ON THE THEORY OF INTERPOLATION.

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Introduction.

Let us denote

(1) $$x_1^{(n)},\ x_2^{(n)},\ \ldots,\ x_n^{(n)}$$

n distinct points in the interval $-1 \leqq x \leqq +1$ and let $f(x)$ be a function defined in the same interval. We investigate in this note the convergence problems of the Lagrange and Hermite interpolation polynomials of the function $f(x)$ corresponding to the "fundamental points" (1). The n^{th} Lagrange interpolation polynomial of $f(x)$ is the unique polynomial of degree $n-1$ at most, assuming the values $f(x_1^{(n)}),\ f(x_2^{(n)}),\ \ldots,\ f(x_n^{(n)})$ at the abscissas $x_1^{(n)},\ x_2^{(n)},\ \ldots,\ x_n^{(n)}$ respectively. This polynomial is given by the formula

(2) $$L_n[f] = \sum_{k=1}^{n} f(x_k^{(n)})\, l_k^{(n)}(x);$$

here

(3) $$l_k^{(n)}(x) = \frac{\omega_n(x)}{\omega_n'(x_k^{(n)})\,(x - x_k^{(n)})}$$

and the polynomial $\omega(x)$ defined by

(4) $$\omega(x) = (x - x_1^{(n)})(x - x_2^{(n)}) \ldots (x - x_n^{(n)}).$$

The n^{th} Hermite interpolation polynomial of $f(x)$ is the unique polynomial of degree at most $2n-1$ which for the values $x_1^{(n)},\ x_2^{(n)},\ \ldots,\ x_n^{(n)}$ assumes, respectively, the values $f(x_1^{(n)}),\ f(x_2^{(n)}),\ \ldots,\ f(x_n^{(n)})$ and whose derivative correspondingly assumes the given values $d_1^{(n)},\ d_2^{(n)},\ \ldots,\ d_n^{(n)}$. The explicite form of this polynomial is given by the formula

(5)
$$H_n[f] \equiv \sum_{k=1}^{n} f(x_k^{(n)})\, h_k^{(n)}(x) + \sum_{k=1}^{n} d_k^{(n)}\, \mathfrak{h}_k^{(n)}(x);$$

here

(6)
$$h_k^{(n)}(x) = v_k^{(n)}(x) \{ l_k^{(n)}(x) \}^2,$$

(7)
$$v_k^{(n)}(x) = 1 - (x - x_k^{(n)}) \frac{\omega_n''(x_k^{(n)})}{\omega_n'(x_k^{(n)})},$$

(8)
$$\mathfrak{h}_k^{(n)}(x) = (x - x_k^{(n)}) \{ l_k^{(n)}(x) \}^2$$

and $\omega_n(x)$, $l_k^{(n)}(x)$ have the same meaning as before.

There exist many investigations for the behavior of the sequence $L_n[f]$ $(n = 1, 2, \ldots)$. We mention here the following negativ results only. For an arbitrarily given

(9)
$$\begin{array}{l} x_1^{(1)} \\ x_1^{(2)},\ x_2^{(2)} \\ \ldots \\ x_1^{(n)},\ x_2^{(n)},\ \ldots,\ x_n^{(n)} \\ \ldots \end{array}$$

system of fundamental points there exists a function $f(x)$ continuous in the interval $-1 \leqq x \leqq +1$ for which the sequence $L_n[f]$ $(n = 1, 2, \ldots)$ is not uniformly convergent in $-1 \leqq x \leqq +1$ [1] and there exists a continuous function $f(x)$ also for which the sequence $L_n[f]$ is divergent.[2] Even in the most regular case of the Tchebycheff fundamental points (i. e.

(10)
$$x_k^{(n)} = \cos(2k - 1)\frac{\pi}{2n} \quad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots);$$

$x_k^{(n)}$ $(k = 1, 2, \ldots, n)$ are the roots of the n^{th} Tchebycheff polynomial $T_n(x) = \cos(n \arccos x)$) there exists a function $f(x)$ continuous in the interval $-1 \leqq x \leqq +1$ for which the sequence $L_n[f]$ is divergent everywhere in

[1] S. BERNSTEIN, Quelques remarques sur l'interpolation, *Comm. Soc. Math. Charkow*, 14 (1914). G. FABER, Über die interpolatorische Darstellung stetiger Funktionen, *Jahresbericht d. D. Math. Vereinigung*, 23 (1914), p. 102—210. A very simple proof is given in L. FEJÉR, Die Abschätzung eines Polynoms . . ., *Math. Zeitschrift*, 32 (1930) p. 426—457.

[2] S. BERNSTEIN, Sur la limitation des valeurs d'un polynome etc., *Bull. de l'Acad. des Science de l'U. R. S. S.*, (1931), p. 1025—1050.

$-1 \leqq x \leqq +1$.[1] For the uniform convergence of the sequence $L_n[f]$ further suppositions on the function $f(x)$ are necessary. E. g. in the Tchebycheff case these suppositions are analogous with the conditions for which the Fourier series of $f(x)$ is convergent.

If we look for a sequence of interpolation polynomials, which is convergent for all continuous functions, then we can select only from those for which the degree of the n^{th} polynomial — which thus assumes for the values $x_1^{(n)}$, $x_2^{(n)}$, $\ldots$, $x_n^{(n)}$ respectively the values $f(x_1^{(n)})$, $f(x_2^{(n)})$, $\ldots$, $f(x_n^{(n)})$ — is higher than $n-1$. It is here a very natural requirement to investigate interpolation polynomials with simple form and not too high degree.[2] In the case of the aequidistant fundamental points (i. e. the fundamental points are the points dividing the interval $-1 \leqq x \leqq +1$ in n equal parts) this problem was solved by De la Vallée Poussin and S. Bernstein.[3] L. Fejér investigated the Hermite interpolation polynomials.[4] We have seen that these classical interpolation polynomials are of simple character and the degree of the n^{th} polynomial is $2n-1$. The degree is higher by n than that of the n^{th} Lagrange polynomial.[5] There is a very important difference between the Lagrange and Hermite interpolation polynomials. All the functions $l_k^{(n)}(x)$ $(k=1, 2, \ldots, n)$ have exactly $n-1$ changes of sign if x runs over all real values; and this is true by arbitrary choice of the fundamental points. On the other hand as the formulas (6), (7), (8) show, the functions

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[1] G. Grünwald, Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome stetiger Funktionen, *Annals of Mathematics*, 37 (1936), p. 908—918. See also G. Grünwald, Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome, *Acta Szeged*, 7 (1935), p. 207—211; J. Marcinkiewicz, Sur la divergence des polynomes d'interpolation, *Acta Szeged*, 8 (1937), p. 131—135.

[2] If the n^{th} interpolation polynomial can have any arbitrary degree, then for an arbitrary everywhere dense pointsystem we can find a sequence of interpolation polynomials which is convergent uniformly for all continuous functions. G. Grünwald, *On Interpolation*, *Bull. of Am. Math. Soc.*, 47 (1941), p. 257—260.

[3] de la Vallée Poussin, Sur la convergence des formules d'interpolation entre coordonées equidistantes, *Bulletin de l'Académie Belgique*, 1908. S. Bernstein, Sur une formule d'Interpolation de M. de la Vallée Poussin, *Comm. Soc. Math. Charkow*, (4) 5. (1932), p. 61—64. The degree of de la Vallée Poussins interpolation polynomials is $6n$; that of Bernsteins is $< 3n$.

[4] Fejérs first note on Hermite interpolation: L. Fejér, Über Interpolation, *Nachrichten d. K. Gesellschaft zu Göttingen* (1916), p. 66—91. For the further investigation see the papers of Fejér cited later.

[5] In the Tchebycheff case the degree of the n^{th} interpolation polynomial must be $n-1+cn$ $(c>0)$ at least, if we desire convergence for all continuous functions. G. Grünwald, On a theorem of S. Bernstein, *Acta Szeged*, in the Press. It is very likely that this is true in the general case, too.

