

ON FUNCTIONS ORTHOGONAL TO INVARIANT SUBSPACES

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Let H^2 denote the usual Hardy class of functions holomorphic in the unit disk, U . Let M denote a closed, invariant subspace of H^2 . The theory of such subspaces is well-known and may be found, for example, in the first three chapters of Hoffman's book [6]; every such M has the form $M = \varphi H^2$, where $\varphi \in H^2$ is an *inner* function, $\varphi = Bs\Delta$ with

$$B(z) = \prod_{\nu=1}^{\infty} \left(-\frac{\bar{a}_\nu}{|a_\nu|} \right) \frac{z - a_\nu}{1 - \bar{a}_\nu z}, \quad s(z) = \exp \left\{ - \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta) \right\}$$

$$\Delta(z) = \exp \left\{ - \sum_{\nu=1}^{\infty} r_\nu \frac{e^{i\theta_\nu} + z}{e^{i\theta_\nu} - z} \right\}$$

where $\{a_\nu\}$ is a Blaschke sequence ($\sum(1 - |a_\nu|) < \infty$) ($\bar{a}_\nu/|a_\nu| \equiv 1$ is understood whenever $a_\nu = 0$), σ is a finite, positive, continuous, singular measure, and $r_\nu \geq 0$, $\sum r_\nu < \infty$.

In this paper we study the subspace $M^\perp = H^2 \ominus M$. Our results may be summarized as follows: we obtain a unitary operator V which maps the sum of three L^2 spaces onto M^\perp . The first, corresponding to the factor B of φ , is the space $L^2(d\sigma_B)$, where σ_B is the measure on the positive integers that assigns a mass $1 - |a_k|$ to the integer k . The second L^2 space is $L^2(d\sigma)$, and the third is the sum of the L^2 spaces of Lebesgue measure on the real intervals of length r_j .

In the special case $\varphi = B$, the functions $h_n(z) = (1 - |a_n|^2)^{\frac{1}{2}} B_n(z)/(1 - \bar{a}_n z)$ (B_n the Blaschke product with zeros a_1, \dots, a_{n-1}) form an orthonormal basis of M^\perp ; cf. [10, p. 305], [1]. From this fact it follows easily that the map

$$V(\{c_n\})(z) = \sum_{n=1}^{\infty} c_n (1 + |a_n|)^{\frac{1}{2}} B_n(z) (1 - \bar{a}_n z)^{-1} (1 - |a_n|) \quad (0.1)$$

carries $L^2(d\sigma_B)$ isometrically onto M^\perp , and this represents one instance of our theorem.

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In the very special case $\varphi = \Delta$, $r_\nu = 0$ ($\nu \neq 1$) and $\theta_1 = 0$, our operator V reduces to the unitary operator defined by Sarason in [8].

When $\varphi = s$, the form of V represents a direct generalization of (0.1):

$$(Vc(\lambda))(z) = \int_0^{2\pi} c(\lambda) \sqrt{2} s_\lambda(z) (1 - e^{-i\lambda} z)^{-1} d\sigma(\lambda), \quad (0.2)$$

where $s_\lambda(z)$ denotes the inner function

$$s_\lambda(z) = \exp \left\{ - \int_0^\lambda \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta) \right\}. \quad (0.3)$$

In Part I of this paper we develop the various properties of V , first, when $\varphi = s$ (Section 1) then, when $\varphi = \Delta$ (Section 2) and finally (Section 3) for the general case. In Section 4, we present for V an inversion formula of the usual type for such transforms.

Part II is devoted to applications. In Section 5 we study the form of the operator V^*TV , where T is the restricted shift

$$Tf = Pzf, \quad f \in M^\perp.$$

Here P denotes the projection⁽¹⁾ of H^2 onto M^\perp . We show, for example, that T is the sum of a normal operator and a Hilbert-Schmidt operator. We also study spectral properties of functions of T . In Section 6 we give applications of our results to the density of certain functions in M^\perp .

We were first led to seek a continuous analogue of (0.1) by consideration of Section 4 of [1]. Reciprocally, the results presented here can be used to give a more transparent approach to the theory developed there. The details of this approach, however, will not be included in the present paper.

Part I

1. *Continuous singular subspaces.* In this section, we prove that the operator defined in (0.2) is a unitary isometry of $L^2(d\sigma)$ onto M^\perp (where $\varphi(z) = s(z)$ is completely determined by σ).

