ABSTRACT FLAT PROJECTIVE DIFFERENTIAL GEOMETRY.

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Introduction. In an earlier paper, Michal[1] has defined an abstract projective curvature form in a Hausdorff space having coordinates in a Banach space with inner product, under the condition that the associated Banach ring of linear functions possess a contraction operation. The basis for a general flat projective geometry under the same restrictions was also sketched in the same paper. More recently the authors[2] have considered a general geometry of paths in which the concept of projective connection and projective curvature form was generalized to geometric spaces having coordinates in Banach spaces without independently postulated inner product or contraction.

In the present paper we study an abstract flat projective geometry from two initial viewpoints. In the first, which is developed in sections one and two, we begin with a general geometric space with postulated allowable and preferred (projective) coordinate systems. We then show that transformations from allowable to projective coordinates determine in their domains the solutions of a characteristic second order differential system. The latter involves a projective linear connection which determines an identically vanishing projective curvature form. Our second approach seeks to characterize locally the projective coordinate systems by means of a second order differential system. In developing this other viewpoint in the third and fourth sections we assume that our geometric space is a Hausdorff topological space, and establish existence theorems for the solution of

[1] Michal III. Roman numerals refer to the bibliography at the end of the paper.
[2] Michal and Mewborn VII.

a certain first order differential system involving a postulated projective connection whose curvature form is identically zero, and whose Fréchet differential has the δ-property. This δ-property is a particularly interesting development of our general treatment, for we show that it may or may not be satisfied for functions in infinite dimensional spaces, whereas it automatically holds for the finite dimensional arithmetic case. In the concluding section we show that the solution of our first order differential system is unique in a restricted neighborhood of each point of our projective coordinate space B_1. Further we show that it is of such form that it determines projective coordinates satisfying the postulates used in our first approach to the problem, and hence that the two methods yield equivalent (local) characterisations of a flat projective geometry.

1. Projective Coordinate Systems and their Differential Properties.

We shall assume that we have a geometric space of points H having allowable coordinates already defined in a Banach space B, and shall consider the geometry of this space from the standpoint of an undefined set of »preferred homogeneous coordinate systems» (hereafter called »projective coordinate systems» or briefly »p. c. s.»), valued in a second Banach space B_1 of couples $X = (x, x^0)$ where x is in B and x^0 is a real number hereafter called the *gauge variable*. These p. c. s. will be subject to the following five postulates[1]:

P 1. *In a p. c. s. there will correspond to each point p of the geometric space H at least one element X of the space B_1, and to each Y in B_1 except* (0, 0) *there will correspond just one point q of H.*

P 2. *Two elements X and Y of B_1 represent the same point p of H if and only if they lie on the same straight line*[2] *through the origin* (0, 0) *of B_1.*

P 3. *Any p. c. s. can be transformed into any other by a linear transformation.*[3]

P 4. *Any homogeneous coordinate system obtained from a p. c. s. by a linear transformation is a p. c. s.*

P 5. *There exists at least one p. c. s.*

From the above postulates it can readily be proved that any transformation between two p. c. s. is a solvable linear transformation.

[1] Veblen and Whitehead I, p. 29.

[2] I. e. if and only if they satisfy a relation of the form $X = \alpha Y$ where α is a real number.

[3] I. e., a transformation $\bar{X} = \bar{X}(X)$ additive and continuous in X and hence homogeneous of degree one.

Definition 1. 1. *Allowable Coordinate System.* Any $(1-1)$ solvable transformation on H to an open subset $B' \subset B$ is an allowable coordinate system and will be denoted by $x(p)$ and its inverse by $p(x)$ where p is in H and x in $B' \subset B$ and $x(p_0) = 0$ in B'.

Definition 1. 2. *Transformation of Coordinates from Allowable to Preferred Coordinates.* This is any transformation from a given allowable coordinate system $x(p)$ to a p. c. s. $U(x(p))$. The range[1] of $U(x(p))$ is the entire space H less the point p_0, and its domain[1] is an open subset B'_1 of B_1.

Definition 1. 3. *Change of Representation.* The simultaneous transformation of allowable coordinates $\bar{x} = \bar{x}(x)$ and the change of gauge variable $\bar{x}^0 = x^0 + \log \varrho(x)$ where $\varrho(x)$ is a positive scalar field valued function of x of class $C^{(3)}$ will be called a change of representation.

Definition 1. 4. *Projective Scalar Field.* By a projective scalar field we shall mean any geometric object whose components $S(X)$ transform according to the law $\bar{S}(\bar{X}) = S(X)$ under the change of representation $\bar{X} = \bar{X}(X)$.

Definition 1. 5. *The Projective Scalar Field $A(X)$.* By this we shall denote the transformation

$$A(X) = e^{x^0} U(x) \tag{1.1}$$

whose domain is the subset $(B'_0, |x^0| < \infty)$ of B_1, and whose range is the subset of B_1 obtained by adjoining to B'_1 all elements of B_1 lying on a straight line through the origin with any element of B'_1. Furthermore $A(X)$

a) is of class $C^{(3)}$ on its domain,
b) is a projective scalar,
c) has a first Fréchet differential $A(X; Y)$ which is a solvable linear function of the projective c. v. Y with inverse $A^{-1}(X, Y)$.

Definition 1. 6. *Projective Contravariant Vector.* A geometric object V associated with the point p whose component undergoes the transformation

$$\bar{V} = \bar{X}(X(p); V) \tag{1.2}$$

under the change of representation $\bar{X}(p) = \bar{X}(X(p))$ will be called a projective contravariant vector associated with the point p.

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[1] The set of values of the indicated independent variable for which a function is defined will be called the *domain* of the function with respect to that variable. The corresponding set of values of the function will be called its *range*, e. g. here the domain of $U(x)$ is B'_0 (the set B' less the zero element), and its range is B'_1.

Definition 1. 7. *Projective Contravariant Vector Field or p. c. v. f.* A set of projective contravariant vectors associated one to each point of some set in H will be called a p. c. v. f.

Definition 1. 8. *Hyperplane through the Origin of B_1.* The set of elements X of B_1 which all satisfy a given numerically valued linear function (not identically zero) equated to zero will be said to lie on a hyperplane through the origin (0, 0) of B_1.

The condition c) of definition 1. 5 implies that the values of $A(X)$ do not lie on a hyperplane through the origin of B_1. For if we assume condition c) and assume that there exists a linear function $L(V) \not\equiv 0$ such that $L(A(X)) = 0$ for all values of $A(X)$, and differentiate, we get

$$L(A(X;\ Y)) = 0. \tag{1. 3}$$

Let $Y = A^{-1}(X,\ W)$ whence $L(W) = 0$ for all W contrary to assumption.

Definition 1. 9. *The Function $\mathfrak{Z}(X)$.* Any solvable linear function $F(S)$ of the projective scalar $A(X)$ will be denoted by

$$\mathfrak{Z}(X) = F(A(X)) = e^{x^0} F(U(x)). \tag{1. 4}$$

By a well known theorem of Banach-Schauder it follows that the inverse $F^{-1}(S)$ of $F(S)$ is also linear in S.

