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Subharmonic functions of completely regular growth in a cone

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1. Introduction

We shall consider domains Γ in the unit sphere $S_1 \subset \mathbf{R}^m$, satisfying the following conditions:

(a) The boundary $\partial\Gamma$ is twice smooth;

(b) The normalized solutions ϕ_j, corresponding to the eigenvalues λ_j with $0<\lambda_1<\lambda_2\leq\ldots$, of the boundary value problem

$$\Delta^*\phi+\lambda\phi=0; \quad \phi\,|_{\partial\Gamma}=0,$$

where Δ^* is the spherical part of the Laplace operator Δ, are twice continuously differentiable functions on the closure of Γ, and the inward normal derivative $\partial\phi_1/\partial n$ is strictly positive on the boundary $\partial\Gamma$;

(c) There exists a point $x_*\in S_1$ such that, if K^Γ is the cone $\{x\in\mathbf{R}^m; x/|x|\in\Gamma\}$, then the closure of the translated cone $K^\Gamma+lx_*$ is contained in K^Γ, for every $l>0$.

Denote by $B(x_0,r)$ the open ball in $\mathbf{R}^m$ with center at x_0 and radius r, $B_r=B(0,r)$ and $S_r=\partial B(0,r)$. Given such a domain Γ and the cone $K=K^\Gamma$ spanned by Γ, we shall use the notations $K_r=K\cap B_r$, $K_{r_1,r_2}=K\cap\{r_1<|x|<r_2\}$, $\Gamma_r=K\cap S_r$ and $\Gamma_{r_1,r_2}=\partial K\cap\partial K_{r_1,r_2}$. Notice that if the function ϕ_1 is homogeneously extended to the cone K, then the functions $|x|^{k_1^\pm}\phi_1(x)$, with $2k_1^\pm=-m+2\pm\sqrt{(m-2)^2+4\lambda_1}$, are harmonic in K and vanish on $\partial K\setminus\{0\}$.

By $SH(K,\varrho)$, $\varrho>0$, we denote all subharmonic functions u in K satisfying the condition

$$\limsup_{t\to\infty}\frac{\widehat{M}_u(t)}{t^\varrho}<\infty,$$

where

$$\widehat{M}_u(t)=\max\{M_u(t),\Phi_u(t)\},$$

$$M_u(t)=\sup_{x\in\Gamma_t}\{u(x)\} \quad \text{and} \quad \Phi_u(t)=\int_\Gamma\phi_1(x)|u(tx)|\,dS_1(x).$$

Here dS_1 denotes the element of $(m-1)$-dimensional Euclidean volume on the unit sphere.

Recall that a set $E\subset\mathbf{R}^m$ is said to be a C_0^{m-1}-set if it can be covered by balls $B(x_j,r_j)$ such that the relation

$$\lim_{t\to\infty}\frac{1}{t^{m-1}}\sum_{|x_j|<t} r_j^{m-1}=0$$

holds. Following Rashkovskiĭ and Ronkin, see [4], we introduce the concept of completely regular growth for subharmonic functions in the cone K.

Definition. A function $u\in SH(K,\varrho)$ is said to be of completely regular growth (CRG) in the closed cone $\overline{K}$ if there exists a C_0^{m-1}-set $E\subset K$ such that

$$\lim_{\substack{|x|\to\infty\\ x\notin E}}|u(x)-h_u^*(x)|/|x|^\varrho=0,$$

where the indicator function

$$h_u^*(x)=\limsup_{y\to x}\limsup_{t\to\infty}u_t(y)\quad\text{with } u_t(y)=u(ty)/t^\varrho.$$

A function $u\in SH(K,\varrho)$ is said to be CRG in the open cone K if it is CRG in every closed cone $\overline{K^{\Gamma'}}$ spanned by $\overline{\Gamma'}\subset\Gamma$.

