THE CLASSIFICATION OF SETS OF POINTS.

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Preface.

The origin of this paper was a desire for a definition of a plane curve which should require a curve to be in some sense in one piece without requiring it to be closed or to be of the very special character of a Jordan curve. To take a simple example, there must be some sense in which a lemniscate deprived of any point but the node is a single curve but a lemniscate deprived of the node is two curves. The discovery of the property of a set which I define in section 20 and describe by saying that a set is united led to the definition of a curve on a surface which is given below in section 38 of the paper,[1] but aroused a fresh discontent, since this definition gave no clue to the distinction in three-dimensional space between a curve and a surface, a distinction which the definition since evolved, given in section 36, enables me to draw.

Although the work was begun for the sake of a theory of dimensions, it is not on account of the theory suggested in the concluding sections that this paper is published; much remains to be done before that theory can be proved valuable or valueless. But of the thirty-nine sections of this paper the first thirty-five are concerned only with ideas which certainly have technical use as well as intrinsic interest; among these, the fundamental idea of which I have found[2] no

[1] These definitions of a united set and of a curve on a surface were given in a short note entitled »Definition of a plane curve» in *The Journal of the Indian Mathematical Society,* Vol. vii. pp. 175—177 (1915), the set being there described as *perfectly connected.*

[2] (Added November 1916) This statement only reveals my ignorance when it was written. Undeniable traces are to be found in SCHOENFLIES' definition (*Math. Annalen*, bd. 58 (1904), s. 210) of a plane set of a special kind as *coherent* if every pair of its members can be joined by a simple path within the set, and in a footnote (*American Journal of Mathematics*, v. 33 (1911)

trace elsewhere is the idea associated here with the word unity, and the paper is essentially the offer of this idea for consideration.

The paper contains no proofs. It seems to me that proofs are almost worthless unless given in fundamental logical symbols, and at present I have not the time to prepare detailed proofs for publication. Moreover, many of the ideas here put forward, and in particular the idea of unity and the derived idea of a cell, are to be justified less by propositions than by examples, and in any case an account of a subject in general terms is a valuable preliminary to a formal development.

Certain acknowledgments of debt I am glad to make: my definition of a congregate of points was suggested in part by JORDAN's treatment of a closed set *d'un seul tenant,* in part by the desire for a definition expressible in a simple form with the symbolism of »Principia Mathematica», the use of this symbolism alone has enabled me to construct proofs of the propositions which I assert and of others that shew the importance of the various definitions, and to the influence of RUSSELL and WHITEHEAD are due among other features the form in which the subject matter of the paper is described in section 2 and the recognition at various points of the relevance of the multiplicative axiom.

This work was done in the first instance in ignorance of FRÉCHET's theory [1] of dimension-types, but a comparison of the definitions of section 36 with his definitions is to be found in the closing section. Since the idea of a united set does not enter into FRÉCHET's theory, my work is altogether independent of his, but probably an earlier acquaintance with FRÉCHET's paper would have led me to develop the present theory as a contribution to the classification rather of abstract aggregates than of sets of points. The universe with which I deal explicitly is less general than FRÉCHET's class L (l. c., s. 145) and more general than his class E (l. c., s. 160), inasmuch as I assume the existence of separating numbers but assume neither that two points separated by the number zero *necessarily* coincide nor that the inequality $xz \overline{<} xy + yz$ is true for *every* set of three elements; exactly how much of my paper is in fact independent of the use of numbers I do not

p. 319) in which LENNES remarks that obviously there are complete connected sets which contain no continuous arcs joining certain pairs of their points, the one writer using »simple path» and the other »continuous arc» according to a precise definition. LENNES' definition, unlike SCHOENFLIES', is applicable to aggregates in general, but neither writer anticipates the kind of use made of the conception in the present paper.

[1] MAURICE FRÉCHET, »*Les dimensions d'un ensemble abstrait*», *Math. Annalen,* bd. lxviii, s. 145—168 (1910): the study of dimensions is now associated so closely with the name of BROUWER that it is as well for me to state definitely that none of his work that I have seen indicates a perception of what I call unity, and that there is therefore nothing in this paper for which I am indebted to Prof. BROUWER.

know, but it is evident that the definitions of *united sets* and of *cells* require only the assumption that such limits as exist are members of the universe, and that if limits are defined by means of neighbourhoods, then *plots* and *confined limiting points* also can be defined, whether or not the definition of a neighbourhood is numerical. On one detail, not of logic but of nomenclature, I incline to disagree with M. FRÉCHET, who wishes to restrict the phrase »of n dimensions» to an aggregate whose type is that of the complete geometrical space with n coordinates; since there is no ordinal similarity between the types of dimensions and the signless real numbers, insistence on a one-one correlation in the particular case in which the numbers are integral is actually misleading, and I prefer to allow that the surface of a sphere may be described legitimately as well as popularly as two-dimensional. Nevertheless, not actually to contradict M. FRÉCHET, I have spoken of the dimension-integer rather than of the number of dimensions of a set.

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1. Introduction.

The vocabulary of the theory of sets of points is evidence that it has been left too much to the writers whose interests are primarily philosophical to insist on the extent to which the nature of the derivative of a set of points Γ depends on the nature of the universe of points V of which Γ is a part. Technical mathematicians, instead of recognising frankly that they need to deal sometimes with a universe of one kind, sometimes with a universe of another kind, have expended a considerable amount of ingenuity in reducing their special universes by a variety of more or less elaborate conventions to a common pattern. What is most surprising is that the chosen pattern is a universe of a different kind from Euclidean space and that the conventional reduction of Euclidian space to the standard form is no easy task.

A universe of points V is said to be limited or closed if every infinite set of points in V has a limiting point in V. Thus the surface of a sphere is a closed universe, and so is the set formed of all the points inside or on the perimeter of a plane triangle: on the other hand, the points inside a sphere do not compose a closed universe, neither do the points inside a triangle, nor, to take the most important cases of all, do the points of Euclidean space or of a Euclidean plane. Mathematicans, fully alive in some connections to the advantages of dealing with open rather than with closed sets, as is shewn in the use made of circular domains in the theory of functions of a complex variable, have nevertheless shewn curious and I think unwarrantable reluctance to contemplate an open universe of points.

The first difficulty for which this reluctance is responsible in the treatment of Euclidean space is in the application of the words closed and open to unlimited sets, that is, to sets tending in some manner to infinity. The obvious interpretation, sanctioned by the effects of establishing, for example, a point-to-point correspondence between the finite points of a plane and all the points except one on a sphere, is to regard an unlimited set as closed or open according as points at infinity have or have not an actual existence. This interpretation does not meet the difficulty: unless by the invention of ideal points existence has been conferred on points at infinity, such limiting points as a set posesses must be in the finite part of the universe, and a set which tends to infinity but has no finite limiting point simply has no limiting point whatever, and cannot be regarded as an open set without a change in the fundamental definitions; moreover, the introduction of ideal points and points at infinity is often accompanied by a change in the definition of distance, and if a point at infinity is to be regarded as a limiting point the definition of a limiting point must be changed from the form which is naturally the first to be adopted.

We must not be supposed to assert that the difficulties in this subject cannot be surmounted, or rather evaded, by a multiplication of conventions. Every student of the theory of functions of a complex variable knows that much can be accomplished by introducing what is really an incomplete symbol z_{∞} or ∞, a symbol of which only the uses are defined, it being agreed for example that to write

$$|z-\infty|<\varrho$$

is to mean

$$|z|>1/\varrho.$$

But even in this case the properties of ∞ as a limit are different from those of any other limit; for example, if (u_n) and (v_n) are two sequences of complex numbers with a common finite limit, u_n-v_n must tend to zero, while if (u_n) and (v_n) both tend to ∞, that is, to infinity, u_n-v_n may behave in any manner as n increases, converging to any finite number, oscillating in any way, or diverging to infinity. It is not, however, on technical troubles of any particular kind that the case for a candid consideration of facts is based: to be content rather to evade a difficulty than to analyse it and to understand it is the mark distinguishing not the technical mathematician from the philosopher but the computer from the mathematician.

