A GENERAL PRIME NUMBER THEOREM.

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Consider a monotone sequence of real positive numbers

$$(1) \qquad 1 < y_1 < y_2 < \cdots < y_n < \cdots .$$

Form all possible products

$$(2) \qquad x = y_{n_1} y_{n_2} \ldots y_{n_k}, \qquad n_1 \leq n_2 \leq \cdots \leq n_k,$$

and arrange them in a non-decreasing sequence

$$(3) \qquad 1 < x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$$

where every x appears as many times as it can be represented by formula (2). The numbers $\{y_n\}$ are called the primes of the sequence $\{x_n\}$. Let $\pi(x)$ denote the number of primes $\leq x$, and $N(x)$ the number of $x_n \leq x$.

This definition of generalized prime numbers is given by BEURLING, who under certain general conditions has derived very interesting relations between the functions $N(x)$ and $\pi(x)$.[1]

In what follows, $\zeta(s)$ denotes the function

$$(4) \qquad \zeta(s) = 1 + x_1^{-s} + x_2^{-s} + \cdots = \int_0^\infty x^{-s}\, d\, N(x). \qquad s = \sigma + it.$$

(For the sake of simplicity, we assume that $N(x)$ has a step equal to 1 at the point $x = 1$.) $\mathrm{Li}(x)$ denotes the logarithmic integral, i. e. the principal value of the integral

$$\int_0^x \frac{dy}{\log y}.$$

[1] A. BEURLING, Analyse de la loi asymptotique de la distribution des nombres premiers généralisés, Acta mathematica, vol. 68.

It is well known that $\mathrm{Li}(x)$ has the following asymptotic expansion:

$$\mathrm{Li}(x) \sim x\left\{\frac{1}{\log x} + \frac{1!}{(\log x)^2} + \frac{2!}{(\log x)^3} + \cdots\right\}.$$

The following theorem will be proved:

Theorem: *The following three statements are equivalent:*

A. *There exists a real number* $a > 0$, *such that*

$$N(x) = a x + O\left\{\frac{x}{(\log x)^n}\right\} \quad \text{as} \quad x \to \infty \tag{5}$$

for every positive n.

B. *To every* $\varepsilon > 0$ *and every non-negative integer* n, *a constant* A[1] *can be chosen such that*

$$|\zeta^{(n)}(s)| < A\,|t|^{\varepsilon}, \tag{6}$$

$$\left|\frac{1}{\zeta(s)}\right| < A\,|t|^{\varepsilon}, \tag{7}$$

uniformly in the region $\sigma > 1$, $|t| \geq \varepsilon$.

C. $\pi(x)$ *has the same asymptotic expansion as* $\mathrm{Li}(x)$, *i. e.*

$$\pi(x) = \mathrm{Li}(x) + O\left\{\frac{x}{(\log x)^n}\right\} \quad \text{as} \quad x \to \infty \tag{8}$$

for every positive n.

This theorem will be proved by the aid of Parseval's formula for Mellin transforms.

From each of the hypothesis A, B and C it follows that the series defining $\zeta(s)$ is absolutely convergent in the half-plane $\sigma > 1$ and can be written there as an Euler-product

$$\zeta(s) = \prod_{1}^{\infty} \frac{1}{1 - y_n^{-s}}.$$

Thus

$$\log \zeta(s) = -\sum_{1}^{\infty} \log(1 - y_n^{-s}) = \int_{1}^{\infty} x^{-s}\, d\,\Pi(x), \tag{9}$$

where

$$\Pi(x) = \pi(x) + \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \tfrac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots. \tag{10}$$

[1] A always denotes a positive constant, possibly depending upon ε and n, but not depending upon σ and t. A can very well have different values in different places.

For the proof we need the following lemmas:

Lemma I: *Let $\varphi(s)$ be a function which is holomorphic in the band $1 < \sigma < 2$ and, for $n = 0, 1, 2, 3, \ldots$, satisfies the following conditions:*

$$|\varphi^{(n)}(s)| < \frac{A}{(\sigma - 1)^{n+1}}, \tag{11}$$

$$|\varphi^{(n)}(s)| < A\,|t|^{k_n}, \tag{12}$$

where $k_n \geq 0$ and

$$\varliminf_{n\to\infty} \frac{k_n}{n} = 0,$$

uniformly in the region $1 < \sigma < 2$, $|t| > t_0 > 0$. Then to every $\varepsilon > 0$ and $n = 0, 1, 2, 3, \ldots$, a constant A can be chosen such that

$$|\varphi^{(n)}(s)| < A\,|t|^{\varepsilon} \tag{13}$$

uniformly in the same region.