Remark. Unlike in the case of functions defined in the whole space, we cannot cancel the integral $\Phi_u(t)$ in the definition of the class $SH(K,\varrho)$. Otherwise the indicator functions may be identical to infinity. Such an example is given by the function $u^0(x_1,x_2)=-x_1$ in the half plane $x_1>0$.

It is known, see [3], that for any function $u\in SH(K,\varrho)$ there exists a real measure ν_u on the boundary ∂K, which is the boundary value of u in the following sense:

For any continuous function ψ on ∂K with $\operatorname{supp}\psi\subset\subset\partial K\setminus\{0\}$, the relation

$$\lim_{l\to+0}\int_{\partial K}\psi(x)u(x+lx_*)\,d\sigma(x)=\int_{\partial K}\psi(x)\,d\nu_u(x)$$

holds, where $d\sigma$ denotes the $(m-1)$-dimensional Euclidean volume element on ∂K. By means of the weak boundary value, Ronkin discussed the relation between CRG functions in an open cone K and in a closed cone $\overline{K}$, and obtained the following result, see [5, Theorem 4.4.6].

Theorem A. *In order that a function $u \in SH(K,\varrho)$ be CRG in $\overline{K}$, it is sufficient and, under the additional assumption* $\sup_{x\in\Gamma}|h_u^*(x)|<\infty$ *also necessary, that u be CRG in K and that*

$$(\mathcal{A}) \qquad \lim_{t\to\infty}\frac{1}{t^{\varrho+m-1}}\int_{\Gamma_{1,t}} d|\nu_u-\nu_{h_u^*}|=0,$$

where $|\nu_u-\nu_{h_u^*}|$ denotes the total variation of the measure $\nu_u-\nu_{h_u^*}$.

The proof of Theorem A was rather long, and an integral representation for subharmonic functions of finite order was used, see [6]. In his book [5, p. 240] Ronkin conjectured that the assumption on boundedness of the indicator h_u^* is unnecessary, and his opinion was based on the following theorem.

Theorem B [4, Theorem 2]. *If $u\in SH(K,\varrho)$ is CRG in the closed cone $\overline{K}$ and can be extended to some large cone $K'=K^{\Gamma'}$ spanned by $\Gamma'\supset\supset\Gamma$ as a function in $SH(K',\varrho)$, then u satisfies Condition $\mathcal{A}$.*

However, Theorem B does not seem to give any indication as to whether or not the assumption on boundedness of the indicator is superfluous, because we have the following improvement.

Theorem 1. *Suppose that $u\in SH(K,\varrho)$ is CRG in the open cone K and can be extended to be a function in $SH(K',\varrho)$ for some cone $K'=K^{\Gamma'}$ spanned by $\Gamma'\supset\supset\Gamma$. Then u satisfies Condition $\mathcal{A}$.*

We have not been able to prove Ronkin's conjecture, but we have found slightly weaker sufficient conditions. First, by the positive homogeneity, it is clear that the boundedness of h_u^* implies

$$|\nu_{h_u^*}|(E)\le\sup_{x\in\Gamma}|h_u^*(x)|\,d\sigma(E)$$

for any subset $E\subset\Gamma_{0,1}$. The following result is therefore slightly stronger than the necessary part in Theorem A, and requires a different method of proof.

Theorem 2. *Suppose that $u\in SH(K,\varrho)$ is CRG in $\overline{K}$ and that there exists a positive constant c such that $|\nu_{h_u^*}|(E)\le c\,d\sigma(E)$ for all subsets E in $\Gamma_{0,1}$. Then u satisfies Condition $\mathcal{A}$.*

On the other hand, since the functions u_t with $t>1$ are uniformly bounded from above in Γ, the boundedness of h_u^* also implies that there exists a positive constant c such that

$$u_t(x)<h_u^*(x)+c$$

for all $x\in\Gamma$. This inequality is in fact enough to ensure Condition $\mathcal{A}$.