2. Points and Separating Numbers.

The one requirement of the theory before us is a one-many relation, subject to certain conditions, between signless real numbers and cardinal couples of individuals. The individuals, in virtue solely of the fact that every member of the converse domain of the fundamental relation is a pair of them, are called points, and the real number associated with a pair of points is called the separating number of the two points, or between the two points, forming the pair, or the number separating one of the points from the other. We use for points the letters v, w, x, y, z and we denote the separating number of x and y by xy; because the separating number of x and y is a number related to the cardinal couple composed of x and y, not to either of the ordinal couples composed of the same individuals, yx is the same as xy, and the further hypotheses as to the nature of the fundamental relation must be enumerated. It is assumed not only that it is between pairs of points that there are separating numbers, but also that if there are separating numbers yv and zw there is a separating number yz, an assumption we express by saying that our points compose a universe. It is assumed that if the two points forming a cardinal couple are identical, the number associated with the couple is zero, that is, that if x is any point, the separating number xx is zero, but it is not assumed that the separating number of two distinguishable points is necessarily different from zero. It is assumed that the class of signless real numbers defined by the criterion that ϱ belongs to the class if for every group of three points x, y, z the separating number xz is equal to or less than the sum of the separating numbers xy, yz provided only that xy and yz are both less than ϱ is a class which has members other than zero: from the form of the class, if σ belongs to it so does every number less than σ; in Euclidean space every real number belongs to this class, and by dealing with the class itself we avoid both the indefinite and unnecessary restriction involved in choosing one of its members and the introduction symbolically of an infinite number. From the assumption just explained it follows that if the separating number yz is zero, then for every point x for which either xy or xz is sufficiently small, xy is equal to xz, and we make the final assumption that in fact if yz is zero then for *every* point x the numbers xy and xz are the same.

It must be realised that the separating number of two points need not be the distance between the points in any space in which they can be represented. As an example of the latitude we allow, the universe might consist of the surfaces of two non-intersecting spheres in Euclidean space, and we might take for the separating number of two points on the same sphere their least geodesic distance apart and for the separating number of two points of which one is on each of the spheres their distance apart in space. Moreover, the separating number regarded as a function of the two points on which it depends, may vary considerably in form without involving any change in the classification of classes of points with reference to any of the properties which it is our object to describe. Indeed, in the case in which the universe is space of a finite number n of dimensions and a point x is determined by its n coordinates $x_1, x_2, \dots x_n$, while some writers use for the separating number of two points $\dot{x}, \ddot{x}$ the distance between them, which if the axes are orthogonal is $\sqrt{\{\Sigma(\dot{x}_r - \ddot{x}_r)^2\}}$, others, following JORDAN, use for the separating number $\Sigma|\dot{x}_r - \ddot{x}_r|$, and others again, including MOSKOWSKI, use[1] the greatest among the numbers of the form $|\dot{x}_r - \ddot{x}_r|$; none of the properties of classes of points which separating numbers are used to described are affected by a choice between these three functions.

An excellent example of the mode of embodying ideas in the definition of separating numbers when these numbers are divorced entirely from distances is to be found in the geometrical treatment of the complex variable. To justify the usual method of dealing with the point at infinity, it is common to suppose a sphere drawn to touch the plane of the variable z at the origin O, and to say that often a value z of the complex variable is associated not with a point z_p of the plane but with a corresponding point z_s of the sphere, z_s being obtained by joining z_p to the point of the sphere diametrically opposite to O; if we wish to concentrate our attention on the plane, we have only to say that limiting points are determined not with reference to distances in the plane but with reference to a distinct system of separating numbers, the separating number of two points $\dot{z}_p$, $\ddot{z}_p$ of the plane being the distance between the corresponding points $\dot{z}_s$, $\ddot{z}_s$ of the sphere or some other number suggested by geometry, but the most satisfactory method is to regard the complex variable itself as a point, and the separating number not as a number separating two points of a plane or of a sphere but as a number separating two values of the complex variable. Probably the most useful number separating the complex numbers $\dot{z}$, $\ddot{z}$ is

$$|\dot{z} - \ddot{z}|/\sqrt{\{(1 + |\dot{z}|^2)(1 + |\ddot{z}|^2)\}},$$

which is the actual distance in space between the corresponding points on a sphere of unit diameter, but the number

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[1] This last choice has the advantage of being far more widely applicable in space of an infinity of dimensions than either the Cartesian measure or JORDAN'S: I have found the consideration of this kind of space with this species of separating number valuable in removing prejudices.

$$(|\dot{x}-\ddot{x}|+|\dot{y}-\ddot{y}|)/\{(1+|\dot{x}|+|\dot{y}|)(1+|\ddot{x}|+|\ddot{y}|)\},$$

constructed on JORDAN's model, is equally effective; denoting either of these separating numbers by $(\dot{z}\ddot{z})$ we can define the uses of the symbol ∞ by the assertion that $(z\infty)$ denotes $1/\sqrt{(1+|z|^2)}$ or $1/(1+|x|+|y|)^2$ and that $(\infty\infty)$ is zero, and ∞ then *exists*, for the statement that the complex number ∞ exists *means* only that the numbers separating ∞ from all complex numbers including itself are defined and satisfy certain conditions.

More interesting is the treatment of RIEMANN surfaces, which we may illustrate by the simple example of the surfaces connected with the representation of $w^{1/2}$. In the primitive form, these are a double plane and a double sphere, with cross-connections along the positive halves of the real axes in one case and along overlying semicircles in the other case. Logically, in the space in which $w^{1/2}$ is a one-valued function each point consists of a complex number z and a variable t which has only two values (most conveniently, 1 and 2); if m and n coincide and if in a plane corresponding to the one variable z the chord joining $\dot{z}_p$ to $\ddot{z}_p$ does not cross the positive half of the real axis, or if m and n are different and if in this plane the chord does cross this half-axis, then the number $\dot{z}_m\ddot{z}_n$ separating the RIEMANN point $\dot{z}_m$ from the RIEMANN point $\ddot{z}_n$ is a number $(\dot{z}\ddot{z})$ such as was defined in the last paragraph, while in other cases the separating number $\dot{z}_m\ddot{z}_n$ is the smaller of the two numbers $(\dot{z}0)+(\ddot{z}0)$, $(\dot{z}\infty)+(\ddot{z}\infty)$, where 0 denotes the complex number zero. Here we have distinct points at zero distance apart, for it is simpler to say that there are two points $0_1, 0_2$ and two points ∞_1, ∞_2 and that $0_1 0_2$ and $\infty_1\infty_2$ are zero than to say that while almost every point of the RIEMANN surface consists of a complex number and a two-valued variable there are certain points which consist of complex numbers alone.

3. Sets and Classes of Sets.

Our subject is the classification of classes of points, and for the sake of emphasis a class of points is called a set of points, or simply a set, while set is not regarded in any other sense as equivalent to class. A set is usually determined by the statement of a group of conditions which a point is to satisfy if it is to belong to the set; if there are no points satisfying simultaneously all the prescribed conditions, the group of conditions is said to determine the null set. For sets in general we use the letters Γ, Δ, Θ, Φ, Ω, the universe is itself a set, which is denoted by V, and the null set is denoted by Λ. For a general class of sets we use γ, while $\varkappa`\Gamma$ and $\tau`\Gamma$ denote classes of sets related in a special manner to a set Γ; thus each of the letters $\varkappa$, τ may be regarded as denoting either a relation between a class of sets and a single set or an operation by which from any set is derived a class of sets having a particular relation to that set. Similarly we use Latin capitals other than V to denote particular relations between one set and another, or operators deriving from any given set certain other sets related to it, while we use $K'(x, \Gamma)$ and $T'(x, \Gamma)$ for special sets connected with a point x and a set Γ.

