ON THE GROWTH OF SLOWLY INCREASING UNBOUNDED HARMONIC FUNCTIONS

BY

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1. For a non-constant real-valued harmonic function $u(z)$ defined in a plane domain Ω let $l(c)$ denote the level set $\{z \in \Omega \mid u(z) = c\}$ and let $\Theta(c) = \int_{l(c)} |*du|$ where this is to have the value zero if $l(c)$ is void. Then if a is a value taken by u the behaviour of the integral $\int_a^b (\Theta(c))^{-1} dc$ as b tends to infinity provides important information about the rate of growth of u as we approach the boundary of Ω. In particular if u is bounded the integral will take the value infinity for a finite value of b. On the other hand, if u is unbounded above the more rapidly the function tends to infinity as we approach the boundary the more slowly the integral increases. Our attention is primarily fastened on those functions for which the integral is finite for all finite b but tends to infinity with b. In a sense they are the most slowly increasing unbounded harmonic functions.

Our principal method is the method of the extremal metric making use of the essential identity of the integral indicated and the module of the curve family made up of the $l(c)$ for $a < c < b$. We should point out the relationship to questions studied first by Hayman [3] and later by Eke [1, 2]. However, their use of the length–area method restricted their study to regular functions. As is usually the case the method of the extremal metric gives new and deeper insights while providing simpler proofs of more general results.

2. In order to avoid possible confusion, we shall state definitions of some terminologies used in the sequel.

By an *arc* we mean a one-to-one continuous mapping φ of one of the intervals [0, 1], [0, 1), (0, 1], (0, 1) into the Riemann sphere. We shall say that it *has an initial point* (or *terminal point*) if the "tail" $T_0 = \bigcap_{\varepsilon>0} \mathrm{Cl}\,\{\varphi(t) \mid 0 < t < \varepsilon\}$ (or $T_1 = \bigcap_{\varepsilon>0} \mathrm{Cl}\,\{\varphi(t) \mid 1-\varepsilon < t < 1\}$, resp.) consists of a single point; here the closure Cl is taken on the Riemann sphere.

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Given a domain Ω and sets A, B on $\mathrm{Cl}\,\Omega$ an arc is said to *join A and B within Ω* if $\varphi(t)\in\Omega$ for $0<t<1$ and $T_0\subset A$, $T_1\subset B$. An arc is said to *separate A and B in Ω* if $\varphi(t)\in\Omega$ for $0<t<1$ and it meets every arc joining A and B within Ω.

By a *triad* (Jenkins [4]) we mean a triple (Ω, z_0, A) where Ω is a simply-connected hyperbolic domain, z_0 is a point in Ω, and A is a set of prime ends which is mapped onto an arc in the unit circle by the Riemann mapping function. *The module of a triad* is by definition the module, i.e., the reciprocal of the extremal length, of the family of arcs with $T_0, T_1\subset A$ separating z_0 in Ω from the set of prime ends of Ω not contained in A.

§ 1. Levels of harmonic functions

3. Let u be a non-constant harmonic function on a domain Ω in the complex plane. Consider the levels

$$l(c)=\{z\in\Omega\,|\,u(z)=c\},$$

$-\infty<c<\infty$, of u. Each $l(c)$ consists of at most a countable number of analytic arcs which may have fork points, and which never cluster to a point in Ω. Introduce the quantity

$$\Theta(c)=\int_{l(c)}|*du|,$$

$-\infty<c<\infty$, under the agreement that $\Theta(c)=0$ if $l(c)=\varnothing$. The value $\Theta(c)=\infty$ is permissible.

For $-\infty\leqslant a<b\leqslant\infty$, consider the family

$$\Gamma(a,b)=\{l(c)\,|\,a<c<b\}.$$

If it is not void, its module will be denoted by the symbol $\mu(a,b)$. We shall not use this symbol for $a\geqslant b$; in other words a sentence containing this symbol shall mean $a<b$ as well. For example the conclusion of Theorem 2 includes $a<u(z)$.

Consider the metric ϱ defined in Ω by

$$\varrho(z)\,|dz|=\frac{|du+i*du|}{\Theta(c)}\quad\text{if}\quad z\in l(c),\ -\infty<c<\infty.$$

It is not difficult to show (cf., e.g., Ohtsuka [7]) that, if $\Gamma(a,b)\neq\varnothing$, the restriction of ϱ to $\Omega(a,b)$ is the extremal metric for $\Gamma(a,b)$ and therefore,

$$\mu(a,b)=\|\varrho\|^2_{\Omega(a,b)},\tag{1}$$

where $\|\varrho\|^2_A=\iint_A\varrho^2\,dx\,dy$, as well as the fact that, if the interval (a,b) is contained in the range $u(\Omega)$ of u, the identity

$$\mu(a, b) = \int_a^b \frac{dc}{\Theta(c)} \tag{2}$$

holds.

