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An algorithm that changes the companion graphs

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Introduction

Knot theory is a rich subject because of its many readily available examples. It has undergone dramatic changes during the last 12 years. A connection between knot theory and graph theory has been established by Reidemeister [R]. Graphs of knots (links) have been repeatedly employed in knot theory [Au], [C] and [KT]. In the recent past L. H. Kauffman [K] has established that "Universes of knots (links) are in one-to-one correspondence with planar graphs". In the proof, he has beautifully given the method of constructing corresponding universe from a given graph. With the introduction of LR-Graphs, one can easily extend the one-to-one correspondence to knots (linked links). The pivotal moves in the theory of knots are the Reidemeister moves. I will view these moves as Reidemeister moves of type I, type II and type III as shown in Figure 1.

Graph theoretic versions of Reidemeister moves has been discussed in detail by Azram [Az2]. Yajima–Kinoshita [YK] have shown that the companion graphs corresponding to a projection are equivalent. I myself have shown via Reidemeister moves that the graph corresponding to black (white) regions of a prime knot as well as that of composite knot is equivalent to its companion [Az1].

Having equivalence of companion graphs, construction (ahead) of the companion graph of a given connected planar LR-Graph and graphic versions of Reidemeister moves, it is natural to consider how one can change a given graph to its companion and vice versa by applying the graphic moves corresponding to the basic Reidemeister moves and without reference of the knot (link).

This article is devoted to write an algorithm that changes the graph corresponding to the black (white) regions of an altenating knot (linked link) with all labels 'L' (or 'R') to its companion graph. Consequently, by constructing the corresponding

Type I:

Type II:

Type III:

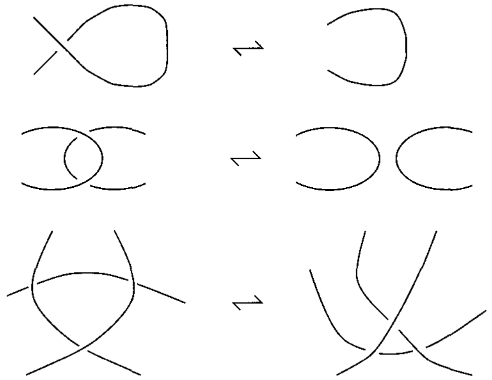


Figure 1.

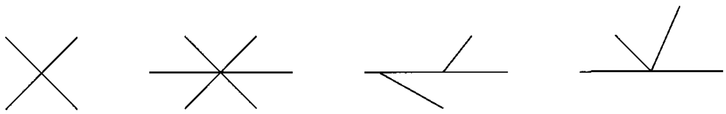
knots (linked links) one has a systematic rather than hit and trial approach of changing an alternating knot (linked link) to its mirror image. If the companion graph turns out to be isomorphic with the given graph then obviously the corresponding knot (linked link) is achiral.

Basic background

A knot is an isotopy class of the embedding of the unit circle in R^3. A link with n components is an isotopy class of the embedding of a collection of n disjoint circles in R^3. If one considers a representation of a link, i.e., an embedding of n disjoint circles in R^3, then an individual simple closed curve of the representation is called a component of a link. A link of just one component is a knot. It is tacitly assumed that the closed curves are piecewise linear, i.e., they consist of a finite number (possibly very large) of straight line segments placed end to end. This is a technical restriction which merely avoids wild knots and links [CF] or [FA]. Restricting the components to being differentiable would do equally well.

A projection of a knot (link) is simply a diagram obtained by mapping the knot (link) under orthogonal projection of R^3 onto R^2 in R^3. The direction of the projection is always chosen so that when the projection of two distinct parts of the knot (link) meet in R^2 they do so transversally at a crossing point, i.e., as in Figure 2(a) and never as in Figure 2(b), (c) or (d).

At a crossing an indication of which of the two arcs corresponds to the upper string and which to the lower string can be given by breaking the line corresponding to the lower string at the crossing. In the future, knots (links) will be confused with

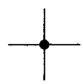


(a) (b) (c) (d)

Figure 2.

their class of projections with crossings indicated unless otherwise stated.