$h_k^{(n)}(x), \mathfrak{h}_k^{(n)}(x)$ of the Hermite interpolation change their signs only once at most. The function $\mathfrak{h}_k^{(n)}(x)$ changes its sign at the point $x_k^{(n)}$. The function $h_k^{(n)}(x)$ changes its sign at most once: at the root of the linear function

$$(11) \qquad v_k^{(n)}(x) = 1 - (x - x_k^{(n)}) \frac{\omega_n''(x_k^{(n)})}{\omega_n'(x_k^{(n)})}.$$

If the roots of $v_k^{(n)}(x)$ $(k = 1, 2, \ldots, n)$ are outside of $-1 \leqq x \leqq +1$, then the functions $h_k^{(n)}(x)$ are all positive. In the Tchebycheff case $v_k^{(n)}(x) > 0$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ and just with the aid of this fact L. Fejér proved the uniform convergence of the Hermite interpolation polynomials of an arbitrary continuous function if the numbers $|d_k^{(n)}|$ are uniformly bounded.[1] The further investigations showed that where the proof of the convergence of the Hermite interpolation polynomials was simple, the condition $v_k^{(n)}(x) > 0$ was satisfied. The systems of fundamental points for which this latter condition is holding shows also in the theory of Lagrange interpolation and other investigations a regular behavior.[2] Because of this, L. Fejér called these systems of fundamental points normal point systems. The scope of this note is the investigation of Lagrange and Hermite interpolation polynomials corresponding to normal pointsystems. The results show that this condition in itself — thus without further specification of the pointsystem — is sufficient to prove very general convergence theorems for the Hermite and Lagrange interpolation polynomials.[3]

This note contains three parts. In the first we investigate the consequences of the conditions $v_k^{(n)}(x) \geqq \varrho > 0$ and $v_k^{(n)}(x) > 0$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ $-1 \leqq x \leqq +1$ for the convergence of the Hermite interpolation polynomials of a continuous function. In the second part under the same conditions we investigate the convergence of the Lagrange interpolation polynomials of a function satisfying a certain Lipschitz condition. In the last part we investigate the behavior of the sum $\sum_{k=1}^{n} \{ l_k^{(n)}(x) \}^2$ if the condition $v_k^{(n)}(x) \geqq 0$ $(k=1, 2, \ldots, n;\ n=1, 2, \ldots)$ is satisfied.

[1] See the note of L. Fejér cited in note 1, p. 220.

[2] See e. g. L. Fejér, Lagrangesche Interpolation und die zugehörigen konjugierten Punkte, *Mathematische Annalen*, 106 (1932), p. 1—55. L. Fejér, On the characterisation of some remarkable systems of points of interpolation by means of conjugate points, *American Math. Monthly*, 41 (1934), p. 1—14. P. Erdős-P. Turán, On Interpolation II, *Annals of Mathematics*, 39 (1938), p. 703—724.

[3] We note that the results are — with few exceptions — the best possibles.

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I. The Convergence of the Hermite Interpolation Polynomials of a Continuous Function Corresponding to Normal and Strongly Normal Pointsystems.

1 §. Definitions and Notations and Elementary Properties of the Interpolation Polynomials.

The numbers $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$,

$$(1) \qquad -1 \leqq x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} \leqq +1,$$

are called the n^{th} *fundamental points* of the interpolation.[1] The set of numbers

$$
\begin{array}{l}
x_1^{(1)} \\
x_1^{(2)},\ x_2^{(2)} \\
\dots \\
x_1^{(n)},\ x_2^{(n)},\ \dots,\ x_n^{(n)} \\
\dots
\end{array}
\tag{2}
$$

is a *fundamental pointsystem* of the interpolation or shortly a *pointsystem* if for all n (1) is satisfied. In the following for the sake of simplicity, we do not use in our formulas the upper index n. This does not lead to a misunderstanding. The n^{th} Lagrange interpolation polynomial of a function $f(x)$ corresponding to the fundamental points (1) is given by the formula

$$L_n[f] = \sum_{k=1}^{n} f(x_k)\, l_k(x), \tag{3}$$

where

$$l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)} \tag{4}$$

and

$$\omega(x) = (x - x_1)(x - x_2)\dots(x - x_n). \tag{5}$$

The polynomials

$$l_1(x),\ l_2(x),\ \dots,\ l_n(x) \tag{6}$$

which are all exactly of degree $n - 1$ will be called *fundamental polynomials of the Lagrange interpolation.* $l_k(x)$ assumes at x_k the value 1 and at $x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_n$ the value 0. So it is clear that $L_n[f]$ assumes at $x_1, x_2, \dots x_n$ the values $f(x_1), f(x_2), \dots, f(x_n)$, respectively. Since the polynomial $L_n[f]$ of degree at most $n - 1$ is determined by the values assumed at n distinct points, (3) is the unique polynomial of degree at most $n - 1$ which at $x_1, x_2, \dots, x_n$ assumes the same values as $f(x)$. Furthermore from the unicity it follows, that for an arbitrary polynomial $P(x)$ of degree at most $n - 1$

$$L_n[P] \equiv P(x) \equiv \sum_{k=1}^{n} P(x_k)\, l_k(x). \tag{7}$$

[1] The notations used are introduced — with few exceptions — by L. Fejér. The results of this §. are due to L. Fejér.

If $P(x) \equiv 1$ (7) gives the fundamental identity

$$(8) \qquad l_1(x) + l_2(x) + \cdots + l_n(x) \equiv 1 .$$

The sequence $L_n[f]$ $n = 1, 2, \ldots$ will be called the sequence of the Lagrange interpolation polynomials of the function $f(x)$ corresponding to the pointsystem (2).

Let $d_1, d_2, \ldots, d_n$ be n arbitrary numbers, then the n^{th} Hermite interpolation polynomial of the function $f(x)$ corresponding to the fundamental points (1) is given by the formula

$$(9) \qquad H_n[f; d] \equiv \sum_{k=1}^{n} f(x_k)\, h_k(x) + \sum_{k=1}^{n} d_k\, \mathfrak{h}_k(x),$$

where

$$(10) \qquad h_k(x) = v_k(x)\, l_k^2(x) \qquad k = 1, 2, \ldots, n,$$

$$(11) \qquad v_k(x) = 1 - (x - x_k)\frac{\omega''(x_k)}{\omega'(x_k)} \qquad k = 1, 2, \ldots, n,$$

$$(12) \qquad \mathfrak{h}_k(x) = (x - x_k)\, l_k^2(x) \qquad k = 1, 2, \ldots, n$$

and $\omega(x)$, $l_k(x)$ have the same meaning as before. It follows from (10), (11) and (12) that

$$(13) \qquad h_k(x_k) = 1,\ h_k(x_i) = 0\ (k \neq i),\ k = 1, 2, \ldots, n;$$

$$(14) \qquad \mathfrak{h}_k(x_i) = 0,\ k = 1, 2, \ldots, n;\ i = 1, 2, \ldots, n.$$

That is, the polynomial (9) assumes the values $f(x_1), f(x_2), \ldots, f(x_n)$ at $x_1, x_2, \ldots, x_n$, respectively. On the other hand the formula

$$(15) \qquad \mathfrak{h}_k'(x) = l_k^2(x) + 2(x - x_k)\, l_k(x)\, l_k'(x)$$

shows that

$$(16) \qquad \mathfrak{h}_k'(x_k) = 1,\ \mathfrak{h}_k'(x_i) = 0\ (i \neq k), \quad k = 1, 2, \ldots, n$$

and the formulas

$$(17) \qquad h_k'(x) = 2\, v_k(x)\, l_k(x)\, l_k'(x) - \frac{\omega''(x_k)}{\omega'(x_k)}\, l_k^2(x),$$

$$(18) \qquad l_k'(x_k) = \frac{1}{2}\,\frac{\omega''(x_k)}{\omega'(x_k)}$$

show that

$$(19) \qquad h_k'(x_i) = 0 \qquad i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, n.$$

The formulas (13), (14) and (16), (19) give that the polynomial (9) at $x_1, x_2, \ldots, x_n$ assumes the values $f(x_1), f(x_2), \ldots, f(x_n)$, respectively and the derivative of this polynomial at $x_1, x_2, \ldots, x_n$ assumes the values $d_1, d_2, \ldots, d_n$, respectively. The polynomial (9) of degree at most $2n-1$ is uniquely determined by these conditions.

