ON THE FOUR ROTATIONS WHICH DISPLACE ONE ORTHOGONAL SYSTEM OF AXES INTO ANOTHER

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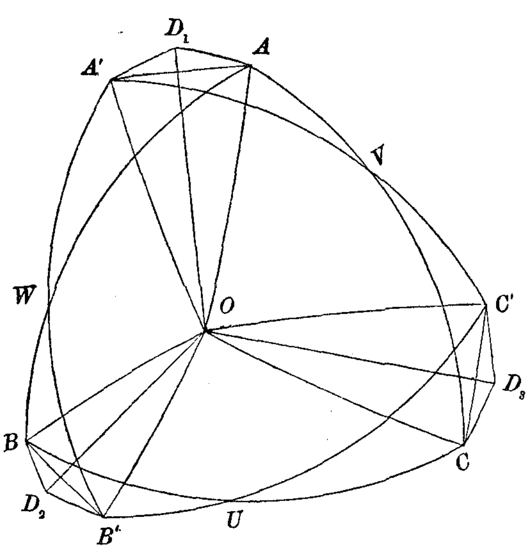
This question has been treated by Herr LIPSCHITZ in a very elegant analytical manner in the first three sections of a memoir in the Acta mathematica (Vol. 24, p. 123). It is the object of the following short note to shew that the question is susceptible of a simple kinematical treatment which brings out, even more clearly perhaps than an algebraical process, the essential space-relations of the configuration involved.

The one kinematical theorem of which repeated use is made, — namely that successive rotations about three radii of a sphere through twice the angles of the corresponding spherical triangle produce no displacement at all — is due originally, I believe, to HAMILTON (*Lectures on Quaternions*, p. 267). It is equivalent to a construction for the resultant of any two rotations about intersecting axes. Moreover from the theorem itself the converse — if successive rotations about two radii OP, OQ of a sphere have OR for the axis of their resultant, then the amplitudes of the two rotations are $2RPQ$ and $2PQR$ (mod 2π) — immediately follows. It may be noted that HAMILTON's theorem is established almost intuitively by drawing Op, Oq, Or perpendicular to the planes QOR, ROP, POQ. For successive rotations through two right angles about Oq and Or is the same as a rotation round OP through $2RPQ$. Hence successive rotations through $2RPQ$, $2PQR$, $2QRP$ about OP, OQ, OR are equivalent to successive rotations through two right angles about Oq, Or, Or, Op, Op, Oq, which clearly give no displacement.

I denote the two systems of orthogonal lines by $AO\bar{A}$, $BO\bar{B}$, $CO\bar{C}$ and $A'O\bar{A}'$, $B'O\bar{B}'$, $C'O\bar{C}'$, and I suppose that a rotation through an angle α in an assigned sense round an axis OD brings the first set to coincidence with the second. Then the resultants of rotations (i) α round OD and π round OA'; (ii) α round OD and π round OB'; (iii) α round OD and π round OC', each bring the first system to coincide with the second. Moreover if R is any rotation that has this effect; then the resultant of R reversed and α round OD brings the second system to coincide with itself. Hence the four rotations specified are the only ones which produce the required displacement.

If the rotation α round OD brings OA, OB, OC to coincidence with OA', OB', OC', then the remaining three rotations in order bring OA, OB, OC to coincidence with (i) OA', $O\bar{B}'$, $O\bar{C}'$, (ii) $O\bar{A}'$, OB', $O\bar{C}'$, and (iii) $O\bar{A}'$, $O\bar{B}'$, OC'.



Let $OA, \ldots, OA', \ldots$, meet a unit sphere centre O in A, B, C, A', B', C'. Join AA', BB' and CC', and bisect the first two arcs at right angles by D_1D and D_2D. The spherical triangles ADB, $A'DB'$ are equal in all respects, and therefore the angles AOA' and BOB' are equal. Hence a rotation round OD through ADA' [1] brings OA and OB to coincidence

[1] The sense of the rotation will be thus denoted; ADA' and $A'DA$ representing equal and opposite rotations.

with OA' and OB'; and therefore necessarily OC to coincidence with OC'. Hence D_3D bisecting CC' at rigth angles passes through D.

Draw AD_1, $A'D_1$ perpendicular respectively to DA and DA'; and similarly BD_2, $B'D_2$, CD_3, $C'D_3$ perpendicular to DB, DB', DC, DC'. Then by HAMILTON's theorem a rotation ADA' round OD followed by a rotation π round OA' has for resultant AD_1A' round D_1. Hence OD, OD_1, OD_2, OD_3 are the axes of the four required rotations and ADA', AD_1A', BD_2B', CD_3C' are the corresponding amplitudes.