A knot (link) diagram can be considered as a planar graph with 4-valent vertices:



We call such a planar graph the universe of a knot (link). At the points corresponding to double points of a given projection of a knot (link), the point with greater 'z' coordinate is an overcrossing and the point with smaller 'z' coordinate is an undercrossing. An alternating knot (link) is a knot (link) where the crossings alternate under-over-under-over, etc. as one travels along each component af the knot (link), (crossing at all crossing). Two knots (links) are equivalent (via Reidemeister moves) if and only if (any of) their projections differ by a finite sequence of Reidemeister moves. We will denote this equivalence by $\sim$, i.e., $K_1 \sim K_2$ means, K_1 is equivalent to K_2 via Reidemeister moves. If in a given knot (link), all of the crossings are reversed, i.e., overcrossings changed to undercrossings and vice versa, then the resulting knot (link) is called its mirror image. A knot (link) is said to be achiral if it is equivalent (via Reidemeister moves) to its mirror image. Let K be a diagram resulting as a projection of a knot (link). Let K^* be its universe. By a region of K, we mean the correponding region of K^*, which is a maximal portion of the plane for which any two points may be joined by a curve such that each point of the curve neither corresponds to a vertex of K^* (view K^* as a planar graph) nor lies on any curve corresponding to an edge of K^*.

A link will be called unlinked if it is equivalent to a link whose projection contains at least two non-empty parts which are contained in disjoint simply connected subsets of the plane, otherwise we will call it a linked link. A knot is an unknotted knot if it is equivalent to a knot that has a projection with no crossings, otherwise it is knotted. A prime knot is a knotted knot which can not be expressed as a sum (Figure 3) of two knotted knots. A composite knot is a knotted knot which is not a prime knot. Note that the sum of two links of more than one component is not well defined unless it is specified which two components are to be banded together.

For a given diagram of a knot (link), shade (checker board shading) its regions

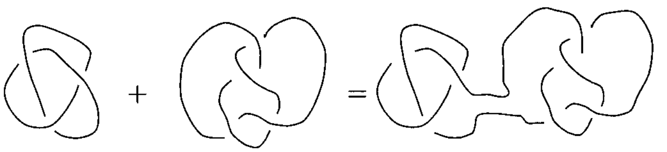


Figure 3.

[R2] white and black so that the unbounded region is white. We will refer to the shaded regions as the black regions and the unshaded regions as the white regions. Now associate a pseudograph to the knot (link) so that the vertices of the graph correspond to the black regions and the edges correspond to the crossing shared by the black regions. We will call such a graph as the graph corresponding to the black regions of the knot (link). Construction of the graph corresponding to the white regions is similar. The LR-Graph corresponding to the black (white) regions of a knot (link) is the graph whose edges are labelled as 'L' or 'R' depending on whether the upper string at the corresponding crossing falls on the left or on the right side when going from either black (white) region to the other black (white) region.

Let G be the graph corresponding to the black (white) regions of a given knot (link). By the companion of G, we mean the graph corresponding to the white (black) regions of the same knot (link). If the given connected loopless bridgeless planar graph corresponds to the black regions of a knot (link) then to construct its companion graph, all one needs to consider is that the vertices of the required graph correspond to the regions of the given graph. The edges correspond to the edges shared by the regions of the given graph and the labelling of the edges is just opposite to the labelling of the corresponding edge in the given graph. Alternatively, one can also construct the companion graph of a given graph as follow.

Fix one of the boundary vertices of the given graph. Consider the other vertices corresponding to the bounded regions of the given graph and then repeat the above construction while considering the fixed vertex as the vertex that corresponds to the unbounded region. It is straightforward to observe that the graphs constructed by these constructions are isomorphic. It may be noted that, if the given connected loopless bridgeless planar graph corresponds to the white regions of a knot (link), then the construction of the correponding companion graph is the same as discussed above.

If in the above construction, the given graph has a loop and/or bridge, then the companion graph can be constructed as follows;

(1) Ignoring the loops/bridges, construct the companion as discussed earlier.