Theorem 1. 1. *Let $U(x)$ be a transformation of coordinates from $x(p)$ to a particular p. c. s. $U(x(p))$. Let $\mathfrak{Z}(x)$ be a transformation of coordinates from the same allowable coordinate system $x(p)$ to any p. c. s. $\mathfrak{Z}(x(p))$. Then the function $\mathfrak{Z}(X)$ (def. 1. 9) satisfies the differential system*

$$\begin{cases} \text{a)}\ \ \mathfrak{Z}(X;\ Y;\ Z) = \mathfrak{Z}(X;\ \Pi(X,\ Y,\ Z)) \\ \text{b)}\ \ \mathfrak{Z}(X;\ (0,\ y^0)) = y^0\, \mathfrak{Z}(X) \end{cases} \tag{1. 5}$$

where

$$\Pi(X,\ Y,\ Z) = A^{-1}(X,\ A(X;\ Y;\ Z)). \tag{1. 6}$$

Proof: Taking two successive Fréchet differentials of equation (1. 4) we obtain

$$\mathfrak{Z}(X;\ Y) = F(A(X;\ Y))$$

and

$$\mathfrak{Z}(X;\ Y;\ Z) = F(A(X;\ Y;\ Z)),$$

whence, from the solvability of $A(X;\ Y)$ and of $F(S)$

$$\mathfrak{Z}(X;\ A^{-1}(X,\ Y)) = F(A(X;\ A^{-1}(X;\ Y))) = F(Y).$$

Now let $Y = A(X;\ Y;\ Z)$, which completes the proof. Q. E. D.

The second order differential system (1. 5) of this theorem can be replaced by the equivalent system of three first order differential equations:

$$(1.\ 7)\qquad \begin{cases} \text{a)}\ \ P(X,\ Y;\ Z) = P(X,\ \Pi(X,\ Y,\ Z)) \\ \text{b)}\ \ \mathfrak{Z}(X;\ Y) = P(X,\ Y) \\ \text{c)}\ \ \mathfrak{Z}(X;\ (0,\ y^0)) = y^0\,\mathfrak{Z}(X). \end{cases}$$

This modification is important, as it is with a differential system of this type that we shall be dealing in section 3. In particular, compare

$$P(X_0,\ (0,\ y^0)) = \mathfrak{Z}(X_0;\ (0,\ y^0)) = y^0\,\mathfrak{Z}(X_0)$$

obtained from b) and c) of (1. 7) with the analogous relation in the initial condition c) of equation (3. 1).

2. The Flat Projective Connection.

The function $\Pi(X, Y, Z)$ defined by equation (1. 6) plays an important role in the geometry of our space, and is the component in the given coordinates of a geometric object which we shall call the *projective connection*[1]. Some of its important properties are exhibited in this section.

Theorem 2. 1. *The projective connection $\Pi(X,\ Y,\ Z)$ is symmetric and bilinear in Y and Z, and satisfies the relation*

$$(2.\ 1)\qquad \Pi(X,\ (0,\ y^0),\ Z) = y^0\,Z.$$

Proof: The symmetry of the function is an immediate consequence of Kerner's theorem on the symmetry of the second Fréchet differential. An application of the before mentioned theorem of Banach-Schauder and the definition of Fréchet differential, shows the function to be bilinear. A direct computation of the differentials in (1. 6) verifies (2. 1). *Q. E. D.*

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[1] For brevity we shall hereafter, if there is no ambiguity, use this term for the *component* of the projective connection. In general we shall similarly apply the name of a geometric object to one of its components.

Theorem 2. 2. *The projective connection is invariant under a solvable linear transformation of* $A(X)$.

Proof: Let $L(V)$ be a solvable linear transformation of V with inverse $L^{-1}(V)$ and let

$$\tilde{A}(X) = L(A(X)).$$

$\tilde{A}(X;\ Y)$ has the inverse $\tilde{A}^{-1}(X,\ Y) = A^{-1}(X, L^{-1}(Y))$, hence $\tilde{A}^{-1}(X, \tilde{A}(X;\ Y;\ Z)) =$ $= A^{-1}(X, A(X;\ Y;\ Z)) = \Pi(X,\ Y,\ Z)$. *Q. E. D.*

Theorem 2. 3. *Under a change of representation the projective connection transforms as a component of a linear connection,*

(2. 2) $$\bar{\Pi}(\bar{X},\ \bar{Y},\ \bar{Z}) = \bar{X}(X;\ \Pi(X,\ Y,\ Z)) + \bar{X}(X;\ X(\bar{X};\ \bar{Y};\ \bar{Z}))$$

when Y *and* Z *are projective contravariant vectors.*

Proof: By definition we have in the new representation

(2. 3) $$\bar{\Pi}(\bar{X},\ \bar{Y},\ \bar{Z}) = \bar{A}^{-1}(\bar{X},\ \bar{A}(\bar{X};\ \bar{Y};\ \bar{Z})).$$

From b) and c) of definition 1. 5 and definition 1. 6 we have

(2. 4) $$\bar{A}(\bar{X};\ \bar{Y}) = A(X;\ Y) = A(X;\ X(\bar{X};\ \bar{Y})).$$

By taking inverses of this,

(2. 5) $$\bar{A}^{-1}(\bar{X},\ \bar{S}) = \bar{X}(X;\ A^{-1}(X,\ S)),$$

showing that $A^{-1}(X,\ S)$ is a p. c. v. f. valued linear form in the projective scalar S.

Differentiating the first and last of (2. 4) we have

(2. 6) $$\bar{A}(\bar{X};\ \bar{Y};\ \bar{Z}) = A(X;\ Y;\ Z) + A(X;\ X(\bar{X};\ \bar{Y};\ \bar{Z}))$$

which, with the aid of (2. 5) and (2. 3) yields

(2. 7) $$\begin{cases} \bar{\Pi}(\bar{X},\ \bar{Y},\ \bar{Z}) = \bar{A}^{-1}(\bar{X},\ \bar{A}(\bar{X};\ \bar{Y};\ \bar{Z})) \\ \qquad = \bar{X}(X;\ A^{-1}(X,\ A(X;\ Y;\ Z) + A(X;\ X(\bar{X};\ \bar{Y};\ \bar{Z})))). \end{cases}$$

Equation (2. 2) follows at once from (2. 7). *Q. E. D.*

It can be shown that the inverse of $A(X; Y)$ is of the form

$$A^{-1}(X, Y) = (e^{-x^0} l(x, Y), \quad e^{-x^0} l^0(x, Y)) \tag{2. 8}$$

where x is in B, $l(x, Y)$ and $l^0(x, Y)$ are linear in Y of B_1 and valued in B and the reals respectively. Using this property of $A^{-1}(X, Y)$ and theorem 2. 3 we obtain

Theorem 2. 4. *The projective connection $\Pi(X, Y, Z)$ is independent of x^0 and is of the form*

$$\Pi(X, Y, Z) = (\Gamma(x, y, z) + y^0 z + z^0 y, \quad \Gamma^0(x, y, z) + y^0 z^0) \tag{2. 9}$$

where, in the notation of (2. 8),

$$\begin{cases} \Gamma(x, y, z) = l(x, U(x; y; z)) \\ \Gamma^0(x, y, z) = l^0(x, U(x; y; z)) \end{cases} \tag{2. 10}$$

are bilinear and symmetric in y and z. If y and z are contravariant vectors, $\Gamma(x, y, z)$ transforms as a component of a linear connection; and $\Gamma^0(x, y, z)$, called the gauge form, is absolute scalar field valued.