Several special functions in $L^2(d\sigma)$ will be of importance. One is s_λ , as defined in (0.3), where $z \in U$ is fixed. Another function is the function $\sigma(\lambda)$ which denotes the σ -measure of the interval $(0, \lambda)$. A third important function for us will be

$$c_{\mu, \eta}(\lambda) = \chi_{\mu, \eta}(\lambda) e^{-\sigma(\lambda)}$$

⁽¹⁾ *Projection* always means orthogonal projection.

where $\chi_{\mu,\eta}$ is the characteristic function of the interval $[\mu, \eta)$. In fact it is easy to compute the norm of $c_{\mu,\eta}(\lambda)$ in $L^2(d\sigma)$.

$$\|c_{\mu,\eta}(\lambda)\|^2 = \int_{\mu}^{\eta} e^{-2\sigma(\lambda)} d\sigma(\lambda) = \frac{1}{2} (e^{-2\sigma(\mu)} - e^{-2\sigma(\eta)}).$$

Concerning the set D , by which we denote the span of all $c_{\mu,\eta}(\lambda)$, for $0 \leq \mu < \eta \leq 2\pi$, we have the following

LEMMA 1.1. *D is dense in $L^2(d\sigma)$.*

Proof. Let $[a, b]$ be an interval in $[0, 2\pi]$, and let $\chi(\lambda)$ denote the characteristic function of $[a, b]$. We will prove that $\chi(\lambda)$ is a limit of functions in D . Now for any interval $(\mu, \eta]$, D contains the function $\chi_{\mu,\eta}(\lambda)e^{-\sigma(\mu,\lambda)} = e^{\sigma(\mu)}c_{\mu,\eta}$. Thus, if $[a, b]$ is divided into subintervals $I_i = [\mu_i, \eta_i)$, such that $\sigma(I_i) < \varepsilon$, then D contains functions which are closer to 1 than $1 - e^{-\varepsilon}$ on I_i and 0 elsewhere. Clearly the sum of such functions approximates $\chi(\lambda)$ uniformly (and hence in $L^2(d\sigma)$).

Now we can prove

THEOREM 1.1. *The operator V defined in (0.2) maps $L^2(d\sigma)$ isometrically onto $(sH^2)^\perp$.*

Proof. It is clear that V is a linear map of $L^2(d\sigma)$ into the space of functions holomorphic in U . We begin by showing that V restricted to D is an isometry into H^2 .

To compute $Vc_{\mu,\eta}$, we notice that, for fixed z , the measure $d(s_\lambda(z)e^{-\sigma(\lambda)})$ is absolutely continuous with respect to σ , and

$$d(s_\lambda(z)e^{-\sigma(\lambda)}) = -2e^{-\sigma(\lambda)}s_\lambda(z)(1 - e^{-i\lambda}z)^{-1}d\sigma(\lambda). \tag{1.1}$$

Integrating (1.1), we obtain

$$\begin{aligned} (Vc_{\mu,\eta})(z) &= \sqrt{2} \int_{\mu}^{\eta} e^{-\sigma(\lambda)}s_\lambda(z)(1 - e^{-i\lambda}z)^{-1}d\sigma(\lambda) = -\frac{1}{\sqrt{2}} \int_{\mu}^{\eta} d(s_\lambda(z)e^{-\sigma(\lambda)}) \\ &= \frac{1}{\sqrt{2}} (s_\mu(z)e^{-\sigma(\mu)} - s_\eta(z)e^{-\sigma(\eta)}). \end{aligned} \tag{1.2}$$

Now, it is easy to verify that the function

$$s_\mu(z)e^{-\sigma(\mu)} - s_\eta(z)e^{-\sigma(\eta)} = s_\mu(z)s_\mu(0) - s_\eta(z)s_\eta(0)$$

is the projection onto $(s_\eta H^2)^\perp \ominus (s_\mu H^2)^\perp$ of the function 1, and hence the square of its norm is given by its inner product with 1, i.e. its value at $z = 0$. This yields

$$\|Vc_{\mu,\eta}\|^2 = \frac{1}{2}(s_\mu(0)^2 - s_\eta(0)^2),$$

and this, as we have seen, is equal to the $L^2(d\sigma)$ -norm of $c_{\mu,\eta}$.

To extend this result to D , note that if $[\mu, \eta]$ and $[\nu, \tau]$ are disjoint intervals, then $c_{\mu,\eta}$ and $c_{\nu,\tau}$ are orthogonal, as are $Vc_{\mu,\eta}$ and $Vc_{\nu,\tau}$. Furthermore, any finite linear combination of such $c_{\mu,\eta}$ can be rewritten so that the terms in the sum are pairwise orthogonal. Thus V maps D isometrically into M^\perp .

It remains only to show that V maps $L^2(d\sigma)$ onto $(sH^2)^\perp$. Let $\zeta \in U$ and define $K_\zeta(z) = (1 - \bar{s}(\zeta)s(z))(1 - \bar{\zeta}z)^{-1}$. Then $K_\zeta \in (sH^2)^\perp$ and, for every $f \in (sH^2)^\perp$, $(f, K_\zeta) = f(\zeta)$. We will show V is onto by showing that every K_ζ is in the range of V . This will prove that the range of V is dense in M^\perp . But since V is an isometry, its range must be closed, so this will prove the theorem.