When we have to deal with classes of sets, the whole theory of selections is applicable to our work, and we have to consider whether or not in a class of sets γ every member Γ has a representative. We say that the universe is a Zermelo universe if in every class of existent sets each set has a representative, or in other words if every class of mutually exclusive existent sets is multipliable, and we say that a set Γ is a Zermelo set if every class of mutually exclusive existent sets contained in Γ is multipliable; in a Zermelo universe every set is a Zermelo set.

4. Bounded Sets and Sets with Separating Numbers Finite; the Span of a Set.

Sets and universes may be classified according to the magnitude of the separating numbers which are possible between points belonging to them. A set is bounded with reference to an origin v if there is a finite upper limit or maximum to the numbers separating its points from v. If there is a number σ, necessarily different from zero, such that if x, y belong to Γ, then xy is less than σ, there is a least number ϱ which is such that if x, y belong to Γ and σ is greater than ϱ, then xy is less than ϱ; this number is an upper limit or maximum to numbers separating pairs of members of Γ, and is called the span of Γ. If xz is never greater than $xy + yz$, a set bounded with reference to any origin has a finite span.

A set is a set with infinite separating numbers or a set with separating numbers finite according as it does or does not contain two points whose separating number is infinite. Every set with separating numbers limited is a set with separating numbers finite, but the converse is not true: for example, in Euclidean space, not subjected to any conventional closing, the distance between any two points is finite, but there is no upper limit or maximum to the distances possible; by replacing the points and lines of Euclidean space by ideal points and lines we can obtain a space in which there are actual points at infinity, and in this space infinite distances are possible.

5. Units.

A set which includes a point x need not include all the points separated from x by the number zero, and we denote by $T'(x, \Gamma)$ the set composed of all the points of Γ separated from the point x of Γ by the number zero, calling $T'(x, \Gamma)$ the unit of Γ containing x; if x does not belong to Γ, the set $T'(x, \Gamma)$ is defined to be null, even if Γ contains points separated from x by the number zero. A set is called a unit of Γ if it is the unit containing some point of Γ, and we denote by $\tau'\Gamma$ the class of units of Γ. If y is a member of $T'(x, \Gamma)$, the units $T'(x, \Gamma)$ and $T'(y, \Gamma)$ are identical, and x and y contribute the same member to the class $\tau'\Gamma$.

If Γ is not null, the class $\tau'\Gamma$ is a class of mutually exclusive existent sets whose sum is Γ; even if Γ is null, the null set is not a member of $\tau'\Gamma$: if Γ is null, $\tau'\Gamma$ is the null class of sets, not the class whose member is the null set.

The set $T'(x, V)$ is a unit of the universe, and while every unit of a set Γ is contained in some unit of the universe, a unit of Γ need not coincide with the unit of V which contains it, and the class $\tau'\Gamma$ need not be contained in the class $\tau' V$.

6. The Cardinal Number and the Reduced Number of a Set.

The number of different points belonging to a set Γ is called the cardinal number of the set and denoted by $Nc'\Gamma$. A set is called finite or infinite according as its cardinal number is or is not inductive, and is called singular if it has only one member, plural if it has more members than one.

For many purposes the number connected with a set Γ which is of greatest importance is not the number of points belonging to Γ but the number of units contained in Γ, that is, the number of sets belonging to $\tau'\Gamma$; this number, $Nc'\tau'\Gamma$, we call the reduced number of Γ and denote by $Ncr'\Gamma$. We say that a set is scattered finitely or infinitely according as its reduced number is or is not inductive, and we call a set a unit set if it has only one unit, a multiple set if it has more units than one.

In formal work unit sets, both singular and plural, require attention out of all proportion to their interest, which is negligible. The universe may itself be finite, in which case every set is finite, or finitely scattered, when every set is finitely scattered, but none of our definitions give any but the most trivial results when applied to sets in a finitely scattered universe.

7. Reduction and Expansion.

To reduce a set is to omit from it any set of points each of which is separated by the number zero from some one of the points retained. Reduction is possible as long as the set contains units which are not singular, and we say that a set or a universe is fully reduced if all its units are singular, that is, if it includes no two distinguishable points whose separating number is zero. If a set is fully reduced its cardinal number is the same as its reduced number, but the converse implication holds only if the set is finite. A set is called a reduced form of a set Γ if it is contained in Γ and includes one and only one member of each unit of Γ; in more technical language, the reduced forms of Γ are the selected sets of the class of units of Γ. To say that reduced forms of Γ exist is to assert that the class of units of Γ is multipliable, and since this assertion can not be made of every set unless the multiplicative axiom is assumed, we content ourselves with describing a set as reducible if there are reduced forms of it. In a fully reduced universe every set is fully reduced, but a reducible universe may contain sets that are not reducible; in a ZERMELO universe all sets are reducible, but we have no reason to suppose that if every set is reducible the universe is a ZERMELO universe, for every class of unit sets might be multipliable while some classes including multiple sets were not multipliable. It is important to notice that every finitely scattered set is reducible.

The converse of reducing a set is adding to it any set each of whose points is separated by the number zero from some point already included, and this process is called expanding the set. Expansion is possible if there are units of the set which do not coincide with the units of the universe containing them, and we say that a set is fully expanded if every pair of points of which one belongs to the set and the other does not has a separating number different from zero. The set obtained by expanding Γ fully is the sum of the class of units of the universe containing points of Γ, and we denote it by $E`\Gamma$. If $\Gamma$ is a set having points in common with another set $\Delta$, to expand $\Gamma$ in $\Delta$ is to add to $\Gamma$ any members of $\Delta$ separated by the number zero from points belonging to both $\Gamma$ and $\Delta$, and we denote by $E_\Delta`\Gamma$ the set obtained by expanding as far as possible in Δ the part of Γ contained in Δ; the set $E_\Delta`\Gamma$ is contained in both $E`\Gamma$ and Δ, but there may be points common to $E`\Gamma$ and $\Delta$ which do not belong to $E_\Delta`\Gamma$.

8. Sequences.

Of infinitely scattered sets the simplest are fully reduced sets whose members can be put into a one-one correlation with the inductive cardinals; if such a correlation has been established, and x_n is the point corresponding to n, then the relation of n to x_n is called a sequence and denoted by (x_n), and the set of points is the converse domain of this relation.

9. Complements.

The points which do not belong to a set Γ compose a set called the complement of Γ, which we denote by $C`\Gamma$. If $\Gamma$ and $\Delta$ are any two sets, the points of $\Delta$ which do not belong to $\Gamma$ form a set called the complement of $\Gamma$ in $\Delta$, which we denote by $C_\Delta`\Gamma$; if Δ is contained in Γ, there is no point of Δ which does not belong to Γ, and $C_\Delta`\Gamma$ is null.

10. Neighbourhoods.

By the neighbourhood of x with radius ϱ we mean the set composed of all points separated from x by numbers less than ϱ. The character of a neighbourhood depends on the nature of the universe; for example, if the numbers separating x from points outside the unit to which it belongs have a lower limit or minimum σ other than o, then for any value of ϱ between o and σ the neighbourhood of x with radius ϱ coincides with the unit which includes x. Whatever the character of the universe, the neighbourhood of any point x with zero radius is null, but every other neighbourhood of x includes at least the one point x, and neighbourhoods with radii that are not zero may be distinguished as existent neighbourhoods.

11. Limiting Points and Limited Sets.

Having agreed rather to classify universes according to their properties than to consider only universes of special kinds, we define the limiting points of a set by the definition that x is a limiting point of Γ if there is a point of Γ outside the unit containing x in every existent neighbourhood of x, and we admit no modification which leads to a change in the content of the set of limiting points of any set whatever. The set whose members are the limiting points of Γ is called the derivative of Γ and denoted by $D`\Gamma$.