Theorem 2. 5. *Under a change of representation*

$$\begin{cases} \bar{X}(X) = (\bar{x}(x), \quad x^0 + \log \varrho(x)) \\ X(\bar{X}) = (x(\bar{x}), \quad \bar{x}^0 - \log \bar{\varrho}(\bar{x})) \end{cases} \tag{2. 11}$$

if Y and Z are projective contravariant vectors, equation (2. 9) *goes over into the $\bar{X}$ representation as*

$$\bar{\Pi}(\bar{X}, \bar{Y}, \bar{Z}) = (\bar{\hat{\Gamma}}(\bar{x}, \bar{y}, \bar{z}) + \bar{y}^0 \bar{z} + \bar{z}^0 \bar{y}, \quad \bar{\hat{\Gamma}}^0(\bar{x}, \bar{y}, \bar{z}) + \bar{y}^0 \bar{z}^0), \tag{2. 12}$$

where

$$\begin{cases} \bar{\hat{\Gamma}}(\bar{x}, \bar{y}, \bar{z}) = \bar{x}(x; \Gamma(x, y, z)) + \bar{x}(x; x(\bar{x}; \bar{y}; \bar{z})) \\ \qquad\qquad + \Phi(x, y)\, \bar{x}(x; z) + \Phi(x, z)\, \bar{x}(x; y), \\ \bar{\hat{\Gamma}}^0(\bar{x}, \bar{y}, \bar{z}) = \Gamma^0(x, y, z) + \frac{1}{2}\{\Phi(x, y; z) + \Phi(x, z; y)\} \\ \qquad\qquad - \Phi(x, \Gamma(x, y, z)) - \Phi(x, y)\, \Phi(x, z), \\ \Phi(x, \nu) = -\dfrac{\varrho(x; \nu)}{\varrho(x)} = -d_\nu^x \log \varrho(x). \end{cases} \tag{2. 13}$$

This theorem may be proved either by a direct computation from (2. 9) or by reversing the order of steps in the proof of a theorem we have given elsewhere[1] in connection with a general, not necessarily flat, projective geometry.

Corollary 2. 1. *If we use the notation*

$$\Pi(X, Y, Z) = (j(x, Y, Z), j^0(x, Y, Z)) \tag{2. 14}$$

then

$$\begin{cases} j(x, (y, 0), (z, 0)) = \Gamma(x, y, z) \\ j^0(x, (y, 0), (z, 0)) = \Gamma^0(x, y, z). \end{cases} \tag{2. 15}$$

Definition 2. 1. *The Projective Curvature Form.* The function $B_{(1)}(X, Y, Z, W)$ defined by the relation

$$\begin{aligned} B_{(1)}(X, Y, Z, W) = {} & \Pi(X, Y, Z; W) - \Pi(X, Y, W; Z) \\ & + \Pi(X, \Pi(X, Y, Z), W) - \Pi(X, \Pi(X, Y, W), Z) \end{aligned} \tag{2. 16}$$

where $\Pi(X, Y, Z)$ is the projective connection of equation (1. 6) will be called the projective curvature form based on $\Pi(X, Y, Z)$.

Theorem 2. 6. *The curvature form* (2. 16) *is a p. c. v. f. valued trilinear form in the projective contravariant vectors Y, Z and W, and vanishes identically.*

Proof: It is easy to verify that $B_{(1)}(X, Y, Z, W)$ satisfies the conditions of definitions 1. 6 and 1. 7 by using theorem 2. 3. The trilinearity property follows from theorem 2. 1 and the definition of Fréchet differential.

We next must show that $B_{(1)}(X, Y, Z, W) \equiv 0$. From equation (1. 6) it follows that

$$\Pi(X, Y, Z) = -A^{-1}(X, A(X; Y); Z) \tag{2. 17}$$

whence

$$\begin{aligned} \Pi(X, Y, Z; W) = {} & -A^{-1}(X, A(X; Y); Z; W) \\ & - A^{-1}(X, A(X; Y; W); Z). \end{aligned} \tag{2. 18}$$

This exists from the conditions of definition 1. 5, and is clearly trilinear in Y, Z, W. From (1. 6) and (2. 17) we have

[1] Michal and Mewborn VII, Theorem 2. 1.

$$\Pi(X, \Pi(X, Y, Z), W) = -A^{-1}(X, A(X; Y; Z); W). \tag{2. 19}$$

Substitutions by means of (2. 18) and (2. 19) in (2. 16) complete the proof of the theorem. *Q. E. D.*

3. Local Characterization of a General Flat Projective Geometry.

We now change our point of view, and set up a converse problem to that treated in section 1. Suppose that we are given the differential system (1. 7) and the *initial conditions*

$$\begin{cases} \text{a)} \ P(X_0, V) = P_0(V), \text{ a linear solvable function of } V, \\ \text{b)} \ \mathfrak{Z}(X_0) = \mathfrak{Z}_0, \\ \text{c)} \ P_0((0, 1)) = \mathfrak{Z}_0, \end{cases} \tag{3. 1}$$

under what restrictions can this be said to characterize a flat projective geometry?

In the present section we impose the needed broad restrictions upon the structure of our space. In the next we develope a number of necessary preliminary results of a general character, and in section 5 we show that, under these restrictions, the solution of (1. 7) exists and satisfies the postulates of section 1 for a p. c. s. Hence we may say that this system actually characterizes our flat projective geometry.

Let *the geometric space be a Hausdorff space with Banach coordinates*, but now by »allowable coordinate systems» we shall mean allowable $K^{(3)}$ coordinate systems.[1] Clearly each geometric domain of such an allowable coordinate system is a metric space whose metric is defined as

$$\delta(p_1, p_2) = \| x(p_1) - x(p_2) \|, \qquad p_i \in H_0 \subset H. \tag{3. 2}$$

Let the Banach space of couples B_1 be as in section 1.

Definition 3. 1. *Change of Representation.* The simultaneous transformation $\bar{x} = \bar{x}(x)$ of allowable $K^{(3)}$ coordinates and the change $\bar{x}^0 = x^0 + \log \varrho(x)$ of gauge variable, where $\varrho(x)$ is as in definition 1. 3, will be termed a change of representation.