The polynomials

$$h_1(x),\ h_2(x),\ \ldots,\ h_n(x) \tag{20}$$

will be called *fundamental polynomials of the first kind*, and the polynomials

$$\mathfrak{h}_1(x),\ \mathfrak{h}_2(x),\ \ldots,\ \mathfrak{h}_n(x) \tag{21}$$

fundamental polynomials of the second kind of the Hermite interpolation.

From the unicity of the interpolation formula it follows that for a polynomial $P(x)$ of degree at most $2n-1$

$$H_n[P;\,P'] \equiv P(x) \equiv \sum_{k=1}^{n} P(x_k)\,h_k(x) + \sum_{k=1}^{n} P'(x_k)\,\mathfrak{h}_k(x). \tag{22}$$

If $P(x) \equiv 1$, (22) gives the fundamental identity

$$h_1(x) + h_2(x) + \cdots + h_n(x) \equiv 1. \tag{23}$$

If $P(x) \equiv x$, (22) gives, with the aid of (23)

$$\sum_{k=1}^{n} \mathfrak{h}_k(x) \equiv \sum_{k=1}^{n} (x - x_k)\,h_k(x). \tag{24}$$

Let the pointsystem (2) be such, that for all n

$$v_k(x) = 1 - (x - x_k)\frac{\omega''(x_k)}{\omega'(x_k)} \geqq \varrho > 0 \quad k = 1, 2, \ldots, n;\ -1 \leqq x \leqq +1. \tag{25}$$

Then the pointsystem will be called *strongly normal*. If we will emphasize the number ϱ in the condition (25) we use the expression ϱ-normal, too. For the number ϱ we have $\varrho < 1$, because $v_k(x_k) = 1$. Let the pointsystem (2) be such, that for all n

$$v_k(x) \geqq 0 \qquad k = 1, 2, \ldots, n;\ -1 \leqq x \leqq +1, \tag{26}$$

then the pointsystem will be called *normal*.

E. g. ϱ-normal pointsystems are the roots of certain Jacobi polynomials.[1] The n^{th} Jacobi polynomial $\mathfrak{h}_n(\alpha, \beta, x)$ has n distinct roots in the interval $-1 \leqq x \leqq +1$ if $\alpha \geqq 0$, $\beta \geqq 0$. If $0 \leqq \alpha < 1/2$, $0 \leqq \beta < 1/2$ then the roots give a ϱ-normal pointsystem, where $\varrho = \min(1-2\alpha,\ 1-2\beta)$. In the case $\alpha = \beta = 1/2$, $\mathfrak{h}_n(1/2, 1/2, x) = n^{\text{th}}$

[1] See FEJÉRS first paper cited in note 2, p. 222.

Legendre polynomial, and the condition (25) is not satisfied for any positive ϱ; but the weaker condition (26) is satisfied and so the pointsystem is normal.

From the definition of the ϱ-normal pointsystem and from the identity (23) follows the important inequality

$$\sum_{k=1}^{n} l_k^2(x) \leqq \frac{1}{\varrho} \qquad -1 \leqq x \leqq +1 . \tag{27}$$

For normal pointsystems, as an elementary geometrical consideration shows, everywhere in the interval $-1 + \varepsilon \leqq x \leqq 1 - \varepsilon$ $(\varepsilon > 0)$

$$v_k(x) \geqq \frac{\varepsilon}{2} \tag{28}$$

and so

$$\sum_{k=1}^{n} l_k^2(x) \leqq \frac{2}{\varepsilon} \qquad -1 + \varepsilon \leqq x \leqq 1 - \varepsilon . \tag{29}$$

2 §. A Convergence Theorem for the Hermite Interpolation Polynomials of a Function with Continuous Derivative.

We prove the following theorem. *Theorem 1. Let $f(x)$ be a function with continuous derivative in the interval $-1 \leqq x \leqq +1$. Then we have for strongly normal pointsystems*

$$\lim_{n \to \infty} H_n[f; f'] = \lim_{n \to \infty} \left(\sum_{k=1}^{n} f(x_k) h_k(x) + \sum_{k=1}^{n} f'(x_k) \mathfrak{h}_k(x) \right) = f(x) \tag{30}$$

and the convergence is uniform in the interval $-1 \leqq x \leqq +1$.

From (27) there follows the inequality

$$\sum_{k=1}^{n} |\mathfrak{h}_k(x)| < \frac{2}{\varrho} \qquad -1 \leqq x \leqq +1 . \tag{31}$$

An easy consequence of the approximation theorem of Weierstrass is that for a given $\varepsilon > 0$ there exists a polynomial $P(x)$ for which we have

$$|f(x) - P(x)| \leqq \varepsilon, \; |f'(x) - P'(x)| \leqq \varepsilon \qquad -1 \leqq x \leqq +1 . \tag{32}$$

If n is sufficiently large, the identity (22) gives that

$$P(x) \equiv \sum_{k=1}^{n} P(x_k) h_k(x) + \sum_{k=1}^{n} P'(x_k) \mathfrak{h}_k(x) . \tag{33}$$

So by (32)

$$(34)\quad \begin{aligned} &|H_n[f;\, f'] - f(x)| = |H_n[f - P;\, f' - P'] + P(x) - f(x)| \leqq \\ &\leqq |H_n[f - P, f' - P']| + \varepsilon \leqq \\ &\leqq \sum_{k=1}^{n} |f(x_k) - P(x_k)|\,|h_k(x)| + \sum_{k=1}^{n} |f'(x_k) - P'(x_k)|\,|\mathfrak{h}_k(x)| + \varepsilon \leqq \\ &\leqq \varepsilon \sum_{k=1}^{n} |h_k(x)| + \varepsilon \sum_{k=1}^{n} |\mathfrak{h}_k(x)| + \varepsilon\,. \end{aligned}$$

Thus it follows from $v_k(x) \geqq \varrho > 0$, (23) and (31) that

$$(35)\qquad |H_n[f;\, f'] - f(x)| < \varepsilon + \frac{2\varepsilon}{\varrho} + \varepsilon = 2\varepsilon\left(1 + \frac{1}{\varrho}\right),$$

which was to be proved.

For normal pointsystems instead of (31) we have

$$(36)\qquad \sum_{k=1}^{n} l_k^2(x) \leqq \frac{2}{\varepsilon} \qquad\qquad -1 + \varepsilon \leqq x \leqq 1 - \varepsilon$$

and so we have convergence in $-1 < x < +1$ only; and uniform convergence in $-1 + \varepsilon \leqq x \leqq 1 - \varepsilon$ $(\varepsilon > 0)$.

3 §. A Sufficient Condition for the Convergence of the Hermite Interpolation Polynomials of a Continuous Function Corresponding to Normal and Strongly Normal Pointsystems.

In § 2 we have seen that for strongly normal pointsystems the sum of the absolute values of the fundamental functions of the second kind is bounded (formula (31)). We will show that the behavior of this sum is decisive in the question of the convergence of Hermite interpolation polynomials corresponding to strongly normal pointsystems. In this § we make the first step in this direction. We prove the theorem: *Theorem 2. Let a pointsystem be normal and let us suppose, that*

$$(37)\qquad \lim_{n \to \infty} \sum_{k=1}^{n} |\mathfrak{h}_k(x)| = 0$$

and the convergence is uniform in $-1 \leqq x \leqq +1$. *Then we have for an arbitrary function* $f(x)$ *continuous in* $-1 \leqq x \leqq +1$

$$(38)\qquad \lim_{n \to \infty} H_n[f;\, d] = \lim_{n \to \infty}\left(\sum_{k=1}^{n} f(x_k)\, h_k(x) + \sum_{k=1}^{n} d_k\, \mathfrak{h}_k(x)\right) = f(x)$$

and the convergence is uniform in the interval $-1 \leqq x \leqq +1$. *The numbers* $|d_k^{(n)}| = |d_k|$ *are arbitrary but uniformly bounded:* $|d_k| < A$, *where* $A > 0$ *is independent from* k *and* n. *If the condition* (37) *is satisfied on the pointset* S, *then* (38) *is holding on* S.