Let U be the point of intersection of BC and $B'C'$. Then a rotation BUB' round OU followed by a rotation round A' through the difference of BU and $B'U$ brings OA, OB, OC to coincidence with OA', OB', OC'; and is therefore equivalent to a rotation ADA' round D. Hence by the converse of HAMILTON's theorem the angle $A'UD$ is one half of BUB', and the angle $A'DU$ is the supplement of one half of ADA'. It follows that D_1D passes through U. Again it is evident from the figure that a rotation BUC' round OU followed by a rotation $BU + UB'$ round OA' brings OA, OB, OC to coincidence with $O\bar{A}'$, OB', $O\bar{C}'$; and is therefore equivalent to a rotation BD_2B' round D_2. Hence by the converse of HAMILTON's theorem the angle D_2UA' is one half of BUC'. Therefore D_2U bisects BUB'; and similary D_3U bisects CUC', so that D_2D_3 passes through U. Moreover since $A'UD$ is half BUB', while $A'UB'$ is a right angle, it follows that DU and D_2U are at right angles. Hence DD_1 and D_2D_3 meet at right angles in U: and similarly DD_2, D_1D_3 meet at right angles in V, and DD_3, D_1D_2 in W. The axes of the four rotations must therefore be such that the planes through any two pairs, into which they may be divided, are at right angles. The well known trigonometrical conditions for this, which may be derived directly from the figure, are

$$\cos DD_1 \cos D_2D_3 = \cos DD_2 \cos D_3D_1 = \cos DD_3 \cos D_1D_2.$$

Further it follows at once from the figure that when D, D_1, D_2, D_3 are given in position on the sphere, subject to these conditions, the corresponding angles of rotation are uniquely determinate. Thus since the two triangles D_2DD_3, BDC have the same angle at D, while BC is a quadrant,

$$\cos D_2D_3 = \cos DD_2 \cos DD_3 - \sin DD_2 \sin DD_3 \cot DB \cot DC.$$

But if α is the angle of rotation round OD, so that

$$BDD_2 = CDD_3 = \frac{\alpha}{2},$$

then

$$\cot DB = \cot DD_2 \sec\frac{\alpha}{2},$$

and

$$\cot DC = \cot DD_3 \sec\frac{\alpha}{2}.$$

Hence

$$\cos D_2D_3 = -\tan^2\frac{\alpha}{2}\cos DD_2 \cos DD_3;$$

or

$$\tan^2\frac{\alpha}{2} = -\frac{\cos D_2D_3}{\cos DD_2 \cos DD_3},$$

which in virtue of the relations connecting the arcs is unaltered by any permutation of the suffixes 1, 2, 3. Similarly if $\alpha_1, \alpha_2, \alpha_3$ are the angles of rotation about the other three axes,

$$\tan^2\frac{\alpha_1}{2} = -\frac{\cos D_2D_3}{\cos D_1D_2 \cos D_1D_3},$$

$$\tan^2\frac{\alpha_2}{2} = -\frac{\cos DD_1}{\cos D_2D \,.\, \cos D_2D_1},$$

$$\tan^2\frac{\alpha_3}{2} = -\frac{\cos DD_1}{\cos D_3D \,.\, \cos D_3D_1}.$$

The angles $\alpha, \alpha_1, \alpha_2, \alpha_3$ are therefore either all real or all imaginary. When the four points D, D_1, D_2, D_3 are given on the sphere, subject to the two conditions,

$$\cos DD_1 \cos D_2D_3 = \cos DD_2 \cos D_3D_1 = \cos DD_3 \cos D_1D_2,$$

the necessary and sufficient conditions that the α's should be real is clearly that either just three or all six of the arcs $DD_1, \ldots,$ should lie between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. When this condition is satisfied the angle α is (mod 2π)

uniquely determined, and the positions of A, B, C, A', B', C' are obtained by drawing DA, DA', ..., making angles $\frac{\alpha}{2}$ with DD_1, ..., and drawing perpendiculars to them from D_1, ...

Lastly from the previous equations, the further system

$$\cos^2 DD_1 = \cot^2\frac{\alpha}{2}\cot^2\frac{\alpha_1}{2}, \ldots,$$

$$\cos^2 D_2D_3 = \cot^2\frac{\alpha_2}{2}\cot^2\frac{\alpha_3}{2}, \ldots,$$

may be immediately deduced. Hence when the amplitudes of the rotations are given (subject for real displacements to certain obvious inequalities) the relative position of the four axes is determinate.