.1 is complete.

Theorem 6.1 shows, for instance, that the asymptotic behavior of the characteristic values of the system

$$y'' + \sum_{k=1}^{m} p_k(x)\,y^{2k+1} = 0, \quad y(a) = y(b) = 0, \tag{6.13}$$

where the function $p_k(x)$ are continuous in $[a,\,b]$, $p_k \geqslant 0$ for $k=1,\,\ldots,\,m$, and $p_m(x) > 0$ (cf. [4]) is identical with that of the characteristic values of

$$y'' + p_m(x)\,y^{2m+1} = 0, \quad y(a) = y(b) = 0,$$

and it may therefore be obtained from Theorem 5.1.

7. We now turn to the consideration of the more general equation (1.2). With a slight change of notation, we write (1.2) in the form

$$y'' + \Lambda\,p(x)\,y + y\,F(y^2,\,x) = 0, \tag{7.1}$$

where Λ is a positive number. In addition to leading to more concise formulation of both proofs and results, this notation is suggestive of the analogy between the Sturm-Liouville problem

$$y'' + \mu p(x) y = 0, \quad y(a) = y(b) = 0, \tag{7.2}$$

and the problem of finding solutions of (7.1) which satisfy the boundary conditions $y(a) = y(b) = 0$.

It is easy to see that there are values of Λ for which the latter type of solution with a prescribed number of zeros in (a, b) does not exist. Indeed, suppose that $\Lambda \geqslant \mu_n$, where μ_n is the nth eigenvalue of the problem (7.2), and that $y_0(x)$ is a solution of (7.1) which vanishes at $x = a$, $x = b$, and at $n - 1$ distinct points of the interval (a, b). If we write $p_1(x) = p(x) + \Lambda^{-1} F(y_0^2, x)$, the function $y_0(x)$ will then be the nth eigenfunction of the Sturm-Liouville problem $y'' + \Lambda p_1(x) y = 0$, $y(a) = y(b) = 0$, and Λ its nth eigenvalue. Since $p_1(x) \geqslant p(x)$ (but not $p_1(x) \equiv p(x)$), it follows from classical results [2] that $\Lambda < \mu_n$, contrary to our assumption.

The condition $\Lambda < \mu_n$ is thus necessary for the existence of such a solution of (7.1). As the following theorem shows, it is also sufficient.

THEOREM 7.1. *Let $F(t, x)$ be subject to the conditions (1.3), and let μ_n denote the nth eigenvalue of the Sturm-Liouville problem (7.2), where $p(x)$ is positive and continuous in $[a, b]$. In order that there exist in $[a, b]$ a solution of the problem*

$$y'' + \Lambda p(x) y + y F(y^2, x) = 0, \quad y(a) = y(b) = 0, \quad y(x) \in C', \tag{7.3}$$

with $n - 1$ zeros in (a, b), it is necessary and sufficient that $\Lambda < \mu_n$.

This solution may also be characterized by the minimum property

$$\lambda_n = H(y) \leqslant H(u), \tag{7.4}$$

where $H(u)$ denotes the functional

$$H(u) = \int_a^b [u^2 F(u^2, x) - G(u^2, x)]\, dx, \tag{7.5}$$

$G(t, x)$ is defined by (2.1), and $u(x)$ ranges over the class of functions with the following properties: $u(x)$ is piecewise continuously differentiable in $[a, b]$; $u(a_\nu) = 0$, where $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ and the values of $a_1, \ldots, a_{n-1}$ are otherwise arbitrary; $u(x)$ satisfies the inequalities

$$\int_{a_{\nu-1}}^{a_\nu} u'^2\, dx \leqslant \Lambda \int_{a_{\nu-1}}^{a_\nu} p(x) u^2\, dx + \int_{a_{\nu-1}}^{a_\nu} u^2 F(u^2, x)\, dx, \quad \nu = 1, \ldots, n. \tag{7.6}$$

We remark that, if the other conditions hold, conditions (7.6) can always be satisfied by multiplying $u(x)$ by suitable positive constants α_ν in the intervals $[a_{\nu-1}, a_\nu]$. If $U(x)$ is defined by $U(x)=\alpha_\nu u(x)$ in $[a_{\nu-1}, a_\nu]$, $U(x)$ will satisfy (7.6), provided

$$\int_{a_{\nu-1}}^{a_\nu} u'^2\,dx \leqslant \Lambda \int_{a_{\nu-1}}^{a_\nu} p(x)\,u^2\,dx + \int_{a_{\nu-1}}^{a_\nu} u^2 F(\alpha_\nu^2 u^2, x)\,dx, \quad \nu=1, \ldots, n. \tag{7.7}$$