For the proof let $P(x)$ be a polynomial for which

$$(39) \qquad |f(x) - P(x)| \leqq \frac{\varepsilon}{3} \qquad -1 \leqq x \leqq +1,$$

where $\varepsilon > 0$ is a given number (the existence of $P(x)$ is assured by the theorem of Weierstrass), and let us denote

$$M = \operatorname*{Max}_{-1 \leqq x \leqq +1} |P'(x)|.$$

From (37) follows that for sufficiently large n

$$(40) \qquad \left| \sum_{k=1}^{n} P'(x_k)\, \mathfrak{h}_k(x) \right| < M \sum_{k=1}^{n} |\mathfrak{h}_k(x)| < \frac{\varepsilon}{3}.$$

Thus for sufficiently large n

$$(41) \qquad \begin{aligned} \left| \sum_{k=1}^{n} f(x_k)\, h_k(x) - f(x) \right| &= \left| \sum_{k=1}^{n} f(x_k)\, h_k(x) - P(x) + P(x) - f(x) \right| = \\ &= \left| \sum_{k=1}^{n} (f(x_k) - P(x_k))\, h_k(x) - \sum_{k=1}^{n} P'(x_k)\, \mathfrak{h}_k(x) + P(x) - f(x) \right| \leqq \\ &\leqq \sum_{k=1}^{n} |f(x_k) - P(x_k)|\, h_k(x) + M \sum_{k=1}^{n} |\mathfrak{h}_k(x)| + |f(x) - P(x)| \leqq \\ &\leqq \frac{\varepsilon}{3} \sum_{k=1}^{n} h_k(x) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

So we have

$$(42) \qquad \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\, h_k(x) = f(x) \qquad -1 \leqq x \leqq +1$$

and the convergence is uniform in the interval $-1 \leqq x \leqq +1$. Since $|d_k| < A$ and

$$(43) \qquad \lim_{n \to \infty} \sum_{k=1}^{n} |\mathfrak{h}_k(x)| = 0$$

our statement follows from (42).

We can evidently prove the second part of the theorem in the same manner.

4 §. The Convergence of the Hermite Interpolation Polynomials of a Continuous Function.

We prove the theorems.

Theorem 3. Let $f(x)$ *be a continuous function in the interval* $-1 \leqq x \leqq +1$ *and let* $d_k = d_k^{(n)}$ *be arbitrarily given numbers for which* $|d_k| < A$ *(A is independent from k and n). Furthermore let the pointsystem*

$$(44) \qquad \begin{array}{l} x_1^{(1)} \\ x_1^{(2)},\ x_2^{(2)} \\ \cdots \\ x_1^{(n)},\ x_2^{(n)},\ \ldots,\ x_n^{(n)} \\ \cdots \end{array}$$

be strongly normal in the interval $-1 \leqq x \leqq +1$, *that is*

$$(45) \qquad v_k(x) \geqq \varrho > 0 \qquad k = 1, 2, \ldots, n;\ n = 1, 2, \ldots;\ -1 \leqq x \leqq +1$$

Then

$$(46) \qquad \lim_{n\to\infty} H_n[f;\ d] = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f(x_k)\,h_k(x) + \sum_{k=1}^{n} d_k\,\mathfrak{h}_k(x)\right) = f(x)$$

and the convergence is uniform in the interval $-1 \leqq x \leqq +1$.

Theorem 4. Let the pointsystem (44) *be normal, that is*

$$(47) \qquad v_k(x) \geqq 0 \qquad k = 1, 2, \ldots, n;\ n = 1, 2, \ldots;\ -1 \leqq x \leqq +1$$

If the other conditions of theorem 3. are satisfied then

$$(48) \qquad \lim_{n\to\infty} H_n[f;\ d] = f(x) \qquad -1 < x < +1$$

and the convergence is uniform in the interval $-1 + \varepsilon \leqq x \leqq 1 - \varepsilon$, *where* $\varepsilon > 0$ *is an arbitrary given number.*

It follows from theorem 2. that it is sufficient to prove in the strongly normal case that

$$(49) \qquad \lim_{n\to\infty} \sum_{k=1}^{n} |\mathfrak{h}_k(x)| = 0 \qquad -1 \leqq x \leqq +1$$

and the convergence is uniform in the interval $-1 \leqq x \leqq +1$, furthermore that in the normal case

(50) $$\lim_{n\to\infty}\sum_{k=1}^{n}|\mathfrak{h}_k(x)|=0 \qquad -1<x<+1$$

and the convergence is uniform in the interval $-1+\varepsilon\leqq x\leqq 1-\varepsilon$ where $\varepsilon>0$ is an arbitrary given number.

Let a be any number of the interval $[-1, 0]$ and

(51) $$H_n[g;\, g']_{x=a}=\sum_{k=1}^{n}g(x_k)\,h_k(a)+\sum_{k=1}^{n}g'(x_k)\,\mathfrak{h}_k(a).$$

Let $g(x)$ be defined by

(52) $$g(x)=\begin{cases}0 & \text{if } -1\leqq x\leqq a\\ (x-a)^{\varrho/2} & \text{if } a\leqq x\leqq +1.\end{cases}$$

We shall first consider in (51) only such values of n for which a does not occur in the n-th row of (44); we shall call them regular n-values. They depend of course on a. We have for regular n-values.

(53) $$\begin{aligned}H_n[g;\, g']_{x=a}&=\sum_{k=1}^{n}g(x_k)\,v_k(a)\,l_k^2(x)+\sum_{k=1}^{n}g'(x_k)\,(a-x_k)\,l_k^2(a)=\\&=\sum_{a>x_k}(x_k-a)^{\varrho/2}\,v_k(a)\,l_k^2(a)+\sum_{a<x_k}\frac{\varrho}{2}(x_k-a)^{\varrho/2-1}(a-x_k)\,l_k^2(a)=\\&=\sum_{a<x_k}(x_k-a)^{\varrho/2}\,l_k^2(a)\left(v_k(a)-\frac{\varrho}{2}\right).\end{aligned}$$

It is important to note, that the terms in the last sum of (53) are positive, since $v_k(a)-\frac{\varrho}{2}\geqq\varrho-\frac{\varrho}{2}=\frac{\varrho}{2}>0$. We shall prove that there exists a number $n_0=n_0(\varepsilon)$, that for regular n-values greater than n_0, and for $-1\leqq a\leqq 0$

$$|H_n(g,\, g')_{x=a}|<\varepsilon.$$

For the proof we define the functions $\nu=1, 2, \ldots$

(54) $$g_\nu(x)=\begin{cases}0 & \text{if } -1\leqq x\leqq a\\ -\nu^2(x-a)^{\varrho/2+2}+2\nu(x-a)^{\varrho/2+1} & \text{if } a\leqq x\leqq a+\frac{1}{\nu}\\ (x-a)^{\varrho/2} & \text{if } a+\frac{1}{\nu}\leqq x\leqq +1.\end{cases}$$

$g_\nu(x)$ is differentiable and its derivative is continuous in the interval $-1\leqq x\leqq +1$. This is a consequence of an easy calculation. We prove that

(55) $$g_\nu(x)\to g(x) \text{ if } \nu\to\infty$$

and the convergence is uniform in the interval $-1 \leqq x \leqq +1$ and $-1 \leqq a \leqq 0$. Let $\varepsilon > 0$ be a given number and ν so large that

$$4\left(\frac{1}{\nu}\right)^{\varrho/2} < \varepsilon, \tag{56}$$

then we have

$$|g_\nu(x) - g(x)| < \varepsilon \qquad -1 \leqq x \leqq +1, \tag{57}$$

since in $-1 \leqq x \leqq a$ and $a + \frac{1}{\nu} \leqq x \leqq 1$ $g_\nu(x) - g(x) = 0$ and in the interval $a \leqq x \leqq a + \frac{1}{\nu}$

$$\begin{aligned} |g_\nu(x) - g(x)| &= |-\nu^2(x-a)^{\varrho/2+2} + 2\nu(x-a)^{\varrho/2+1} - (x-a)^{\varrho/2}| \leqq \\ &\leqq \nu^2\left(\frac{1}{\nu}\right)^{\varrho/2+2} + 2\nu\left(\frac{1}{\nu}\right)^{\varrho/2+1} + \left(\frac{1}{\nu}\right)^{\varrho/2} = 4\left(\frac{1}{\nu}\right)^{\varrho/2} < \varepsilon. \end{aligned} \tag{58}$$

It follows from the theorem 1. that for a fixed ν

$$\lim_{n\to\infty} H_n[g_\nu; g'_\nu] = g_\nu(x) \tag{59}$$

and the convergence is uniform in the interval $-1 \leqq x \leqq +1$. As the proof of theorem 1. shows, the convergence is also uniform for $-1 \leqq a \leqq 0$.

