A GENERALISATION OF A THEOREM OF M. PICARD WITH REGARD TO INTEGRALS OF THE FIRST KIND OF TOTAL DIFFERENTIALS

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To the integrals connected with a plane curve, which are associated with the name of ABEL, correspond two distinct classes of integrals connected with an algebraic surface, viz. double integrals and integrals of total differentials. The latter were introduced into mathematical science by M. PICARD and a large part of what is at present known about them is due to him[1].

If a surface of order n,

$$f(x, y, z, w) = 0, \tag{1}$$

admits of an integral of the *first* kind, it is necessary that four homogeneous polynomials, $\theta_1, \theta_2, \theta_3, \theta_4$, of order $n - 3$, should exist, which satisfy the identity

$$\theta_1 f_x + \theta_2 f_y + \theta_3 f_z + \theta_4 f_w = 0, \tag{2}$$

and that the determinants of order $n - 2$, belonging to the array

$$\begin{Vmatrix} \theta_1, & \theta_2, & \theta_3, & \theta_4 \\ x, & y, & z, & w \end{Vmatrix},$$

[1] M. PICARD's first important memoir on the subject appeared in LIOUVILLE's Journal, sér. IV, t. 1 (1885); the chief results are to be found in the *Théorie des fonctions algébriques de deux variables indépendantes,* which he published in 1897 in conjunction with M. SIMART. All the results which I use are contained in chapter V of this book.

should vanish at every singular point of the surface. They must also satisfy further conditions, at present imperfectly known, at points of higher multiplicity.

It is also known that, if an integral of the first kind exists, the surface must have at least one singular point. The object of this note is to generalize this result.

Let us take two points (P, Q) in space with coordinates $(\lambda, \mu, \nu, \varpi)$ and $(\lambda', \mu', \nu', \varpi')$; then if we avoid special positions we can take ∞^4 positions of PQ such that the tangent planes through PQ touch the surface in n' distinct points, which do not lie on any singular line or at any singular points of the surface; n' is then the *class* of the surface.

The coordinates of these n' points satisfy the equations

$$\lambda f_x + \mu f_y + \nu f_z + \varpi f_w = 0 \tag{4}$$

$$\lambda' f_x + \mu' f_y + \nu' f_z + \varpi' f_w = 0 \tag{5}$$

as well as

$$nf \equiv x f_x + y f_y + z f_z + w f_w = 0. \tag{6}$$

Also, by hypothesis, f_x, f_y, f_z, f_w do not all vanish at these points; hence eliminating these differential coefficients between (4), (5), (6) and the identical relation (2), we have:

$$F \equiv \begin{vmatrix} \theta_1, & \theta_2, & \theta_3, & \theta_4 \\ x, & y, & z, & w \\ \lambda, & \mu, & \nu, & \varpi \\ \lambda', & \mu', & \nu', & \varpi' \end{vmatrix} = 0. \tag{7}$$

This is a surface of order $n - 2$, on which the n' points also lie.

Thus the n' points lie on each of the surfaces (4), (5), (7); but these surfaces cannot meet in more than $(n-1)^2(n-2)$ points, unless they have a common curve.

If possible let these three surfaces have a common curve; then if this curve also lie on $f = 0$ it follows from (4) and (5) that the tangent plane at every point of it passes through PQ, which is impossible unless it be a double (or multiple) curve on $f=0$. We may therefore assume that along this curve, assumed not to be a multiple curve on $f=0$,

$$(8) \qquad x f_x + y f_y + z f_z + w f_w = k,$$

where $k \neq 0$, except at a finite number of points where the curve meets $f=0$.

Solving for f_w from (2), (4), (5) and (8) we have

$$f_w . F = k \begin{vmatrix} \theta_1, & \theta_2, & \theta_3 \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix}.$$

But $F=0$ along the curve, therefore also along it

$$(9) \qquad \begin{vmatrix} \theta_1, & \theta_2, & \theta_3 \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix} = 0.$$

Thus the curve in question is some part of the intersection of the surfaces (5) and (9); but these are independent of ϖ, so that the curve remains fixed as ϖ varies continuously; accordingly it lies on all the surfaces given by (4) as ϖ varies continuously; hence it lies on $f_w = 0$. Similarly it lies on $f_x = 0$, $f_y = 0$, $f_z = 0$; it is therefore a double curve on $f=0$.