Since, in view of (1.3 c), the right-hand side of (7.7) can be made arbitrarily large by choosing α_ν large enough, the assertion follows. We further remark that, for the purpose of solving the extremal problem of Theorem 7.1, the constants α_ν should be given the smallest values compatible with the conditions (7.7). If β is a positive constant and $H(u)$ is the functional (7.5), it follows from (3.11) that

$$\begin{aligned} H(\beta u) &= \int_a^b [\beta^2 u^2 F(\beta^2 u^2, x) - G(\beta^2 u^2, x)]\,dx \\ &\leqslant \int_a^b [\beta^2 u^2 F(\beta^2 u^2, x) - G(u^2, x) - (\beta^2-1)\,u^2 F(u^2, x)]\,dx, \end{aligned}$$

i.e.,

$$H(\beta u) \leqslant H(u) + \beta^2 \int_a^b u^2 [F(\beta^2 u^2, x) - F(u^2, x)]\,dx. \tag{7.8}$$

This inequality is evidently also valid for any subinterval of $[a, b]$. If $\beta < 1$, it follows from (1.3 c) and (7.8) that $H(\beta u) < H(u)$. We now suppose that the constant α_k may be replaced by the smaller constant $\beta\alpha_k$ without violating condition (7.7) (for $\nu = k$). If $H_k(u)$ denotes the integral (7.5) taken over $[a_{k-1}, a_k]$, we have $H_k(\beta\alpha_k u) < H_k(\alpha_k u)$, and this shows that $H(u_1) < H(u)$, where $u_1 = \beta u$ in $[a_{k-1}, a_k]$ and $u_1 = u$ elsewhere. For the purpose of minimizing the functional (7.5) it is thus indeed sufficient to take the smallest values of α_ν compatible with (7.7).

Inequality (7.7) may be true for $\alpha_\nu = 0$ in some of the intervals $[a_{\nu-1}, a_\nu]$, but not in all of them. In the latter case we would have, by classical results [2],

$$\mu_n \leqslant \max_\nu \int_{a_{\nu-1}}^{a_\nu} u'^2\,dx \left[\int_{a_{\nu-1}}^{a_\nu} p u^2\,dx\right]^{-1} \leqslant \Lambda,$$

contrary to our assumption.

We now turn to the proof of Theorem 7.1, considering first the case $n=1$, i.e., $\Lambda<\mu_1$, where μ_1 is the lowest eigenvalue of (7.2). Since the equation $y''+\mu_1 p(x) y=0$ has a solution which vanishes for $x=a$ and $x=b$ but not for $x \in (a, b)$, it follows from the Sturm comparison theorem [3] that the equation

$$\sigma''+\Lambda\, p(x)\, \sigma=0 \tag{7.9}$$

has a solution $\sigma(x)$ which is positive in $[a, b]$. It is therefore possible to apply the transformation

$$y(x)=\sigma(x)\, v(t), \quad t=\int_a^x [\sigma(\xi)]^{-2}\, d\xi \tag{7.10}$$

to the differential system (7.3). Carrying out the computation, we obtain

$$\ddot{v}+\sigma^3[\sigma''+\Lambda p \sigma]+\sigma^4 v F(\sigma^2 v^2, x)=0, \quad (\ddot{v}=d^2 v/d t^2),$$

or, in view of (7.9), $$v+\sigma^4 v F(\sigma^2 v, x)=0.$$

The function $F_1(v^2, t)=\sigma^4 F(\sigma^2 v^2, x)$ satisfies the conditions (1.3) (with obvious changes in the notation), and we may therefore apply Theorem 2.1 to the problem

$$\ddot{v}+v F_1(v^2, t)=0, \quad v(0)=v(T)=0, \quad T=\int_a^b [\sigma(x)]^{-2}\, d x, \tag{7.11}$$

into which problem (7.3) transforms. This shows that (7.3) indeed has a solution which does not vanish in (a, b). It may also be noted that this argument remains valid if Λ is a *negative number since, in this case,* (7.9) *certainly has a solution which is* positive in $[a, b]$.