We want to find $k_\zeta \in L^2(d\sigma)$ such that

$$Vk_\zeta = K_\zeta. \quad (1.3)$$

If such a k_ζ exists, it must satisfy $(Vc)(z) = (c, k_\zeta)$, for every $c \in L^2(d\sigma)$, and thus, from (0.2) we can "guess" that $k_\zeta(\lambda)$ must be given by

$$\bar{k}_\zeta(\lambda) = \sqrt{2}s_\lambda(\zeta)/(1 - e^{-i\lambda}\zeta). \quad (1.4)$$

We therefore define $k_\zeta(\lambda)$ by (1.4) and prove that (1.3) holds. By definition,

$$(Vk_\zeta)(z) = 2 \int_0^{2\pi} s_\lambda(z)\bar{s}_\lambda(\zeta)(1 - e^{-i\lambda}z)^{-1}(1 - e^{i\lambda}\bar{\zeta})^{-1}d\sigma(\lambda).$$

Now $(1 - e^{-i\lambda}z)^{-1}(1 - e^{i\lambda}\bar{\zeta})^{-1} = \frac{1}{2}(1 - \bar{\zeta}z)^{-1}h(\lambda),$

where $h(\lambda) = (e^{-i\lambda} + \bar{\zeta})(e^{-i\lambda} - \bar{\zeta})^{-1} + (e^{i\lambda} + z)(e^{i\lambda} - z)^{-1}.$

Furthermore $\bar{s}_\lambda(\zeta)s_\lambda(z) = \exp\{-\int_0^\lambda h(\theta)d\sigma(\theta)\}$, so that

$$\begin{aligned} (Vk_\zeta)(z) &= -(1 - \bar{\zeta}z)^{-1} \int_0^{2\pi} \exp\left\{-\int_0^\lambda h(\theta)d\sigma(\theta)\right\} (-h(\lambda))d\sigma(\lambda) \\ &= -(1 - \bar{\zeta}z)^{-1} \int_0^{2\pi} d\left(\exp\left\{-\int_0^\lambda h(\theta)d\sigma(\theta)\right\}\right) \\ &= \left(1 - \exp\left\{-\int_0^{2\pi} h(\theta)d\sigma(\theta)\right\}\right) (1 - \bar{\zeta}z)^{-1} \\ &= (1 - \bar{s}(\zeta)s(z))(1 - \bar{\zeta}z)^{-1} = K_\zeta(z), \end{aligned}$$

and this completes the proof.

2. *Atomic measures.* In this section, we assume $\varphi(z)$ is of the form $\varphi(z) = \Delta(z)$. This is the same as assuming $\varphi(z) = s(z)$, where σ instead of being continuous, is a purely atomic measure, with masses r_j at the points $e^{i\theta_j}$.

Let us define a measure τ on $(0, \infty)$, by $d\tau(\lambda) = r_{N+1} d\lambda$ on the real interval $[N, N + 1]$. Let $\Delta_\lambda(z)$ denote the inner function

$$\Delta_\lambda(z) = \exp \left\{ - \sum_{j=1}^N r_j \frac{e^{i\theta_j} + z}{e^{i\theta_j} - z} - (\lambda - N) r_{N+1} \frac{e^{i\theta_{N+1}} + z}{e^{i\theta_{N+1}} - z} \right\},$$

where N is the integral part of λ . Thus $\Delta_0 \equiv 1$, and $\Delta_\infty = \Delta$. Finally, we define $h(z, \lambda) = \sqrt{2} (1 - e^{-i\theta_{N+1}} z)^{-1}$, where again N is the integral part of λ .

THEOREM 2.1. *The operator V defined by*

$$(Vc)(z) = \int_0^\infty c(\lambda) \Delta_\lambda(z) h(z, \lambda) d\tau(\lambda) \tag{2.1}$$

maps $L^2(d\tau)$ isometrically onto $(\Delta H^2)^\perp$.

The proof could be accomplished by appealing to the results of Sarason [8], but we prefer to outline a proof which is entirely analogous to that of Theorem 1.1.

Proof. Choose $\mu < \eta$ and suppose $N \leq \mu < \eta \leq N + 1$ for some non-negative integer N , and consider

$$c_{\mu, \eta}(\lambda) = \chi_{\mu, \eta}(\lambda) \Delta_\lambda(0).$$

One checks that $(Vc_{\mu, \eta})(z) = \frac{1}{\sqrt{2}} (\Delta_\mu(0) \Delta_\mu(z) - \Delta_\eta(0) \Delta_\eta(z)).$

Just as before, this last function is the projection of 1 on $(\Delta_\eta H^2)^\perp \ominus (\Delta_\mu H^2)^\perp$, so that one may verify that $\|Vc_{\mu, \eta}\| = \|c_{\mu, \eta}\|$. If $[\mu, \eta]$ and $[\nu, \tau]$ are disjoint intervals of the above type, then $c_{\mu, \eta}$ and $c_{\nu, \tau}$ are orthogonal, as are $Vc_{\mu, \eta}$ and $Vc_{\nu, \tau}$. We may conclude that V maps a dense linear subspace of $L^2(d\tau)$ isometrically into $(\Delta H^2)^\perp$, and it follows, as above that V maps $L^2(d\tau)$ isometrically into $(\Delta H^2)^\perp$.

