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Invariant subspaces in Bergman spaces and Hedenmalm's boundary value problem

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Abstract. A function G in a Bergman space A^p, $0<p<\infty$, in the unit disk D is called A^p-inner if $|G|^p-1$ annihilates all bounded harmonic functions in D. Extending a recent result by Hedenmalm for $p=2$, we show (Thm. 2) that the unique compactly-supported solution Φ of the problem

$$\Delta\Phi=\chi_D(|G|^p-1),$$

where χ_D denotes the characteristic function of D and G is an arbitrary A^p-inner function, is continuous in C, and, moreover, has a vanishing normal derivative in a weak sense on the unit circle. This allows us to extend all of Hedenmalm's results concerning the invariant subspaces in the Bergman space A^2 to a general A^p-setting.

1. Introduction

For $0<p<\infty$, the *Bergman space* $A^p(\mathbf{D})$, $\mathbf{D}=\{z:|z|<1\}$ consists of all functions f analytic in $\mathbf{D}$ for which

$$\|f\|_p^p:=\int_{\mathbf{D}}|f(z)|^p\,dA<\infty.$$

Here, dA is the area measure. As is well-known, $\|\ \|_p^p$ makes A^p into a Banach space for $1\le p<\infty$ and a complete metric space for $0<p<1$. A closed subspace $I\subset A^p$ is called an *invariant subspace* if $zf\in I$ for all $f\in I$. Let the function $G\in I$ be a solution of the extremal problem

$$\sup\{\operatorname{Re} g^{(m)}(0):g\in I,\ \|g\|_p\le 1\}, \tag{1.1}$$

where m is the order of the common zero at the origin for functions in I. For $p>1$, the existence of G is an easy corollary of Fatou's lemma and a normal family argument. For $p=1$ it follows from the well-known fact that A^1 can be identified with a dual of the little Bloch space (cf. [Z]). For $0<p<1$, we do not know whether the extremal function in (1.1) exists for the most general subspaces. However, if we in addition assume that the invariant subspace is *weakly* closed, i.e., $f_n \in I$, $\|f_n\|_p \le \text{const}$ and $f_n \to f$ uniformly on compact subsets of $\mathbf{D}$ imply that $f \in I$, then, as before, the existence of G for $p{:}0<p<1$ follows from Fatou's lemma and Montel's theorem. Note that all *zero subspaces*, i.e., $I=\{f\in A^p : f(\zeta_j)=0,\ j=1,\dots\}$, are weakly closed. (Here, $\{\zeta_j\}_1^\infty$ is a zero set of an A^p-function.) *Uniqueness* of G is known to hold for $1\le p<\infty$, while for an arbitrary I it remains an open problem for $0<p<1$ (cf. [DKSS1] , [DKSS2]). Let Φ denote a (distributional) solution in $\mathbf{R}^2$ of the problem

$$\Delta\Phi=\chi_{\mathbf{D}}(|G|^p-1), \tag{1.2}$$

where $\chi_{\mathbf{D}}$ is the characteristic function of $\mathbf{D}$. Problem (1.2) has been introduced by H. Hedenmalm in [H1] for $p=2$. Since a simple variational argument (cf. [H1], [DKSS1], [DKSS2]) shows that $|G|^p-1$ annihilates all bounded harmonic functions in $\mathbf{D}$ $(:=L_h^\infty)$, i.e., $\int_{\mathbf{D}}(|G|^p-1)u\,dA=0$ for all $u\in L_h^\infty$, one solution Φ of (1.2) has the integral representation

$$\Phi(z)=\frac{1}{2\pi}\int_{\mathbf{D}}(|G|^p-1)\log|z-\zeta|\,dA(\zeta). \tag{1.3}$$

Henceforth, Φ shall always denote that solution of (1.2) given by (1.3). Since $g:=|G|^p-1$ only belongs to $L^1(\mathbf{D})$, one cannot expect a priori anything more than $\Phi\in VMO(\mathbf{C})$—cf. [IK]. However, in [H1], for $p=2$, using Hilbert space techniques and explicit calculations with power series, Hedenmalm was able to show much more.

Theorem 1 ([H1], for $p=2$).