Let us suppose that $\alpha_n \geq 0$ is the least number such that, for every $\varepsilon > 0$,

$$|\varphi^{(n)}(s)| < A\,|t|^{\alpha_n + \varepsilon}$$

uniformly in the above region. By (12), $\alpha_n \leq k_n$. Suppose that $\sigma \leq \frac{3}{2}$ and choose σ' so that $\sigma < \sigma' < 2$. For $|t| > t_0$ we have, by (11),

$$|\varphi^{(n)}(\sigma + it)| \leq |\varphi^{(n)}(\sigma' + it)| + \int_{\sigma+it}^{\sigma'+it} |\varphi^{(n+1)}(s)|\,|ds| \leq \frac{A}{(\sigma' - \sigma)^{n+1}} + (\sigma' - \sigma)A\,|t|^{\alpha_{n+1}+\varepsilon}.$$

Putting $\sigma' = \sigma + A\,|t|^{-\frac{\alpha_{n+1}}{n+2}}$, where A is chosen so that $\sigma' < 2$ for all σ in the interval $\left(1, \frac{3}{2}\right)$ and $|t| > t_0$, we obtain

$$|\varphi^{(n)}(\sigma + it)| < A\,|t|^{\alpha_{n+1}\frac{n+1}{n+2}+\varepsilon}$$

uniformly for $1 < \sigma \leq \frac{3}{2}$, $|t| > t_0$. By (11), an inequality of the same form evidently holds even for $1 < \sigma < 2$. Thus

$$\alpha_n \leq \frac{n+1}{n+2}\,\alpha_{n+1}$$

and

$$\frac{\alpha_n}{n+1} \leq \frac{\alpha_{n+1}}{n+2} \leq \cdots \leq \frac{\alpha_{n+p}}{n+p+1} \leq \frac{k_{n+p}}{n+p+1}.$$

Since we may choose p arbitrarily large, it follows that $\alpha_n = 0$ for all n, and (13) is proved.

Lemma II: *Let $\varphi(s)$ and $\psi(s)$ be two functions, which for $\sigma > 1$ may be represented by the absolutely convergent integrals*

$$\varphi(s) = \int_{1-0}^{\infty} x^{-s}\, d\,S(x), \tag{14}$$

$$\psi(s) = \int_{1}^{\infty} x^{-s}\, d\,T(x) \tag{15}$$

where $S(x)$ is non-decreasing, $S(x+0) = S(x)$, and $0 \leq T'(x) \leq A$. Let us put

$$\frac{d^k}{d\,s^k}\left\{\frac{\varphi(s) - \psi(s)}{s}\right\} = \theta_k(s)$$

and suppose that

$$\int_{-\infty}^{\infty} |\theta_k(\sigma + i\,t)|^2\, d\,t \tag{16}$$

is uniformly bounded for $\sigma > 1$ for a fixed $k \geq 0$. Then the relation

$$S(x) = T(x) + o\left\{\frac{x}{(\log x)^n}\right\} \quad \text{as} \quad x \to \infty \tag{17}$$

is valid for $n \leq \frac{2}{3} k$.

By the proof, we can obviously assume that $S(1-0) = 0$ and $T(1) = 0$. Let $\sigma_0 > 1$. The inequality

$$\varphi(\sigma_0) \geq \int_{1-0}^{x} y^{-\sigma_0}\, d\,S(y) = \frac{S(x)}{x^{\sigma_0}} + \sigma_0 \int_{1}^{x} \frac{S(y)}{y^{1+\sigma_0}}\, d\,y \geq \frac{S(x)}{x^{\sigma_0}}$$

yields

$$S(x) \leq \varphi(\sigma_0)\, x^{\sigma_0}.$$

Thus (14) may be integrated by parts for $\sigma > \sigma_0$, i. e. for $\sigma > 1$, since we may choose σ_0 arbitrarily near to 1. Thus

$$\frac{\varphi(s)}{s} = \int_{1}^{\infty} x^{-s}\, \frac{S(x)}{x}\, d\,x.$$

Combining this formula and the analogous formula for $\psi(s)$, we obtain

$$\frac{\varphi(s)-\psi(s)}{s}=\int_1^\infty x^{-s}\frac{S(x)-T(x)}{x}\,dx, \qquad \sigma>1.$$

Differentiating k times, we obtain, for $\sigma>1$,

$$(-1)^k\theta_k(s)=\int_1^\infty x^{-s}\frac{S(x)-T(x)}{x}(\log x)^k\,dx.$$

From Parseval's formula for Mellin transforms, it follows that, for $\sigma>1$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\theta_k(\sigma+it)|^2\,dt=\int_1^\infty\left|\frac{S(x)-T(x)}{x}(\log x)^k\right|^2 x^{1-2\sigma}\,dx.$$