By Theorem 2.1, the corresponding solution $v(t)$ of (7.11) is characterized by the *minimum* property

$$\int_0^T [\dot{v}^2-G_1(v^2, t)]\, d t=\min. \tag{7.12}$$

under the admissibility conditions $v(0)=v(T)=0$ and

$$\int_0^T \dot{v}^2\, d t=\int_0^T v^2 F_1(v^2, t)\, d t, \tag{7.13}$$

where, in accordance with (2.1),

$$G_1(v^2, t) = \int_0^{v^2} F_1(s, t)\,ds = \sigma^4 \int_0^{v^2} F(\sigma^2 s, x)\,ds$$

$$= \sigma^2 \int_0^{\sigma^2 v^2} F(s, x)\,ds = \sigma^2 G(\sigma^2 v^2, x) = \sigma^2 G(y^2, x).$$

By (7.10)
$$\int_0^T v^2 F_1(v^2, t)\,dt = \int_a^b y^2 F(y^2, x)\,dx$$

and
$$\int_0^T \dot v^2\,dt = \int_a^b \left[y' - \frac{\sigma'}{\sigma} y\right]^2 dx = \int_a^b \left[y'^2 + \frac{\sigma''}{\sigma} y^2\right] dx = \int_a^b [y'^2 - \Lambda p y^2]\,dx,$$

the last step following from (7.9). These identities show that (7.13) is equivalent to (7.6) for $y = u$, $a_{\nu-1} = a$, $a_\nu = b$, and the sign of equality), and that the functional (7.12) coincides with the expression $H(u)$ defined in (7.5). This completes the proof of Theorem 7.1 in the case $n = 1$.

If $n > 1$ we carry out the minimization of the functional (7.5) in two stages. We first choose a fixed set $a_0, \ldots, a_n$ such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ and minimize $H(u)$ under the admissibility conditions related to this set $\{a_\nu\}$. The second step will then consist in letting the set $\{a_\nu\}$ vary in order to obtain the smallest possible minimum. If $\mu(a_{\nu-1}, a_\nu)$ denotes the lowest eigenvalue of the Sturm-Liouville problem

$$y'' + \mu p(x) y = 0, \quad y(a_{\nu-1}) = y(a_\nu) = 0, \tag{7.14}$$

the character of the function minimizing $H(u)$ will, in each interval $[a_{\nu-1}, a_\nu]$, be different according as $\Lambda < \mu(a_{\nu-1}, a_\nu)$ or $\Lambda \geqslant \mu(a_{\nu-1}, a_\nu)$. If $\Lambda < \mu(a_{\nu-1}, a_\nu)$ then, as just shown, the extremal function will be in $[a_{\nu-1}, a_\nu]$ a non-trivial solution of (7.1) for which $y(a_{\nu-1}) = y(a_\nu) = 0$, and $y(x) \neq 0$ in $(a_{\nu-1}, a_\nu)$. The value of $\min H_\nu(a)$, where the subscript ν indicates that the functional (7.5) refers to the interval $[a_{\nu-1}, a_\nu]$, is in this case positive. If $\Lambda \geqslant \mu(a_{\nu-1}, a_\nu)$ it is not difficult to see that $\min H_\nu(u) = 0$. Indeed, let $w(x)$ denote the first eigenfunction of (7.14), normalized by the condition $w'(a_\nu) = 1$. Since

$$\int_{a_{\nu-1}}^{a_\nu} w'^2\,dx = \mu(a_{\nu-1}, a_\nu) \int_{a_{\nu-1}}^{a_\nu} p w^2\,dx \leqslant \Lambda \int_{a_{\nu-1}}^{a_\nu} p w^2\,dx,$$

$w(x)$ satisfies (7.7), and the same is evidently true of the function $\beta w(x)$, where β is an arbitrary non-zero constant. Hence,

$$\min H_\nu(u) \leqslant H_\nu(\beta w) = \int_{a_{\nu-1}}^{a_\nu} [\beta^2 w^2 F(\beta^2 w^2, x) - G(\beta^2 w^2, x)]\,dx \leqslant \beta^2 \int_{a_{\nu-1}}^{a_\nu} w^2 F(\beta^2 w^2, x)\,dx.$$