(i) *Φ is continuous in $\mathbf{C}$.*

(ii) *$\partial\Phi/\partial n=0$ weakly on $\mathbf{T}=\partial\mathbf{D}$, i.e.,*

$$\lim_{r\to1}\int_{r\mathbf{T}}\frac{\partial\Phi}{\partial n}s(z)\,d\sigma=0,$$

for any C^2-smooth test function $s(z)$. (Here, $\partial/\partial n$ is the outer normal derivative and $d\sigma$ is the arclength.)

(iii) *$\Phi=0$ in $\mathbf{C}\setminus\mathbf{D}$ and*

$$0\le\Phi\le\tfrac14(1-|z|^2)\quad \text{in } \overline{\mathbf{D}}.$$

From Theorem 1, by simply applying Green's formula, Hedenmalm obtained the following identity

$$\int_{\mathbf{D}}(|G|^2-1)|f|^2\,dA=4\int_{\mathbf{D}}\Phi|f'|^2\,dA \tag{1.4}$$

for any polynomial f. The Corollary, which immediately follows from (1.4), namely, that

$$\|Gf\|_2\geq\|f\|_2 \quad \text{for all } f\in H^\infty, \tag{1.5}$$

is crucial in Hedenmalm's construction of contractive zero-divisors in A_2. In [DKSS2] (also, cf. [DKSS1]), we have been able to circumvent the boundary problem (1.2) by proving instead of (1.4) the following ($0<p<\infty$):

$$\int_{\mathbf{D}}(|G|^p-1)|f|^p\,dA=\iint_{\mathbf{D}\times\mathbf{D}}\Delta_\zeta(|G|^p)\Delta_z(|f|^p)\Gamma(z,\zeta)\,dA_z\,dA_\zeta, \tag{1.6}$$

for all polynomials f, where Γ is the biharmonic Green kernel

$$\Gamma(z,\zeta):=\frac{1}{16\pi}\left\{|z-\zeta|^2\log\left|\frac{z-\zeta}{1-\bar\zeta z}\right|^2+(1-|z|^2)(1-|\zeta|^2)\right\}.$$

Since Γ is positive, by using (1.6) instead of (1.4) one can extend (1.5) to arbitrary $0<p<\infty$.

In this note we extend Hedenmalm's original approach via a boundary-value problem (1.2) and Theorem 1 to all p, $0<p<\infty$. A general proof of the A^p-version of Thm. 1 (Thm. 2) we offer here is still somewhat simpler than Hedenmalm's original proof for $p=2$ in [H1]. In the last section we give a number of corollaries, extending the results in [H1] to a general A^p-setting, and discuss some open problems.

2. Extension of Theorem 1 to A^p-spaces for $0<p<\infty$

Let us restate Thm. 1 in a general A^p-setting, adopting the concept of an *A^p-inner function* recently suggested by B. Korenblum (in view of (1.5)).

Definition. A function $G\in A^p$ is called A^p-inner (or simply, inner), if $|G|^p-1$ is orthogonal to all bounded harmonic functions in $\mathbf{D}$.

(Note that all extremal functions for problems (1.1) are inner.)

Theorem 2. *Let G be an A^p-inner function, $0<p<\infty$, and let Φ be a solution of* (1.2) *defined by* (1.3). *Then*

(i) *Φ is continuous in* $\mathbf{C}$.

(ii) *$\partial\Phi/\partial n=0$ weakly on* $\mathbf{T}$, *i.e.,*

$$\lim_{r\to 1}\int_{rT}\frac{\partial\Phi}{\partial n}s(z)\,d\sigma=0$$

for any C^2-smooth test function s.

(iii) *$\Phi=0$ in $\mathbf{C}\setminus\mathbf{D}$ and*

$$0\le\Phi\le\tfrac{1}{4}(1-|z|^2)\quad \textit{in } \overline{\mathbf{D}}.$$

Proof. The major difficulty (technical) lies in proving (i). So, let us assume (i) for a moment, and derive (ii) and the second inequality in (iii).