Let now $\varepsilon > 0$ be a given number and ν so large, that

$$|g_\nu(x) - g(x)| < \frac{\varepsilon}{3} \qquad -1 \leqq x \leqq +1 \tag{60}$$

(the existence of such a ν follows from (55)) and

$$\frac{6}{\varrho}\left(\frac{1}{\nu}\right)^{\varrho/2} < \frac{\varepsilon}{3} \tag{61}$$

Let us fix this ν and let n_0 be so large that for $n > n_0$

$$|H_n[g_\nu; g'_\nu]_{x=a} - g_\nu(a)| = |H_n[g_\nu; g'_\nu]_{x=a}| < \frac{\varepsilon}{3} \tag{62}$$

(this is possible as it follows from (59)). (60), (61), (62) give for regular n-values greater than n_0.

$$\begin{aligned} &|H_n[g; g']_{x=a} - g(a)| = |H_n[g; g']_{x=a}| = \\ &= |H_n[g - g_\nu; g' - g'_\nu]_{x=a} + H_n[g_\nu; g'_\nu]_{x=a}| \leqq \\ &\leqq |H_n[g - g_\nu; g' - g'_\nu]_{x=a}| + \frac{\varepsilon}{3}, \end{aligned} \tag{63}$$

$$
\begin{aligned}
&|H_n[g-g_\nu;\ g'-g'_\nu]_{x=a}| = \\
(64)\qquad &= \left| \sum_{k=1}^{n} (g(x_k)-g_\nu(x_k))\, v_k(a)\, l_k^2(a) + \sum_{k=1}^{n} (g'(x_k)-g'_\nu(x_k))\,(a-x_k)\, l_k^2(a) \right| \leqq \\
&\leqq \frac{\varepsilon}{3} \sum_{k=1}^{n} v_k(a)\, l_k^2(a) + \sum_{k=1}^{n} |g'(x_k)-g'_\nu(x_k)|\, |a-x_k|\, l_k^2(a),
\end{aligned}
$$

furthermore, since in $-1 \leqq x \leqq a$ and $a + \frac{1}{\nu} \leqq x \leqq +1$ $g'(x_k) = g'_\nu(x_k)$

$$
\begin{aligned}
&\sum_{k=1}^{n} |g'(x_k)-g'_\nu(x_k)|\, |a-x_k|\, l_k^2(a) = \sum_{a<x_k<a+\frac{1}{\nu}} |g'(x_k)-g'_\nu(x_k)|\, |a-x_k|\, l_k^2(a) = \\
&= \sum_{a<x_k<a+\frac{1}{\nu}} \left| \frac{\varrho}{2}(x_k-a)^{\varrho/2-1} + \nu^2 (x_k-a)^{\varrho/2+1}\left(\frac{\varrho}{2}+2\right) - 2\nu (x_k-a)^{\varrho/2}\left(\frac{\varrho}{2}+1\right) \right| \cdot \\
(65)\qquad &|a-x_k|\, l_k^2(a) \leqq \sum_{a<x_k<a+\frac{1}{\nu}} \left(\frac{\varrho}{2}\left(\frac{1}{\nu}\right)^{\varrho/2-1} + \nu^2\left(\frac{\varrho}{2}+2\right)\left(\frac{1}{\nu}\right)^{\varrho/2+1} + 2\nu\left(\frac{\varrho}{2}+1\right)\left(\frac{1}{\nu}\right)^{\varrho/2} \right) \cdot \\
&\cdot \frac{1}{\nu}\, l_k^2(a) < \sum_{k=1}^{n} (4+2\varrho)\left(\frac{1}{\nu}\right)^{\varrho/2} l_k^2(a) < (4+2\varrho)\left(\frac{1}{\nu}\right)^{\varrho/2} \frac{1}{\varrho} < \frac{6}{\varrho}\left(\frac{1}{\nu}\right)^{\varrho/2} < \frac{\varepsilon}{3}
\end{aligned}
$$

(63), (64) and (65) give for regular n-values greater than n_0 and for $-1 \leqq a \leqq 0$

$$
(66)\qquad |H_n[g;\ g']_{x=a}| = \sum_{a<x_k} (x_k-a)^{\varrho/2}\, l_k^2(a) \left(v_k(a) - \frac{\varrho}{2} \right) < \varepsilon
$$

and this was to be proved. Since for non-regular n-values each term of the sum in (66) vanishes and $v_k(a) - \frac{\varrho}{2} \geqq \frac{\varrho}{2} > 0$, (66) gives uniformly for $-1 \leqq a \leqq 0$

$$
(67)\qquad \lim_{n\to\infty} \sum_{a \leqq x_k} (x_k-a)^{\varrho/2}\, l_k^2(a) = 0 .
$$

For $0 \leqq a \leqq 1$ we obtain (67) with a suitable modification of the definition of $g_\nu(x)$, in the same manner, with the aid of the function

$$
(68)\qquad G(x) = \begin{cases} (a-x)^{\varrho/2} & \text{if} \quad -1 \leqq x \leqq a \\ 0 & \text{if} \quad a \leqq x \leqq +1 \end{cases}
$$

we can prove

$$
(69)\qquad \lim_{n\to\infty} \sum_{x_k \leqq a} (a-x_k)^{\varrho/2}\, l_k^2(a) = 0 .
$$

It follows from (67), (69) that

$$\lim_{n\to\infty}\sum_{k=1}^{n}|a-x_k|^{\varrho/2}\, l_k^2(a)=0. \tag{70}$$

Let $\delta>0$ be fixed, then we have

$$\lim_{n\to\infty}\sum_{|a-x_k|>\delta} l_k^2(a)=0 \tag{71}$$

thus if $\varepsilon>0$ is a given number

$$\begin{aligned}\sum_{k=1}^{n}|a-x_k|\, l_k^2(a) &= \sum_{|a-x_k|\leqq\varepsilon\varrho}|a-x_k|\, l_k^2(a)+\sum_{|a-x_k|>\varepsilon\varrho}|a-x_k|\, l_k^2(a)< \\ &<\varepsilon\varrho\sum_{k=1}^{n} l_k^2(a)+2\sum_{|a-x_k|>\varepsilon\varrho} l_k^2(a)<\varepsilon\varrho\,\frac{1}{\varrho}+2\sum_{|a-x_k|>\varepsilon\varrho} l_k^2(a)\to\varepsilon.\end{aligned} \tag{72}$$

That is

$$\lim_{n\to\infty}\sum_{k=1}^{n}|a-x_k|\, l_k^2(a)=0. \tag{73}$$

Since a is an arbitrary point in $-1\leqq x\leqq+1$ we have for strongly normal pointsystems

$$\lim_{n\to\infty}\sum_{k=1}^{n}|\mathfrak{h}_k(x)|=0 \qquad -1\leqq x\leqq+1. \tag{74}$$

(74) is uniformly holding in $-1\leqq x\leqq+1$ since our estimates were independent of a.

So the proof of theorem 3. is completed. The proof of (50) with the aid of (28) and (29) runs analogously and so the theorem 4. is established, too.

5 §. The Convergence of the Hermite Interpolation Polynomials of a Continuous Function if the Derivatives of the Polynomials are not Bounded.

In this and the following § we shall generalize the theorem 3. L. Fejér proved that in the case of the Tchebycheff pointsystem the Hermite interpolation polynomials of an arbitrary continuous function $f(x)$ are convergent when the numbers $d_k^{(n)}$ are bounded if $n\to\infty$. Even if

$$d_k^{(n)}=\varepsilon_k^{(n)}\frac{n}{\log n} \tag{75}$$

where $\varepsilon_k^{(n)} \to 0$ when $n \to \infty$, then

(76) $$\lim_{n\to\infty} H_n[f; d] = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f(x_k)\, h_k(x) + \sum_{k=1}^{n} d_k\, \mathfrak{h}_k(x)\right) = f(x)$$

and the convergence is uniform in the interval $-1 \leqq x \leqq +1$.[1]

We prove for ϱ-normal pointsystems the theorem:

Theorem 5. Let $f(x)$ be an arbitrary continuous function in the interval $-1 \leqq x \leqq +1$ and let $d_k = d_k^{(n)}$ be given numbers for which

(77) $$|d_k^{(n)}| < n^{\varrho-\varepsilon} \qquad k = 1, 2, \ldots, n;\ n = 1, 2, \ldots,$$

if $n \to \infty$ and $\varepsilon > 0$ is an arbitrarily given number. Furthermore let the pointsystem be ϱ-normal, that is for all n

(78) $$v_k(x) \geqq \varrho > 0 \qquad k = 1, 2, \ldots, n;\ -1 \leqq x \leqq +1.$$

Then

(79) $$\lim_{n\to\infty} H_n[f; d] = f(x)$$

and the convergence is uniform in the interval $-1 \leqq x \leqq +1$.