To show that V maps $L^2(d\tau)$ onto $(\Delta H^2)^\perp$, we try, as before, to find $k_\zeta \in L^2(d\tau)$ such that

$$(Vk_\zeta)(z) = (1 - \bar{\Delta}(\zeta) \Delta(z)) (1 - \bar{\zeta} z)^{-1}. \tag{2.2}$$

Once again, such a k_ζ , if it exists, must be given by $k_\zeta(\lambda) = \bar{\Delta}_\lambda(\zeta) \bar{h}(\zeta, \lambda)$. To check that this k_ζ satisfies (2.2), we consider

$$\int_N^{N+1} \Delta_\lambda(z) h(z, \lambda) \bar{\Delta}_\lambda(\zeta) \bar{h}(\zeta, \lambda) d\tau(\lambda).$$

This integral is equal to

$$(\bar{\Delta}_N(\zeta) \Delta_N(z) - \bar{\Delta}_{N+1}(\zeta) \Delta_{N+1}(z))(1 - \bar{\zeta}z)^{-1},$$

and hence

$$\int_0^{N+1} \bar{\Delta}_\lambda(\zeta) \Delta_\lambda(z) \bar{h}(\lambda, \zeta) h(\lambda, z) d\tau(\lambda) = (1 - \bar{\Delta}_{N+1}(\zeta) \Delta_{N+1}(z))(1 - \bar{\zeta}z)^{-1}. \tag{2.3}$$

Letting $N \rightarrow \infty$ in (2.3), yields (2.2) and this completes the proof.

3. The general case. Let φ be a general inner function, of the form $\varphi = Bs\Delta$, where B is a Blaschke product with zeros $\{a_k\}$, s is a singular inner function whose associated measure $\sigma = \sigma_s$ is continuous, and Δ is a singular inner function with a purely atomic measure having masses r_j at the points $e^{i\theta_j}$, $j = 1, 2, \dots$

Let σ_B be the measure on the positive integers that assigns the mass $1 - |a_k|$ to the integer k , and let σ_Δ denote the measure τ of the last section, i.e. $d\sigma_\Delta = r_{N+1} d\lambda$ on $[N, N + 1]$. As we have seen, we have an isometry V_B of $L^2(d\sigma_B)$ onto $(BH^2)^\perp$ given by (0.1); an isometry V_s of $L^2(d\sigma_s)$ onto $(sH^2)^\perp$, given by (0.2); and an isometry V_Δ of $L^2(d\sigma_\Delta)$ onto $(\Delta H^2)^\perp$, given by (2.1).

THEOREM 3.1. *Define*

$$V: L^2(d\sigma_B) \times L^2(d\sigma_s) \times L^2(d\sigma_\Delta) \rightarrow (Bs\Delta H^2)^\perp$$

by
$$V(c_B, c_s, c_\Delta) = V_B c_B + BV_s c_s + BsV_\Delta c_\Delta. \tag{3.1}$$

Then V is an isometry onto $(Bs\Delta H^2)^\perp$.

Proof. The proof is an immediate consequence of Theorems 1.1, 2.1 and the following easy

LEMMA 3.1. *If φ_1, φ_2 are inner functions,*

$$(\varphi_1\varphi_2 H^2)^\perp = (\varphi_1 H^2)^\perp \oplus \varphi_1(\varphi_2 H^2)^\perp.$$

(Here \oplus denotes orthogonal direct sum.)

Proof. Recalling that multiplication by an inner function is an isometry, we have

$$H^2 = (\varphi_2 H^2) \oplus (\varphi_2 H^2)^\perp,$$

so that

$$\varphi_1 H^2 = \varphi_1 \varphi_2 H^2 \oplus \varphi_1(\varphi_2 H^2)^\perp$$

i.e.,

$$H^2 = \varphi_1 H^2 \oplus (\varphi_1 H^2)^\perp = \varphi_1 \varphi_2 H^2 \oplus \varphi_1(\varphi_2 H^2)^\perp \oplus (\varphi_1 H^2)^\perp,$$

and the conclusion of the lemma is clear.

4. *Inversion formula.* In this section we prove the following inversion formula for the operator V .

THEOREM 4.1. *If $f \in (\varphi H^2)^\perp$, then*

$$V^* f (= V^{-1}f) = (c_B, c_s, c_\Delta),$$

where
$$c_B(n) = \int_0^{2\pi} f(e^{i\theta}) \bar{B}_n(e^{i\theta}) (1 - a_n e^{-i\theta})^{-1} (1 + |a_n|)^{\frac{1}{2}} d\theta;$$

$c_s(\lambda)$ is the limit in $L^2(d\sigma_s)$, as $r \rightarrow 1 - 0$, of the functions

$$V_{s,r}^* f = \int_0^{2\pi} f(e^{i\theta}) \bar{B}(e^{i\theta}) \bar{s}_\lambda(re^{i\theta}) (1 - e^{i\lambda} re^{-i\theta})^{-1} \sqrt{2} d\theta; \tag{4.1}$$

and $c_\Delta(\lambda)$ is the limit in $L^2(d\sigma_\Delta)$, as $r \rightarrow 1 - 0$, of the functions

$$V_{\Delta,r}^* f = \int_0^{2\pi} f(e^{i\theta}) \bar{B}(e^{i\theta}) \bar{s}(e^{i\theta}) \bar{\Delta}_\lambda(re^{i\theta}) \bar{h}(re^{i\theta}, \lambda) \sqrt{2} d\theta. \tag{4.2}$$