Further we assume that there exists a function $\Pi(X, Y, Z)$ with arguments and values in B_1 and having the following properties when x is in the coordinate domain in B of each allowable $K^{(3)}$ coordinate system:

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[1] Subject to postulates I—IV page 5, Michal-Hyers (II).

$$
(3.\ 3)\quad \begin{cases}
\text{a) } \Pi(X,\ Y,\ Z) = \Pi(X,\ Z,\ Y); \\
\text{b) } \Pi(X,\ Y,\ Z) \text{ is bilinear in } Y,\ Z; \\
\text{c) } \Pi(X, (0,\ y^0),\ Z) = y^0 Z; \\
\text{d) } \Pi(X,\ Y,\ Z) \text{ is of class } C^{(1)} \text{ locally uniformly}^{1} \text{ such that the differential } \Pi(X,\ Y,\ Z;\ W) \text{ has the } \delta\text{-property (definition 4. 1) with respect to } Y \text{ for each } Z. \\
\text{e) Under change of representation, } \Pi(X,\ Y,\ Z) \text{ transforms formally as a component of a linear connection whenever } Y,\ Z \text{ are projective contravariant vectors.} \\
\text{f) The curvature form } B_{(1)}(X,\ Y,\ Z,\ W) \text{ based on } \Pi(X,\ Y,\ Z) \text{ is identically zero.}
\end{cases}
$$

4. Theorems on Differentials.

Definition 4. 1. *The δ-property.* Let $f(x, y)$ have arguments and values in Banach spaces (not necessarily the same). The Fréchet differential $f(x_0, y; z)$ of $f(x, y)$ at $x = x_0$ is said to have the δ-property (with respect to y) if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon, x_0) > 0$ independent of y such that

$$
(4.\ 1)\quad ||f(x_0 + z,\ y) - f(x_0,\ y) - f(x_0,\ y;\ z)|| \leq \varepsilon ||z|| \text{ for } ||z|| < \delta(\varepsilon,\ x_0) \text{ and } ||y|| < 1.
$$

Theorem 4. 1. *If $f(x, y)$, linear in y, has a differential with the δ-property at $x = x_0$, then* (4. 1) *is satisfied for $||z|| < \delta'(\varepsilon, x_0) = \frac{1}{b}\delta(\varepsilon, x_0)$ and $||y|| < b$, where b is an arbitrary positive number. Conversely if* (4. 1) *holds for $||y|| < b$, any chosen positive number, then the differential has the δ-property.*

Definition 4. 2. *The Banach Ring R_1.* The set of all linear transformations with the domain B_1 and ranges in B_1 under suitable definitions of operations and norm[2] form a Banach ring which we shall call R_1.

In general, we shall denote the R_1 correspondent of a B_1 valued function, linear in X_i,

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[1] Michal-Hyers (II).

[2] Michal (III) p. 547, but note carefully that in the present discussion no inner product or contraction is postulated.

$$\Phi(X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_s),$$

considered as a linear function of its B_1 valued argument X_i, by

$$\boldsymbol{\Phi}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_s).$$

To avoid ambiguity, however, the following two exceptions will be made to this notations:

(A) $\Pi(X, *, Z; W)$ and $\Phi(X, *; Z)$ in R_1 will be respectively the correspondents of $\Pi(X, Y, Z; W)$ and any $\Phi(X, Y; Z)$ in B_1 considered as linear functions of Y.

(B) $\boldsymbol{\Pi}(X, Z; W)$ and $\boldsymbol{\Phi}(X; Z)$ in R_1 will mean the Fréchet differential of $\boldsymbol{\Pi}(X, Z)$ and $\boldsymbol{\Phi}(X)$ in R_1 respectively, and *not* the correspondents of $\Pi(X, Y, Z; W)$ and $\Phi(X, Y; Z)$ of B_1.

Theorem 4. 2. *Let $\boldsymbol{\Pi}(X, Z)$ be any function with arguments in B_1 and values in R_1 and linear in Z. Then, if it is of class $C^{(1)}$ in X uniformly*[1] *on $(X_0)_a$ in B_1 and satisfies*

$$(4.2) \quad \boldsymbol{B}_{(1)}(X, Z, W) = \boldsymbol{\Pi}(X, Z; W) - \boldsymbol{\Pi}(X, W; Z) + \boldsymbol{\Pi}(X, W)\boldsymbol{\Pi}(X, Z) - \boldsymbol{\Pi}(X, Z)\boldsymbol{\Pi}(X, W) = 0$$

for X in $(X_0)_a$, the differential system

$$(4.3) \quad \begin{cases} \text{a)} \ \boldsymbol{P}(X; Z) = \boldsymbol{P}(X)\boldsymbol{\Pi}(X, Z) \\ \text{b)} \ \boldsymbol{P}(X_0) = \boldsymbol{P}_0 \end{cases}$$

where $\boldsymbol{P}_0$ is an arbitrarily chosen element of R_1, has a unique solution

$$(4.4) \quad \boldsymbol{P}(X) = \lim_{n\to\infty} \boldsymbol{P}_n(X)$$

for X in $(X_0)_a$, where $\boldsymbol{P}_n(X)$ is defined recurrently by

$$(4.5) \quad \boldsymbol{P}_{n+1}(X) = \boldsymbol{P}_0 + \int_0^1 \boldsymbol{P}_n(X_0 + s(X - X_0))\,\boldsymbol{\Pi}(X_0 + s(X - X_0), X - X_0)\,ds.$$

Proof: The condition of complete integrability for the system (4. 3) can readily be shown equivalent to the condition (4. 2) of the hypothesis. Hence

[1] Lemma 3 p. 651, Michal-Hyers (IV).

this theorem becomes a particular case of a known theorem on completely integrable differential equations.[1] *Q. E. D.*

Corollary 4. 1. *The unique solution of system* (4. 3) *is given by*

$$
(4.\,6)\qquad \begin{cases} \boldsymbol{P}(X) = \boldsymbol{P}_0 \{\boldsymbol{I} + \sum_{i=1}^{\infty} \int_0^1 \underset{(i)}{\cdots} \int_0^1 t_i^{i-1} t_{i-1}^{i-2} \ldots t_2 \cdot \\ \cdot \boldsymbol{\Pi}(X_0 + t_i \ldots t_1 (X - X_0),\ X - X_0) \ldots \cdot \\ \cdot \boldsymbol{\Pi}(X_0 + t_1 (X - X_0),\ X - X_0)\, d t_1 \ldots d t_i\} \end{cases}
$$

where $\boldsymbol{I}$, the unit element of R_1, is the correspondent of the linear function $L(X) \equiv X$.

Corollary 4. 2. *The function*

$$
(4.\,7)\quad \boldsymbol{\Phi}(X) = \boldsymbol{I} + \sum_{i=1}^{\infty} \int_0^1 \underset{(i)}{\cdots} \int_0^1 t_i^{i-1} \ldots t_2 \boldsymbol{\Pi}(X_0 + t_i \ldots t_1 (X - X_0),\ X - X_0) \ldots
$$

$$
\boldsymbol{\Pi}(X_0 + t_1 (X - X_0),\ X - X_0)\, d t_1 \ldots d t_i = \boldsymbol{I} + \boldsymbol{C}(X_0,\ X - X_0)
$$

is Fréchet differentiable in X for X in $(X_0)_a$.

Proof: If we choose the $\boldsymbol{P}_0$ of (4. 3) to be $\boldsymbol{I}$, the corollary follows at once, since by Corollary 4. 1

$$
\boldsymbol{P}(X) = \boldsymbol{\Phi}(X). \qquad Q. E. D.
$$

Corollary 4. 3. *The function $\boldsymbol{C}(X_0,\ X - X_0)$ of* (4. 7) *satisfies*

$$
(4.\,8)\qquad \|\boldsymbol{C}(X_0,\ X - X_0)\|_{R_1} < 1
$$

for X in a sufficiently small neighborhood $(X_0)_b$.