The fundament of the proof is the following approximation theorem: Let $1 > \varrho' > 0$ and

(80) $$g(x) = \begin{cases} 0 & \text{if } \quad -1 \leqq x \leqq a \\ (x-a)^{\varrho'} & \text{if } \quad a \leqq x \leqq +1, \end{cases}$$

then there exists a polynomial $P(x)$ of degree n so that in $-1 \leqq x \leqq +1$

(81) $$|g(x) - P(x)| < c \log n \cdot n^{-\varrho'}, \quad |\varrho' g(x) - (x-a) P'(x)| < c \log n \cdot n^{-\varrho'},$$

where c denotes a positive absolute constant. We shall prove this theorem in the next §.

Let $\varrho' = \varrho - \frac{\varepsilon}{2}$, where $\varepsilon > 0$ arbitrary but $< \varrho$. It follows from (81) that for sufficiently large n

(82) $$\begin{aligned} &|g(x) - P(x)| < c\, n^{-\varrho+\varepsilon} \\ &|\varrho' g(x) - (x-a) P'(x)| < c\, n^{-\varrho+\varepsilon} \end{aligned} \qquad -1 \leqq x \leqq +1.$$

If $x_k \neq a$ we have (see (53))

(83) $$H_n[g; g']_{x=a} = \sum_{a \leqq x_k} (x_k - a)^{\varrho'}\, l_k^2(a)\, (v_k(a) - \varrho')$$

[1] See FEJÉRS first paper cited in note 2, p. 222.

and for sufficiently large n

$$P(a) = \sum_{k=1}^{n} P(x_k)\, h_k(a) + \sum_{k=1}^{n} P'(x_k)\, \mathfrak{h}_k(a). \tag{84}$$

It follows from (82), (84) and (27)

$$\begin{aligned} |H_n[g;\, g']_{x=a} - P(a)| &= \left| \sum_{k=1}^{n} (g(x_k) - P(x_k))\, h_k(a) + \sum_{k=1}^{n} (g'(x_k) - P'(x_k))\, \mathfrak{h}_k(a) \right| \leqq \\ &\leqq c\, n^{-\varrho+\varepsilon} \sum_{k=1}^{n} h_k(a) + c\, n^{-\varrho+\varepsilon} \sum_{k=1}^{n} l_k^2(a) \leqq c\, n^{-\varrho+\varepsilon} + \frac{c}{\varrho'} n^{-\varrho+\varepsilon} < c\, n^{-\varrho+\varepsilon}. \end{aligned} \tag{85}$$

Since $|P(a)| < c\, n^{-\varrho+\varepsilon}$ (this follows from (82) and from $g(a) = 0$)

$$|H_n[g;\, g']_{x=a}| < c\, n^{-\varrho+\varepsilon}. \tag{86}$$

(83), (86) and $v_k(a) - \varrho' \geqq \frac{\varepsilon}{2} > 0$ give

$$\sum_{x_k \geqq a} (x_k - a)^{\varrho'}\, l_k^2(a)\, \frac{\varepsilon}{2} < c\, n^{-\varrho'+\varepsilon} \tag{87}$$

that is

$$\sum_{x_k \geqq a} \left(\frac{x_k - a}{2}\right)^{\varrho'} l_k^2(a) < c\, n^{-\varrho'+\varepsilon}. \tag{88}$$

$\left(\frac{x_k - a}{2}\right)^{\varrho'} > \frac{x_k - a}{2}$ since $\varrho' = \varrho - \frac{\varepsilon}{2} < 1$ and $\frac{x_k - a}{2} \leqq 1$ so it follows from (88)

$$\sum_{x_k \geqq a} (x_k - a)\, l_k^2(a) < c\, n^{-\varrho+\varepsilon}. \tag{89}$$

An analogous argumentation, with the aid of the function

$$G(x) = \begin{cases} (a - x)^{\varrho'} & \text{if } \; -1 \leqq x \leqq a \\ 0 & \text{if } \; a \leqq x \leqq +1 \end{cases} \tag{90}$$

gives

$$\sum_{x_k \leqq a} (a - x_k)\, l_k^2(a) < c\, n^{-\varrho+\varepsilon} \tag{91}$$

and so

$$\sum_{k=1}^{n} |\mathfrak{h}_k(a)| = \sum_{k=1}^{n} |a - x_k|\, l_k^2(a) < c\, n^{-\varrho+\varepsilon}. \tag{92}$$

Thus

(93) $$\lim_{n\to\infty} n^{\varrho-2\varepsilon}\sum_{k=1}^{n}|\mathfrak{h}_k(a)| = 0 .$$

It is clear that (93) holds uniformly in a so we have proved that for an arbitrary $\varepsilon > 0$ uniformly in the interval $-1 \leqq x \leqq +1$

(94) $$\lim_{n\to\infty} n^{\varrho-\varepsilon}\sum_{k=1}^{n}|\mathfrak{h}_k(x)| = 0 .$$

Theorem 5. follows from (77), (94) and

(95) $$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\, h_k(x) = f(x) .$$

6 §. **An Approximation Theorem.**

We prove the approximation theorem enunciated in § 5. The function $\varrho' g(x)$ satisfies in the interval $-1 \leqq x \leqq +1$ a Lipschitz condition with the exponent ϱ' i e.

(96) $$|\varrho' g(x') - \varrho' g(x'')| < c\,|x' - x''|^{\varrho'} \qquad -1 \leqq x',\ x'' \leqq +1$$

where c is an absolute constant. The existence of a polynomial $Q(x)$ of degree n for which

(97) $$|\varrho' g(x) - Q(x)| < c\, n^{-\varrho'} \qquad -1 \leqq x \leqq +1$$

follows from a wellknown theorem.[1]

In the interval $-1 \leqq x \leqq a + \frac{1}{n^3}$

(98) $$|Q(x)| < c\, n^{-\varrho'},$$

since in the same interval

(99) $$|\varrho' g(x)| < c\, n^{-\varrho'} .$$

Thus

(100) $$|\varrho' g(x) - (Q(x) - Q(a))| < c\, n^{-\varrho'} \qquad -1 \leqq x \leqq +1 .$$

---

[1] We mention the investigations of DE LA VALLÉE POUSSIN, LEBESGUE, S. BERNSTEIN, D. JACKSON.

It follows from the theorem of Markoff that

(101) $$|Q'(x)| < cn^2 \qquad -1 \leqq x \leqq +1,$$

since $|Q(x)| \leqq c$ in $-1 \leqq x \leqq +1$.