Theorem 3. *Suppose that $u \in SH(K,\varrho)$ is CRG in $\overline{K}$ and that there exists $c>0$, such that*

$$u_t(x) < h_u^*(x)+c$$

for all $x\in\Gamma$ and sufficiently large t. Then u satisfies condition $\mathcal{A}$.

In our opinion, for subharmonic functions, the hypothesis in Theorem 3 is more natural than the necessary part in Theorem A. For some functions, such inequalities follow for instance from the Hartogs lemma, see [2, Theorem 1.31]. The proof of Theorem 3 is essentially parallel to the proof of Theorem 2 given below, and will therefore be omitted.

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2. Proofs of Theorems 1 and 2

Proof of Theorem 1. As it is mentioned in [4], under the assumption of Theorem 1, the weak boundary values ν_u and $\nu_{h_u^*}$ are equal to u and h_u^* respectively on $\partial K\setminus\{0\}$. Hence Condition $\mathcal{A}$ takes the form

$$\lim_{t\to\infty}\frac{1}{t^{\varrho+m-1}}\int_{\Gamma_{1,t}}|u(x)-h_u^*(x)|\,d\sigma(x)=0. \tag{1}$$

To obtain (1) we first show the equality

$$\lim_{t\to\infty}\int_{\Gamma_{1/2,1}}|u_t(x)-h_u^*(x)|\,d\sigma(x)=0. \tag{2}$$

It is enough to show that for any sequence $t_j\to\infty$ there exists a subsequence such that the equality (2) holds for such a subsequence. To do this, we choose a domain D such that $\Gamma_{1/2,1}\subset\subset D\subset\subset K'$. It follows from Theorem 4.1.9 in [1] that the family $\{u_{t_j}\}$ is relatively compact in $L^1_{\mathrm{loc}}(D)$. So there exists a subsequence $t_{j_k}\to\infty$ such that $u_{t_{j_k}}$ converges to some subharmonic function g in $L^1_{\mathrm{loc}}(D)$. Since u is *CRG* in K, u_t converges to h_u^* in the distribution sense, see [5, Theorem 4.4.3]. By Theorem 4.1.9 in [1] we then have that u_t converges to h_u^* in $L^1_{\mathrm{loc}}(K)$, and hence $g=h_u^*$ in $D\cap K$. This implies that $h_u^*\,d\sigma=g\,d\sigma$ on $\partial K\cap D\supset\Gamma_{1/2,1}$. Now for $x\in D$, using Riesz's theorem, we write

$$u_{t_{j_k}}(x)=-\int_D G(x,y)\,d\mu_{u_{t_{j_k}}}(y)+\Phi_k(x)$$

and

$$g(x) = -\int_D G(x,y)\,d\mu_g(y) + \Phi(x),$$

where Φ_k and Φ are the smallest harmonic majorants of $u_{t_{j_k}}$ and g in D respectively, and G is the Green function of D. Since $u_{t_{j_k}}$ converges to g in $L^1_{\mathrm{loc}}(D)$, it follows that Φ_k converges uniformly to Φ in $\Gamma_{1/2,1}$ and $d\mu_{u_{t_{j_k}}}$ converges to $d\mu_g$ as a distribution in D. Therefore, $d\mu_{u_{t_{j_k}}}$ converges to $d\mu_g$ in the weak topology of measures in D. On the other hand, we have that

$$\int_{\Gamma_{1/2,1}} G(x,y)\,d\sigma(x)$$

is a continuous function of y in D and vanishes on the boundary ∂D. So

$$\begin{aligned}\lim_{k\to\infty}\int_{\Gamma_{1/2,1}} (u_{t_{j_k}}(x) - h_u^*(x))\,d\sigma(x) &= \lim_{k\to\infty}\int_{\Gamma_{1/2,1}} (u_{t_{j_k}}(x) - g(x))\,d\sigma(x)\\ &= \lim_{k\to\infty}\int_D (d\mu_g(y) - d\mu_{u_{t_{j_k}}}(y))\int_{\Gamma_{1/2,1}} G(x,y)\,d\sigma(x)\\ &\quad + \lim_{k\to\infty}\int_{\Gamma_{1/2,1}} (\Phi_k(x) - \Phi(x))\,d\sigma(x) = 0.\end{aligned}$$