(ii) Fix $r<1$ and let $s_r(z)$ denote the solution of the Dirichlet problem for the Laplacian in $r\mathbf{D}$ with data s. Applying Green's formula in $r\mathbf{D}$ we obtain (Φ is obviously C^∞ inside $\mathbf{D}$—see (1.3)!)

$$\int_{r\mathbf{T}}\frac{\partial\Phi}{\partial n}s_r(z)\,d\sigma=\int_{r\mathbf{T}}\Phi\frac{\partial s_r(z)}{\partial n}\,d\sigma+\int_{r\mathbf{D}}g(z)s_r(z)\,dA(z)\tag{2.1}$$

(recall, $g:=|G|^p-1$). As $r\to 1$, the first term in the right-hand side of (2.1) tends to 0, since $\Phi=0$ on $\mathbf{T}$ (as $g=|G|^p-1\perp L^\infty_h$, $\Phi\equiv 0$ in $\mathbf{C}\setminus\mathbf{D}$), while $\partial s_r/\partial n$ remains bounded (in fact, it tends to $\partial s_1/\partial n$, where s_1 is a solution of the Dirichlet problem in $\mathbf{D}$ with data s). The second term tends to

$$\int_{\mathbf{D}}g(z)s_1(z)\,dA(z)=0,$$

since s_1 is harmonic in $\mathbf{D}$ and $g\perp L^\infty_h$. From this (ii) follows.

(iii) Consider $\psi=\frac{1}{4}(1-|z|^2)-\Phi$. By (i) $\psi\in C(\overline{\mathbf{D}})$, and

$$\Delta\psi=-1-(|G|^p-1)=-|G|^p<0$$

in $\mathbf{D}$. So ψ is superharmonic in $\mathbf{D}$, continuous in $\overline{\mathbf{D}}$, and $\psi|_{\mathbf{T}}=0$. Hence, $\psi\ge 0$ in $\mathbf{D}$ and the second inequality in (iii) follows.

To prove (i), we need a lemma.

Lemma. *The measure* $|g|\,dA$ *is a Carleson measure in* $\mathbf{D}$.

Proof. Since $g=|G|^p-1$ annihilates L_h^∞, $|G|^p\,dA$ is a representing measure for bounded harmonic functions at the origin. In particular, for

$$u=u_\lambda=\operatorname{Re}\left(\frac{1+\lambda z}{1-\lambda z}\right)=\frac{1-|\lambda|^2|z|^2}{|1-\lambda z|^2},$$

$\lambda\in\mathbf{D}$, we have

$$\int_{\mathbf{D}}|G|^p\frac{1-|\lambda|^2|z|^2}{|1-\lambda z|^2}\,dA(z)=1.$$

Hence (see, e.g., [G; p. 239, Lemma 3.3]), $|G|^p\,dA$ is a Carleson measure, and the lemma follows because $|g|\le|G|^p+1$.

Proof of (i). Since g is orthogonal to L_h^∞, taking any $a\in\mathbf{D}$ we can rewrite (1.3) in the form

$$\Phi(a)=-\frac{1}{2\pi}\int_{\mathbf{D}}g(\zeta)\log\left|\frac{a'-\zeta}{a-\zeta}\right|dA(\zeta),\tag{2.2}$$

where a', $|a'|>1$, lies on the ray joining 0 to a, and $|a'-a|=2(1-|a|)$. Set $w=(a'-\zeta)/(a-\zeta)$. Let

$$\begin{aligned}\Omega_a&=\{\zeta\in\mathbf{D}:|w-1|<\sqrt{1-|a|}\,\}=\left\{\zeta\in\mathbf{D}:\left|\frac{a'-\zeta}{a-\zeta}-1\right|<\sqrt{1-|a|}\right\}\\&=\{\zeta\in\mathbf{D}:|a-\zeta|>2\sqrt{1-|a|}\,\}.\end{aligned}$$

Then (cf. (2.2)),

$$-2\pi\Phi(a)=\int_{\Omega_a}+\int_{\mathbf{D}\setminus\Omega_a}.\tag{2.3}$$

Claim. $\left|\int_{\Omega_a}\right|\le\text{const}\sqrt{1-|a|}$, *and therefore, tends to* 0 *when* $|a|\to1$.