In the sequel we shall be mainly dealing with functions u for which there exists $a \in u(\Omega)$ such that

$$\int_a^b \frac{dc}{\Theta(c)} < \infty \quad \text{for every } b > a.$$

This condition is readily seen to be equivalent to the existence of a such that $[a, \infty) \subset u(\Omega)$, namely unboundedness above of u.

4. As a principle the growth of the quantity $\mu(a, u(z))$ expresses that of $u(z)$, for $\mu(a, b)$ is an increasing continuous function of b.

If a function u is subject to a restriction, the relationship between $u(z)$ and $\mu(a, u(z))$ becomes more explicit. Observe the relation

$$(b-a)^2 \leq \int_a^b \Theta(c)\, dc \int_a^b \frac{dc}{\Theta(c)} = D_{\Omega(a,b)}[u]\, \mu(a, b)$$

for $a < b$, where $D_A[u]$ stands for the Dirichlet integral of u over A. If $D_\Omega[u] < \infty$ then

$$u(z) - a \leq D_\Omega[u]^{\frac{1}{2}} \mu(a, u(z))^{\frac{1}{2}}.$$

If there exist constants σ and τ such that

$$D_{\Omega(a,b)}[u] \leq \sigma(b-a) + \tau \tag{3}$$

for every $a < b$, then

$$u(z) - a \leq \sigma\mu(a, u(z)) + \tau'$$

with another constant τ'. For example, if f is an areally mean p-valent regular function without zeros, then $u = \log |f|$ satisfies (3) with $\sigma = 2\pi p$; see Hayman [3] and observe

$$\int_{R_1}^{R_2} \frac{dR}{Rp(R)} = 2\pi \int_a^b \frac{dc}{\Theta(c)} \quad \text{if } a = \log R_1 \text{ and } b = \log R_2.$$

5. Given an open subset G of Ω, denote by $\mu_G(a, b)$ the module of the family $\Gamma_G(a, b) = \{l(c) \cap G \mid a < c < b\}$, provided it is not void. As before, the metric ϱ_G defined on Ω by

$$\varrho_G(z)\,|dz| = \begin{cases} \dfrac{|du + i * du|}{\Theta_G(c)} & \text{if } z \in l(c) \cap G, \ -\infty < c < \infty \\ 0 & \text{if } z \in \Omega - G \end{cases}$$

satisfies

$$\mu_G(a, b) = \|\varrho_G\|^2_{\Omega(a,b)} \tag{1'}$$

and, therefore, its restriction to $\Omega(a, b) \cap G$ is the extremal metric for the family $\Gamma_G(a, b)$. If $(a, b) \subset u(G)$, then

$$\mu_G(a, b) = \int_a^b \frac{dc}{\Theta_G(c)}, \tag{2'}$$

where $\Theta_G(c) = \int_{l(c) \cap G} |*du|$.

The following observation shows that the module $\mu_G(a, b)$ can be used to compare the growth of u in different directions. For G_1 and G_2, as $b \to \infty$, $\mu_{G_1}(a, b)$ grows faster than $\mu_{G_2}(a, b)$ if u grows more slowly in G_1 than in G_2 in the sense that $\Omega(a, b) \cap G_1$ expands more rapidly than $\Omega(a, b) \cap G_2$.

6. We collect some identities and inequalities needed in the sequel. Norms and inner products are considered on $\Omega(a, b)$, $-\infty \leqslant a < b \leqslant \infty$. The assumption $(a, b) \subset u(G)$ is made for (4)–(6), and $(a, b) \subset u(G_j)$, $j = 1, \ldots, k$, for (7)–(11).

First, from $\Theta \geqslant \Theta_G$, or else from the fact that ϱ_G is admissible also for $\Gamma(a, b)$,

$$\|\varrho\| \leqslant \|\varrho_G\|. \tag{4}$$

By direct computation we have $(\varrho_G, \varrho) = \int_a^b \Theta^{-1} dc$ so that

$$(\varrho_G, \varrho) = \|\varrho\|^2, \tag{5}$$

which implies if $\|\varrho\| < \infty$

$$\|\varrho_G - \varrho\|^2 = \|\varrho_G\|^2 - \|\varrho\|^2. \tag{6}$$

With respect to mutually disjoint $G_1, \ldots, G_k$, consider

$$\tilde{\varrho}_{G_1, \ldots, G_k} = \frac{1}{k} \sum_{j=1}^{k} \varrho_{G_j},$$

which satisfies

$$\|\tilde{\varrho}\|^2 = \frac{1}{k^2} \sum_{j=1}^{k} \|\varrho_{G_j}\|^2. \tag{7}$$