(2) Suppose there is a loop at say vertex V_i. Join that part of the graph enclosed by the loop to the part of the graph incident at V_i by a bridge and label this bridge opposite to the labelling of the loop. Now, change the graph at both

ends of this bridge to its companion. The same can be repeated to all the loops.

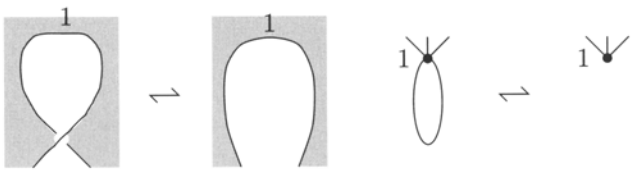
(3) Let there be a bridge adjacent to vertices V_i and V_j of a given graph. Change the graph at either end of this bridge to the companion and then contract V_i on V_j or vice versa and then place an oppositely labelled loop at $V_i = V_j$ enclosing one of the parts of the graph that was incident to vertex V_i or V_j before the contraction.

The following are the graphic moves corresponding to the Reidemeister moves studied on the black regions. For more details and observations, see [Az2].

Type I move Graphic move

Case I:

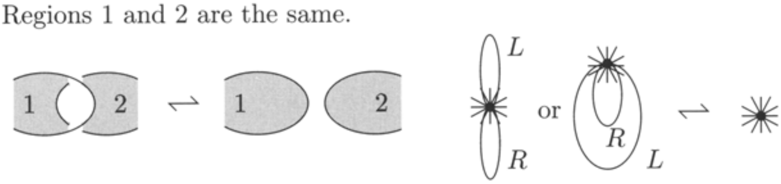
Case II:



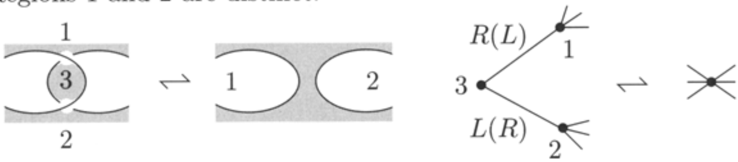
Type II move Graphic move

Case I: Regions 1 and 2 are distinct.

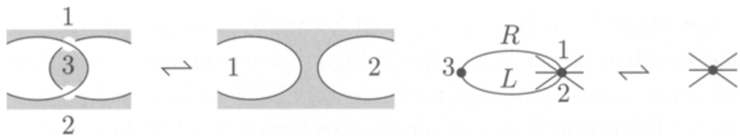
Case II: Regions 1 and 2 are the same.



Case III: Regions 1 and 2 are distinct.



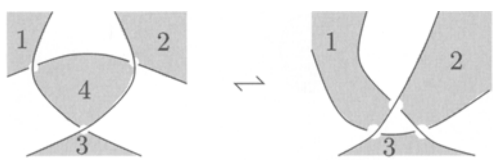
Case IV: Regions 1 and 2 are the same.



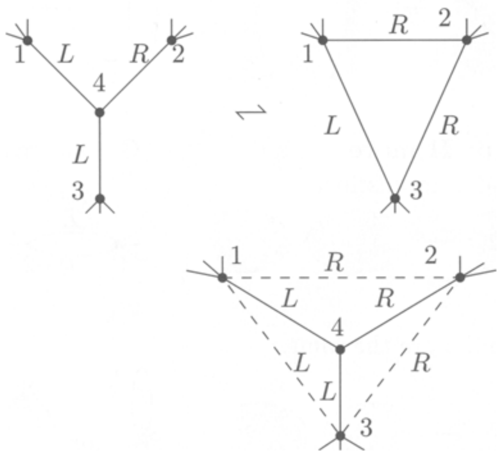
Type III move

Case I: Regions 1, 2, and 3 are distinct.