Hence, using the same method as in the proof of Lemma 2.1.4 in [5], we obtain

$$\lim_{k\to\infty}\int_{\Gamma_{1/2,1}} |u_{t_{j_k}}(x) - h_u^*(x)|\,d\sigma(x) = 0,$$

and this completes the proof of equality (2).

We now know that for any $\varepsilon > 0$ there exists a constant $t_0 > 1$ such that

$$\int_{\Gamma_{t/2,t}} |u(x) - h_u^*(x)|\,d\sigma(x) \le \varepsilon t^{\varrho+m-1} \quad \text{for } t \ge t_0.$$

So we have

$$\begin{aligned}\int_{\Gamma_{1,t}} |u(x) - h_u^*(x)|\,d\sigma(x) &\le \int_{\Gamma_{1,t_0}} |u(x) - h_u^*(x)|\,d\sigma(x)\\ &\quad + \sum_{k=1}^{[(\ln(t/t_0))/\ln 2]+1} \int_{\Gamma_{2^{k-1}t_0,2^k t_0}} |u(x) - h_u^*(x)|\,d\sigma(x)\\ &= o(t^{\varrho+m-1}) + \varepsilon \sum_{k=1}^{[(\ln(t/t_0))/\ln 2]+1} (2^k t_0)^{\varrho+m-1}\\ &= o(t^{\varrho+m-1}) + \varepsilon O(t^{\varrho+m-1}), \quad \text{as } t \to \infty.\end{aligned}$$

This implies Condition $\mathcal{A}$, and hence the proof is complete.

To prove Theorem 2 we need the following lemma for CRG functions in the open cone K, see [5, Theorem 4.4.5].

Lemma. *Suppose that* $u\in SH(K,\varrho)$ *is* CRG *in the open cone* K. *Then we have*

$$\int_{\overline{K}_{1/t,3}} \psi(x)\left(\phi_1(x)\,d\mu_{u_t}(x)-\frac{1}{\theta_m}\frac{\partial\phi_1}{\partial n}\,d\nu_{u_t}(x)\right) \longrightarrow \int_{\overline{K}_3} \psi(x)\left(\phi_1(x)\,d\mu_{h_u^*}(x)-\frac{1}{\theta_m}\frac{\partial\phi_1}{\partial n}\,d\nu_{h_u^*}(x)\right), \qquad as\ t\to\infty,$$

for any function $\psi\in C(\overline{K}_3)$, *where* μ_g *denotes the Laplacian* $\theta_m^{-1}\Delta g$, *the constant* $\theta_m=(m-2)\int_{S_1} dS_1$ *for* $m>2$ *and* $\theta_2=2\pi$.

Actually, in [5] this result was obtained for the functions ψ in $C(\overline{K}_1)$, but by homogeneity it is equivalent to take ψ in $C(\overline{K}_3)$.

Proof of Theorem 2. Using the same argument as in the above proof, we only need to show that for any sequence $t_j\to\infty$ there exists a subsequence $t_{j_k}\to\infty$, such that

$$\lim_{k\to\infty}\frac{1}{t_{j_k}^{\varrho+m-1}}\int_{\Gamma_{t_{j_k}/2,t_{j_k}}} d|\nu_u-\nu_{h_u^*}|=0. \tag{3}$$

For simplicity, we consider the whole family $t>0$.