If $\beta^2 < 1$, it follows from (1.3c) that

$$\min H_\nu(u) \leqslant \beta^2 \int_{a_{\nu-1}}^{a_\nu} w^2 F(w^2, x)\,dx$$

But β^2 may be taken arbitrarily small, and we thus have $\min H_\nu(u) \leqslant 0$. On the other hand, $H_\nu(u) \geqslant 0$ since, by (3.11) (for $t=0$), the integrand in (7.5) is non-negative. Hence, $\min H_\nu(u) = 0$. In view of $H_\nu(0) = 0$, the function $y(x) \equiv 0$ will thus minimize $H_\nu(u)$ in an interval $[a_{\nu-1}, a_\nu]$ for which $\mu(a_{\nu-1}, a_\nu) \leqslant \Lambda$.

It was shown above that for $\mu(a_{\nu-1}, a_\nu) > \Lambda$, the problem of minimizing $H_\nu(u)$ is equivalent to a minimum problem of the type described in Theorem 2.1. The function $\lambda(a_{\nu-1}, a_\nu) = \min H_\nu(u)$ has therefore the properties enumerated in Lemma (3.1). In particular, $\lambda(a_{\nu-1}, a_\nu)$ is a positive continuous function of $a_{\nu-1}$ and a_ν, which decreases as the interval $[a_{\nu-1}, a_\nu]$ expands. As just shown, $\lambda(a_{\nu-1}, a_\nu) = 0$ if $\mu(a_{\nu-1}, a_\nu) \leqslant \Lambda$. In order to prove that $\lambda(a_{\nu-1}, a_\nu)$ is a continuous function of $a_{\nu-1}$ and a_ν without any restrictions on the length of the interval, it is thus sufficient to show that $\lim \lambda(a_{\nu-1}, a_\nu) = 0$ if $\mu(a_{\nu-1}, a_\nu) - \Lambda$ approaches zero through positive values. We derive here a slightly more accurate description of the limiting behavior of $\lambda(a_{\nu-1}, a_\nu)$, which will be needed later.

If $\mu(a_{\nu-1}, a_\nu) = \Lambda$ *and* δ *is a small positive number, then*

$$\begin{aligned} \lambda(a_{\nu-1}, a_\nu - \delta) &= o(\delta), \\ \lambda(a_{\nu-1} + \delta, a_\nu) &= o(\delta). \end{aligned} \tag{7.15}$$

Since an elementary transformation interchanges the roles of $a_{\nu-1}$ and a_ν, it is sufficient to prove the first relation (7.15). By classical results [2], $\mu(a_{\nu-1}, a_\nu)$ is a continuous function of a_ν and $a_{\nu-1}$. A standard computation shows that

$$\mu(a_{\nu-1}, a_\nu - \delta) = \mu(a_{\nu-1}, a_\nu) + \delta\left[\int_{a_{\nu-1}}^{a_\nu} p w^2\,dx\right]^{-1} + o(\delta^2),$$

where $w(x)$ is the first eigenfunction of (7.14), normalized by the condition $w'(a_\nu)=1$. Hence,

$$\mu(a_{\nu-1}, a_\nu-\delta)=\Lambda+\delta A+o(\delta^2)=\Lambda+A'\delta,$$

where A is a positive constant, and $A'=A+o(\delta^2)$. We now denote by $r(x)$ the first eigenfunction of (7.14) for the interval $[a_{\nu-1}, a_\nu-\delta]$, normalized by $r'(a_\nu-\delta)=1$, and determine a positive constant β by the equation

$$A'\delta\int_{a_{\nu-1}}^{a_\nu-\delta} pr^2\,dx=\int_{a_{\nu-1}}^{a_\nu-\delta} r^2F(\beta^2r^2, x)\,dx. \tag{7.16}$$

Setting $R(x)=\beta r(x)$, we have

$$\int_{a_{\nu-1}}^{a_\nu-\delta} R'^2\,dx=(\Lambda+\delta A')\int_{a_{\nu-1}}^{a_\nu-\delta} pR^2\,dx=\Lambda\int_{a_{\nu-1}}^{a_\nu-\delta} pR^2\,dx+\int_{a_{\nu-1}}^{a_\nu-\delta} R^2F(R^2, x)\,dx.$$