Type III move

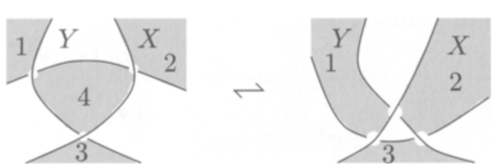


Graphic move

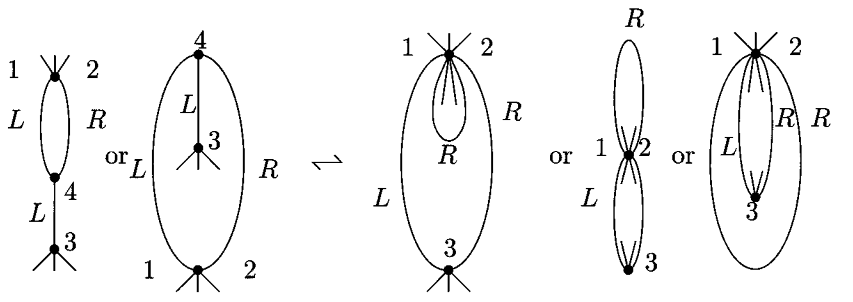


Case II: Two of the regions 1, 2, and 3 are the same.

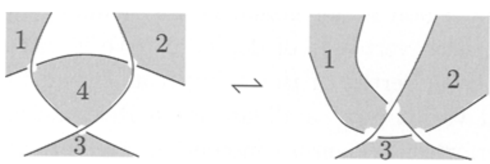
Type III move



Graphic move



Case III: Regions 1, 2, and 3 are the same.



Graphic move

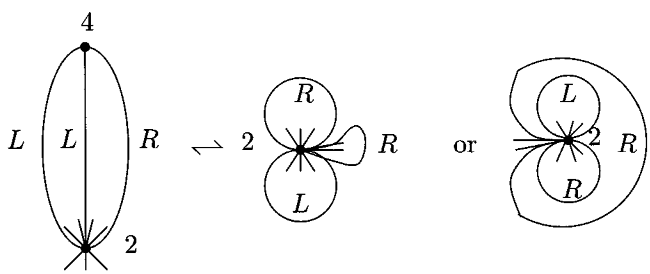


Figure 4.

Algorithm

In the next few pages an algorithm will be given that changes the graph corresponding to the black (white) regions of an alternating knot (linked link) with all labels 'L' (or 'R') to its companion. Before applying any graphic move, one may construct the companion of the given graph and can select either the graph or its companion to be changed to the other. The one with a smaller number of bounded regions will be preferred. This is not necessary but it helps to reduce the work.

Case I

The given graph is loopless and bridgeless.

Subcase I

The given graph is loopless, bridgeless and has no cut-vertex, i.e., it is a single loopless block.

Step I

Without loss of generality, let all the edges be labelled as 'R'. Note that a region bounded by only two edges will not be considered as a polygonal region. Among all the regions of the given graph that share their boundaries with the unbounded region, choose the one which shares the greatest number of edges with the unbounded region. In the case of more than one such region, choose any one.

Step II

(a) If the selected region is polygonal, i.e., is bounded by a polygon with at least three edges and each vertex is of degree 2. We introduce an edge labelled 'L' incident to any of the vertex of the polygon as shown in Figure 5(a). As a consequence, the said vertex is a candidate for a Reidemeister move of type III (graphic version). Perform it. Then a consecutive vertex is now a candidate for a similar move. Continuing consecutively, one arrives at the required graph.

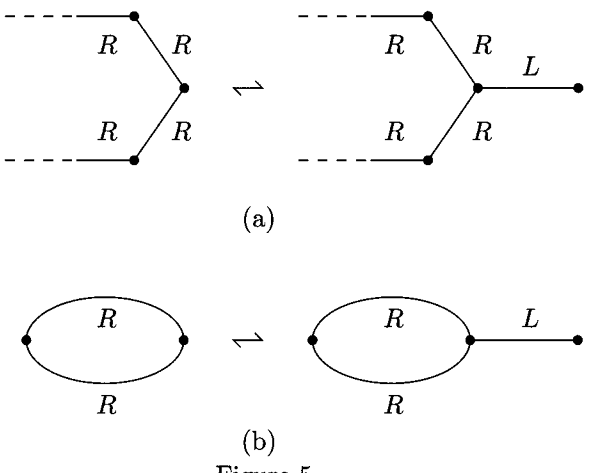


Figure 5.

(b) If the selected region is like the left part of Figure 5(b) then introduce an edge labelled L as shown in that figure, and then perform a move of type III, and then a move of type I to get the required graph.