Proof: By corollary 4. 2, $\boldsymbol{C}(X_0,\ X - X_0)$ is differentiable and hence continuous for X in $(X_0)_b$. Since $\boldsymbol{\Pi}(X, Z)$ is linear in its second argument, $\boldsymbol{C}(X_0,\ X_0 - X_0) = 0$. The norm in the Banach ring R_1 (denoted by $\|\cdots\|_{R_1}$) is defined as the modulus of the correspondent linear function in B_1 hence there exists a b, $0 < b \leq a$, such that (4. 8) holds for X in $(X_0)_b$.

Corollary 4. 4. *The function $\boldsymbol{\Phi}(X)$ has a unique inverse of the form*

[1] Theorem 3. 1, p. 85, Michal-Elconin (IX).

$$\boldsymbol{\Phi}^{-1}(X) = \boldsymbol{I} - \boldsymbol{\Phi}(X) + [\boldsymbol{\Phi}(X)]^2 - \cdots \tag{4. 9}$$

for each X in the neighborhood $(X_0)_b$ *of corollary* 4. 3.

Proof: By the definition of the product of elements of the Banach ring R_1, the use of corollary 4. 3, and an obvious modification of a theorem of Michal and Martin[1] we obtain (4. 9). *Q. E. D.*

Corollary 4. 5. *If, in addition to the hypotheses of Theorem* 4. 2, *we assume that* $\boldsymbol{P}_0$ *has an inverse* $\boldsymbol{P}_0^{-1}$ *then there exists a number* b, $0 < b \leq a$, *such that for all* X *in* $(X_0)_b$ *the unique solution* $\boldsymbol{P}(X)$ *of the differential system* (4. 3) *has an inverse*

$$\boldsymbol{P}^{-1}(X) = \boldsymbol{\Phi}^{-1}(X)\,\boldsymbol{P}_0^{-1}. \tag{4. 10}$$

Theorem 4. 3. *A necessary and sufficient condition that a function* $\boldsymbol{F}(Z)$ *on* B_1 *to* R_1 *be linear in* Z *is that the correspondent* $F(Y, Z)$ *be bilinear in* Y *and* Z.

Proof: Sufficiency: From the hypotheses and definitions there exist numbers M_{YZ} and $M_Y = ||\boldsymbol{F}(Z)||_{R_1}$ for each Z such that

$$||F(Y, Z)|| \leq M_Y ||Y|| \leq M_{YZ} ||Z|| \cdot ||Y|| \text{ for all } Y.$$

This implies that $||\boldsymbol{F}(Z)||_{R_1} \leq M_{YZ} ||Z||$ which is equivalent to the condition for continuity of $\boldsymbol{F}(Z)$ at $Z = 0$. The additivity of $\boldsymbol{F}(Z)$ is clear from its definition.

Necessity: By definition $F(Y, Z)$ is linear in Y, hence

$$||F(Y, Z)|| \leq M_Y ||Y|| = ||\boldsymbol{F}(Z)||_{R_1} ||Y|| \leq M ||Z|| \cdot ||Y||$$

by the hypothesis on continuity of $\boldsymbol{F}(Z)$. The additivity of $F(Y, Z)$ in Z follows from the linearity of $\boldsymbol{F}(Z)$, which completes the proof. *Q. E. D.*

Theorem 4. 4. *If* $\boldsymbol{\Psi}(X)$ *is Fréchet differentiable at* $X = X_0$, *then* $\Psi(X_0, *; Z)$ *exists and*

$$\boldsymbol{\Psi}(X_0;\ Z) = \Psi(X_0, *;\ Z). \tag{4. 11}$$

Proof: By hypothesis, for any $\varepsilon > 0$ there exists a δ_1 such that

$$||\boldsymbol{\Omega}_0||_{R_1} = ||\boldsymbol{\Psi}(X_0 + Z) - \boldsymbol{\Psi}(X_0) - \boldsymbol{\Psi}(X_0;\ Z)||_{R_1} \leq \varepsilon ||Z|| \text{ for } ||Z|| < \delta_1.$$

[1] Theorem 5. II, p. 77, Michal-Martin (V).

But

$$\| \Omega_0(Y) \| \leq M \| Y \| = \| \boldsymbol{\Omega}_0 \|_{R_1} \| Y \| \leq \varepsilon \| Z \| . \| Y \| \text{ for } \| Z \| < \delta_1$$

where

$$\Omega_0(Y) = \Psi(X_0 + Z,\ Y) - \Psi X_0,\ Y) - \Psi'(X_0,\ Y,\ Z)$$

and $\Psi'(X_0, Y, Z)$ is the B_1 correspondent of $\boldsymbol{\Psi}(X_0; Z)$ and is bilinear in Y, Z by theorem 4. 3. Hence for any $\varepsilon > 0$ there exists a $\delta_2(\varepsilon, Y)$ such that

$$\| \Omega_0(Y) \| \leq \varepsilon \| Z \| \text{ for } \| Z \| < \delta_2(\varepsilon,\ Y).$$

From this, $\Psi'(X_0, Y, Z)$ evidently satisfies the definition for $\Psi(X_0, Y; Z)$, which completes the proof. *Q. E. D.*

Theorem 4. 5. *Let $\Psi(X, Y)$ be linear in Y, and have its arguments and values in B_1; and let $\boldsymbol{\Psi}(X)$ be its correspondent in R_1. Then a necessary and sufficient condition that $\boldsymbol{\Psi}(X; Z)$ exist at $X = X_0$ and that* (4. 11) *hold is that the Fréchet differential $\Psi(X_0, Y; Z)$ of $\Psi(X, Y)$ exist and have the δ-property at $X = X_0$.*

Proof: We shall establish the sufficiency of the condition as follows, and since the steps are all reversible, this proof also holds for the necessity.

The δ-condition inequality can be rewritten in the form

$$\| \tilde{\Omega}_0(Y') \| = \| \Psi(X_0 + Z,\ Y') - \Psi(X_0,\ Y') - \Psi(X_0,\ Y';\ Z) \| \leq \varepsilon \| Z \| \frac{\| Y' \|}{\Theta}$$

for $\| Z \| < \delta_1(\varepsilon, X_0)$ and where Y' is now *any* element of B_1 and $0 < \Theta < 1$.

By the modular condition there exists a *least* number M satisfying

$$\| \tilde{\Omega}_0(Y) \| \leq M \| Y \|.$$

Hence

$$M = \| \tilde{\boldsymbol{\Omega}}_0 \|_{R_1} \leq \varepsilon \frac{\| Z \|}{\Theta}, \quad \text{for } \| Z \| < \delta_1$$

where

$$\tilde{\boldsymbol{\Omega}}_0 = \boldsymbol{\Psi}(X_0 + Z) - \boldsymbol{\Psi}(X_0) - \Psi(X_0,\ *;\ Z),$$

which implies that

$$\| \tilde{\boldsymbol{\Omega}}_0 \|_{R_1} \leq \varepsilon \| Z \| \text{ for } \| Z \| < \delta_2 = \Theta \delta_1.$$

From this, $\Psi(X_0, *; Z)$ clearly satisfies the definition for $\boldsymbol{\Psi}(X_0; Z)$ which completes the proof. *Q. E. D.*

Corollary 4.6. *Let $\Psi(X, Y)$ be linear in Y. A necessary and sufficient condition that $\Psi(X_0, Y; Z)$ exist and have the δ-property with respect to Y is that $\Psi(X_0; Z)$ exist.*

Proof: Use Theorems 4.5 and 4.6.