As $\sigma\to 1$, the right-hand member is non decreasing and thus has a limit, which, by (16), is finite. By monotone convergence we thus get

$$\int_1^\infty\left|\frac{S(x)-T(x)}{x}(\log x)^k\right|^2\frac{dx}{x}<\infty.$$

Let us put $S(x)-T(x)=\delta(x)$. Then

$$\int_1^\infty|\delta(x)|^2\cdot\frac{(\log x)^{2k}}{x^3}\,dx<\infty. \tag{18}$$

Since $S(x)$ is non-decreasing and $0\le T'(x)\le A$, we have

$$\delta(y)\ge\frac{\delta(x)}{2}\quad\text{for}\quad x\le y\le x+\frac{\delta(x)}{2A}\quad\text{if}\quad\delta(x)>0,$$

$$-\delta(y)\ge\frac{-\delta(x)}{2}\quad\text{for}\quad x+\frac{\delta(x)}{2A}\le y\le x\quad\text{if}\quad\delta(x)<0.$$

If $\delta(x)>0$, we thus get

$$\int_x^{x+\frac{\delta(x)}{2A}}|\delta(y)|^2\frac{(\log y)^{2k}}{y^3}\,dy>\frac{1}{A}\cdot\left\{\frac{\delta(x)}{2}\right\}^3\frac{(\log x)^{2k}}{\left\{x+\dfrac{\delta(x)}{2A}\right\}^3}=$$

$$=\left\{\frac{\delta(x)}{x}(\log x)^n\right\}^3\cdot\frac{(\log x)^{2k-3n}}{A\left\{2+\dfrac{\delta(x)}{Ax}\right\}^3}.$$

By (18), this integral must have the limit 0 as $x \to \infty$. If we choose $n \leq \frac{2}{3}k$ it follows that

$$\overline{\lim_{x \to \infty}} \frac{\delta(x)}{x} (\log x)^n \leq 0.$$

A quite analogous argument shows that $\underline{\lim} \geq 0$. Thus the lemma is proved.

A implies B. Integrating (4) by parts, we get

$$\frac{\zeta(s)}{s} = \int_1^\infty x^{-s} \frac{N(x)}{x} dx.$$

Combining this formula and

$$\frac{a}{s-1} = \int_1^\infty a x^{-s} dx,$$

we obtain

(19) $$\frac{\zeta(s)}{s} - \frac{a}{s-1} = \int_1^\infty x^{-s} \frac{N(x) - ax}{x} dx.$$

These formulae are valid for $\sigma > 1$. However, by (5), it follows that the integral in (19) is absolutely and uniformly convergent for $\sigma \geq 1$. Thus the left-hand member of (19) is continuous in the closed half-plane $\sigma \geq 1$. If $g(s)$ denotes the integral in (19), we can write

(20) $$\zeta(s) = a + \frac{a}{s-1} + s g(s).$$

Thus

(21) $$\zeta^{(n)}(s) = \frac{(-1)^n a n!}{(s-1)^{n+1}} + s g^{(n)}(s) + n g^{(n-1)}(s),$$

where

$$g^{(n)}(s) = (-1)^n \int_1^\infty x^{-s} \frac{N(x) - ax}{x} (\log x)^n dx.$$

By (5), this integral is absolutely and uniformly convergent for $\sigma \geq 1$. Thus all derivatives $g^{(n)}(s)$ are continuous and bounded for $\sigma \geq 1$. Consequently, it follows from (20) and (21) that $\zeta(s)$ satisfies the conditions of lemma I with all $k_n = 1$

and t_0 arbitrarily small. This lemma thus yields (6). The function $\zeta(s)$ satisfies the inequality

$$|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 1,$$

due to Hadamard. Using this and (6), a classical argument[1] gives (7).

B implies C. The formula

$$\log\frac{s}{s-1} = \int_1^\infty x^{-s}\,dp(x), \qquad \sigma > 1, \tag{22}$$

where

$$p(x) = \int_1^x \frac{1-\frac{1}{y}}{\log y}\,dy = \mathrm{Li}(x) - \log\log x + A, \tag{23}$$

is easily proved. We can now use lemma II with

$S(x) = \Pi(x)$ (cf. (9)!) and $T(x) = p(x)$, since an inequality of the form

$$\left|\frac{d^k}{ds^k}\left\{\frac{\log\zeta(s) - \log\frac{s}{s-1}}{s}\right\}\right| < \frac{A}{1+|t|^{1-\varepsilon}}$$

is valid for $\sigma > 1$ and $k = 0, 1, 2, \ldots$ For, carrying out the differentiations, every term will be of the form