(c) If the selected region is as shown in Figure 6(a) where the number of edges bounding it can be as small as 2, we let i be a vertex of degree greater than 2 that is

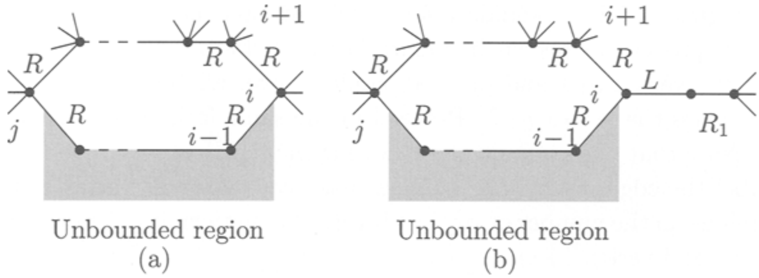


Figure 6.

adjacent to a vertex of degree 2 (if $i-j$ path consists of more than one edge shared with the unbounded region, otherwise, pick either of the vertices incident to the edge in common with the unbounded region) in the selected region. Now, perform a Reidemeister move of type II at vertex i as shown in Figure 6(b), where the edge labelled R_1 means that it is labelled as R but is a result of the very first move of this nature. Performing a Reidemeister move of type III at vertex i, the result is shown in Figure 7(a).

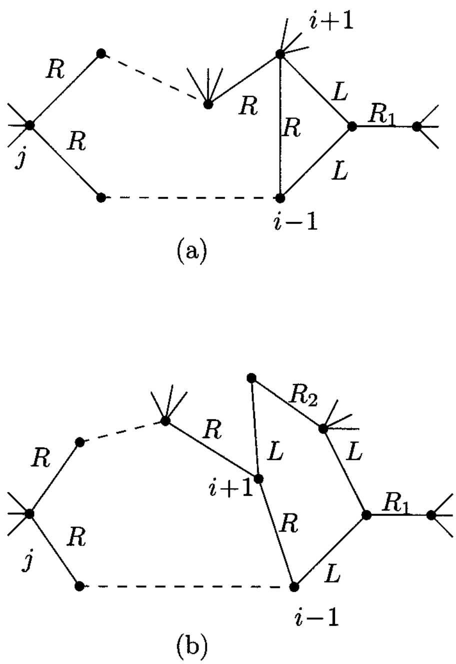


Figure 7.

If the vertex $i+1$ is a candidate for a Reidemeister move of type III, perform it; otherwise, proceed as shown in Figure 7(b). Perform a Reidemeister move of type III at the vertex $i+1$ and proceed similarly over all the consecutive vertices until one reaches the vertex $i-2$. Proceed in the same fashion at vertex $i-2$ and then stop. Note that one must do the following in step II(c).

(i) Label the edges as $R_1, R_2, ..., R_n$ as has been shown at vertex i and vertex $i+1$ (n depends on the number of vertices involved in performing the move of type II as at vertex i and vertex $i+1$).

(ii) While performing the moves, especially of type III at a vertex, if it results in any of the situations shown in Figure 8, perform the required move before proceeding to the next vertex.

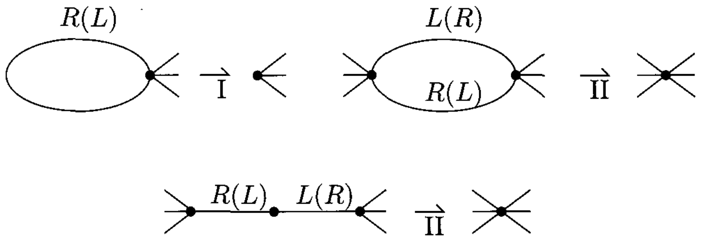


Figure 8.

(iii) If, during step II(c), any vertex turns out to be a vertex of degree 2 with both incident edges labelled R then perform a move of type III at that vertex by introducing an edge labelled L incident to it as is done in step II(a).