Since $\tilde{\varrho}$ is admissible for $\Gamma_{G_1 \cup \ldots \cup G_k}(a, b)$,

$$\|\varrho\| \leqslant \|\varrho_{G_1 \cup \ldots \cup G_k}\| \leqslant \|\tilde{\varrho}\|. \tag{8}$$

It is equivalent to

$$\int_a^b \frac{dc}{\Theta(c)} \leqslant \int_a^b \frac{dc}{\Theta_{G_1 \cup \ldots \cup G_k}(c)} \leqslant \frac{1}{k^2} \sum_{j=1}^{k} \int_a^b \frac{dc}{\Theta_{G_j}(c)} \tag{8'}$$

being verified directly on using the relation

$$\Theta^{-1} \leqslant (\Theta_{G_1 \cup \ldots \cup G_k})^{-1} = (\sum \Theta_{G_j})^{-1} \leqslant k^{-2} \sum (\Theta_{G_j})^{-1}.$$

The identity (5) implies

$$(\tilde{\varrho}, \varrho) = \|\varrho\|^2 \tag{9}$$

and, therefore, if $\|\varrho\| < \infty$

$$\|\tilde{\varrho} - \varrho\|^2 = \|\tilde{\varrho}\|^2 - \|\varrho\|^2. \tag{10}$$

Finally, if $\|\varrho\| < \infty$, for $i \neq j$

$$\|\varrho\|^2 \leq \|\varrho_{G_i}\|(\|\varrho_{G_j}\|^2 - \|\varrho\|^2)^{\frac{1}{2}} \leq \|\varrho_{G_i}\| \cdot \|\varrho_{G_j}\|. \tag{11}$$

In fact, from (5) we have

$$\|\varrho\|^2 = (\varrho_{G_i}, \varrho) = (\varrho_{G_i}, \varrho)_{\Omega(a,b)\cap G_i} \leq \|\varrho_{G_i}\| \; \|\varrho\|_{\Omega(a,b)\cap G_i}$$

and

$$\|\varrho\|^2_{\Omega(a,b)\cap G_i} = \|\varrho_{G_j} - \varrho\|^2_{\Omega(a,b)\cap G_i} \leq \|\varrho_{G_j} - \varrho\|^2 = \|\varrho_{G_j}\|^2 - \|\varrho\|^2$$

by (6).

7. If u has the property that $(-\infty, b) \subset u(\Omega)$ and $\Theta(a)$ is constant for all sufficiently small a, then

$$\tilde{\mu}(b) = \frac{a}{\Theta(a)} + \mu(a, b)$$

does not depend on such a. This quantity will be called the *reduced module* of the family $\Gamma(-\infty, b)$. We may consider $\tilde{\mu}(u(z))$ for the study of the growth of $u(z)$.

For example, suppose u is a harmonic function on Ω with a finite number of singularities at z_l where $u(z) - \lambda_l \log|z - z_l|$ $(\lambda_l > 0)$ is harmonic, $l = 1, \ldots, n$. For sufficiently small a, the level $l(a)$ consists of n mutually disjoint closed curves encircling $z_1, \ldots, z_n$ respectively. Then $\Theta(a)$ coincides with the total flux $2\pi(\lambda_1 + \ldots + \lambda_n)$ independent of a. The reduced module is therefore considered and is equal to

$$\tilde{\mu}(b) = \frac{a}{2\pi(\lambda_1 + \ldots + \lambda_n)} + \mu(a, b).$$

In particular, for $n = 1$, a simple calculation results in another expression

$$\tilde{\mu}(b) = \frac{u_1}{2\pi\lambda_1} + \lim_{\varepsilon \to 0}\left(\frac{1}{2\pi}\log\varepsilon + \mu_{\Omega_\varepsilon}(-\infty, b)\right),$$

where $u_1 = \lim_{z \to z_1}(u(z) - \lambda_1 \log|z - z_1|)$ and $\Omega_\varepsilon = \Omega - \{z \mid |z - z_1| \leq \varepsilon\}$.

§ 2. Upper bound of values

8. The following is a refined version of Hayman [3, Theorem 2.2] for $u = \log|f|$; see Jenkins–Oikawa [6]:

THEOREM 1. *If u is a harmonic function on the annulus $\Omega = \{z \mid r_0 < |z| < 1\}$, then*

$$\left|\int_{u(r_1 e^{i\theta})}^{u(r_2 e^{i\theta})} \frac{dc}{\Theta(c)}\right| \leq \frac{1}{\pi}\log\frac{1 - r_1}{1 - r_2} + \frac{1}{\pi}\log 2 + \frac{\pi}{4} \tag{12}$$

for arbitrary θ with $0 \leq \theta \leq 2\pi$ and r_1, r_2 with $1 - \frac{1}{2}e^{-\pi/2}(1 - r_0) = r^* \leq r_1 < r_2 < 1$.