Indeed, $\left|\int_{\Omega_a}\right|\le\|g\|_{L^1}\|\log|w|\,\|_{L^\infty(\Omega_a)}$. Since

$$|\log|w|\,|=\left|\log\left|1+\left(\frac{a'-\zeta}{a-\zeta}-1\right)\right|\right|$$

and on Ω_a

$$\left|\frac{a'-\zeta}{a-\zeta}-1\right|<\sqrt{1-|a|}\to0\quad\text{when }|a|\to1,$$

we have for $\zeta\in\Omega_a$

$$|\log|w||=\left|\log\left|1+\left(\frac{a'-\zeta}{a-\zeta}-1\right)\right|\right|$$
$$=O\left(\left|\frac{a'-\zeta}{a-\zeta}-1\right|\right)\le O(\sqrt{1-|a|})$$

and the Claim follows. To estimate $\left|\int_{\mathbf{D}\setminus\Omega_a}\right|$ in (2.3), set

$$\Delta_a=\{\zeta:|\zeta-a|<(1-|a|)^3\}\subset\mathbf{D}\setminus\Omega_a.$$

Then,

$$\left|\int_{\mathbf{D}\setminus\Omega_a}\right|\le\left|\int_{\Delta_a}\right|+\left|\int_{\mathbf{D}\setminus\Omega_a\setminus\Delta_a}\right|. \tag{2.4}$$

Let $E_a=\mathbf{D}\setminus\Omega_a\setminus\Delta_a$. $|a-\zeta|\ge(1-|a|)^3$ on E_a, hence for $\zeta\in E_a$ we have

$$\left|\frac{a'-\zeta}{a-\zeta}\right|\le\frac{2(1-|a|)^{1/2}+2(1-|a|)}{(1-|a|)^3}\le\frac{4}{(1-|a|)^{5/2}}.$$

So,

$$\log\left|\frac{a'-\zeta}{a-\zeta}\right|\le C\log\frac{1}{1-|a|},$$

where C is a constant. Thus,

$$\left|\int_{E_a}\right|=\left|\int_{E_a}g(\zeta)\log\left|\frac{a'-\zeta}{a-\zeta}\right|dA\right|\le C\log\frac{1}{1-|a|}\left(\int_{E_a}|g(\zeta)|\,dA\right). \tag{2.5}$$

Clearly, E_a belongs to a Carleson square of size $C\sqrt{1-|a|}$, with some absolute constant C. So, from (2.5) and the Lemma it follows that

$$\left|\int_{E_a}\right|\le\text{const}\sqrt{1-|a|}\log\frac{1}{1-|a|}\to 0$$

when $|a|\to 1$. Finally, it remains to estimate $\left|\int_{\Delta_a}\right|$ in (2.4). For this, we need the assertion:

Assertion. $|g(\zeta)|=||G(\zeta)|^p-1|=O(1/(1-|\zeta|))$.

Assume the Assertion and estimate

$$\left|\int_{\Delta_a}g(\zeta)\log\left|\frac{a'-\zeta}{a-\zeta}\right|dA\right|.$$

$\Delta_a=\{\zeta:|\zeta-a|<(1-|a|)^3\}$, so $1-|\zeta|\geq\frac{1}{2}(1-a)$, since we can always assume $|a|\geq\frac{1}{2}$ in Δ_a. From the above assertion it follows then that

$$|g(\zeta)|\leq\frac{\text{const}}{1-|a|}\quad\text{in }\Delta_a. \tag{2.6}$$

Also, $|a'-\zeta|\geq 1-|a|$ for each $\zeta\in\Delta_a$. So,

$$|\log|a'-\zeta||=\log\frac{1}{|a'-\zeta|}\leq\log\frac{1}{1-|a|}\quad\text{on }\Delta_a. \tag{2.7}$$

Thus, from (2.6) and (2.7) we obtain

$$\begin{aligned}\left|\int_{\Delta_a} g(\zeta)\log\left|\frac{a'-\zeta}{a-\zeta}\right|dA\right| &\leq\frac{\text{const}}{1-|a|}\int_{\Delta_a}\left(\log\frac{1}{|a-\zeta|}+\log\frac{1}{1-|a|}\right)dA\\ &\leq\frac{\text{const}}{1-|a|}\left((1-|a|)^6\log\frac{1}{1-|a|}\right)\to 0\end{aligned}$$

as $|a|\to 1$. The last estimate follows from a direct calculation:

$$\begin{aligned}\int_{\Delta_a}\log\frac{1}{|a-\zeta|}\,dA &=\int_0^{2\pi}\int_0^{(1-|a|)^3}\log\frac{1}{\varrho}\,\varrho\,d\varrho\,d\theta\\ &=\pi\left[\varrho^2\log\frac{1}{\varrho}\Big|_0^{(1-|a|)^3}+\int_0^{(1-|a|)^3}\varrho\,d\varrho\right]\\ &\leq\text{const}\left[(1-|a|)^6\log\frac{1}{(1-|a|)^3}\right].\end{aligned}$$

Thus, (i) is proved modulo Assertion.