Proof. The instance $\varphi = B$ of Theorem 4.1 is obvious from the remarks in the introduction and the simple properties of orthonormal bases. Let us begin, therefore, by considering the case $\varphi = s$. Denote by $k(re^{i\theta}, \lambda)$ the function

$$\sqrt{2} s_\lambda(re^{i\theta}) / (1 - re^{-i\lambda} e^{i\theta})$$

and define a map $V_r: L^2(d\sigma) \rightarrow H^2$, by

$$(V_r c)(e^{i\theta}) = \int_0^{2\pi} c(\lambda) k(re^{i\theta}, \lambda) d\sigma(\lambda).$$

Then V_r converges to V strongly as $r \rightarrow 1$, hence V_r^* converges weakly to V^* as $r \rightarrow 1$. Now we are considering $V^*: H^2 \rightarrow L^2(d\sigma)$, so that $V^* \equiv 0$ on sH^2 and $V^* = V^{-1}$ on $(sH^2)^\perp$. The adjoint of V_r is easily computed to be the operator $V_{s,r}^*$ defined in (4.1) above (of course here $B(e^{i\theta}) \equiv 1$). Thus to prove the theorem in the case $\varphi = s$, it suffices to prove that $V_{s,r}^*$ converges strongly to V^* , as $r \rightarrow 1$.

To prove this, let W_ζ be defined by $W_\zeta(z) = (1 - \bar{\zeta}z)^{-1}$. It is easy to calculate that if $\zeta \in U$,

$$(V_r^* W_\zeta)(\lambda) = \bar{k}(r\zeta, \lambda) = k_{r\zeta}(\lambda).$$

It is also clear that $k_{r\zeta}$ converges to k_ζ in L^2 norm (even in the uniform norm) as r converges to 1. Thus the V_r^* have uniformly bounded norms and converge strongly on a complete

set in H^2 . It follows that $\lim_{r \rightarrow 1} V_r^*$ exists in the strong operator topology. Of course the limit must be V^* , and this completes the proof in the case $\varphi = s$.

Next consider the case $\varphi = \Delta$. If we let $k(z, \lambda) = \Delta_\lambda(z)h(z, \lambda)$ and $V_r: L^2(d\tau) \rightarrow H^2$ be given by

$$(V_r c)(e^{i\theta}) = \int_0^\infty c(\lambda) k(re^{i\theta}, \lambda) d\tau(\lambda),$$

then $V_r \rightarrow V$ strongly as $r \rightarrow 1$, so V_r^* tends to V^* weakly. Again V_r^* is given by (4.2) above (if one again recalls that this time $B = s = 1$). Thus we must again prove V_r^* converges strongly to 0 on ΔH^2 and to V^{-1} on $(\Delta H^2)^\perp$.

As in the case $\varphi = s$, one checks that $W_\zeta(z) = (1 - \bar{\zeta}z)^{-1}$ satisfies $V_r^* W_\zeta$ tends to k_ζ strongly. Since the V_r^* are norm bounded and converge strongly on a dense set, $\lim_{r \rightarrow 1} V_r^* = V^*$ exists in the strong operator topology. That $V^* = 0$ on ΔH^2 and $V^* = V^{-1}$ on $(\Delta H^2)^\perp$ follow as in the case $\varphi = s$.

The proof of Theorem 4.1 in the general case now follows from

LEMMA 4.1. *Let P_B, P_s, P_Δ be the projections onto $(BH^2)^\perp, (sH^2)^\perp$ and $(\Delta H^2)^\perp$. If $f \in (\varphi H^2)^\perp$, then*

$$f = P_B f + B P_s \bar{B} f + B s P_\Delta \bar{B} s f.$$

Proof. We know from Lemma 3.1 that $f = f_1 + B f_2 + B s f_3$, where $f_1 \in (BH^2)^\perp, f_2 \in (sH^2)^\perp$ and $f_3 \in (\Delta H^2)^\perp$. That $P_B f = f_1$ is clear. Multiply by \bar{B} to get

$$\bar{B} f = \bar{B} f_1 + f_2 + s f_3.$$

Apply P_s and obtain $f_2 = P_s \bar{B} f$. Now multiply by \bar{s} and apply P_Δ to get $f_3 = P_\Delta \bar{B} s f$. This proves the lemma and completes the proof of Theorem 4.1.

Part II

5. *The restricted shift.* Let $\varphi = B s \Delta$ be an inner function, and V the isometry of $L^2(d\sigma_B) \times L^2(d\sigma_s) \times L^2(d\sigma_\Delta)$ onto $(\varphi H^2)^\perp$ described in the last sections. We consider the restricted shift operator T on $(\varphi H^2)^\perp$ defined by

$$Tf = Pz f, \quad f \in (\varphi H^2)^\perp$$

where P is the projection onto $(\varphi H^2)^\perp$. We wish to determine the operator $V^* T V$, unitarily equivalent to T under V .

The results below are to be compared with Sarason's [8] for the case $\varphi = \Delta, r_j = 0, j = 2, 3, \dots, \theta_1 = 0$, except that Sarason considers $(I + T)^{-1}$ and, in our case, $I + T$ may not, in general, possess a bounded inverse.