In view of (7.6), $R(x)$ is therefore an admissible function for the problem $H_\nu(u)=\min$ (where H_ν now refers to the interval $[a_{\nu-1}, a_\nu-\delta]$). Hence,

$$0<\lambda(a_{\nu-1}, a_\nu-\delta)\leqslant H_\nu(\beta w)=\int_{a_{\nu-1}}^{a_\nu-\delta}[\beta^2F(\beta^2r^2, x)-G(\beta^2r^2, x)]\,dx\leqslant\beta^2\int_{a_{\nu-1}}^{a_\nu-\delta}F(\beta^2r^2, x)\,dx,$$

where, by (7.16),

$$0<\lambda(a_{\nu-1}, a_\nu-\delta)\leqslant\beta^2A'\delta\int_{a_{\nu-1}}^{a_\nu-\delta} pr^2\,dx. \tag{7.17}$$

If $\delta\to0$, $r(x)$ tends to $w(x)$ uniformly in x. Since $A'=A+o(\delta^2)$, (7.16) shows therefore that $\beta\to0$ for $\delta\to0$. In view of (7.17), this proves first relation (7.15).

We now consider the problem of minimizing the value of $\phi=\phi(a_0, a_1, \dots, a_n)$ $=\sum_{\nu=1}^{n}\lambda(a_{\nu-1}, a_\nu)$, where the set $\{a_\nu\}$ is subject to the conditions $a=a_0<a_1<\cdots<a_n=b$ and is otherwise unrestricted. As shown above, $\phi(a_0, \dots, a_n)$ is a continuous function of the arguments $a_1, \dots, a_{n-1}$, and the existence of a minimizing set $\{a_\nu\}$ is thus assured. We also note that the minimizing set $\{a_\nu\}$ consists of $n+1$ distinct points since, by Lemma 3.1, ϕ would tend to infinity if two adjacent points a_ν were to approach each other.

The intervals $[a_{\nu-1}, a_\nu]$ which correspond to the set—or, one of the sets—$\{a_\nu\}$ minimizing the expression $\phi(a_0, \ldots, a_n)$ may be divided into two classes, according as $\mu(a_{\nu-1}, a_\nu) > \Lambda$ or $\mu(a_{\nu-1}, a_\nu) \leqslant \Lambda$. In the first case, the extremal function $y(x)$ is in $[a_{\nu-1}, a_\nu]$ a nontrivial solution of (7.1) for which $y(a_{\nu-1}) = y(a_\nu) = 0$ and $y(x) \not\equiv 0$ in $(a_{\nu-1}, a_\nu)$; in the second case, we have $y(x) \equiv 0$ throughout $[a_{\nu-1}, a_\nu]$. We shall show, however, that the existence of an interval of the second type is incompatible with the extremal character of the set $\{a_\nu\}$.

We first remark that there must be at least one interval of the first type; otherwise the inequality $\mu_n \leqslant \max \mu(a_{\nu-1}, a_\nu)$ [2] would contradict the assumption $\mu_n > \Lambda$. Hence, if there are any intervals of the second type, at least one of these, say $[a_{k-1}, a_k]$, must be adjacent to an interval of the first type. Without loss of generality the latter may be assumed the interval $[a_k, a_{k+1}]$. If it were true that $\mu(a_{k-1}, a_k) < \Lambda$, we could choose a small positive number δ such that $\mu(a_{k-1}, a_k - \delta) < \Lambda$ and $\mu(a_k - \delta, a_{k+1}) > \Lambda$. But this would imply that $\lambda(a_{k-1}, a_k - \delta) = 0$ and $\lambda(a_k - \delta, a_{k+1}) < \lambda(a_k, a_{k+1})$, i.e., the substitution of $a_k - \delta$ for a_k would decrease the value of $\sum_{\nu=1}^{n} \lambda(a_{\nu-1}, a_\nu)$.