Again, since F is a linear combination of the determinants (3), the surface $F=0$ passes, with a certain multiplicity, through the multiple points and curves of $f=0$; let us suppose that these singularities absorb q of the intersections of (4), (5), (7), so that the remaining points of intersection are diminished to

$$(n-1)^2(n-2)-q.$$

We have thus the inequality

$$(10) \qquad n' \leqq (n-1)^2(n-2)-q.$$

But for a non singular surface

$$n' = (n-1)^2 n,$$

so that there must be enough singularities to diminish the class of the surface by at least

$$2(n-1)^2 + q.$$

We can obtain a second inequality of a similar character by considering the number of points of intersection of one of the polars, say (4), with $f = 0$, $F = 0$. By similar reasoning we can shew that these three surfaces can have no common curve other than a multiple curve on $f = 0$, so that the number of points of intersection distinct from singularities is $n(n-1)(n-2) - r$, where r is the number of intersections of the three surfaces absorbed by the singularities of $f = 0$. We thus obtain

$$n' \leqq n(n-1)(n-2) - r, \tag{11}$$

so that there must be enough singularities to diminish the class by at least

$$n(n-1) + r.$$

In the case of the simplest kinds of singular points and singular lines the numbers q and r can be calculated without difficulty; but in the more complicated cases I do not know of any methods that are generally applicable. Accordingly I only illustrate these inequalities by some very simple cases.

If any multiple point of $f = 0$ is equivalent to the same number of intersections of $f = 0$ with two polars on the one hand, and with one polar and $F = 0$ on the other hand, its presence effects both sides of (11) equally. This is the case with an ordinary conical point of order 2, which diminishes the class by 2, and with a biplanar point of the simplest kind, which diminishes the class by 3 and also counts triply as an intersection of $F = 0$ with $f = 0$ and a polar, since it can easily be shewn that $F = 0$, like a polar, has a tangent plane passing through the intersection of the two tangent planes to the surface at the biplanar point. It follows that if the only singularities of the surface are double points of these two species, the inequality (11) is impossible. We thus get the result:

a surface, the only singularities of which are double points which diminish the class by 2 *or* 3, *can have no integral of the first kind of a total differential.*

Let us next suppose that the only singularity is a nodal double curve, reducible or otherwise, of order m, with h apparent double points and t actual triple points; then if there are no further singularities on the curve, other than those which result necessarily from these characteristics, it is known[1] that the curve diminishes the class of $f=0$ by

$$m(7n-4m-8)+8h+9t.$$

Also since $F=0$ and the two polars pass through this curve it absorbs at least

$$q\equiv m(n-1+n-1+n-2-m-1)+2h$$

of the points of intersection of the three surfaces[2].

Similarly the curve absorbs at least

$$r\equiv m\{n+2(n-1)+2(n-2)-2m-2\}+4h$$

of the points of intersection of $F=0$, a polar and $f=0$.

Substituting in (10) and (11) we have the inequalities

(12) $$m(4n-3m-3)+6h+9t\geqq 2(n-1)^2$$

and

(13) $$2m(n-m)+4h+9t\geqq n(n-1).$$

These formulæ may be illustrated by the cases of quartic and quintic surfaces.

In the case of a quartic surface $(n=4)$, if $m>2$, the surface is rational or reducible; rejecting these cases we see that the only admissible solution of these inequalities is given by

$$m=2,\qquad h=1,\qquad t=0.$$

The nodal curve accordingly consists of two non-intersecting straight lines; and it is known that this quartic does admit of an integral of the first kind.

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[1] SALMON's *Geometry of three Dimensions*, § 94. I follow the notation of § 386, which is different from that of this article.

[2] *Ib.* § 386.

In the case of a quintic surface ($n = 5$), we can exclude for the same reason as before the cases of $m > 5$; the inequalities reduce to

$$m(17 - 3m) + 6h + 9t \geqq 32$$

and

$$m(10 - 2m) + 4h + 9t \geqq 20.$$

The inequalities obviously cannot be satisfied by $m = 1$ or $m = 2$. If $m = 3$, then

$$6h + 9t \geqq 8, \qquad 4h + 9t \geqq 8,$$

whence

$$h \geqq 2; \quad \text{or} \quad t \geqq 1.$$

In the former case we have a conic and a straight line, or three straight lines which are not coplanar, and in either case it is easily shewn that the surface is rational or reducible; in the latter case we have three straight lines meeting in a point.

If $m = 4$, then

$$6h + 9t \geqq 12, \qquad 4h + 9t \geqq 12,$$

whence

$$h \geqq 3, \quad \text{or} \quad h \geqq 1, t \geqq 1, \quad \text{or} \quad t \geqq 2.$$

It is easy to verify that in all these cases the quintic must be rational or reducible.

If $m = 5$, then

$$6h + 9t \geqq 22, \qquad 4h + 9t \geqq 20,$$

whence

$$h \geqq 5, \quad \text{or} \quad h \geqq 3, t \geqq 1, \quad \text{or} \quad h \geqq 1, t \geqq 2, \quad \text{or} \quad t \geqq 3.$$

It is again easy to verify that in all cases except the first the quintic must be rational or reducible if it can exist at all; and that we have left the case in which the double curve is an irreducible quintic with 5 apparent double points. I have verified by other methods that such a quintic effectively possesses an integral of the first kind.

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