Since u is CRG in $\overline{K}$, there exists a C_0^{m-1}-set $E\subset K$ such that

$$\lim_{\substack{|x|\to\infty\\ x\notin E}} |u(x)-h_u^*(x)|/|x|^{\varrho}=0. \tag{4}$$

Hence, using the definition of the weak boundary value, we can write

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t^{\varrho+m-1}}\int_{\Gamma_{t/2,t}} d|\nu_u-\nu_{h_u^*}| \\
&\quad\le \limsup_{t\to\infty}\limsup_{l\to+0}\frac{1}{t^{\varrho+m-1}}\int_{\Gamma_{t/2,t}} |u(x+lx_*)-h_u^*(x+lx_*)|\,d\sigma(x) \\
&\quad\le \limsup_{t\to\infty}\limsup_{l\to+0}\frac{1}{t^{\varrho+m-1}}\int_{A_t} |u(x+lx_*)|\,d\sigma(x) \\
&\qquad+\limsup_{t\to\infty}\limsup_{l\to+0}\frac{1}{t^{\varrho+m-1}}\int_{A_t} |h_u^*(x+lx_*)|\,d\sigma(x) \\
&\qquad+\limsup_{t\to\infty}\limsup_{l\to+0}\frac{1}{t^{\varrho+m-1}}\int_{\Gamma_{t/2,t}\setminus A_t} |u(x+lx_*)-h_u^*(x+lx_*)|\,d\sigma(x) \\
&\quad\stackrel{\text{def}}{=} I+II+III,
\end{aligned} \tag{5}$$

where $A_t=\{x\in\Gamma_{t/2,t}; x+lx_*\in E$, for some $0<l<1\}\subset\partial K$.

It follows from (4) that $III=0$. Since E is a C_0^{m-1}-set, there exist balls $B(x_j,r_j)\subset\mathbf{R}^m$ such that $E\subset\bigcup_j B(x_j,r_j)$ and

$$\lim_{t\to\infty}\frac{1}{t^{m-1}}\sum_{|x_j|<t} r_j^{m-1}=0. \tag{6}$$

We denote $B_{tj}=\{x\in\Gamma_{t/2,t}; x+lx_*\in B(x_j,r_j)$, for some $0<l<1\}$. Then

$$A_t\subset\bigcup_{|x_j|<2t} B_{tj}$$

for large enough t.

We now define a projection $P\colon K\to\partial K$ as follows. For all $x\in K$, we claim that the intersection $\partial K\cap(x+\mathbf{R}x_*)$ consists of exactly one point. Then we let $P(x)$ be this point. To justify this claim, suppose that $a,b\in\partial K\cap(x+\mathbf{R}x_*)$. Then $a-b=l\,x_*$, for some constant l, and we can assume $l\geq 0$. But since $\overline{K}+l\,x_*\subset K$ for every $l>0$, we obtain $l=0$ and hence $a=b$.

Since $\partial\Gamma$ is compact, there exists a positive constant c, independent of j and t, such that $B_{tj}\subset P(B(x_j,r_j)\cap K)\subset B'(x_j',cr_j)$, where $B'(x_j',cr_j)$ denotes the intersection $\partial K\cap B(x_j',cr_j)$ and $x_j'=P(x_j)$. It follows that

$$A_t\subset\bigcup_{|x_j|<2t} B'(x_j',cr_j) \tag{7}$$

for large enough t. Furthermore, since $\partial\Gamma$ is smooth, we can also find another constant $c'>0$, such that $d\sigma\,(B'(x_j',cr_j))\leq c'r_j^{m-1}$ for all j.