It is a fundamental proposition that every set which has a limiting point is infinitely scattered; the converse is not true. There are universes in which every infinitely scattered set has a limiting point, and such universes are said to be limited; for many purposes the distinction between a limited universe and an unlimited universe is the most important distinction between one universe and another. In any universe a set is called unlimited or limited according as it does or does not contain an infinitely scattered set without a limiting point, and a limited universe may be defined as a universe in which every set is limited. It was proved by WEIERSTRASS and is said by YOUNG to have been known to earlier writers that in the simplest types of space every infinitely scattered bounded set has a limiting point. But it is easy to show that this proposition is not true for all spaces: if the universe consists of all signless rational numbers in a finite stretch and separating numbers are arithmetical differences, then every set is bounded but no set is limited which in a wider universe would have irrational limits; again, if the point is a progression of numbers $(x_1, x_2, \ldots\ldots)$ and the separating number $\dot{x}\ddot{x}$ is the greatest among the differences $|\dot{x}_1 - \ddot{x}_2|, |\dot{x}_1 - \ddot{x}_2|, \ldots$, the sequence $(1, 0, 0, \ldots)$, $(0, 1, 0, \ldots)$, $(0, 0, 1, \ldots)$, ... in which the nth point has its nth coordinate unity and its other coordinates zero, is an infinitely scattered bounded set, with span unity, but with no limiting points.

12. Simple Sets.

A set which is the converse domain of a sequence may have no limiting points, or any finite number of limiting points; indeed, as is shewn by the familiar correlations of the rational numbers in a finite interval with the natural numbers, the cardinal number of the derivative of such a set may be an infinite number greater than the cardinal number of the set itself. A set is said to be simple if it is the converse domain of a sequence and has not more than one limiting point, and a sequence is said to be simple if its converse domain is simple. The use of simple sets comes from the theorem that each limiting point of any ZERMELO set Γ is the unique limiting point of some simple set contained in Γ, from which it follows that a ZERMELO universe is limited if every simple set has a limiting point.

In Euclidean space of any number of dimensions, if (y_n) is a sequence whose converse domain is simple and unlimited and x is any point of the universe, and if the separating number of two points is the distance between them, then xy_n tends to infinity, but this property is not common to all unlimited universes.

13. Adherences and Coherences, Isolated Points and Internal Points, Edges and Boundaries.

There is no general relation of inclusion between a set and its derivative. Points which belong to Γ but not to $D^{\text{‘}}\Gamma$ are called isolated points of Γ and compose a set known as the adherence of Γ, and points which belong both to Γ and to $D'\Gamma$ form a set called the coherence of Γ. Points of Γ which are limiting points of the complement of Γ constitute the edge of Γ, and points of Γ which do not belong to the edge of Γ are called internal points of Γ. Of the points of the derivative $D'\Gamma$, those which belong to Γ can of course be distinguished from those which belong to $C'\Gamma$, but it is unnecessary to invent names for the two sets indicated by this distinction, for the points common to $D'\Gamma$ and Γ compose the coherence of Γ, and the points common to $D'\Gamma$ and $C'\Gamma$ form the edge of the complement of Γ. Here we may add that the sum of the edges of Γ and $C'\Gamma$ is called the boundary of Γ.

If x is an isolated point of Γ, every point of the unit of Γ containing x is an isolated point, and the unit itself is called an isolated unit of Γ. If x is a limiting point of Γ, every point separated from x by the number zero is a limiting point of Γ, and therefore every derivative is a fully expanded set.

14. Complete Sets and Closed Sets.

We say that a set of points, limited or unlimited, is complete if it contains all its limiting points; thus in Euclidean space hyperbolas and helices are no less complete than circles, and a universe is necessarily complete. It is only to sets that are both complete and limited that we give the description of closed; a set is open if it is either incomplete or unlimited. Since a universe can not be incomplete, to say that a universe is closed is the same as to say that it is limited; moreover, in a limited universe all sets are limited, and therefore in a limited universe there is no difference between closed sets and complete sets.

15. Extension and Completion.

Extension of a set is addition to it of any set of its limiting points which it does not include originally, that is, addition of any part of the edge of the complement. If to a set Γ is added the whole of the edge of $C'\Gamma$, the set Γ is said to be completed or fully extended, and we denote the set obtained, which may be described simply as the sum of Γ and $D'\Gamma$, by $G'\Gamma$. If Γ is not limited, neither is $G'\Gamma$, and therefore $G'\Gamma$ is not necessarily closed, but the limiting points of $G'\Gamma$ are those of Γ itself and belong to $G'\Gamma$, whence it follows that $G'\Gamma$ is in all circumstances complete. The adherence of $G'\Gamma$ is the same as the adherence of Γ, and so $G'\Gamma$ need not be fully expanded, and for some purposes we have to use the set $E'G'\Gamma$ obtained by expanding $G'\Gamma$ fully, which is the same as the set $G'E'\Gamma$ obtained by completing $E'\Gamma$.

16. Dense Sets and Perfect Sets.

If every point of a set is a limiting point of the set, the set is said to be dense, and a set which is both complete and dense is called a perfect set, whether or not it is limited. A dense universe is necessarily perfect, but although the commonest and most important universes are dense, there is no logical necessity for a universe to be dense even if it is infinitely scattered. The simplicity of a dense universe comes chiefly from the fact that in a dense universe if Γ is any set whatever every point is a limiting point either of Γ or of $C'\Gamma$.

Any set obtained by extending a dense set is itself dense, and therefore the set obtained by completing any dense set is perfect. Although a dense universe must contain perfect sets, it is not every dense universe that contains sets both perfect and limited.

17. Inseparability.

We say that two sets are inseparable if the corresponding completed and fully expanded sets have at least one point in common; thus Γ and Δ are inseparable if there is a point which belongs to both $E'G'\Gamma$ and $E'G'\Delta$, that is, if there is a unit of the universe which contains members of both $G'\Gamma$ and $G'\Delta$. With this definition the null set is not inseparable from any set.

18. Congregates of Points.

A set in any universe we call a congregate of points if in every expression of it as the sum of two existent sets the two components are inseparable. If two sets have a common point they are certainly inseparable, and therefore although it is only a division into mutually exclusive sets that it is natural to contemplate when considering whether or not a set is a congregate, there is no need to complicate the definition of a congregate by requiring the sets considered to be mutually exclusive; since, however, the null set is not inseparable from any other set, it is necessary to insist that in every case neither of the components is null. Removing the word inseparable we may say that a set is a congregate if however it is expressed as the sum of two existent components the completed and fully expanded sets corresponding to the two components have at least one common point.

19. Connected Sets.

American writers have adopted a definition of considerable interest, which implies cohesion of a higher order than is essential to a congregate: a set is said to be connected if in every expression of it as the sum of two existent complementary components one of these components includes a limit of the other; the definition recalls the Dedekindian axiom of continuity. Obviously every connected set is a congregate, and it is easy to shew that every complete congregate is connected, but the set formed of those points of a sphere which do not lie on a particular great circle presents a simple example of a congregate that is not connected.

The step from a congregate to a connected set introduces precisely the condition essential for a theorem that is both interesting and valuable: a connected set that includes both a point that belongs to a set Δ and a point that does not belong to Δ necessarily includes a point on the boundary of Δ.

20. United Sets.

We come now to an idea[1] which appears to be of fundamental importance. We say that a set Γ is united if every pair of members of Γ is contained in a closed congregate contained in the fully expanded set $E^{\prime}\Gamma$. It can be shewn that every closed congregate is united, but while a united set is a congregate in the original sense it may be incomplete or unlimited and may indeed be both incomplete and unlimited. Closed congregates are necessary as a means to the definition and study of united sets, but the properties of closed congregates which are valuable and arise from the combination of their qualities seem to be precisely those which belong equally to all united sets.