Corollary 4.7. *This theorem 4.5 holds if we replace $\| Y \| < 1$ in definition 4.1 by $\| Y \| \leq 1$.*

Proof: Exactly as before except that $0 < \Theta \leq 1$.

Clearly theorems 4.3, 4.4, 4.5 and corollaries 4.6 and 4.7 can be generalized in an obvious way to the case B_1 is any Banach space and R_1 is its associated Banach ring of linear transformations.

Let us consider now how this theory applies in certain finite and infinite dimensional cases and illustrates its use in a general Banach space.

First, suppose that B_1 is the $(n+1)$-dimensional arithmetic space of elements

$$X = ((x^i)) = (x^0, x^1, \ldots, x^n)$$

x^i a real number such that

$$\| X \| = \| x^i \|_i = \sqrt{\sum_{i=0}^{n} (x^i)^2},$$

and that R_1 is its associated Banach ring of linear functions

$$\boldsymbol{L} \text{ corresponding to } L(X) = ((\lambda^i_j x^j)),\ \lambda^i_j \text{ a real number,}$$

such that $\| \boldsymbol{L} \|_{R_1} = \boldsymbol{M} =$ the modulus of $L(X)$.

Lemma. *If $L(X) = ((\lambda^i_j x^j))$ as above, then*

$$\| \boldsymbol{L} \|_{R_1} = \boldsymbol{M} = (n+1)^{-\frac{1}{2}} \sqrt{\sum_{j=0}^{n} \sum_{i=0}^{n} (\lambda^i_j)^2}. \tag{4.12}$$

Proof: By hypothesis we have $\| L(X) \| \leq M \| X \|$ for all X, hence in particular for

$$X_k = ((\delta^i_{(k)})), \quad \| X_k \| = 1, \quad k = 0, 1, 2, \ldots, n,$$

we have $\| L(X_k) \|^2 \leq M^2$, whence by summing on k

$$\sqrt{\sum_{k=0}^{n} \| L(X_k) \|^2} = \sqrt{\sum_{k=0}^{n} \sum_{i=0}^{n} (\lambda^i_k)^2} \leq \sqrt{n+1}\, M. \tag{4.13}$$

Now since any arbitrary X can be expressed as

$$X = \sum_{k=0}^{n} x^k X_k = (x^0,\ x^1,\ \ldots,\ x^n),$$

$$(4.\ 14) \qquad \| L(X) \| = \left\| \sum_{k=0}^{n} x^k L(X_k) \right\| \leq \sum_{k=0}^{n} \| x^k L(X_k) \|$$

$$= \sum_{k=0}^{n} | x^k | \cdot \| L(X_k) \| \leq \| X \| \sqrt{\sum_{k=0}^{n} \| L(X_k) \|^2},$$

whence

$$(4.\ 15) \qquad M \leq \sqrt{\sum_{k=0}^{n} \| L(X_k) \|^2}.$$

But the least number M which will satisfy both (4. 13) and (4. 15) is

$$M = (n + 1)^{-\frac{1}{2}} \sqrt{\sum_{k=0}^{n} \| L(X_k) \|^2}$$

which is equal to the M of (4. 12). *Q. E. D.*

Theorem 4. 6. *Any function $\Psi(X,\ Y)$ with arguments and values in the arithmetic B_1 space, of the form*

$$(4.\ 16) \qquad \Psi(X,\ Y) = ((a^i_j(X)\, y^j)),$$

necessarily has a differential in X at $X = X_0$ with the δ-property if the $a^i_j(X)$ are differentiable in X at $X = X_0$.

Proof: By the hypotheses, for any $\varepsilon > 0$ there exists a $\delta(X_0,\ Y,\ \varepsilon)$ such that

$$\left\| \left\{ a^i_j(X_0 + Z) - a^i_j(X_0) - \frac{\partial\, a^i_j(X_0)}{\partial\, x^k}\, z^k \right\} y^j \right\|_i \leq \varepsilon \| Z \|$$

for $\| Z \| < \delta(X_0,\ Y,\ \varepsilon)$,

or, to define a briefer notation

$$\| \Omega_0(X,\ Y,\ Z) \| = \| b^i_j(X_0,\ Z)\, y^j \|_i \leq \varepsilon \| Z \|$$

for $\| Z \| < \delta(X_0,\ Y,\ \varepsilon)$, and all Y.

Now let $Y_k = ((\delta^i_{(k)}))$ so

$$|| b^i_{(k)}(X_0,\ Z) ||_i \leq \varepsilon || Z ||, \qquad k = 0,\ 1,\ \ldots,\ n$$

$$\text{for } || Z || < \delta(X_0,\ \varepsilon) = \min_k \delta(X_0,\ Y_k,\ \varepsilon).$$

By squaring, summing on k and applying the lemma we get

$$(4.\ 17) \qquad || \Omega_0(X,\ Y,\ Z) || \leq M || Y || \leq \varepsilon || Z || \cdot || Y ||$$

for $|| Z || < \delta(X_0,\ \varepsilon)$ and all Y. From this it can be shown that

$$\Psi(X,\ Y;\ Z) = \left(\left(\frac{\partial a^i_j(X_0)}{\partial x^k} y^j z^k\right)\right)$$

which, from the hypothesis on $a^i_j(X)$, exists and has the δ-property. *Q. E. D.*

Next we take our B_1 space to be a Hilbert space, and exhibit two instances of solvable linear functions whose Fréchet differentials exist but *do not* have the δ-property.

Example A. Let $[x, y]$ denote the Hilbert inner product, then the function

$$f(x,\ y) = 2\,[x,\ y]\,x + [x,\ x]\,y = d\overset{x}{\underset{y}{}}\{[x,\ x]\,x\}$$

is linear in y and has the inverse

$$f^{-1}(x,\ y) = \frac{y}{[x,\ x]} - \frac{2\,[x,\ y]\,x}{3\,[x,\ x]^2} \qquad x \neq 0$$

also linear in y. The partial Fréchet differential

$$d\overset{x}{\underset{z}{}} f(x,\ y) = 2\,[z,\ y]\,x + 2\,[x,\ y]\,z + 2\,[x,\ z]\,y$$

does not have the δ-property.

Example B. The function

$$f(x,\ y) = 2\,[x,\ y]\,x + [x,\ x]\,y + [y,\ a]\,x + [x,\ a]\,y + [a,\ a]\,y$$

$$= d\overset{x}{\underset{y}{}}\{[x,\ x]\,x + [x,\ a]\,x + [a,\ a]\,x + b\}$$

has the inverse $f^{-1}(0,\ y) = || a ||^{-2}\, y$ at $x = 0$ if $a \neq 0$. Its differential

$$\left\{d\overset{x}{\underset{z}{}} f(x,\ y)\right\}_{x=0} = [y,\ a]\,z + [z,\ a]\,y \text{ at } x = 0$$

likewise lacks the δ-property.