Now we estimate II. For large enough t we have

$$\begin{aligned}\int_{A_t}|h_u^*(x+lx_*)|\,d\sigma(x)&\leq\sum_{|x_j|<2t}\int_{\Gamma_{t/2,t}\cap B'(x_j',cr_j)}|h_u^*(x+lx_*)|\,d\sigma(x)\\&\stackrel{\text{def}}{=}\sum_{|x_j|<2t}D_{tjl}.\end{aligned} \tag{8}$$

Suppose that $h_u^*(x+lx_*)\leq a|x|^\varrho$ for all $0<l<1$ and $x\in K\setminus B_1$. Take a continuous function ψ_1 on ∂K such that $0\leq\psi_1\leq 1$ on ∂K, $\operatorname{supp}\psi_1\subset\Gamma_{t/3,2t}$ and $\psi_1\equiv 1$ in $\Gamma_{t/2,t}$. Then for each j, by the definition of $\nu_{h_u^*}$, we have

$$\begin{aligned}D_{tjl}&\leq\int_{\Gamma_{t/2,t}}(at^\varrho-h_u^*(x+lx_*))\,d\sigma(x)+\int_{\Gamma_{t/2,t}}at^\varrho\,d\sigma(x)\\&\leq\int_{\Gamma_{t/3,2t}}\psi_1(x)(a2^\varrho t^\varrho-h_u^*(x+lx_*))\,d\sigma(x)+at^\varrho\,d\sigma(\Gamma_{t/2,t})\\&\longrightarrow\int_{\Gamma_{t/3,2t}}\psi_1(x)(a2^\varrho t^\varrho\,d\sigma(x)-d\nu_{h_u^*}(x))+at^\varrho\,d\sigma(\Gamma_{t/2,t}),\qquad\text{as } l\to+0.\end{aligned} \tag{9}$$

So for any fixed $t>2$, the integrals D_{tjl} are uniformly bounded for small enough l, and all j. Hence the Fatou lemma implies that

$$\limsup_{l\to+0}\sum_{|x_j|<2t} D_{tjl}\le\sum_{|x_j|<2t}\limsup_{l\to+0} D_{tjl}. \tag{10}$$

Choose again a continuous function ψ_2 on ∂K satisfying the conditions: $0\le\psi_2\le1$ on ∂K, $\operatorname{supp}\psi_2\subset B'(x_j',2cr_j)$ and $\psi_2\equiv1$ in $B'(x_j',cr_j)$. In analogy with (9), we get

$$\begin{aligned}\limsup_{l\to+0} D_{tjl}&\le(a2^\varrho t^\varrho\,d\sigma-\nu_{h_u^*})(\Gamma_{t/3,2t}\cap B'(x_j',2cr_j))\\&\quad+a2^\varrho t^\varrho\,d\sigma(\Gamma_{t/2,t}\cap B'(x_j',cr_j))\\&\le|\nu_{h_u^*}|(\Gamma_{t/3,2t}\cap B'(x_j',2cr_j))+a2^{\varrho+1}t^\varrho\,d\sigma(\Gamma_{t/3,2t}\cap B'(x_j',2cr_j))\end{aligned}\tag{11}$$

for each j. Together with (8) and (10), we have

$$\begin{aligned}\limsup_{l\to+0}\frac{1}{t^{\varrho+m-1}}\int_{A_t}|h_u^*(x+lx_*)|\,d\sigma(x)&\\=\frac{1}{t^{\varrho+m-1}}\sum_{|x_j|<2t}|\nu_{h_u^*}|(\Gamma_{t/3,2t}\cap B'(x_j',2cr_j))&\\+O\biggl(\frac{1}{t^{m-1}}\sum_{|x_j|<t}r_j^{m-1}\biggr),\qquad\text{as }t\to\infty.&\end{aligned}\tag{12}$$

Since $|\nu_{h_u^*}|(E)\le c_1\,d\sigma(E)$ for any subset $E\subset\Gamma_{0,1}$, and h_u^* is positively homogeneous of degree ϱ, we obtain

$$\frac{1}{t^{\varrho+m-1}}\sum_{|x_j|<2t}|\nu_{h_u^*}|(\Gamma_{t/3,2t}\cap B'(x_j',2cr_j))=O\biggl(\frac{1}{t^{m-1}}\sum_{|x_j|<2t}r_j^{m-1}\biggr),\qquad\text{as }t\to\infty.$$

It then follows from (6) and (12) that $II=0$.