Proof of the Assertion. $|g(\zeta)|\leq|G|^p+1$, which is subharmonic, and by the Lemma $(|G|^p+1)\,dA$ is a Carleson measure. Let D_ζ be a Carleson box of size $C(1-|\zeta|)$, such that $D_\zeta\supseteq\{z:|z-\zeta|<1-|\zeta|\}$, C is a constant. Then the subharmonicity of $|G|^p+1$ and the Lemma imply

$$|g(\zeta)|\leq\frac{1}{\pi(1-|\zeta|)^2}\int_{D_\zeta}(|G|^p+1)\,dA\leq\frac{\text{const}}{(1-|\zeta|)^2}(1-|\zeta|)=\frac{\text{const}}{1-|\zeta|}.$$

Thus, (i) is proved.

Remark. Note that (iii) implies a better estimate of $\Phi(a)$ near $\mathbf{T}$ than the one we obtained in the above proof of (i). However, (i) is needed to establish (iii).

Finally, let us establish the remaining inequality in (iii) by showing that $\Phi \geq 0$ in $\mathbf{D}$. For that we need the key integration formula (1.6) proved in [DKSS2]. Note that in fact (1.6) holds for an arbitrary, say C^2-smooth, function s, not merely $|f|^p$ (cf. [DKSS2]). Let us rewrite (1.6) as follows ($s \in C^2(\overline{\mathbf{D}})$):

$$\int_{\mathbf{D}} (|G|^p - 1) s \, dA = \int_{\mathbf{D}} (\Delta\Phi) s \, dA = \int_{\mathbf{D}} \Delta s(\zeta) \left\{ \int_{\mathbf{D}} \Delta^2 \Phi(z) \Gamma(z, \zeta) \, dA_z \right\} dA_\zeta. \tag{2.8}$$

Now applying Green's formula to $r\mathbf{D}$, $0<r<1$, using (i) and (ii) of Theorem 2 and letting $r \to 1$, we obtain from (2.8):

$$\begin{aligned} \int_{\mathbf{D}} (\Delta\Phi) s \, dA &= \lim_{r\to 1} \int_{r\mathbf{D}} (\Delta\Phi) s \, dA \\ &= \lim_{r \to 1} \left\{ \int_{r\mathbf{D}} \Phi \Delta s \, dA + \int_{r\mathbf{T}} \left[\Phi \frac{\partial s}{\partial n} - s \frac{\partial \Phi}{\partial n} \right] d\sigma \right\} = \int_{\mathbf{D}} \Phi \Delta s \, dA. \end{aligned} \tag{2.9}$$

Hence from (2.8), (2.9), it follows that $\Phi(\zeta) - \int_{\mathbf{D}} \Delta^2 \Phi(z) \Gamma(z, \zeta) \, dA_z$ annihilates $\Delta s(\zeta)$, for all $s \in C_0^2(\mathbf{R}^2)$. But those functions (restricted to $\overline{\mathbf{D}}$) are obviously dense in $C(\overline{\mathbf{D}})$. Thus,

$$\Phi(\zeta) = \int_{\mathbf{D}} \Delta^2 \Phi(z) \Gamma(z, \zeta) \, dA_z \geq 0. \tag{2.10}$$

The proof of the Theorem is now complete.

3. Some corollaries and open questions

As above, let $I \subset A^p$ be an invariant subspace, and G $(=G_I)$, Φ $(=\Phi_I)$ be ·ed by (1.1) and (1.2). We can now (cf. (2.10)) rewrite (1.6) in the form

$$\|Gf\|_p^p = \|f\|_p^p + \int_{\mathbf{D}} \Phi \Delta(|f|^p) \, dA, \tag{3.1}$$

where f is a polynomial. As in [H1] for $p=2$, define *the space* $\mathcal{A}_0$ $(=\mathcal{A}_0^{I,p})$ *as the closure of the polynomials with respect to the norm*