As simple an example as there is of an open united set is the set formed of all the points on one side of a straight line in a reduced Euclidean plane, the distance between two points being chosen for their separating number. In Euclidean space of any number of dimensions, if y and z are any two points the set formed of y and z and all points between them on the straight line through them is called the closed chord joining them; this set is closed and connected. If y and z lie on the same side of any straight line in a plane every point of the closed chord joining them lies on the same side of that line. Hence in the set described, the closed chord joining any two points of the set is a closed congre-

[1] This idea seems narrowly to have escaped formulation by Prof. R. Baire, but I have failed to find an account of it in any of his writings that I have seen.

gate containing the points and contained in the set, and the set is therefore united, although it is unlimited because it extends to infinity and incomplete because the points of the straight line used in defining it are limiting points which it does not include. In Euclidean space a set Γ is called convex when if y and z are any two points belonging to Γ every point of the closed chord joining y and z belongs to Γ, and after what has been said it is hardly necessary to point out that if the separating numbers are distances every convex set is united.

The nature of unity may be illustrated further by means of a reduced Euclidean circle deprived of one point or of two points, the separating numbers again being distances. If Γ is such a circle and v, w are distinct points of Γ, then Γ itself, the set obtained by removing v from Γ, and the set obtained by removing from Γ both v and w, are all limited congregates, but only the first of them is complete.. The second of these three sets is, however, united, since if y, z are any two of its points the arc of Γ which has y and z for its end points and does not include v is a closed congregate containing y and z and contained in the set in question. But the third set is not united, for there are two arcs of Γ which have v and w for end points, and if y is any point distinct from v and w in one of these arcs and z is any point distinct from v and w in the other, every complete congregate contained in Γ and containing both y and z includes either v or w, and there is therefore no closed congregate containing y and z and contained in the set obtained by depriving Γ of v and w.

An example even more instructive than the last is given by a lemniscate or any other figure which might be described as a figure-of-eight. This is a closed connected figure and the set obtained by removing from it any one point is a congregate although it is incomplete; if the point removed is any other point than the node, or if the curve is supposed to have two distinct points at the node and only one of these is removed, the set remaining is united, but if the node is regarded as one point and is removed, or if all the points at zero distance from the node are removed, there remain two distinct sets, each in itself united, which do not together form a united whole.

A multiple united set is necessarily connected, but an example proves that the converse is not true, and that even a complete connected set may fail to be united. As simple an example as any is given by a branch of a hyperbola in a Euclidean plane together with any stretch contained in one of the asymptotes and a sequence of lines in which every line is parallel to this asymptote and cuts the curve and the sequence has the asymptote for sole limit: this set is connected and is complete if the stretch consists of the whole asymptote but every connected component which includes both a point in the asymptote and a point not on that line is unbounded and is therefore open since the space is Euclidean. In this example it is to be remarked that the set deprived of the points belonging to the asymptote is a united set, whence follows the interesting theorem that extension, whether partial or full, of a united set may result in a loss of the unity.

Since every multiple united set is connected, such a set cannot leave any other set without crossing the boundary of that set. In a dense universe, if a unit contains both points of a set Γ and points of the complement $C'\Gamma$, either all the points of Γ in the unit or all the points of $C'\Gamma$ in the unit belong to the boundary of Γ, and therefore in a dense universe the theorem just stated can be enunciated of all united sets and not of such only as are multiple. More generally, if a united set which is not contained in an isolated unit of the universe includes both a point of a set Γ and a point of the complement of Γ, it includes a point of the boundary of Γ.

To illustrate the way in which the result enunciated in the last paragraph may be used to endorse the conclusions of common sense, let the universe be a reduced Euclidean plane, let Δ be a straight line in this plane, and let Θ be the set formed of points on one side of this straight line and Φ the set formed of points on the other side, separating numbers being distances. Then as already remarked, Θ and Φ are both united. If, however, y is a point of Θ and z a point of Φ, then since z belongs to the complement of Θ and every closed congregate is connected, every closed congregate containing both y and z contains a point of the boundary of Θ, that is, a point of Δ. If then Γ is the sum of Θ and Φ, there are points y and z belonging to Γ such that every closed congregate containing y and z contains a point of the complement of Γ, or in other words such that there does not exist a closed congregate containing y and z and contained in Γ; that is to say, the complement of a straight line in a Euclidean plane is the sum of two united sets but is not itself united. If, however, to the set Γ we add a single point x belonging to the line Δ, then if y and z are any two points of the resulting set, the sum of the closed chords joining y to x and x to z is a closed congregate containing y and z and contained in the set considered, and this set is therefore united.

21. The Separating Number of Two Sets.

From this point we propose a digression to compare the qualities we indicate by calling a set a congregate with the properties associated with the word connex. The digression, interesting in itself, begins with the introduction of a notion that can be made to serve many useful purposes.

If Γ and Δ are any two sets, the class of signless real numbers each of which is not less than the number separating some point of Γ from some point of Δ is such that if σ belongs to it so does every number greater than σ; this class has therefore a lower limit or minimum, and this lower limit or minimum we call the separating number of the two sets Γ and Δ. If the separating number is a true minimum, that is, an attained minimum, of the class by which it is defined, then there exist points y, z belonging one to each of the sets Γ, Δ, such that the separating number of the two sets is the separating number yz of the two points, but if the separating number of the two sets is an unattained lower limit, then even to be entitled to assert that either there are a point y belonging to one of the sets and a simple sequence (z_n) whose converse domain is contained in the other set such that the separating number yz_n tends to the separating number of the sets, or there are simple sequences (y_n), (z_n) whose converse domains are contained one in each of the sets Γ, Δ such that $y_n z_n$ tends to the separating number of the sets, we must know that the sets are Zermelo sets. It is convenient to modify the conclusion just reached as to Zermelo sets by using the fact that a simple sequence (x_n) such that for every value of n the point x_n belongs to a set Θ is either limited or unlimited and if limited has a limiting point belonging to $D' \Phi$. We deduce that if two sets Γ, Δ are Zermelo sets, and their separating number is ϱ, then either there are points y, z belonging one to each of the completed sets $G'\Gamma$, $G'\Delta$ such that the separating number yz is equal to ϱ, or there are a point y belonging to the set obtained by completing one of the sets Γ, Δ and a sequence (z_n) whose converse domain is contained in the other of the sets Γ, Δ and has no limiting point, such that yz_n tends to ϱ, or there are sequences (y_n), (z_n) whose converse domains are contained one in each of the sets and have no limiting points, such that $y_n z_n$ tends to ϱ. This is a result to which it is easy to apply knowledge of distinctive properties of the sets Γ, Δ or of the universe; for example, in Euclidean space of any finite number of dimensions with distances for separating numbers the second case cannot occur, while in any limited Zermelo universe if the separating number of two sets is ϱ, there are points y, z belonging one to each of the corresponding completed sets such that yz is equal to ϱ. It is noteworthy that extension and expansion, partial or full, are without effect on the separating number of two sets.

An example in which the separating number of two sets is an unattained limit will prove useful for reference. In a Euclidean plane with distances for separating numbers, let Γ be a branch of a hyperbola and Δ a line parallel to one of the asymptotes of the curve, at distance ϱ from this asymptote, and on such a side of the asymptote if ϱ is not zero that it does not intersect Γ. Then the separating number of Γ and Δ is ϱ, but Γ and Δ are both complete and the distance of any point of Γ from any point of Δ is greater than ϱ. This example emphasises the difficulty of giving a satisfactory description of such features as this by a process beginning with a conventional closing of the universe; there are different methods of closing the Euclidean plane, but in all of them parallel lines meet at infinity, and so it must be agreed if the universe is closed either that in this example the separating number of Γ and Δ is zero whatever the value of ϱ, on the ground that Γ and Δ have a common point, or that a point at infinity may be at a distance greater than zero from itself.

By means of the general idea of the separating number of two sets it is possible to simplify particular definitions in many important cases. For example, if Δ has only one member, x, the separating number of Δ and Γ is called the number separating x from Γ, and if this number is zero either there is a point of Γ in the unit of the universe which contains x or x is a limiting point of Γ. Again, the number separating x from the set obtained by depriving Γ of all its points in the unit of the universe containing x is the isolating radius of x and Γ; if y is any point separated from x by a number which is not zero but is less than this isolating radius, y belongs to the complement of Γ, and if x belongs to Γ, the unit of Γ containing x is an isolated unit if the isolating radius is not zero but x belongs to the coherence of Γ if the isolating radius is zero.