Step III

Choose the bounded region whose bounding edges have the edge labelled R_1 in it. If no such region exists or R_1 was eliminated during step II, or R_1 is the only bounding edge labelled R, then proceed to edge R_2 and so forth. Repeat step II(c) for this region starting at the vertex incident to R_1 with all the other incident edges labelled as R till the last vertex which is incident to a boundary edge of this region and is labelled as R. Let $R_{11}, R_{12}, ..., R_{1a_1}$ be the resulting edges like $R_1, R_2, ..., R_n$ in step II(c), where $a_1 < \infty$. Continue similarly to $R_2, ..., R_n$ and label the resulting edges as

$$\begin{array}{cccc} R_{21}, & R_{22}, & ..., & R_{2a_2}. \\ R_{31}, & R_{32}, & ..., & R_{3a_3}. \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ R_{n1}, & R_{n2}, & ..., & R_{na_n}. \end{array}$$

Step IV

Repeat step II(c) and step III until all the edges labelled R are changed into the edges labelled L. Note that in the final stages one may have;

(i) A vertex of degree 3 that itself is a candidate for a move of type III. If so perform it.

(ii) A vertex of degree 2 with both incident edges labelled as R. If so, proceed as in step II(iii).

(iii) A vertex of degree 2 with oppositely labelled edges. If so, perform a move of type II.

Subcase II

The given graph has no loop or bridge but has cut-vertices.

Let the vertex V be a cut-vertex of the given graph. Change each block of the graph incident to vertex V to its companion as has been done in subcase I, i.e., apply the algorithm of subcase I consecutively to each individual block at vertex V. If there is more than one cut-vertex, proceed similarly at each one.