Lastly, we show by examples that the δ-property is not vacuous in a general Banach space.

Let $A(x, y) = l(y)\,\alpha(x)$ be a function on $B_1 B_2$ to B_3 (arbitrary Banach spaces), where $l(y)$ is linear on B_1 to R (the real numbers) and $\alpha(x)$ is on B_2 to B_3, differentiable at $x = x_0$. Hence for any $\varepsilon > 0$ there exists a $\delta(x_0, \varepsilon)$ such that

$$(4.\ 18) \qquad \|\alpha(x_0 + z) - \alpha(x_0) - \alpha(x_0; z)\| \leq \varepsilon \|z\| \text{ for } \|z\| < \delta(x_0, \varepsilon).$$

By the modular condition $|l(y)| \leq M\|y\|$ for some M, and hence there exists an $N = \frac{1}{M}$ such that $\|y\| < N$ implies $|l(y)| \leq 1$. Multiplying (4. 18) by this inequality we obtain

$$\|A(x_0 + z, y) - A(x_0, y) - l(y)\,\alpha(x_0; z)\| \leq \varepsilon\|z\|$$
$$\text{for } \|z\| < \delta(x_0, \varepsilon) \text{ and } \|y\| < N.$$

But this, together with theorem 4. 1, implies the existence of $A(x_0, y; z) = l(y)\,\alpha(x_0; z)$ with the δ-property.

This example may be modified by placing the differentiability condition on $A(x, y)$ instead of on $\alpha(x)$ since this implies that the latter is also differentiable.

A somewhat more general example is that of functions of the type

$$A(x, y) = \sum_{i=0}^{t} l_i(y)\,\alpha_i(x)$$

where the $l_i(y)$ and $\alpha_i(x)$ are subject to the same restrictions as $l(y)$ and $\alpha(x)$ above.

5. Solution of the Local Characterization Problem for a Flat Projective Geometry.

The results established in the preceding section now enable us to show the existence and exhibit the form of the solution of the complete differential system consisting of equations (1. 7) and initial conditions (3. 1). Furthermore, we show that this solution satisfies the postulational system of section 1, and hence the above complete differential system is a (differential) characterization of this geometry. *In the hypotheses of all theorems of this section we shall assume that all conditions of section 3 are satisfied.*

Theorem 5. 1. *There exists a neighborhood* $(X_0)_b$ *of each* X_0 *of any coordinate domain such that*

$$(5.1)\qquad \begin{cases} \text{a)}\ P(X,\ Y;\ Z) = P(X,\ \Pi(X,\ Y,\ Z)) \\ \text{b)}\ P(X_0,\ V) = P_0(V),\ \textit{a linear solvable function of } V, \end{cases}$$

has a unique solution $P(X,\ Y)$ *linear and solvable in* Y, *for all* X *in* $(X_0)_b$.

Proof: By conditions b) and d) of (3. 3) and theorem 4. 5

$$(5.2)\qquad \boldsymbol{\Pi}(X,\ Z;\ W) = \Pi(X,\ *,\ Z;\ W).$$

Let

$$\boldsymbol{F}(X,\ Z,\ W) = \Pi(X,\ *,\ Z;\ W) - \Pi(X,\ *,\ W;\ Z) \\ + \boldsymbol{\Pi}(X,\ W)\boldsymbol{\Pi}(X,\ Z) - \boldsymbol{\Pi}(X,\ Z)\boldsymbol{\Pi}(X,\ W),$$

and let $\boldsymbol{B}_{(1)}(X,\ Z,\ W)$ be defined by the first part of equation (4. 2). Then by condition f) of (3. 3) and (5. 2)

$$\boldsymbol{B}_{(1)}(X,\ Z,\ W) = \boldsymbol{F}(X,\ Z,\ W) = 0.$$

The hypotheses of theorem 4. 2 being satisfied, system (4. 3), which is now equivalent to system (5. 1), has a unique solution $\boldsymbol{P}(X)$ for X in $(X_0)_a$.

Hence $P(X,\ Y)$, the correspondent of $\boldsymbol{P}(X)$, is the unique solution of (5. 1), and by corollary 4. 5 is solvable and linear in Y for X in $(X_0)_b$. *Q. E. D.*

Theorem 5. 2. *Let* $P(X,\ Y)$ *be the solution of system* (5. 1), *then the system*

$$(5.3)\qquad \begin{cases} \text{a)}\ \mathfrak{Z}(X;\ Y) = P(X,\ Y), \\ \text{b)}\ \mathfrak{Z}(X_0) = \mathfrak{Z}_0, \qquad P_0((0,\ 1)) = \mathfrak{Z}_0 \end{cases}$$

has a unique solution of the form

$$(5.4)\qquad \mathfrak{Z}(X) = \mathfrak{Z}_0 + \int_0^1 P(X_0 + \sigma(X - X_0),\ X - X_0)\, d\sigma$$

for X *in the neighborhood* $(X_0)_b$ *of theorem* 5. 1.

Proof: The condition for complete integrability of (5. 3) is

$$P(X,\ Y;\ Z) = P(X,\ Z;\ Y),$$

which is clearly satisfied from our hypothesis a) of (5. 1) and condition a) of (3. 3).

Hence a special application of a known theorem[1] on completely integrable differential equations completes the proof. *Q. E. D.*

Corollary 5. 1. *The unique solution $\mathfrak{Z}(X)$ of theorem 5. 2 satisfies*

$$\mathfrak{Z}(X;\ (0,\ y^0)) = y^0\,\mathfrak{Z}(X). \tag{5. 5}$$

Proof: If we let $\Delta = X - X_0$ for X in $(X_0)_b$ then we can write[2]

$$\left\{\begin{aligned} P(X_0 + \Delta,\ Y) - P(X_0,\ Y) &= \int_0^1 P(X_0 + s\Delta,\ Y;\ \Delta)\,ds \\ &= \int_0^1 P(X_0 + s\Delta,\ \Pi(X_0 + s\Delta,\ Y,\ \Delta))\,ds. \end{aligned}\right. \tag{5. 6}$$

Now let $Y = (0, y^0)$ in (5. 6) and we have from property c) of (3. 3)

$$P(X_0 + \Delta,\ (0,\ y^0)) = P(X_0,\ (0,\ y^0)) + \int_0^1 P(X_0 + s\Delta,\ y^0\Delta)\,ds$$

whence by linearity, a) of (5. 3) and (5. 4)

$$\mathfrak{Z}(X;\ (0,\ y^0)) = y^0\left[\mathfrak{Z}_0 + \int_0^1 P(X_0 + s(X - X_0),\ X - X_0)\,ds\right] = y^0\,\mathfrak{Z}(X).$$

Q. E. D.

Theorem 5. 3. *The solution $\mathfrak{Z}(X)$ of theorem 5. 2 is of the form*

$$\mathfrak{Z}(X) = e^{x^0 - x^0_0}\,U(x), \tag{5. 7}$$

where $X = (x,\ x^0)$, $X_0 = (x_0,\ x^0_0)$ and $U(x) = \mathfrak{Z}((x,\ x^0_0))$.