22. Connex Sets.

The first definition of a connex set was given by Cantor, according to whom a set Γ is connex if given any signless number ϱ other than zero and any two points y, z of Γ, there can be formed a set Δ including y and z, contained in Γ, and consisting of a finite number n of points which can be correlated with the first n natural numbers in such a way that if x_m corresponds to m, the first point x_1 is y, the last point x_n is z, and every pair of consecutive points of Δ has a separating number not greater than ϱ.

A definition of an entirely different kind was proposed by Jordan, who drew attention to the property possessed by a limited complete set if the separating number of every pair of existent sets of which it can be expressed as the sum is zero. Such a set is, in our sense of the word, united, and it is evident from

JORDAN's work[1] that it is this characteristic that he wishes to emphasise, though he defines it only for closed sets. If we detach the property which JORDAN describes from the elementary properties which he adds in order to obtain a united set, we are led to consider the nature of any set which is such that the separating number of every pair of existent sets of which it is the sum is zero, and we have no difficulty in shewing that the sets which have this property are the sets which are connex according to CANTOR's definition.

23. The Relation between Connex Sets and Congregates.

In any universe a set which is a congregate is connex, but an unlimited connex set may fail to be a congregate, and it is only in a limited universe that every connex set is a congregate.

An example that we have already used will serve again. Let the universe be a reduced Euclidean plane with distances for separating numbers, let Γ be a branch of a hyperbola and Δ an asymptote, and consider the set obtained by adding Γ and Δ. If this set is expressed as the sum of two existent sets Θ, Φ, either Γ contains both a point of Θ and a point of Φ, or Δ contains both a point of Θ and a point of Φ, or Θ coincides with one of the sets Γ, Δ and Φ coincides with the other. In the first two cases, from the properties of straight lines and hyperbolas, it is true both that $G'\Theta$ and $G'\Phi$ have a common point and that the separating number of Θ and Φ is zero; in the last case the separating number is zero, but the sets are not inseparable. Thus there is a division of the whole set into two separable sets but there is no division into two sets whose separating number is not zero, and therefore the set is not a congregate but by JORDAN's criterion it is connex. The same example will serve to illustrate CANTOR's definition. Of any pair of points of the set, either both members belong to Γ, or both members belong to Δ, or one member belongs to Γ, and the other to Δ. If y, z are any two points of Γ or are any two points of Δ, let λ_{yz} denote the length of the curve or line between y and z, and if σ is any signless real number let $[\sigma]$ denote the integer next below σ (which is $\sigma - 1$ if σ is itself an integer). Then if ϱ is any signless number other than zero and if y and z are distinct points of Γ or distinct points of Δ, the curve or line between y and z can be divided into $[\lambda_{yz}/\varrho] + 1$ parts of equal length, and the end points of these parts, $[\lambda_{yz}/\varrho] + 2$ in number, compose a set of the required form; if, however, with the same value of ϱ, we have to consider two points y, z of which y

[1] Above all, from his phrase *d'un seul tenant*.

belongs to Γ and z to Δ, the first step is to select a point v of Γ and a point w of Δ whose distance apart is not greater than ϱ, a selection that is possible because Δ is an asymptote of Γ, and we can then find a set of the required kind with the finite number $[\lambda_{yv}/\varrho] + [\lambda_{zw}/\sigma] + 4$ of points.

We have said that a limited connex set is necessarily a congregate. Since in practice mathematicians have dealt almost exclusively with limited sets, it is impossible at present to pronounce on the importance of connexity when possessed by a set which is not a congregate. But there is no doubt that in many cases it is easier to apply the criterion of inseparability than to utilise either CANTOR's criterion or JORDAN's, and I think it is safe to predict that for logical developments the hypothesis that the separating number of two sets is zero will prove to be conveniently analysed into the form that either the sets are inseparable or they are separable but their separating number is zero and one at least of them is unlimited.

It should be added that the idea of congregates as distinct from connex sets is not necessary to the definition of united sets, since in that definition closed congregates only are used and there is no extensional difference between closed congregates and closed connex sets. For the theory of sets of points it might seem more elegant, because more economical, to define united sets by means of connex sets and not by means of congregates, but for the course we have adopted there is a double justification that the definition of a congregate lends itself more readily than the definition of a connex set to symbolic treatment and that while the notion of a connex set is inapplicable except to sets of points the idea of a congregate can be extended to any universe in which aggregates have derivatives of their own type. Here concludes our digression.

24. Continuous Sets and Continua.

A study of the formal definitions reveals that every unit set is united. This result is in fact desirable, but there are many properties of multiple united sets which do not belong to unit sets: to mention the simplest, a multiple united set is dense, as is any multiple congregate, but a unit set is not dense. On account of the differences of which this is a typical example, a distinctive epithet is given to a united set which has more than one unit: a multiple united set is said to be continuous.

There appears to be no agreement among mathematicians as to the use of the word continuum; in simple cases, to some writers a continuum is essentially complete, while to others a continuum must have a complete complement; to all writers a continuum is a multiple set, and therefore the universe is the only set which is not inherently incapable of satisfying every definition that has been used. To us it would seem natural to define a continuum in the terms we have just used of a continuous set, but in spite of the absence of unanimity in its present employers we hesitate to appropriate the name; the sets which one class of writers calls continua can be described with sufficient brevity as complete continuous sets, and the sets which writers of the other class find it convenient to use add to continuity a property which we shall presently associate with the word domain and can be called continuous domains.

25. The Cells of a Set.

When the nature of a united set has been grasped, a valuable analysis of any set presents itself, following naturally on some of the examples we have given. If x is any point of a set Γ, what is conveyed in untechnical language by saying that y is a point which we can reach from x without leaving Γ is that there is a closed congregate containing x and y and contained in Γ. The points which we can reach from a point x of a set Γ without leaving Γ compose a set which we call the cell of Γ containing x and denote by $K^{\backprime}(x, \Gamma)$. If y is a point of the cell of Γ containing x, the cell containing y is identical with the cell containing x, and we call a set a cell of Γ if it is the cell of Γ containing some point of Γ; the class whose members are the cells of Γ we denote by $\varkappa^{\backprime}\Gamma$. Every cell of Γ is a united set contained in Γ, no two cells have a common point, and every point of Γ belongs to one cell; by means of its cells, Γ is expressed as the sum of a class of mutually exclusive sets, and the members of this class $\varkappa^{\backprime}\Gamma$ are precisely the sets which common sense regards as the distinct pieces of which Γ is composed. The number of the cells of a set is a number of considerable importance; for example if Γ is a plane set this number is related to the connectivity of the complement of Γ, while a united set is a set with only one cell and properties are not wanting which are peculiar to sets with other specific cell-numbers.

The definitions of this section may be illustrated by examples some of which we have already used. A circle deprived of two units, a hyperbola, a figure of eight deprived of its node, the set complementary to a straight line, are all sets in a Euclidean plane which have two cells, while a circle deprived of one unit and the set obtained by adding to the complement of a straight line any existent set of points contained in the line itself are sets with only one cell; more generally, by depriving a circle of n units we obtain a set with n limited cells, while by depriving a Euclidean straight line of n units we obtain a set with $n+1$ cells of which 2 are unlimited and $n-1$ are limited.

It is to be noticed that a limiting point of one cell of a set may actually belong to another cell of the same set; in particular, the individual cells of a universe may be incomplete, although the universe as a whole is necessarily complete.

26. Sets Continuous in Every Part.

A cell of a set may be either unit or multiple and therefore every cell is either a unit cell or a continuous set. A set of which every cell is continuous is said to be continuous in every part. A set may be continuous in every part without being continuous as a whole, and without being complete or limited, but a set continuous in every part is necessarily dense.