We begin with the case in which $\varphi = s$ is a singular inner function whose associated measure σ is continuous. Define an integral operator $K_s: L^2(d\sigma) \rightarrow L^2(d\sigma)$, by

$$(K_s c)(\lambda) = 2 \int_0^\lambda e^{\sigma(t) - \sigma(\lambda)} c(t) d\sigma(t). \tag{5.1}$$

Define M_s to be the multiplication operator on $L^2(d\sigma)$ given by

$$(M_s c)(\lambda) = e^{i\lambda} c(\lambda).$$

We have the following

THEOREM 5.1. *If $\varphi = s$, $V^* T V = (I - K_s) M_s$.*

Proof. Take $c \in L^2(d\sigma)$; we show that $(T V)c = V(I - K_s) M_s c$. First, note that, writing down $(V K_s M_s)c$ from (0.2) and (5.1), and reversing the order of integration, we obtain

$$(V K_s M_s)c = 2\sqrt{2} \int_0^{2\pi} \left[\int_t^{2\pi} e^{-\sigma(\lambda)} s_\lambda(z) (1 - e^{-i\lambda} z)^{-1} d\sigma(\lambda) \right] e^{\sigma(t) + it} c(t) d\sigma(t).$$

Now, from (1.2), the inside integral in this equation is equal to

$$\frac{1}{2} (s_t(z) e^{-\sigma(t)} - s_{2\pi}(z) e^{-\sigma(2\pi)}).$$

It follows that $(V K_s M_s c)(z) = \sqrt{2} \int_0^{2\pi} s_t(z) e^{it} c(t) d\sigma(t) + \alpha s(z),$ (5.2)

α a constant. In addition, from (0.2),

$$(V M_s c)(z) = \sqrt{2} \int_0^{2\pi} e^{i\lambda} c(\lambda) s_\lambda(z) (1 - e^{-i\lambda} z)^{-1} d\sigma(\lambda). \tag{5.3}$$

Subtracting (5.2) from (5.3), it follows easily that $V(I - K_s) M_s c = z V c - \alpha s$, and the conclusion follows.

Next we consider a singular inner function Δ , or equivalently an $s(z)$ with the measure σ replaced by a purely atomic measure σ_Δ with masses r_j at the points $e^{i\theta_j}$. Let $d\tau$ be the measure on $[0, \infty]$ defined in Section 2 above. We define an operator $K_\Delta: L^2(d\tau) \rightarrow L^2(d\tau)$, by

$$(K_\Delta c)(\lambda) = 2 \int_0^\lambda c(t) (\Delta_\lambda(0)/\Delta_t(0)) d\tau(t),$$

and a multiplication (= diagonal) operator $M_\Delta: L^2(d\tau) \rightarrow L^2(d\tau)$, by

$$(M_\Delta c)(\lambda) = e^{i\theta_{N+1}} c(\lambda) \quad \text{for } N \leq \lambda < N + 1.$$

If $V: L^2(d\tau) \rightarrow (\Delta H^2)^\perp$ is the isometry described in Theorem 2.1, then we have

THEOREM 5.2. *If $\varphi = \Delta$, $V^*TV = (I - K_\Delta)M_\Delta$.*

Proof. The proof is formally the same as that of Theorem 5.1, and hence will be omitted. In place of (1.2) it uses the equation

$$\int_t^\infty \Delta_t(0) \Delta_\lambda(z) h(z, \lambda) d\tau(\lambda) = \frac{1}{\sqrt{2}} [\Delta_t(0) \Delta_t(z) - \Delta(0) \Delta(z)]$$

which was obtained in the course of the proof of Theorem 2.1.

Next we consider the case of a Blaschke product B , with zeros $\{a_k\}$. We assume, for simplicity that $a_k \neq 0$, $k = 1, 2, \dots$. Let σ_B be the purely atomic measure considered above: $\sigma(\{k\}) = 1 - |a_k|$. This time we define $K_B: L^2(d\sigma_B) \rightarrow L^2(d\sigma_B)$ by

$$(K_B c)(n) = \sum_{j=1}^n c(j) B_n(0)/B_j(0) (1 + |a_j|) |a_j|^{-2} (1 - |a_j|),$$

and $M_B: L^2(d\sigma_B) \rightarrow L^2(d\sigma_B)$, by

$$(M_B c)(n) = a_n c(n).$$

Then, if V is defined by (0.1), we have

THEOREM 5.3. *If $\varphi = B$, $V^*TV = (I - K_B)M_B$.*

Proof. Of course, the proof proceeds as in the proof of Theorem 5.1. One needs to know that

$$\sum_{n=j}^\infty B_n(0) (1 + |a_n|)^{\frac{1}{2}} B_n(z) (1 - \bar{a}_n z)^{-1} (1 - |a_n|) = (1 + |a_j|)^{-\frac{1}{2}} [B_j(0) B_j(z) - B(0) B(z)];$$

a fact that easily can be verified.