$$\|f\|_{\mathcal{A}_0} = \left(\|f\|_{A^p}^p + \int_{\mathbf{D}} \Phi \Delta(|f|^p) \, dA \right)^{1/p}, \tag{3.2}$$

for $1 \leq p < \infty$. (That (3.2) is in fact a norm on $\mathcal{A}_0$ follows at once from (3.1).) For p:$0<p<1$, we define $\mathcal{A}_0$ similarly as the closure of polynomials with respect to the metric

$$d(f, g) = \|f-g\|_{A^p} + \int_{\mathbf{D}} \Phi \Delta(|f-g|^p) \, dA. \tag{3.3}$$

Furthermore, define the space

$$\mathcal{A}(=\mathcal{A}^{I,p}):=\left\{f\in A^p : \int_{\mathbf{D}} \Phi\Delta(|f|^p)\,dA<\infty\right\}.$$

Then, (3.1) yields the following Corollary (for $p=2$, cf. [H1, Cor. 4.2]).

Corollary 1. *Multiplication by G is an isometry of $\mathcal{A}_0$ into A^p.*

In view of Thm. 2(iii),

$$\int_{\mathbf{D}} \Phi\Delta(|f|^p)\,dA\le\frac{1}{4}\int_{\mathbf{D}}(1-|z|^2)\Delta(|f|^p)\,dA \tag{3.4}$$

for all polynomials f. As is well-known, the right-hand side of (3.4) is equivalent to the H^p-norm in $H^p/\mathbf{C}$. (To see this, it suffices to note that $(1-|z|^2)\sim\log(1/|z|)$ near $\mathbf{T}$, replace $\frac{1}{4}(1-|z|^2)$ by const$\cdot\log(1/|z|)$ in the right integral in (3.4), and apply Green's formula.) Thus, the right-hand side of (3.4) is finite for all $f\in H^p$ (i.e., $H^p\subset\mathcal{A}_0$), and we have the following result.

Corollary 2. *For any invariant subspace $I\subset A^p$, G_I is a bounded multiplier of $H^p(\mathbf{D})$ into A^p. In particular, $G=G_I$ satisfies in $\mathbf{D}$ the estimate*

$$|G(z)|\le\text{const}(1-|z|)^{-1/p}, \tag{3.5}$$

i.e., G has more severe growth restrictions than an arbitrary A^p-function f, which is only known to satisfy $|f(z)|\le\text{const}(1-|z|)^{-2/p}$.

Remark. The estimate (3.5), of course, also follows directly from the Assertion in the proof of Thm. 2.

Fix an A^p-inner function G and let $I(G)$ denote the A^p-closure of the polynomial multiples of G. Clearly, $I(G)\subset I$.

Corollary 3. *$I(G)=G\cdot\mathcal{A}_0$ (i.e., $I(G)$ is an (isometric) image of $\mathcal{A}_0$ in A^p under multiplication by G), and $G=G_{I(G)}$, i.e., it is the unique extremal function for $I(G)$ with respect to* (1.1).

Proof. Let $g\in I(G)$, i.e., there is a sequence of polynomials $\{f_n\}$ such that $Gf_n\xrightarrow{A_p}g$. Then $\{f_n\}$ is a Cauchy sequence in $\mathcal{A}_0$ (cf. (3.1) or (3.2)), and hence $f_n\xrightarrow{\mathcal{A}_0}f$. But f_n also converges to g/G pointwise in $\mathbf{D}$. Hence, $f=g/G$ and $\|f\|_{\mathcal{A}_0}=\|Gf\|_{A^p}=\|g\|_{A^p}$. So, $I(G)\subset G\mathcal{A}_0$. Conversely, if $\{f_n\}$ are polynomials and $f_n\xrightarrow{\mathcal{A}_0}f$, then $\{Gf_n\}$ is a Cauchy sequence in A^p and $\{Gf_n\}$ converges pointwise to Gf.