In the interval $-1 \leqq x \leqq a + \frac{1}{n^3}$ also

$$\left| \int_{-1}^{x} \frac{Q(x) - Q(a)}{x - a} dx \right| =$$

(102) $$= \left| \int_{-1}^{a-\frac{1}{n^3}} \frac{Q(x) - Q(a)}{x - a} dx + \int_{a-\frac{1}{n^3}}^{x} \frac{Q(x) - Q(a)}{x - a} dx \right| \leqq$$

$$\leqq \int_{-1}^{a-\frac{1}{n^3}} \frac{c \cdot n^{-\varrho'}}{|x-a|} dx + \int_{a-\frac{1}{n^3}}^{x} cn^2 dx < cn^{-\varrho'} \log n + cn^{-1} < cn^{-\varrho'} \log n.$$

Thus we have in the interval $-1 \leqq x \leqq a + \frac{1}{n^3}$

(103)
$$\left| \int_{-1}^{x} \frac{\varrho' g(x)}{x - a} dx - \int_{-1}^{x} \frac{Q(x) - Q(a)}{x - a} dx \right| = \left| g(x) - \int_{-1}^{x} \frac{Q(x) - Q(a)}{x - a} dx \right| < cn^{-\varrho'} \log n.$$

On the other hand it follows from (100) if $a + \frac{1}{n^3} \leqq x \leqq +1$

(104) $$\left| g(x) - \int_{a+\frac{1}{n^3}}^{x} \frac{Q(x) - Q(a)}{x - a} dx \right| < cn^{-\varrho'} \int_{a+\frac{1}{n^3}}^{x} \frac{1}{x - a} dx < cn^{-\varrho'} \log n$$

and so in the same interval by (102) and (104)

(105) $$\left| \int_{-1}^{x} \frac{\varrho' g(x)}{x - a} dx - \int_{-1}^{x} \frac{Q(x) - Q(a)}{x - a} dx \right| =$$

$$= \left| g(x) - \left(\int_{-1}^{a+\frac{1}{n^3}} \frac{Q(x) - Q(a)}{x - a} dx + \int_{a+\frac{1}{n^3}}^{x} \frac{Q(x) - Q(a)}{x - a} dx \right) \right| \leqq cn^{-\varrho'} \log n$$

Collecting the results we have

$$\left| g(x) - \int_{-1}^{x} \frac{Q(x) - Q(a)}{x - a} dx \right| < c n^{-\varrho'} \log n \qquad -1 \leqq x \leqq +1 \tag{106}$$

and this proves the approximation theorem with

$$P(x) = \int_{-1}^{x} \frac{Q(x) - Q(a)}{x - a} dx. \tag{107}$$

II. The Convergence of the Lagrange Interpolation Polynomials of a Function Corresponding to Normal and Strongly Normal Pointsystems.

1 §. The Results of L. Fejér.

It is known that if a function $f(x)$ satisfies a Lipschitz condition with an exponent greater than $\frac{1}{2}$, then the sequence of the Lagrange interpolation polynomials of $f(x)$ corresponding to strongly normal pointsystems converges uniformly to $f(x)$.[1] We shall improve this result. The problem is to find an upper bound for

$$\sum_{k=1}^{n} |l_k(x)|. \tag{108}$$

Indeed if we know e. g.

$$\sum_{k=1}^{n} |l_k(x)| < c n^{\alpha}, \quad -1 \leqq x \leqq 1, \quad 0 < \alpha < 1, \tag{109}$$

where $c > 0$ is an absolute constant, it follows from a wellknown argumentation that the Lagrange polynomials of a function satisfying a Lipschitz condition with an exponent greater than α are uniformly convergent.[2] Our tool is the inequality for the fundamental functions of the second kind of Hermite interpolation, proved in § 5 of Part I. There exists namely a connection between (108) and $\sum_{k=1}^{n} |\mathfrak{h}_k(x)|$.

[1] See FEJÉRS first paper cited in note 2, p. 222.

[2] The idea of the proof due to LEBESGUE, HAAR, FABER. See FEJÉRS first cited paper in note 2, p. 222.

Indeed the inequality of Cauchy gives for $x \neq x_k$

$$\sum_{k=1}^{n} |l_k(x)| \leqq \sqrt{\sum_{k=1}^{n} \frac{1}{|x - x_k|} \sum_{k=1}^{n} |\mathfrak{h}_k(x)|}. \tag{110}$$

2 §. A Convergence Theorem of the Lagrange Interpolation Polynomials of a Function Satisfying a Certain Lipschitz Condition.

Let $\delta > 0$ be a fixed number and x an arbitrary point in $-1 + \delta \leqq x \leqq 1 - \delta$. Then we have for ϱ-normal pointsystems

$$S = \sum_{k=1}^{i-1} \frac{1}{x - x_k} + \sum_{k=i+2}^{n} \frac{1}{x_k - x} < cn \log n \qquad x_i \leqq x \leqq x_{i+1}. \tag{111}$$

Erdős-Turán proved[1], that for a ϱ-normal pointsystem

$$\theta_{\nu-1} - \theta_\nu > \frac{c}{n} \qquad \nu = 2, \ldots, n-1, n, \tag{112}$$

where $x_\nu = \cos \theta_\nu$. (111) is an easy consequence of (112).

From (27) follows

$$|l_i(x)| + |l_{i+1}(x)| < \frac{2}{\sqrt{\varrho}} \tag{113}$$

and so

$$\begin{aligned} \sum_{k=1}^{n} |l_k(x)| &< \frac{2}{\sqrt{\varrho}} + \sum_{\substack{k=1 \\ k \neq i,\, i+1}}^{n} |l_k(x)| < \\ &< \frac{2}{\sqrt{\varrho}} + \sqrt{S \sum_{k=1}^{n} |\mathfrak{h}_k(x)|} < \\ &< \frac{2}{\sqrt{\varrho}} + \sqrt{cn \log n \sum_{k=1}^{n} |\mathfrak{h}_k(x)|}. \end{aligned} \tag{114}$$

Thus (92) gives for an arbitrary but fixed $\varepsilon > 0$ and sufficiently large n

$$\sum_{k=1}^{n} |l_k(x)| < c \cdot n^{\frac{1-\varrho}{2} + \varepsilon} \qquad -1 + \delta \leqq x \leqq 1 - \delta. \tag{115}$$

[1] See the paper of. P. Erdős-P. Turán cited in note 2, p. 222.

So from the theorem mentioned in § 1 follows the *theorem 6. Let* $f(x)$ *be a function defined in* $-1 \leqq x \leqq +1$ *which satisfies a Lipschitz condition with an exponent greater than* $\frac{1-\varrho}{2}$, *that is*

$$(116) \qquad |f(x') - f(x'')| < c\,|x' - x''|^{\alpha} \qquad -1 \leqq x', x'' \leqq +1, \; \alpha > \frac{1-\varrho}{2}.$$

Furthermore let the pointsystem of the fundamental points of the Lagrange interpolation be ϱ*-normal*:

$$(117) \qquad v_k(x) \geqq \varrho > 0 \qquad k = 1, 2, \ldots, n; \; n = 1, 2, \ldots; \; -1 \leqq x \leqq +1.$$

Then the sequence of the Lagrange interpolation polynomials of the function $f(x)$ *is convergent in the interval* $-1 < x < +1$ *and the convergence is uniform in the interval* $-1 + \delta \leqq x \leqq 1 - \delta$, *where* $\delta > 0$ *is an arbitrary but fixed number.*

III. On the Sum $\sum_{k=1}^{n} l_k^2(x)$.

1 §. Preliminaries.

We have seen that the sum

$$(118) \qquad l_1^2(x) + l_2^2(x) + \cdots + l_n^2(x)$$

is important in the investigation of the interpolation polynomials. L. Fejér investigated this sum for Jacobi pointsystems. In the Tchebycheff case he proved[1], that

$$(119) \qquad \lim_{n \to \infty} \sum_{k=1}^{n} l_k^2(x) = \begin{cases} 2 & \text{if } \quad x = -1 \\ 1 & \text{if } \quad -1 < x < +1 \\ 2 & \text{if } \quad x = +1. \end{cases}$$

In the case of the pointsystem corresponding to the roots of the n^{th} Jacobi polynomial with the parameter values α, β

[1] L. Fejér, Bestimmung derjenigen Abscissen eines Intervalles, für welche die Quadratsumme der Grundfunktionen der Lagrangeschen Interpolation im Intervalle ein möglichst kleines Maximum besitzt, *Annali della R. Scuola Normale Superiore di Pisa* (1932), p. 3—16.

$$\lim_{n\to\infty}\sum_{k=1}^{n} l_k^2(x) = \begin{cases} \frac{1}{1-2\beta} & \text{if} \quad x = -1 \\ 1 & \text{if} \quad -1 < x < +1 \qquad 0 \leqq \alpha \leqq \frac{1}{2} \\ \frac{1}{1-2\alpha} & \text{if} \quad x = +1 \qquad 0 \leqq \beta \leqq \frac{1}{2} \end{cases} \tag{120}$$

We shall prove *that for normal pointsystems*

$$\lim_{n\to\infty}\sum_{k=1}^{n} l_k^2(x) = 1 \qquad -1 < x < +1 \tag{121}$$

and the convergence is uniform in the interval $-1+\varepsilon \leqq x \leqq 1-\varepsilon$, *where* $\varepsilon > 0$ *arbitrary fixed number.*

2 §. Proof of the Convergence of $\sum_{k=1}^{n} l_k^2(x)$ if $n \to \infty$ for Normal Pointsystems.