Next we want to show $I=0$. Repeating the above process, we have

$$\begin{aligned}\limsup_{l\to+0}\frac{1}{t^{\varrho+m-1}}\int_{A_t}|u(x+lx_*)|\,d\sigma(x)&\\\le\limsup_{l\to+0}\frac{1}{t^{\varrho+m-1}}\sum_{|x_j|<2t}\int_{\Gamma_{t/2,t}\cap B'(x_j',cr_j)}|u(x+lx_*)|\,d\sigma(x)&\\=\frac{1}{t^{\varrho+m-1}}\sum_{|x_j|<2t}|\nu_u|(\Gamma_{t/3,2t}\cap B'(x_j',2cr_j))+o(1)&\\\overset{\text{def}}{=}\frac{1}{t^{\varrho+m-1}}\sum_{|x_j|<2t}G_{tj}+o(1),\qquad\text{as }t\to\infty,&\end{aligned}\tag{13}$$

and we can also choose, for any $\varepsilon>0$, a sequence $t_k\to\infty$ such that

$$(14)\qquad \frac{1}{t_k^{m-1}}\sum_{|x_j|<2t_k}\left(d\sigma+\frac{1}{t_k^{\varrho}}|\nu_{h_u^*}|\right)(\Gamma_{t_k/4,3t_k}\cap B'(x_j',3cr_j))\le\frac{\varepsilon}{2^k}\qquad\text{for all } k.$$

Clearly, for each k we have

$$(15)\qquad \begin{aligned}\frac{1}{t_k^{\varrho+m-1}}\sum_{|x_j|<2t_k}G_{t_kj}&=\sum_{|x_j|<2t_k}|\nu_{u_{t_k}}|\left(\Gamma_{1/3,2}\cap B'\left(\frac{x_j'}{t_k},\frac{2cr_j}{t_k}\right)\right)\\&\le\sum_{i=1}^{\infty}\sum_{|x_j|<2t_i}|\nu_{u_{t_k}}|\left(\Gamma_{1/3,2}\cap B'\left(\frac{x_j'}{t_i},\frac{2cr_j}{t_i}\right)\right).\end{aligned}$$

But in view of the lemma, all terms in the sum (15) are uniformly upper bounded for large enough k, and so the Fatou lemma implies that

$$(16)\qquad \begin{aligned}&\limsup_{k\to\infty}\frac{1}{t_k^{\varrho+m-1}}\sum_{|x_j|<2t_k}G_{t_kj}\\&\qquad\le\sum_{i=1}^{\infty}\sum_{|x_j|<2t_i}\limsup_{k\to\infty}|\nu_{u_{t_k}}|\left(\Gamma_{1/3,2}\cap B'\left(\frac{x_j'}{t_i},\frac{2cr_j}{t_i}\right)\right).\end{aligned}$$

If we can show that there exists a constant $c_2>0$ such that for each i and j we have

$$(17)\qquad \begin{aligned}&\limsup_{k\to\infty}|\nu_{u_{t_k}}|\left(\Gamma_{1/3,2}\cap B'\left(\frac{x_j'}{t_i},\frac{2cr_j}{t_i}\right)\right)\\&\qquad\le c_2(d\sigma+|\nu_{h_u^*}|)\left(\Gamma_{1/4,3}\cap B'\left(\frac{x_j'}{t_i},\frac{3cr_j}{t_i}\right)\right),\end{aligned}$$

it then follows from (14) and (15) that

$$\limsup_{k\to\infty}\frac{1}{t_k^{\varrho+m-1}}\sum_{|x_j|<2t_k}G_{t_kj}\le c_2\varepsilon.$$

This implies $I=0$, and hence the equality (3) holds.