Hence, $Gf_n \xrightarrow{A^p} Gf$, so $G \cdot \mathcal{A}_0 \subset I(G)$. To show that G is extremal, simply note that for any polynomial q we have, in view of (3.1), $\|Gq\|_p \geq \|q\|_p$. Hence,

$$\frac{|(Gq)(0)|}{\|Gq\|_p} \leq |G(0)| \frac{|q(0)|}{\|q\|_p} \leq G(0)$$

($|q|^p$ is a subharmonic function!). Moreover, since $\|q\|_p = \|Gq/G\|_p \leq \|Gq\|_p$, for all polynomials q, G is a *contractive divisor* for $I(G)$, and therefore is the *unique* solution of the extremal problem (1.1). Indeed, suppose H is another solution. Then,

$$1 = \left|\frac{H(0)}{G(0)}\right| \leq \left\|\frac{H}{G}\right\|_p \leq \|H\|_p = 1.$$

Since $|H/G|^p$ is subharmonic in $\mathbf{D}$ it is a constant, and hence, $H=G$.

One of the most celebrated results in the Hardy space theory is Beurling's Theorem on invariant subspaces. In the present context it can be stated as follows: *every invariant subspace $I \subset H^p$ has the form $I=I(G)$, where G is a solution of the extremal problem* (1.1) (posed, of course, with respect to the H^p-metric). Unfortunately, the direct analogue of Beurling's Theorem cannot hold in A^p for the following reason. Every invariant subspace I of type $I=I(G)$ has the *so-called codimension* 1 *property*: $\dim(I/zI)=1$ (cf. [R]). (Indeed, if $I \ni F = \lim_n Gf_n$, where f_n are polynomials, then $f_n(0)$ converges to some complex number c and $f = \lim_n Gf_n = \lim_{n\to\infty} G[f_n - f_n(0)] + cG$, where $G[f_n - f_n(0)] \in zI$.) On the other hand, in [BFP] it was shown that for any integer $n \geq 0$ there exists an invariant subspace $I \subset A^2$, such that $\dim(I/zI)=n$. Recently, much simpler, constructive examples of such subspaces have been given by Hedenmalm [H2]. Nevertheless, for zero-invariant subspaces there is a good chance that a Beurling-type theorem does hold.

Corollary 4. *Let $I = \{f \in A^p : f(\zeta_j)=0, j=1,\ldots\}$, where $\{\zeta_j\}$ is a zero-set of an A^p-function and $G=G_I$ be the corresponding extremal function. Then, $I = G \cdot \mathcal{A}$.*

Proof. Let $g \in I$. It follows from results in [DKSS1], [DKSS2] that $g=Gh$, $h \in A^p$. As before, denote by G_n the extremal function (1.1) for the "cut-off" subspace $I_n := \{f \in A^p : f(\zeta_j)=0, j=1,\ldots,n\}$. Let $f_n = g/G_n$. We know that $f_n \in A^p$, $G_n \xrightarrow{A^p} G$, and hence, $f_n \to h$ pointwise in $\mathbf{D}$. Moreover, since all G_n's are analytic across $\partial\mathbf{D}$ ([DKSS1], [DKSS2]), the corresponding functions Φ_n defined by (1.3) are real analytic across $\partial\mathbf{D}$, and hence (3.1), with G_n, Φ_n, holds for all $f \in A^p$! So,

$$\|g\|_{A^p}^p = \|G_n f_n\|_{A^p}^p = \|f_n\|_{A^p}^p + \int_{\mathbf{D}} \Phi_n \Delta(|f_n|^p)\, dA. \tag{3.6}$$

Now, since $|G_n|^p-1\to|G|^p-1$ in $L^1(\mathbf{D})$, Φ_n (defined in accordance with (1.3)) tend to Φ in $L^1(\mathbf{D})$. (In fact, looking over the proof of Thm. 2 in Section 2, it is easy to see that $\Phi_n\to\Phi$ uniformly in $\mathbf{D}$.) Therefore, we can assume that $\Phi_n\to\Phi$ pointwise in $\mathbf{D}$. Thus, since $f_n\to h$ uniformly on compact subsets in $\mathbf{D}$, applying Fatou's lemma to (3.6) we obtain

$$\int_{\mathbf{D}}\Phi\Delta(|h|^p)\,dA\le\varliminf\int_{\mathbf{D}}\Phi_n\Delta(|f_n|^p)\,dA\le\|g\|_p^p,$$

i.e., $h\in\mathcal{A}$.

The following question then, is crucial.

Question. *Is $\mathcal{A}_0=\mathcal{A}$?*

If so, the Corollaries 3 and 4 imply the following.