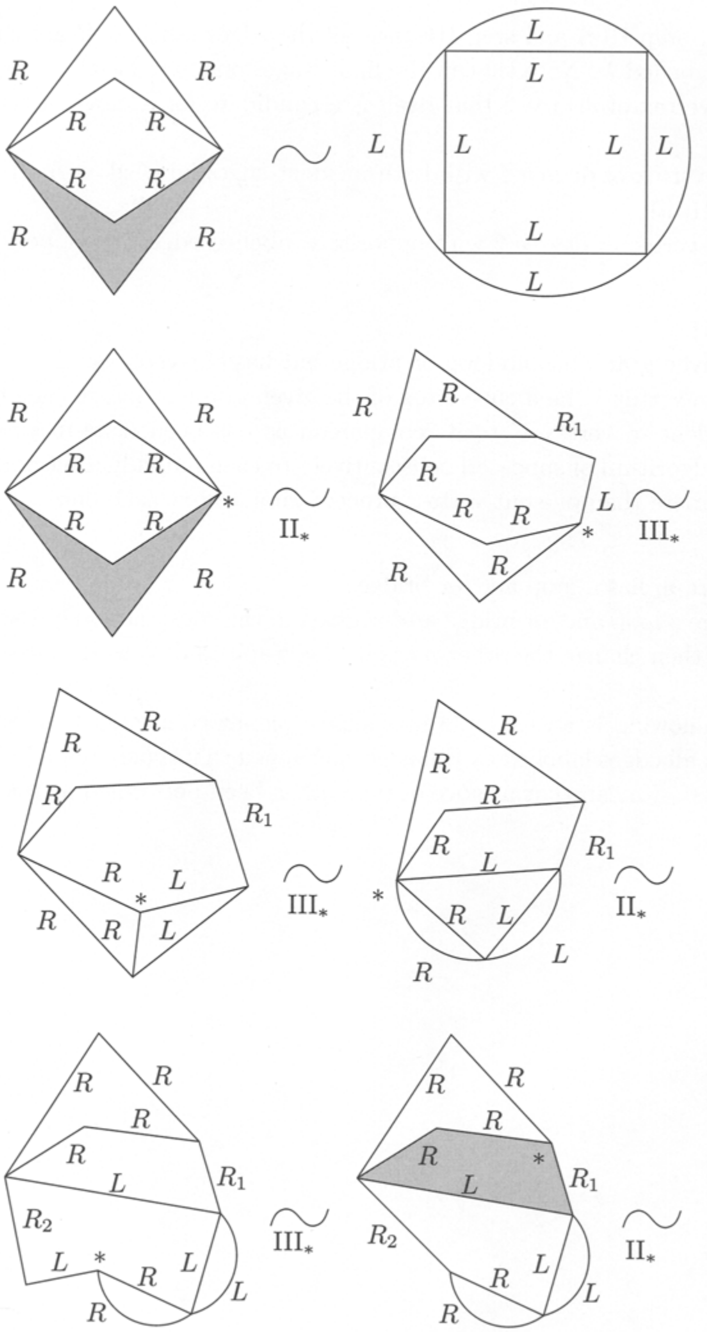
Case II

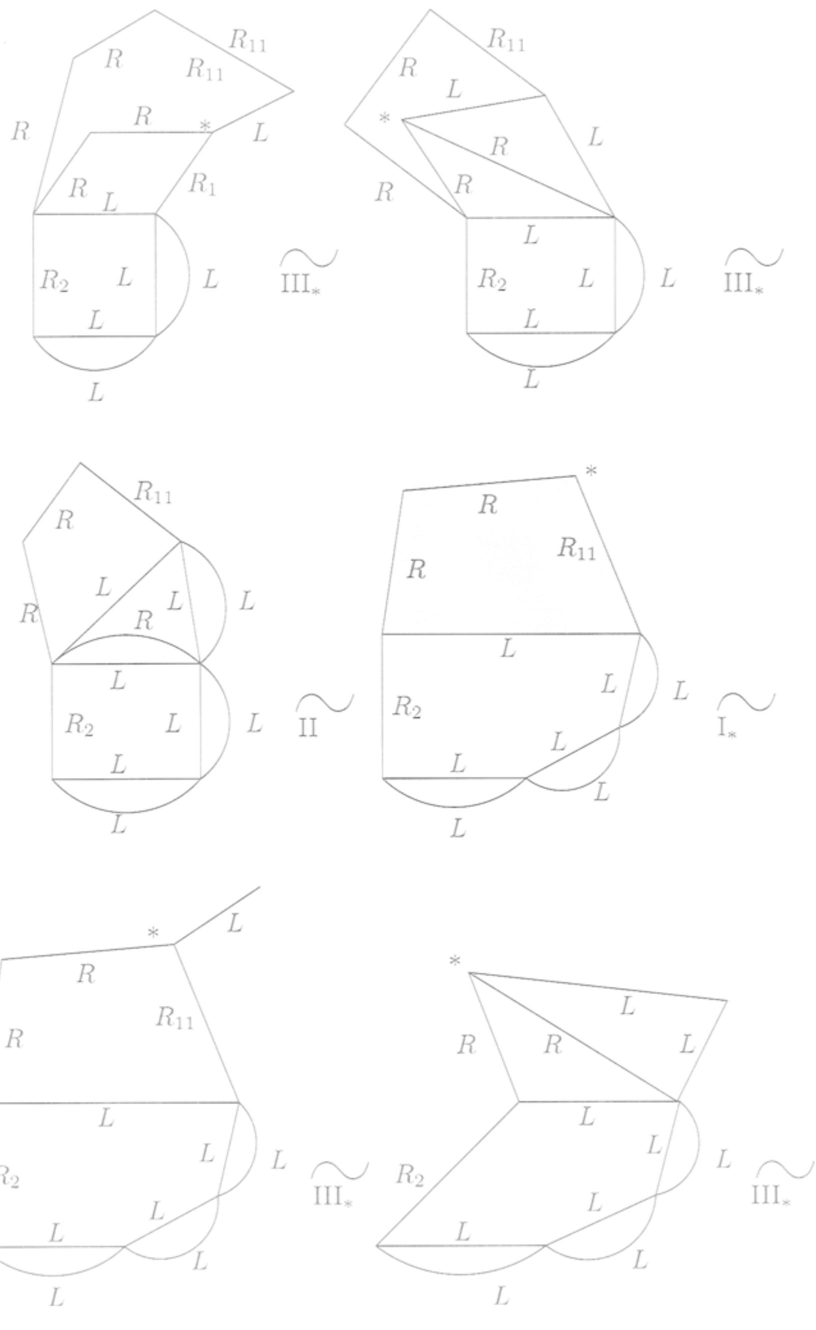
The graph has a loop and/or bridge.