In order to show that the assumption $\mu(a_{k-1}, a_k) = \Lambda$ is likewise incompatible with the extremal character of the set $\{a_\nu\}$, we estimate the variation of $\lambda(a_{k-1}, a_k) + \lambda(a_k, a_{k+1})$ if a_k is replaced by $a_k - \delta$. An estimate for $\lambda(a_k - \delta, a_{k+1})$ is obtained from the inequality (3.19) which, as pointed out, remains valid if the positive constant ε in (1.3 c) is replaced by 0. In particular, (3.19) is applicable to the equation $y'' + yF_1(y^2, x) = 0$, where $F_1(t, x) = \Lambda p(x) + F(t, x)$, i.e., to equation (7.1). In view of the minimum property of $\lambda(a_k - \delta, a_{k+1})$, we thus may apply (3.20) (with appropriate changes in the notation). Combining this with the estimate (7.15) for $\lambda(a_{k-1}, a_k - \delta)$, we obtain

$$\lambda(a_{k-1}, a_k - \delta) + \lambda(a_k - \delta, a_{k+1}) \leqslant \lambda(a_{k-1}, a_k) + \lambda(a_k, a_{k+1}) - \frac{\delta}{2} y'^2(a_k) + o(\delta),$$

where $y(x)$ is a non-trivial solution of (7.1) for which $y(a_k) = 0$. As pointed out at the end of section 1, we necessarily have $y'(a_k) \neq 0$, and the last inequality shows therefore that $\lambda(a_{k-1}, a_k - \delta) + \lambda(a_k - \delta, a_{k+1})$ will be smaller than $\lambda(a_{k-1}, a_k) + \lambda(a_k, a_{k+1})$ if the positive number δ is taken sufficiently small. But this again conflicts with the extremal character of the set $\{a_\nu\}$, and we may therefore conclude that, in the extremal case, all of the intervals $[a_{\nu-1}, a_\nu]$ are such that $\mu(a_{\nu-1}, a_\nu) > \Lambda$. Accordingly, the extremal function will in each interval $[a_{\nu-1}, a_\nu]$ coincide with a non-trivial solution $y(x)$ of (7.1) for which $y(a_{\nu-1}) = y(a_\nu) = 0$ and $y(x) \neq 0$ in $(a_{\nu-1}, a_\nu)$.

We now consider two adjacent intervals $[a_{k-1}, a_k]$ and $[a_k, a_{k+1})$, and apply to a_k the small variation described in section 3. If a_k' is the varied value of a_k and δ is the small positive number controlling the variation, we have, in view of (3.15) and the remark made at the end of section 3,

$$\lambda(a_{k-1}, a_k') + \lambda(a_k', a_{k+1}) \leqslant \lambda(a_{k-1}, a_k) + \lambda(a_k, a_{k+1}) - \frac{1}{2}\delta[y_+'(a_k) - y_-'(a_k)]^2 + o(\delta),$$

provided $y_+'(a_k)$ and $y_-'(a_k)$ are of the same sign. Since the extremal property of $y(x)$ is not affected by substituting $-y(x)$ for $y(x)$ in $[a_{k-1}, a_k]$, the latter may be assumed to be the case. If it were true that $y_+'(a_k) \neq y_-'(a_k)$, the value of $-\delta[y_+'(a_k) - - y_-'(a_k)]^2 + o(\delta)$ could be made negative by taking δ sufficiently small, and the last inequality would contradict the extremal character of the set $\{a_\nu\}$. Hence, $y_+'(a_k) = y_-'(a_k)$ for $k = 1, \ldots, n-1$, i.e., both $y(x)$ and $y'(x)$ are continuous in $[a, b]$. Since the extremal function $y(x)$ was shown to be a non-trivial solution of (7.1) in each interval $[a_{\nu-1}, a_\nu]$, this shows that $y(x)$ is a solution of (7.1) in the entire interval $[a, b]$. In view of the fact that $y(a) = y(b) = 0$ and that $y(x)$ has precisely $n-1$ zeros in (a, b)—at $x = a_\nu$, $\nu = 1, \ldots, n-1$— this completes the proof of Theorem 7.1.

8. In this section we consider equations

$$y'' + yF(y^2, x) = 0, \tag{8.1}$$

which remain unchanged if x is replaced by $x + \omega$, where ω is a positive constant. The function $F(t, x)$ will thus be assumed to satisfy, in addition to conditions (1.3) (for $a = 0$, $b = \omega$), the periodicity condition

$$F(t, x + \omega) = F(t, x) \tag{8.2}$$

for all x and all non-negative values of t. We shall establish the following result.