Conjecture. *If $I\subset A^p$ is an invariant subspace defined by zeros, then $I=I(G)$, where $G=G_I$ is the solution of* (1.1).

The technical problem of extending the p-analogue of (1.5) to all $f\in A^p$ is of fundamental importance: for I being a *zero subspace*, this has been done in [DKSS1], [DKSS2]. For arbitrary invariant subspaces, the question is still open. It is not hard to see that (1.5) can easily be violated if we allow f to be *any* holomorphic function in $\mathbf{D}$. Indeed, let I be the closed subspace in, say, A^2, generated by the polynomial multiples of the inner function $\varphi=\exp((z+1)/(z-1))$. Then, it is easy to show (cf. [Sh]) that I is a *proper* subspace, and, moreover, all $f\in I$ decay exponentially along the radius. Thus, in particular, this holds for $G=G_I$, the extremal function in (1.1). Hence, $G^{-1}\notin A^2$ since it is well-known that the A^2-functions satisfy the (trivial) growth estimate $|f(z)|\le\|f\|_2(1-|z|)^{-1}$. On the other hand, $\|GG^{-1}\|_2=\|1\|_2=\pi<\|G^{-1}\|_2=\infty$. Nevertheless, the following Corollary shows that a counterexample to (1.5), if it exists, may be quite difficult to construct. Let $I\subset A^p$, $G=G_I$, be as above.

Corollary 5. (1.5) *holds for all $f\in N^+$.*

The Corollary follows at once from (3.1), the monotone convergence theorem, and the following simple (but important) Proposition due to V. I. Smirnov [S]. (For the definition and properties of the Smirnov class N^+, see, e.g., [D].)

Proposition. *For every $f\in N^+$, there exists a sequence of H^∞-functions $\{f_n\}$ such that $f_n\to f$ pointwise in $\mathbf{D}$, while $|f_n|\uparrow|f|$. Conversely, if f is a pointwise limit of bounded analytic functions with increasing moduli, then $f\in N^+$.*

For the reader's convenience we include a proof.

Proof of the Proposition. Since $f \in N^+$, it can be written as

$$f(z) = h(z) \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log^+ |f(e^{i\theta})| \, d\theta\right),$$

where $|h| \le 1$ in $\mathbf{D}$. Set

$$f_n(z) = h(z) \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} [\log^+ |f(e^{i\theta})|]^n \, d\theta\right),$$

where

$$[g]^n := \begin{cases} g, & g \le n \\ n, & g > n. \end{cases}$$

is the truncated function, and the assertion follows.

To prove the converse, first note that if $f = \lim f_n$, $f_n \in H^\infty$, where convergence is pointwise and $\{|f_n(z)|\}$ increases with n for each z, then $f = f_n/(f_n/f)$ is a quotient of two bounded functions, and hence belongs to the Nevanlinna class N. Let $f = S_1 F/S_2$ be a canonical factorization of f, where S_1, S_2 are inner functions (in the H^p-sense, of course) and F is an outer function. Since $|f| = |F|$ almost everywhere on $\mathbf{T}$ and $|f(e^{i\theta})| = \lim_{r \to 1} |f(re^{i\theta})|$ for almost all θ while $|f(re^{i\theta})| \ge |f_n(re^{i\theta})|$ for all n, it follows that $|f| \ge |f_n|$ almost everywhere on $\mathbf{T}$ for all n. Let F_n denote the outer part of f_n. Then, $|f| = |F| \ge |F_n|$ almost everywhere on $\mathbf{T}$ for all n. Now for a fixed $z = re^{i\theta}$ in $\mathbf{D}$ we have ($f_n \in H^\infty$!):

$$\begin{aligned}
\log |f(re^{i\theta})| &= \lim_{n \to \infty} \log |f_n(re^{i\theta})| \le \lim_{n \to \infty} \log |F_n(re^{i\theta})| \\
&= \lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)} \log |F_n(e^{i\varphi})| \, d\varphi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)} \log |F(e^{i\varphi})| \, d\varphi = \log |F(re^{i\theta})|.
\end{aligned}$$

Hence, $|f| \le |F|$ in $\mathbf{D}$ and so $|S_1/S_2| \le 1$. Thus, $S_1/S_2 \in H^\infty$, and therefore $S_2 \equiv \text{const}$.

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