Proof: The abstract Volterra integral equation of the second kind

$$\mathfrak{Z}(X) = U(x) + \int_{x^0_0}^{x^0} \mathfrak{Z}((x,\ t))\,dt \tag{5. 8}$$

is equivalent to the system (5. 3).

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[1] Theorem 3. 2, p. 87, Michal and Elconin (IX).

[2] Definition 1. 7 and Theorem 1. 7, pp. 74—76, Michal-Elconin (IX).

This non-homogeneous equation has a *unique* continuous solution for each x in B. The continuous function (5. 7) satisfies (5. 8) and (5. 5) or its equivalent[1]

$$y^0 \frac{\partial \mathfrak{Z}((x, x^0))}{\partial x^0} = d\,{}^{x^0}_{y^0} \mathfrak{Z}((x, x^0)) = y^0 \mathfrak{Z}(X),$$

hence it is the form of the required unique solution. *Q. E. D.*

The results of the theorems and corollary in this section can be collected in the single

Theorem 5. 4. *If the conditions of section 3 are satisfied, then there exists a number $b > 0$ for each x_0 of any allowable coordinate domain such that the differential system*

$$(5.\ 9) \qquad \begin{cases} \text{a)} & \mathfrak{Z}(X;\ Y;\ Z) = \mathfrak{Z}(X;\ \Pi(X,\ Y,\ Z)) \\ \text{b)} & \mathfrak{Z}(X;\ (0,\ y^0)) = y^0 \mathfrak{Z}(X), \\ \text{c)} & \mathfrak{Z}(X_0) = \mathfrak{Z}_0, \\ \text{d)} & \mathfrak{Z}(X_0;\ V) = P_0(V), \text{ a linear solvable function of } V, \\ \text{e)} & P_0((0,\ 1)) = \mathfrak{Z}_0 \end{cases}$$

has a unique solution $\mathfrak{Z}(X)$ for X in $(X_0)_b$. This solution has the form $e^{x^0 - x^0_0}\,U(x)$, and its differential $\mathfrak{Z}(X;\ Y)$ is a solvable linear function of Y.

All that now remains to show that the system (5. 9) affords a differential characterization of the geometry of section 1, is to verify that its solution establishes p. c. s. which satisfy the five postulates. We therefore consider these postulates one by one in connection with (5. 7).

P 1. a) If x is in some allowable coordinate domain of B, then it is the unique correspondent of some geometric point p of H. But by the form of $\mathfrak{Z}(X)$ there is at least one value $\mathfrak{Z}$ of B_1 for this x, and hence for the point p.

b) There exists a b_1, $0 < b_1 \leq b$, such that $\mathfrak{Z}(X)$ does not take on the value $(0, 0)$ for X in $(X_0)_{b_1}$. For, since $\mathfrak{Z}_0 \neq 0$, if we had $\mathfrak{Z}_1 = \mathfrak{Z}(X_1) = 0$ for X_1 in $(X_0)_b$, we can always find a neighborhood $(X_0)_{b_1}$ not containing $\mathfrak{Z}_1$.

c) There exist numbers b'_1 and $b_2 \neq 0$, $0 < b'_1 \leq b_1 \leq b$, such that $\mathfrak{Z}(X)$ for X in $(X_0)_{b'_1}$ has a unique solution $X = X(\mathfrak{Z})$ for $\mathfrak{Z}$ in $(\mathfrak{Z}_0)_{b_2}$. This local solvability follows from the general implicit function theorem[2], since the differential $\mathfrak{Z}(X;\ Y)$ is solvable linear for X in $(X_0)_b$.

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[1] P. 74, Michal and Elconin (IX).

[2] Theorem 4, p. 150, Hildebrandt and Graves (VI).

d) Since by definition the allowable coordinate system $x(q)$ is a homeomorphism, and by b) above $\mathfrak{Z}(X) \neq 0$ for X in $(X_0)_{b'_1}$, the correspondence $q \longleftrightarrow \mathfrak{Z}$ is biunique in the specified neighborhoods.

Note that these neighborhoods $(X_0)_{b'_1}$ and $(\mathfrak{Z}_0)_{b_2}$ remain the same for the verification of the remaining postulates.

P 2. If a point q in H corresponds to two elements

$$\mathfrak{Z}_1 = e^{x_1^0 - x_0^0}\, U(x(q)) \text{ and } \mathfrak{Z}_2 = e^{x_2^0 - x_0^0}(U(x(q))$$

of $(\mathfrak{Z}_0)_{b_2}$, then $\mathfrak{Z}_1 = e^{x_1^0 - x_2^0}\mathfrak{Z}_2$.

P 3. If $P_0(V)$ and $\overline{P}_0(V)$ are two choices (distinct or not) of the arbitrary function in d) of (5. 9) giving rise to two solutions $\mathfrak{Z}(X)$ and $\overline{\mathfrak{Z}}(X)$ (correspondingly distinct or not), then there exists a solvable linear function

$$L(X) = \overline{P}_0(P_0^{-1}(X)) \text{ with inverse}$$

$$L^{-1}(X) = P_0(\overline{P}_0^{-1}(X)) \text{ such that}$$

$$\overline{P}_0(V) = L(P_0(V)).$$

P 4. Conversely, any solvable linear transformation of a $P_0(V)$ yields a $\overline{P}_0(V)$ which is solvable linear and hence gives rise to a $\overline{\mathfrak{Z}}(X)$ having the same properties as $\mathfrak{Z}(X)$, i. e. a p. c. s.

P 5. This postulate is evidently satisfied.

Bibliography.

I. O. Veblen and J. H. C. Whitehead. »The Foundations of Differential Geometry.» Cambridge Tract No. 29, 1932.

II. A. D. Michal and D. H. Hyers. »Theory and Applications of Abstract Normal Coordinates in a General Differential Geometry.» Annali d. R. Scuola Normale Superiore di Pisa. Series II Vol. VII, pp. 1—19, 1938.

III. A. D. Michal. »General Projective Differential Geometry.» Proceedings N. A. S. Vol. 23, pp. 546—548, 1937; »General Differential Geometry and Related Topics.» Bulletin of Amer. Math. Soc., vol. 45, pp. 529—563 (1939).

IV. A. D. Michal and D. H. Hyers. »Second Order Differential Equations with Two-point Boundary Conditions in General Analysis.» American Journal of Mathematics. Vol. 58, pp. 646—660, 1936.

V. A. D. Michal and R. S. Martin. »Some Expansions in Vector Space.» Journal de Math. pures et appliquées. Vol. 13, pp. 69—91. 1934.

VI. T. H. HILDEBRANDT and L. M. GRAVES. »Implicit Functions and their Differentials in General Analysis.» Transactions American Math. Soc. Vol. 29, pp. 127—153, 1927.

VII. A. D. MICHAL and A. B. MEWBORN »General Projective Geometry of Paths.» Compositio Mathematica (in press).

VIII. O. VEBLEN. »Projective Relativitätstheorie» Ergebnisse der Mathematik. Vol. 2, 1933.

IX. A. D. MICHAL and V. ELCONIN. »Completely Integrable Differential Equations in Abstract Spaces.» Acta mathematica, Vol. 68, pp. 71—107 (1937).

California Institute of Technology, Pasadena, California, July, 1939.