Let $1 > \lambda > 0$ be a fixed number then for such indizes k for which $-1+\lambda \leqq x_k \leqq 1-\lambda$ we have

$$\left|\frac{\omega''(x_k)}{\omega'(x_k)}\right| < \frac{1}{\lambda} \tag{122}$$

and

$$v_k(x) \leqq 1 + \frac{2}{\lambda}. \tag{123}$$

Indeed either $(1-x_k)\frac{\omega''(x_k)}{\omega'(x_k)}$ or $(-1-x_k)\frac{\omega''(x_k)}{\omega'(x_k)}$ is positive thus it follows from $v_k(+1) \geqq 0$, $v_k(-1) \geqq 0$

$$\left|\frac{\omega''(x_k)}{\omega'(x_k)}\right| \leqq \max\left(\frac{1}{|1-x_k|}, \frac{1}{|1+x_k|}\right) \leqq \frac{1}{\lambda} \tag{124}$$

since $1 - x_k \geqq \lambda$, $1 + x_k \geqq \lambda$.

Furthermore

$$v_k(x) = 1 - (x - x_k)\frac{\omega''(x_k)}{\omega'(x_k)} \leqq 1 + |x - x_k| \left|\frac{\omega''(x_k)}{\omega'(x_k)}\right| \leqq 1 + \frac{2}{\lambda}. \tag{125}$$

We need the relation

$$(126)\qquad \lim_{n\to\infty} \sum_{|x-x_k|>\delta}^{n} l_k^2(x) = 0 \qquad -1 < x < +1,\ \delta > 0,$$

which is an easy consequence of (50).

Let $f(x)$ be a function defined by

$$(127)\qquad f(x) = \begin{cases} 0 & \text{if } -1 \leqq x \leqq -1 + \frac{\lambda}{2} \\ \frac{2}{\lambda}x + \frac{2-\lambda}{\lambda} & \text{if } -1 + \frac{\lambda}{2} \leqq x \leqq -1 + \lambda \\ 1 & \text{if } -1 + \lambda \leqq x \leqq 1 - \lambda \\ -\frac{2}{\lambda}x + \frac{2-\lambda}{\lambda} & \text{if } 1 - \lambda \leqq x \leqq 1 - \frac{\lambda}{2} \\ 0 & \text{if } 1 - \frac{\lambda}{2} \leqq x \leqq +1 \end{cases}$$

$f(x)$ is continuous in $-1 \leqq x \leqq +1$. It follows from the last remark in § 2 Part I. that

$$(128)\qquad \lim_{n\to\infty} H_n[f;0] = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\, h_k(x) = f(x) \qquad -1 < x < +1$$

and the convergence is uniform in an inner interval of $-1 \leqq x \leqq +1$. Let x_0 be an arbitrary but fixed number in the interval $-1 + \lambda < x < 1 - \lambda$. There exists a $\delta > 0$ so that $f(x) = 1$ in $x_0 - \delta \leqq x \leqq x_0 + \delta$.

Let

$$(129)\qquad \sum = \sum_{k=1}^{n} f(x_k)\, h_k(x_0) = \sum_{|x_0-x_k|\leqq\delta} f(x_k)\, h_k(x_0) + \sum_{|x_0-x_k|>\delta} f(x_k)\, h_k(x_0) = \sum\nolimits_1 + \sum\nolimits_2 .$$

In $\sum_1$ only such k occur for which $f(x_k) = 1$, thus

$$(130)\qquad \sum\nolimits_1 = \sum_{|x_0-x_k|\leqq\delta} v_k(x_0)\, l_k^2(x_0) = \sum_{|x_0-x_k|\leqq\delta} l_k^2(x_0) - \sum_{|x_0-x_k|\leqq\delta} (x_0 - x_k)\frac{\omega''(x_k)}{\omega'(x_k)}\, l_k^2(x_0).$$

Since here $-1 + \lambda \leqq x_k \leqq 1 - \lambda$ the inequality (124) holds and so

$$(131)\qquad \sum_{|x_0-x_k|\leqq\delta} |x_0 - x_k| \left|\frac{\omega''(x_k)}{\omega'(x_k)}\right| l_k^2(x_0) \leqq \frac{1}{\lambda} \sum_{|x_0-x_k|\leqq\delta} |x_0 - x_k|\, l_k^2(x_0).$$

Because of

$$\lim_{n\to\infty}\sum_{k=1}^{n}|\mathfrak{h}_k(x_0)| = 0 \tag{132}$$

(131) and (130) give

$$\sum\nolimits_1 = \sum_{|x_0-x_k|\leqq\delta} l_k^2(x_0) + \varepsilon_n, \tag{133}$$

where $\varepsilon_n \to 0$ if $n \to \infty$.

On the other hand if $-1 \leqq x_k \leqq -1 + \frac{\lambda}{2}$ or $1 - \frac{\lambda}{2} \leqq x_k \leqq 1$ $f(x_k) = 0$ so in $\sum_2$ only such x_k occurs for which $-1 + \frac{\lambda}{2} \leqq x_k \leqq 1 - \frac{\lambda}{2}$. That is (123) holds with $\frac{\lambda}{2}$ instead of λ

$$v_k(x_0) \leqq 1 + \frac{4}{\lambda}. \tag{134}$$

Thus

$$\sum\nolimits_2 = \sum_{|x_0-x_k|>\delta} f(x_k)\, h_k(x_0) < \left(1 + \frac{4}{\lambda}\right) \sum_{|x_0-x_k|>\delta} l_k^2(x_0). \tag{135}$$

From (126) and (135) follows that $\sum_2 \to 0$ if $n \to \infty$. (129), (133) and the last remark give

$$1 = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\, h_k(x_0) = \lim_{n\to\infty}\sum_{|x_0-x_k|\leqq\delta} l_k^2(x_0) = \lim_{n\to\infty}\sum_{k=1}^{n} l_k^2(x_0), \tag{136}$$

which is our theorem.

3 §. A Convergence Theorem for Certain Interpolation Polynomials.

An interesting consequence of the theorem proved in the preceeding § is the following: *If $f(x)$ is a continuous function in the interval $-1 \leqq x \leqq +1$, then we have for strongly normal pointsystems*

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\, l_k^2(x) = f(x) \qquad -1 < x < +1 \tag{137}$$

and the convergence is uniform in $-1 + \lambda \leqq x \leqq 1 - \lambda$, where $\lambda > 0$.

The proof is very simple. Let x be a fixed point in the interval $-1 < x < +1$ and let $\varepsilon > 0$ be given. Then $|f(x) - f(x_k)| < \varepsilon$ when $|x - x_k| < \delta$. Also

$$\sum_{k=1}^{n} f(x_k)\, l_k^2(x) = \sum_{k=1}^{n} (f(x_k) - f(x))\, l_k^2(x) + f(x) \sum_{k=1}^{n} l_k^2(x) =$$

(138)

$$= \sum_{|x-x_k| \leqq \delta} (f(x_k) - f(x))\, l_k^2(x) + \sum_{|x-x_k| > \delta} (f(x_k) - f(x))\, l_k^2(x) + f(x) \sum_{k=1}^{n} l_k^2(x) = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

It follows from (136) that

(139) $$\sum\nolimits_3 \to f(x)$$

and from (126) that

(140) $$\sum\nolimits_2 < 2\, M \sum_{|x-x_k| > \delta} l_k^2(x) \to 0,$$

where $M = \operatorname*{Max}_{-1 \leqq x \leqq +1} |f(x)|$

For $\sum_1$ we have

(141) $$\sum\nolimits_1 < \varepsilon \sum_{k=1}^{n} l_k^2(x) \leqq \frac{\varepsilon}{\varrho}.$$

(138), (139), (140), (141) give for sufficiently large n

(142) $$\left| \sum_{k=1}^{n} f(x_k)\, l_k^2(x) - f(x) \right| < \frac{2\,\varepsilon}{\varrho}$$

and the theorem is established.