27. Plots.

If x is any point and Γ any set, there may be no value of ϱ such that the points of Γ separated from x by numbers less than ϱ form a united set, and even if this set is united for a particular value σ of ϱ there may be values of ϱ both smaller and larger than σ for which the set is not united. But if x belongs to Γ, then for every value of ϱ other than zero x belongs to the set formed of points of Γ in the neighbourhood of x with radius ϱ, and the cell of this set containing x we call the plot of Γ with centre x and radius ϱ. If x does not belong to Γ, or if ϱ is zero, the plot of Γ with centre x and radius ϱ is null. The plot of Γ with centre x and radius ϱ must not be confounded with the part of the cell of Γ containing x which lies within the neighbourhood of x with centre Γ; the plot is contained in the set last described, but this set is not necessarily united.

Given any point x and any number ϱ, there is a plot of the universe itself which has centre x and radius ϱ; this plot, which is not null unless ϱ is zero, may be only a part of the neighbourhood with centre x and radius ϱ, and only a part of the cell of the universe containing x. Every plot of the universe is fully expanded, but a plot of any other set Γ is contained in Γ and if Γ is not fully expanded some of its plots are not fully expanded.

28. Confined Limiting Points.

We say that a point x is a confined limiting point of a set Γ with respect to a set Δ if there is a point of Γ outside the unit containing x in every existent plot of Δ which has x for centre, and we denote the set of limiting points of Γ confined with respect to Δ by $D_\Delta{}^{\backprime}\Gamma$. It follows at once from the definition that whatever Δ may be, the confined derivative $D_\Delta{}^{\backprime}\Gamma$ is contained in the derivative $D^{\backprime}\Gamma$, but we note that even $D_V{}^{\backprime}\Gamma$, the derivative confined with respect to the universe, is not necessarily the same as $D^{\backprime}\Gamma$. The derivative of Γ confined with respect to Δ is the same as the derivative confined with respect to Δ of the product of Γ and Δ, that is, of the set composed of the points common to Γ and Δ.

29. Brinks and Borders.

The set formed of the points of a set Γ which belong to a set Δ and are confined limiting points of $C^{\backprime}\Gamma$ with respect to Δ we call the brink of Γ in Δ, and the sum of the brinks in Δ of Γ and $C^{\backprime}\Gamma$ we call the border of Γ in Δ and denote by $B^{\backprime}{}_\Delta\Gamma$. The brink and the border of Γ in the universe are called simply the brink and the border of Γ, or if a contrast is felt to be desirable the absolute brink and the absolute border. Since the brink and the border in Δ of Γ are identical with the brink and the border in Δ of the part of Γ contained in Δ, it is an easy matter to pass from theorems concerning absolute brinks and borders to theorems concerning brinks and borders in sets other than the universe. The absolute brink of any set is contained in the edge of the set, and the absolute border is contained in the boundary, but before proceeding with abstract work we describe cases in which the brink differs from the edge and the border from the boundary, and we shall see in these examples the kind of part that confined limiting points may play.

Suppose that in Euclidean space of three dimensions v, w are two points and Ω is a set contained in the chord joining v and w and such that both Ω and its complement are dense in this chord. First let the universe V be the family of concentric spheres whose centre is v and whose radii are the distances from v of the points of Ω, let distances be separating numbers, let Σ be one of these spheres, let Θ be a great circle on this sphere, and let Γ, Δ be the unbounded hemispheres lying one on each side of Θ. Every point of Σ is a limiting point of sets contained in the other spheres, and therefore is a limiting point of $C^{\backprime}\Gamma$; hence Γ is contained in $D^{\backprime}C^{\backprime}\Gamma$ and Γ is its own edge. But if there is no connection between Σ and the spheres composing $C'\Sigma$, the cell of V containing any point of Σ is Σ itself,[1] and $D'_V C'\Gamma$ is the sum of Δ and Θ, so that Γ has no brink, but Θ is the brink of $C'\Gamma$ and the border of Γ. If, however, we add to the universe just considered all the points of the chord joining v and w, still using distances for separating numbers, and if this chord cuts Σ in a point y, then if x is any point of Σ and ϱ is not greater than xy, the plot of the universe which has x for centre and ϱ for radius is contained in Σ, and if x is distinct from y then x is a confined limiting point of a set Φ if and only if x is a limiting point of the part of Φ in Σ, but if x coincides with y then x is a confined limiting point of $C'\Sigma$ and of the complement of every set contained in Σ, and every set contained in Σ and including y has y upon its brink. In the case of the unbounded hemispheres Γ and Δ, if y lies in Δ or Θ the brink and the border of Γ are alike unaffected by the addition to the universe, but if y lies in Γ then the brink of Γ contains this one point only, the brink of $C'\Gamma$ is the great circle Θ as before, and the border of Γ is the set obtained by adding the point to the circle.

30. The Relation between Unity and Borders.

Far more fundamental than the relation between connected sets and boundaries is the relation between united sets and borders, expressible in the form that if a united set contained in a set Δ but not in an isolated unit of Δ includes both a point which belongs to a set Γ and a point which does not belong to Γ, then it includes at least one point of the border of Γ in Δ. Whenever the border of a set differs from the boundary, and in particular in the cases in which a set is contained in its boundary but not in its border, the present theorem gives results of greater value than the similar theorem enunciated earlier.

An important application of the theorem of this section is to the determination of the criterion for the existence of a border; it is readily proved that an existent set Γ in a universe V has an existent border if there is a cell of Γ which does not coincide with the cell of V in which it lies and is not contained

[1] We notice that if y, z are any two points of this universe the points of V belonging to the closed chord joining y and z form a set which in V is complete and connected, but this set is unlimited, for it contains sequences which have no limiting points in V; to meet such cases as this, we should have to make unity depend on *closed* congregates and not on *complete* congregates, even if we had no examples in Euclidean space of complete congregates that are not united.

in an isolated unit of V, and to pass from this result to a theorem giving the condition for the existence of a border of any one set Γ in any other set Δ requires little more than verbal modifications.

31. Partitions and Permeation.

Deserving of passing notice is an idea closely related to ideas involved in the theorem of the last section. A set Θ is said to part two sets Γ, Δ or to be a partition between them if no point of Γ or Δ belongs to Θ but every united set containing a point of Γ and a point of Δ necessarily includes a point of Θ. For example, in a reduced Euclidean plane a straight line parts any two sets of which one lies wholly on one side of the line and the other wholly on the other side. We recognise from this example that a point of a partition between two sets may be a common limiting point of the sets. One set is said to permeate another if there is no partition possible between them, and inseparable sets which do not permeate each other have a common limiting point.

32. Fronts.

The border $B_V^2{}^{\text{‘}}\Gamma$ of the border of a set Γ is not necessarily identical with the original border $B_V{}^{\text{‘}}\Gamma$, but is a set which we call the front[1] of Γ, and similarly the border in Δ of the border of Γ in Δ, which we call the front of Γ in Δ, may differ from $B_\Delta{}^{\text{‘}}\Gamma$. For example, if the universe is a Euclidean plane with distances for separating numbers, and Γ is contained in a circular area every point of which is a limiting point of both Γ and $C{}^{\text{‘}}\Gamma$, the border $B_V{}^{\text{‘}}\Gamma$, which in this case is the boundary, consists of the whole circular area together with the circumference, but the front is the circumference alone.

It is not difficult to prove that in any universe the border of any set is a complete set and the front and the border of any complete set coincide; from these propositions it follows that the border of every front is the front itself, so that for every value of n greater than two $B_\Delta^n{}^{\text{‘}}\Gamma$ is identical with $B_\Delta^2{}^{\text{‘}}\Gamma$, and it is this theorem which gives unique value to the front.

[1] Jordan gave the name of *frontière de Γ* to the set we are calling the boundary of Γ, and both boundary and frontier are in use in English as equivalent to his word *frontière*. It is unfortunate that in a subject requiring so extensive a vocabulary as does the theory of sets of points two expressive terms have been consecrated to a single idea, but an attempt to recover the word boundary for fresh service, leaving frontier to fulfil its original function, would lead to confusion.