$$\frac{A_\nu}{s^{k-\nu+1}}\frac{d^\nu}{ds^\nu}\left\{\log\zeta(s) - \log\frac{s}{s-1}\right\}, \qquad \nu = 0, 1, \ldots, k,$$

and, if $\nu > 0$,

$$\left|\frac{d^\nu}{ds^\nu}\log\zeta(s)\right| = \left|\frac{P_\nu(s)}{\{\zeta(s)\}^\nu}\right| < A\,|t|^\varepsilon$$

for $|t| \geq \varepsilon$ by (6) and (7), since $P_\nu(s)$ is a sum of products of $\zeta(s)$ and its ν first derivatives. Further, by (7) and (20), the left-hand member of the above inequality is continuous for $\sigma \geq 1$. Thus the lemma gives

$$\Pi(x) = p(x) + O\left\{\frac{x}{(\log x)^n}\right\} \quad \text{as} \quad x \to \infty,$$

for every n. (8) will then follow from (10) and (23).

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[1] Cf. A. E. Ingham, The distribution of prime numbers, p. 29 and 30.

C implies B. Integrating (9) by parts, we obtain

$$\frac{\log \zeta(s)}{s} = \int_1^\infty x^{-s} \frac{\Pi(x)}{x}\, dx.$$

Combining this formula and formula (22), integrated by parts, we get

$$\frac{\log \zeta(s)}{s} - \frac{1}{s} \log \frac{s}{s-1} = \int_1^\infty x^{-s} \frac{\Pi(x) - p(x)}{x}\, dx.$$

If $h(s)$ denotes the integral, we can write

$$\log \zeta(s) = \log \frac{s}{s-1} + s\, h(s). \tag{24}$$

Since $\pi(x)$ satisfies (8), it follows from (10) that $\Pi(x)$ also satisfies (8). Thus $h(s)$ is absolutely and uniformly convergent for $\sigma \geq 1$. It follows that $\zeta(s)$ is continuous and $\neq 0$ for $\sigma \geq 1$, with the exception of the point $s = 1$. Differentiating (24) n times, we obtain

$$\frac{d^n}{d s^n} \log \zeta(s) = (-1)^{n-1} (n-1)! \left\{ \frac{1}{s^n} - \frac{1}{(s-1)^n} \right\} + s\, h^{(n)}(s) + n\, h^{(n-1)}(s), \tag{25}$$

where

$$h^{(n)}(s) = (-1)^n \int_1^\infty x^{-s}\, \frac{\Pi(x) - p(x)}{x} (\log x)^n\, dx.$$

By (8), this integral is absolutely and uniformly convergent for $\sigma \geq 1$. Thus all derivatives $h^{(n)}(s)$ are continuous and bounded for $\sigma \geq 1$. Consequently, it follows from (25) that the function $\frac{d}{ds} \log \zeta(s)$ satisfies the conditions of lemma I with all $k_n = 1$ and t_0 arbitrarily small. Thus

$$\left| \frac{d^n}{d s^n} \log \zeta(s) \right| < A\, |t|^\varepsilon, \qquad n = 1, 2, 3, \ldots, \tag{26}$$

uniformly in the region $\sigma > 1$, $|t| \geq \varepsilon$. From (24) and (26), it follows that

$$|\log \zeta(\sigma + i t)| \leq |\log \zeta(\sigma' + i t)| + \int_{\sigma + i t}^{\sigma' + i t} \left| \frac{d}{ds} \log \zeta(s) \right| |ds| <$$

$$< \log \frac{1}{\sigma' - \sigma} + A\, (\sigma' - \sigma)\, |t|^\varepsilon, \qquad 1 < \sigma < \sigma'.$$

Putting $\sigma' = \sigma + |t|^{-\varepsilon}$, we obtain

$$|\log \zeta(\sigma + it)| < \log |t|^{\varepsilon} + A.$$

Thus

$$(27) \qquad |\zeta(s)| < A\,|t|^{\varepsilon}, \quad \left|\frac{1}{\zeta(s)}\right| < A\,|t|^{\varepsilon}$$

uniformly in the considered region. By carrying out the differentiations in (26) and using (27), we can prove (6) by induction.

B implies A. Let us put $a = e^{h(1)} > 0$ (cf. (24)!) and $S(x) = N(x)$, $T(x) = ax$ in lemma II. As on page 305, it follows from (6), (7) and (24) that

$$\left| \frac{d^k}{ds^k} \left\{ \frac{\zeta(s) - \frac{a}{s-1}}{s} \right\} \right| < \frac{A}{1 + |t|^{1-\varepsilon}}$$

is valid for $\sigma > 1$ and $k = 0, 1, 2, \ldots$, and (5) follows.