THEOREM 8.1. *If $F(t, x)$ satisfies conditions* (1.3) *and* (8.2), *then the equation* (8.1) *has an infinity of distinct solutions with the period ω. For each positive integer n there exists at least one such solution with exactly $2n$ zeros in the interval* $(0, \omega]$.

To prove Theorem 8.1, we choose a number a and note that, according to Theorem 3.2, there exists a solution $y = y(x, a)$ of (8.1) in $[a, a+\omega]$ which vanishes for $x = a$ and $x = a + \omega$, and has $2n - 1$ zeros in $(a, a+\omega)$. This solution minimizes the functional

$$H(y, a) = \int_a^{a+\omega} [y'^2 - G(y^2, x)]\,dx \tag{8.3}$$

under the side conditions indicated in the statement of Theorem 3.2. Denoting the minimum value of this functional by $\lambda_{2n}(a)$, we now consider the problem of minimizing $\lambda_{2n}(a)$ with respect to a. Since, in view of (8.2), $\lambda_{2n}(a+\omega)=\lambda_{2n}(a)$, it is sufficient to let a vary in the interval $[0, \omega]$.

According to the remark made at the end of section 3, $\lambda_{2n}(a)$, is a continuous function of a, and there will therefore exist at least one value a_0 in $[0, \omega]$ at which $\lambda_{2n}(a)$ attains its minimim. If α denotes the first zero of $y(x, a_0)$ in the interval $(a_0, a_0+\omega)$, we define a function $y_1(x)$ in $[\alpha, \alpha+\omega]$ by

$$\begin{aligned} y_1(x) &= y(x, a_0), \quad x \in [\alpha, a_0+\omega], \\ y_1(x) &= y(x-\omega, a_0), \quad x \in [a_0+\omega, \alpha+\omega]. \end{aligned} \tag{8.4}$$

It may be noted that, since $y(x)$ has $2n+1$ zeros in $[a_0, a_0+\omega]$, the function $y_1(x)$ changes its sign at the point $x=a_0+\omega$. It is easily confirmed that $H(y_1, \alpha)=H(y, a_0)$, where H is the functional (8.3), and that $y_1(x)$ satisfies the admissibility conditions for the minimum problem (3.17). We thus conclude from Theorem 3.2 that

$$\lambda_{2n}(\alpha) \leqslant H(y_1, \alpha) = H(y, a_0) = \lambda_{2n}(a_0).$$

Since, by assumption, $\lambda_{2n}(a_0) \leqslant \lambda_{2n}(\alpha)$, it follows that $\lambda_{2n}(\alpha)=H(y_1, \alpha)$, i.e., $y_1(x)$ is a solution of the extremal problem of Theorem 3.2 (with the appropriate changes of notation). Since $y_1(x)$ changes its sign at each of its zeros, $y_1(x)$ will therefore have a continuous derivative in $[\alpha, \alpha+\omega]$. In particular, we have $y_1'(a_0+\omega+0) = y_1'(a_0+\omega-0)$, i.e., by virtue of (8.4), $y'(a_0+\omega)=y'(a_0)$. In view of the fact that $y(a_0) = y(a_0+\omega)=0$ and that $y(x+\omega)$ is likewise a solution of (8.1), the original solution $y(x)$ may thus be continued into the interval $[a_0+\omega, a_0+2\omega]$ by setting $y(x+\omega)=y(x)$. The proof of Theorem 8.1 now follows by repeating the same argument for successive intervals $[a_0+m\omega, a_0+(m+1)\omega]$, $m=\pm 1, \pm 2, \ldots$.

We remark that a similar method of proof may be used in the case of the characteristic value $\lambda_{2n+1}(n=1, 2, \ldots)$. The only difference is that $y(x)$ has now an even number of zeros in $[a_0, a_0+\omega]$ and that $y'(a_0)$ and $y'(a_0+\omega)$ will therefore be of opposite sign. Accordingly, the function $y_1(x)$ will have a continuous derivative in $[\alpha, \alpha+\omega]$ if the second definition (8.4) is changed to $y_1(x)=-y(x-\omega)$. This leads to the following result.

THEOREM 8.2. *Under the assumptions of Theorem* 8.1, *equation* (8.1) *has an infinity of distinct solutions* $y(x)$ *for which*

$$y(x+\omega) = -y(x).$$

For each positive integer n *there exists at least one such solution with exactly* $2n+1$ *zeros in the interval* $(0, \omega]$.

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