In the general case $\varphi = Bs\Delta$, V^*TV takes the form

$$V^*TV(c_B, c_s, c_\Delta) = (M_B c_B, M_s c_s, M_\Delta c_\Delta) + K(c_B, c_s, c_\Delta), \tag{5.4}$$

where K is a quasi-nilpotent operator of Hilbert-Schmidt class. Neither the precise form of K nor the complete proof of (5.4) will be given here. The derivation is similar to that of Section 3 above. The remainder of this section will be devoted to applications of (5.4).

Let $u(e^{i\theta})$ be a continuous function on ∂U . The operator $T_u: (\varphi H^2)^\perp \rightarrow (\varphi H^2)^\perp$ is defined by

$$T_u f = P u f, \quad f \in (\varphi H^2)^\perp.$$

The following applications of (5.4) are based upon the fact that the analogue of (5.4) above holds for T_u ; i.e., that $V^*T_u V(c_B, c_s, c_\Delta)$ is given by

$$(u(M_B)c_B, u(M_s)c_s, u(M_\Delta)c_\Delta) + K'(c_B, c_s, c_\Delta), \tag{5.5}$$

where K' is compact, and where we assume u has been extended in some way to a continuous function $u(z)$ in $|z| \leq 1$.

To prove (5.5), choose polynomials $p_n(z, \bar{z})$ which tend uniformly to $u(z)$ in $|z| \leq 1$. The analogue of (5.5) with u replaced by p_n is easily seen to follow from (5.4); and then (5.5) follows by taking uniform limits.

For our first application, let $\text{sp}_F(S)$ denote the Fredholm spectrum of an operator S , and let $\text{supp } \varphi$ denote the support of the inner function φ . That is, $\text{supp } \varphi$ is the closure of the union of the set of zeros of B , the points $e^{i\theta}$, and the support of σ .

COROLLARY 5.1. $\text{sp}_F(T_u) = \text{sp}_F(u(M)) = u(\text{supp } \varphi) \cap \partial U$.

The corollary is a straightforward consequence of (5.5). Through similar reasoning, we may obtain several of the results of Moeller [7], Foias and Mlak [4] and Fuhrmann [5]. If, in addition, u is a trigonometric polynomial, then K' in (5.5) is Hilbert-Schmidt and so we may obtain information on when $T_u \in c_p$ with $p \geq 2$; for similar results, see Clark [2].

As a final application of (5.5), we mention

THEOREM 5.4. T_u is compact if and only if $u(e^{i\theta}) = 0$ for $e^{i\theta} \in \text{supp } \varphi \cap \partial U$.

A curious corollary is obtained by noting that, if $u \in A$, the algebra of continuous, analytic functions in \bar{U} , T_u is compact if and only if $u\bar{\varphi} \in H^\infty + C$; [9], where C denotes the space of continuous functions on ∂U .

COROLLARY 5.2. If $u \in A$, then $u\bar{\varphi} \in H^\infty + C$ if and only if $u\bar{\varphi} \in C$.

6. Classes of functions which span $(\varphi H^2)^\perp$. In this section, we consider two classes of functions in $(\varphi H^2)^\perp$. If $\varphi = Bs\Delta$, let F_1 denote the set of all inner functions ψ having one of the forms $\psi = 1$, $\psi = B_n$, $n = 1, 2, \dots$, $\psi = Bs_\lambda$, $0 \leq \lambda \leq 2\pi$, or $\psi = Bs\Delta_\lambda$, $0 \leq \lambda \leq \infty$. Let F_2 denote the set of functions Y^*f , where $f \in F_1$, and Y is the shift operator on all of H^2 :

$$Yg = zg, \quad g \in H^2.$$

We will prove

THEOREM 6.1. The set PF_1 of projections on $(\varphi H^2)^\perp$ of the functions in F_1 spans $(\varphi H^2)^\perp$.

THEOREM 6.2. The set F_2 spans $(\varphi H^2)^\perp$.

Theorem 6.2 has a very simple corollary which was conjectured by Douglas, Shapiro and Shields [3]:

COROLLARY 6.1. *If φ is not of the form $\varphi = e^{in\theta}$, then $\{Y^*\psi\}$ spans $(\varphi H^2)^\perp$, for ψ a divisor of φ , $\psi \neq \varphi$.*

Clearly, if $\varphi = e^{in\theta}$, the function $Y^*\varphi = e^{i(n-1)\theta}$ is orthogonal to $Y^*\psi$ for $\psi = e^{im\theta}$, $0 \leq m < n$.

Theorem 6.1 is a corollary of the proofs of Theorems 1.1, 2.1 and 3.1. Theorem 6.2 follows easily from Theorem 6.1 as we shall see below.

Proof of Theorem 6.1. As usual, we consider first the case $\varphi = s$. In that case, the projection of the function $s_\lambda(0)s_\lambda(z)$ on $(sH^2)^\perp$ is

$$s_\lambda(0)s_\lambda(z) - s_{2\pi}(0)s_{2\pi}(z) = Vc_{\lambda, 2\pi}. \quad (6.1)$$

An obvious modification of Lemma 1.1 shows that the $c_{\lambda, 2\pi}$ span $L^2(d\sigma)$ and hence the result follows from the fact that V is an isometry.