33. Domains.

An existent set is called a domain if the corresponding fully expanded set has no brink. The simplest domains are fully expanded, and the characteristic property of a fully expanded domain may be expressed in a variety of ways; for example, a fully expanded domain is an existent set of which the border is contained in the complement, and a fully expanded set is a domain if it exists and its complement is complete, note being made that a null set is formally complete.

In the definitions of the brink and the border of a set Γ in a set Δ it is irrelevant whether or not Γ is contained in Δ; we do not, however, say that Γ is a domain in Δ unless not only does the fully expanded set $E'\Gamma$ exist and have no brink in Δ but Γ itself is contained in Δ. In general, if there are points of Γ in Δ but $E'\Gamma$ has no brink in Δ, it is the part of Γ in Δ which is a domain in Δ.

If Γ is a domain and x is any point of Γ, there is a number ϱ such that every point separated from x by a number less than ϱ is separated by the number zero from some point of Γ, while if Γ is a fully expanded domain and x is any point of Γ there is a number ϱ such that every point separated from x by a number less than ϱ actually belongs to Γ.

34. Capacious Sets.

If the set $E'\Gamma$ obtained by expanding fully a set Γ is contained in its own brink, there are no domains contained in Γ, but if $E'\Gamma$ has points not belonging to its brink then whether or not Γ is a domain there are domains contained in Γ; we call[1] a set capacious if it contains domains, and we call one set Γ capacious in another set Δ if there is a set contained in Γ which is a domain in Δ.

35. Extreme Points.

An important idea derived from that of a domain is that of the extremity of a set. A point is an extreme point[2] of a set Γ if there is no domain in Γ to which it belongs, and the extreme points of Γ compose a set we call the extremity of Γ. To determine whether or not a point is an extreme point of Γ it is not necessary to consider Γ as contained in a more comprehensive set.

[1] What JORDAN calls a *domaine* is what we are calling a capacious set; our use of domain is the use common in English mathematical writings, but we require a phrase to embody JORDAN's idea.

[2] In the case of a plane curve an end point as defined by W. H. and G. C. YOUNG, The Theory of Sets of Points, p. 221 (1906) is not necessarily an extreme point in our sense.

36. The Dimension-Integer of a Set, and the Definitions of Curves and Surfaces.

We do not need the definition of a domain in order to define a curve, a surface, and inductively a set of any finite dimension-integer. A set Γ is a curve if it is united, if it is not a unit set, and if every existent cell of the front in Γ of every set contained in Γ is a unit cell. A set Γ is a surface if it is united, if it is neither a unit set nor a curve, and if every existent cell of the front in Γ of every set contained in Γ is either a unit set or a curve. In general a united set has the dimension-integer n if it has a dimension-integer not smaller than n and if every existent cell of the front in Γ of every set contained in Γ has a dimension-integer smaller than n, a unit set being regarded as having dimension-integer zero; and if a set is not united its dimension-integer is the greatest number occurring among the dimension-integers of its cells.

It is interesting to notice that extension, partial or full, may increase indefinitely the dimension-integer of a set; perhaps the most important classification of sets with a common dimension-integer is based on the increase effected by completion, the simplest sets of dimension-integer n being those whose completions also have dimension-integer n. From a theorem stated in section 19 it follows that extension even if full may increase the number of cells of a set, so that extension of a curve or a surface may result in a set which is not a single curve or a single surface even if it does not result in an increase of the dimension-integer.

37. Pure Sets and Mixed Sets.

Reference must be made to the distinction between a pure set and a mixed set of dimension-integer n, a distinction that in contrast to the notion of the dimension-integer involves an idea derived from that of a domain, namely, the idea of a capacious set. The cylindroid is a locus furnished analytically which is united but in some sense consists of a surface together with curves which do not lie in the surface. We say that a set Γ of dimension-integer n is a pure set if the front in Γ of every fully expanded set capacious in Γ has dimension-integer $n-1$, and we describe a set as mixed if it is not pure. It is easy to verify that with this definition a cylindroid is a mixed set.

38. Curves on Surfaces.

It can be proved that if Γ is such that the set $E_\Delta{}^{\backprime}\,\Gamma$ obtained by the full expansion of Γ in Δ is contained in its own border in Δ, then the dimension-integer of the part of Γ in Δ is smaller than the dimension-integer of Δ. This principle enables us to define a curve in any set which is known to be a surface in terms simpler than those of the general definition of a curve; a curve on a surface is a multiple united set contained in the surface such that the corresponding set fully expanded in the surface is contained in its own border in the surface, that is, is identical with its own brink in the surface. For example, if the universe is a reduced plane a curve is a multiple united set of points contained in its own boundary. But a multiple united set contained in its own border in a reduced space whose dimension-integer is 3 may be either a curve or a surface, and to deal with dimensions in general by successive reduction from the universe not only assumes the universe to have a finite dimension-integer but also requires a method of discrimination between a decrease by unity and any greater decrease in a dimension-integer, and this discrimination is rendered difficult by the existence in any set with dimension-integer not less than 2 of sets that are not pure.

39. Dimension-integers and Dimension-Types.

The definitions proposed in the last three paragraphs differ from the definitions in other theories of dimension in being independent of any comparison of one set with another, but although the property described in the last paragraph as characteristic of a curve on a surface appears to be the fundamental property that any system of definitions must reproduce, the utility of the definitions in section 36 doubtless depends on the possibility of ascertaining dimension-integers by comparison. A correlation of one set Γ with another set Δ in general connects classes of members of Γ with classes of members of Δ; if to each point of Γ corresponds one and only one point of Δ, the correlation of Γ with Δ is many-one, and if the correlation of Δ with Γ is many-one, the correlation of Γ with Δ is one-many; a correlation which is both many-one and one-many is one-one. Let Θ be a set contained in Γ, and let Φ be the corresponding set in Δ, a many-many correspondence existing between Γ and Δ; to the part of the derived set $D‘\Theta$ which is contained in Γ there corresponds some set contained in Δ, but this set has not necessarily any points in common with $D‘\Phi$; if the correlate in Δ of the part belonging to Γ of the derivative of *every* set Θ contained in Γ is the part contained in Δ of the derivative of the correlate in Δ of Θ, the correspondence of Γ with Δ is continuous from Γ to Δ; a correspondence which is continuous both from Γ to Δ and from Δ to Γ is a bicontinuous correlation of Γ and Δ. A one-one correlation is not necessarily bicontinuous, nor is every bicontinuous correlation one-one, but the correlations which are valuable in the theory of dimensions are correlations which are both bicontinuous and one-one, and if a correlation of this kind exists between two sets the sets are said to be homomorphous and each is called an image of the other.

In FRÉCHET's theory of dimension-types, the type of Γ is said to be not lower than the type of Δ if Γ contains an image of Δ, and two sets are said to have the same type if each contains an image of the other; it is easy to prove that with our definitions two sets between which exists a bicontinuous one-one correlation have the same dimension-integer, and the dimension-integer of a set Δ can not be greater than the dimension-integer of any set in which Δ is contained, and it follows that our ideas are not at variance with FRÉCHET's. FRÉCHET, however, can assign the integers arbitrarily to any set of ascending types, and for him the assertion that n-dimensional Euclidean space, — the space in which a point x is the ordered class $(x_1, x_2, \ldots x_n)$ of n independent real numbers and the separating number $\dot{x}\ddot{x}$ is either $\sqrt{\{\Sigma(\dot{x}_r - \ddot{x}_r)^2\}}$ or $\Sigma|\dot{x}_r - \ddot{x}_r|$, — is of dimension-type n, while requiring for its justification the fundamental dimension-theorem that no bicontinuous one-one correspondence is possible between two complete spaces of this kind which do not depend on the same number of coordinates, is nevertheless an arbitrary definition. We have to recognise that if the definitions of section 36 are to be adopted, despotic assignment of dimension-integers is impossible, and the dimension-integer of Euclidean space with n coordinates is intrinsically determinate: if this integer proves to be n, and if every set with dimension-integer n contains an image of the complete Euclidean space with n coordinates, the relation of the present theory to the theory of dimension-types is perfect.