Change a loop and/or bridge as discussed in the construction of the companion graph and then change the other parts of the graph as discussed in case I (subcase I and II).

The following is an example in which a planar connected loopless bridgeless graph with all edges labelled as R has been changed to its companion by the graphic moves. By $\tilde{a}_*$ I mean that a move of type a has been performed at the $*$ location.

Example





III*
III*
II
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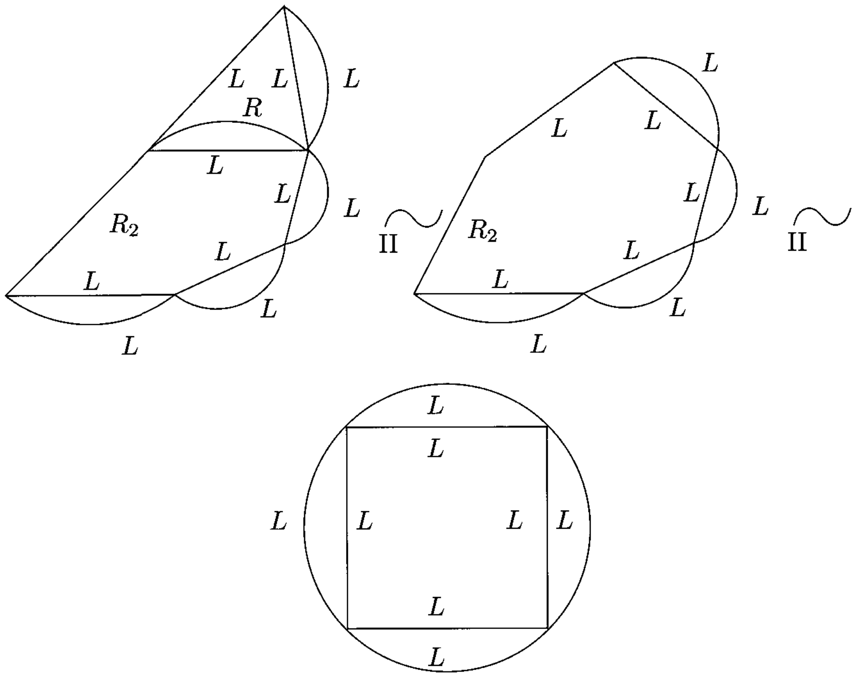
L
R
R_2
II

References

[Au] AUMANN, R. J., Asphericity of alternating knots, *Ann. of Math.* **64** (1956), 374–392.

[Az1] AZRAM, M., Equivalence of the companion graph, *Sci. Khyber* **4** (1991), 51–60.

[Az2] AZRAM, M., Graph theoretic versions of Reidemeister moves, *Sci. Khyber* **6** (1993), 197–228.

[C] CROWELL, R. H., Genus of alternating link types, *Ann. of Math.* **3** (1959), 101–120.

[CF] CROWELL, R. H. and FOX, R. H., *Introduction to Knot Theory, Graduate Texts in Math.* **57**, Springer-Verlag, Berlin–Heidelberg–New York, 1963.

[FA] FOX, R. H. and ARTIN, E., Some wild cells and spheres in 3-dimensional space, *Ann. of Math.* **49** (1948), 979–990.

[K] KAUFFMAN, L. H., New invariants in the theory of knots, *Amer. Math. Monthly* **95** (1988), 195–242.

[KT] KINOSHITA, S. and TERASAKA, H., On unions of knots, *Osaka Math. J.* **9** (1957), 131–153.

[R] REIDEMEISTER, K., *Knotentheorie, Ergebnisse der Mathematik und Ihrer Grenzgebiete, (Alte Folge), Band* **1**, *Heft* **1**, Springer, Berlin, 1932. Reprint: Springer-Verlag, Berlin–New York, 1974. English transl.: B. C. S. Moscow (U. S. A.), Chelsea, New York, 1983

[YK] YAJIMA, T. and KINOSHITA, S., On the graphs of knots, *Osaka Math. J.* **9** (1957), 155–163.

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