Now we need to show (17). Let c_3 be a positive constant such that $u_r(x)\le c_3$ for all $x\in K_4$ and $r\ge1$. So $c_3\,d\sigma-\nu_{u_r}$ and $c_3\,d\sigma-\nu_{h_u^*}$ are positive measures in $\Gamma_{0,3}$. We choose a domain $G\subset B_3-B_{1/4}$ and a continuous function ψ_3 in $\overline{K}_3$ satisfying the following conditions:

(i) $\Gamma_{1/3,2}\cap B'(x_j'/t_i,2cr_j/t_i)\subset\overline{G}\cap\Gamma_{0,3}\subset\Gamma_{1/4,3}\cap B'(x_j'/t_i,3cr_j/t_i)$;

(ii) $\int_{G\cap K_3}\phi_1(x)\,d\mu_{h_u^*}(x)<d\sigma(\Gamma_{1/4,3}\cap B'(x_j'/t_i,3cr_j/t_i))$;

(iii) $0\le\psi_3\le1$ in $\overline{K}_3$, and $\psi_3(x)\equiv1$ in $\Gamma_{1/3,2}\cap B'(x_j'/t_i,2cr_j/t_i)$;

(iv) $\operatorname{supp}\psi_3\subset\overline{G\cap K_3}$.

Hence we have

$$|\nu_{u_{t_k}}|\left(\Gamma_{1/3,2}\cap B'\left(\frac{x'_j}{t_i},\frac{2cr_j}{t_i}\right)\right)\le\int_{\overline{G}\cap\Gamma_{0,3}}\psi_3(x)(c_3\,d\sigma(x)-d\nu_{u_{t_k}}(x)) \tag{18}$$
$$+c_3\,d\sigma\left(\Gamma_{1/3,2}\cap B'\left(\frac{x'_j}{t_i},\frac{2cr_j}{t_i}\right)\right).$$

Since the function ϕ_1 is homogeneous in K and $\partial\phi_1/\partial n$ is a positive continuous function on $\partial\Gamma$, there exists a constant $c_4>0$ such that the integral in (18) can be estimated by

(19)

$$c_4\,d\sigma(\overline{G}\cap\Gamma_{0,3})-c_4\int_{\overline{G}\cap\Gamma_{0,3}}\psi_3(x)\frac{1}{\theta_m}\frac{\partial\phi_1}{\partial n}\,d\nu_{u_{t_k}}(x)\le c_4\,d\sigma\left(\Gamma_{1/4,3}\cap B'\left(\frac{x'_j}{t_i},\frac{3cr_j}{t_i}\right)\right)$$
$$+c_4\int_{\overline{G\cap K_3}}\psi_3(x)\left(\phi_1(x)\,d\mu_{u_{t_k}}(x)-\frac{1}{\theta_m}\frac{\partial\phi_1}{\partial n}\,d\nu_{u_{t_k}}(x)\right).$$

Since u is CRG in K, it follows from the lemma that, when $k\to\infty$, the last integral tends to

(20)

$$\int_{\overline{G\cap K_3}}\psi_3(x)\left(\phi_1(x)\,d\mu_{h_u^*}(x)-\frac{1}{\theta_m}\frac{\partial\phi_1}{\partial n}\,d\nu_{h_u^*}(x)\right)\le d\sigma\left(\Gamma_{1/4,3}\cap B'\left(\frac{x'_j}{t_i},\frac{3cr_j}{t_i}\right)\right)$$
$$+\sup_{\overline{G}\cap\Gamma_{0,3}}\left(\frac{1}{\theta_m}\frac{\partial\phi_1}{\partial n}\right)|\nu_{h_u^*}|\left(\Gamma_{1/4,3}\cap B'\left(\frac{x'_j}{t_i},\frac{3cr_j}{t_i}\right)\right).$$

So (17) follows from (18)–(20), and therefore (3) holds. Hence we complete the proof of Theorem 2.

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