In case $\varphi = \Delta$, set $c_\mu(\lambda) = \chi_{(\mu, \infty)}(\lambda)\Delta_\lambda(0)$. Then, if N is the integral part of μ , we have

$$c_\mu(\lambda) = c_{\mu, N+1}(\lambda) + \sum_{j=N+1}^{\infty} c_{j, j+1}(\lambda),$$

in the notation of Section 2 above. Thus,

$$\begin{aligned} Vc_\mu(\lambda) &= \frac{1}{\sqrt{2}} \left[\Delta_\mu(0)\Delta_\mu - \Delta_{N+1}(0)\Delta_{N+1} + \sum_{j=N+1}^{\infty} (\Delta_j(0)\Delta_j - \Delta_{j+1}(0)\Delta_{j+1}) \right] \\ &= \frac{1}{\sqrt{2}} [\Delta_\mu(0)\Delta_\mu - \Delta(0)\Delta], \end{aligned}$$

i.e., $Vc_\mu(\lambda)$ is the projection of $(1/\sqrt{2})\Delta_\mu(0)\Delta_\mu$ on $(\Delta H^2)^\perp$. Since the c_μ clearly span $L^2(d\tau)$, the result follows.

For the case $\varphi = B$, let $B_n(z)$ denote the product of the first $n-1$ factors of B . Then, as is seen from (0.1) and a simple computation, $PB_n = V\{c_j\}$, where

$$c_j = \begin{cases} 0 & \text{if } j < n \\ (1 + |a_j|)^\frac{1}{2} & \text{if } j = n \\ \prod_{\nu=1}^{j-1} |a_\nu| (1 + |a_\nu|)^\frac{1}{2} & \text{if } j > n. \end{cases}$$

Hence the characteristic function of the integer n is given by the inverse image under V of the function $(1 + |a_n|)^{-\frac{1}{2}}[B_n(z) - |a_n|B_{n+1}(z)]$. Since those characteristic functions span $L^2(d\sigma_B)$, it follows that the PB_n span $(BH^2)^\perp$.

For the general case of Theorem 6.1, let P_φ , P_B , P_s and P_Δ be the projections on

$(\varphi H^2)^\perp$, $(BH^2)^\perp$, $(sH^2)^\perp$ and $(\Delta H^2)^\perp$ respectively. If $g \in (\varphi H^2)^\perp$ and $(g, P_\varphi \psi) = 0$ for all ψ in F_1 , then, in particular,

$$0 = (g, P_\varphi B s \Delta_\lambda) = (g, B s P_\Delta \Delta_\lambda) = (P_\Delta \overline{B s g}, P_\Delta \Delta_\lambda),$$

so that $P_\Delta \overline{B s g} = 0$. By Lemma 4.1, we may thus assume that $g \in (B s H^2)^\perp$. With this assumption and $g \perp P_\varphi F_1$, we have

$$0 = (g, P_\varphi B s_\lambda) = (g, B P_s s_\lambda) = (P_s \overline{B g}, P_s s_\lambda)$$

and this implies $P_s \overline{B g} = 0$. Again by Lemma 4.1, we may now assume $g \in (B H^2)^\perp$. But then,

$$0 = (g, P_\varphi B_n) = (g, P_B B_n)$$

and this implies $g = 0$, by the case $\varphi = B$ above. This proves Theorem 6.1.

Proof of Theorem 6.2. Let F_3 denote the set of functions of the form $g(z) + c\varphi$, where $g \in (\varphi H^2)^\perp$ and c is an arbitrary constant. By (6.1), Theorem 6.1 states that F_3 is the H^2 closure of the span of F_1 . Since Y^* is a contraction map, it follows that the span of $Y^* F_1 = F_2$ is dense in $Y^* F_3$. Now the set $Y^* F_3$ exactly covers $(\varphi H^2)^\perp$. In fact, if T (the restricted shift on $(\varphi H^2)^\perp$) has no nullspace, then $Y^* M^\perp$ covers M^\perp , and if T does have a nullspace then that nullspace $(= (\varphi H^2)^\perp \ominus Y^* (\varphi H^2)^\perp)$ is equal to the one-dimensional span of the vector $Y^* \varphi$. In either case, F_2 spans $(\varphi H^2)^\perp$.

Proof of Corollary 6.1. It suffices, by Theorem 6.2 to prove that $Y^* \varphi$ lies in the closed linear span of $\{Y^* \psi\}$. If φ is a finite Blaschke product, this may easily be accomplished directly, as in Section 3.2.1 of Douglas, Shapiro and Shields [3]. If φ is not a finite Blaschke product, we will prove the stronger statement that φ lies in the closure of the set of its proper divisors. In fact, in that case, the functions $B_n s_\lambda \Delta_\mu$ tend to $\varphi = B s \Delta$ in H^2 as $n \rightarrow \infty$, $\lambda \rightarrow 2\pi$ and $\mu \rightarrow \infty$, as is easily seen. This completes the proof of Corollary 6.1.

Added in proof. T. L. Kriete, III has independently obtained close analogues of the results of Sections 3 and 5 above. In work as yet unpublished, Kriete uses a somewhat different representation for $s(z)$ and obtains a unitary map from $(sH^2)^\perp$ to the L^2 space of Lebesgue measure on a certain interval.

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