To prove this theorem we shall map the unit disc $|z|<1$ conformally onto the strip $|\operatorname{Im}\zeta|<\pi/2$ and shall apply the following:

LEMMA 1. *Let $u(\zeta)$ be a harmonic function on the domain $\{\zeta \mid \xi^*<\operatorname{Re}\zeta,\ |\operatorname{Im}\zeta|<\pi/2\}$. If ξ_1 and ξ_2 are real numbers with $\xi^*+\pi/2\leqslant\xi_1<\xi_2$, then*

$$\left|\int_{u(\xi_1)}^{u(\xi_2)}\frac{dc}{\Theta(c)}\right|\leqslant\frac{\xi_2-\xi_1}{\pi}+\frac{\pi}{4}.$$

Proof of Lemma 1. First assume $u(\xi_1)<u(\xi_2)$. Consider $D=\{\zeta \mid \xi_1<\operatorname{Re}\zeta<\xi_2,\ |\operatorname{Im}\zeta|<\pi/2\}\cup\{\zeta \mid |\zeta-\xi_1|<\pi/2\}\cup\{\zeta \mid |\zeta-\xi_2|<\pi/2\}$. For every c with $u(\xi_1)<c<u(\xi_2)$, the level $l(c)$ contains an arc which meets the interval $[\xi_1,\xi_2]$ and whose tails T_0, T_1 are on the boundary of D. It has length not less than π. Accordingly the metric ϱ_0 defined by $\varrho_0=\pi^{-1}$ on D and $\varrho_0=0$ elsewhere is admissible for the family $\Gamma(u(\xi_1), u(\xi_2))$. We obtain $\mu(u(\xi_1), u(\xi_2))\leqslant\|\varrho_0\|^2=(\xi_2-\xi_1)\pi^{-1}+\pi/4$. If $u(\xi_1)>u(\xi_2)$, then the same estimate for $\mu(u(\xi_2), u(\xi_1))$ is available.

9. *Proof of Theorem 1.* Map $|z|<1$ onto $|\operatorname{Im}\zeta|<\pi/2$ by

$$\zeta=\log\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} \tag{13}$$

so that $z=e^{i\theta}$ corresponds to $\zeta=+\infty$. If the neighborhood $U(\theta,s)=\{z \mid |z|<1,\ |z-e^{i\theta}|<s\}$ is contained in Ω, then $u(z(\zeta))$ is defined on $\{\zeta \mid \log(2s^{-1})<\operatorname{Re}\zeta,\ |\operatorname{Im}\zeta|<\pi/2\}$. If $1-2^{-1}e^{-\pi/2}s\leqslant r_1<r_2$ the points $r_1e^{i\theta}$, $r_2e^{i\theta}$ are mapped onto ξ_1, ξ_2, respectively, satisfying $\log(2s^{-1})+\pi/2\leqslant\xi_1<\xi_2$. On taking $s=1-r_0$ we see that the left-hand side of (12) is dominated by

$$\frac{1}{\pi}\log\frac{1+r_2}{1-r_2}-\frac{1}{\pi}\log\frac{1+r_1}{1-r_1}+\frac{\pi}{4},$$

which does not exceed the right-hand side of (12).

10. For a sector $S=S(\varphi,\delta)=\{z\in\Omega \mid |\arg z-\varphi|<\delta\}$, $0<\delta\leqslant\pi$, a similar estimate for the module μ_S is obtained:

THEOREM 1'. *Let u be as in Theorem 1. Given $S=S(\varphi,\delta)$ and η $(0<\eta<\delta)$, there exists an r^* $(r_0<r^*<1)$ such that*

$$\int_{u(r_1e^{i\theta})}^{u(r_2e^{i\theta})}\frac{dc}{\Theta_S(c)}\leqslant\frac{1}{\pi}\log\frac{1-r_1}{1-r_2}+\frac{1}{\pi}\log 2+\frac{\pi}{4} \tag{14}$$

for $r^*\leqslant r_1<r_2<1$ and $|\theta-\varphi|\leqslant\delta-\eta$.

For the proof use the map (13) with φ instead of θ. We have to let $U(\theta,s)$ with $s=\sin\eta$ be in S. A possible value for r^* is $1-2^{-1}e^{-\pi/2}\cdot\min(1-r_0,\sin\eta)$.

11. Instead of (13) we may map the unit disc incised along the radius $\arg z = \theta + \pi$ conformally onto the strip $|\operatorname{Im} \zeta| < \pi/2$ by

$$\zeta = \frac{1}{2} \log \frac{ze^{-i\theta}}{(1-ze^{-i\theta})^2}. \tag{15}$$

Notice that the value $\Theta(c)$ is unchanged by incision. We obtain (12) and (14) with the right-hand side replaced by

$$\frac{1}{\pi} \log \frac{1-r_1}{1-r_2} + \frac{1}{2\pi} \log \frac{1}{r_1} + \frac{\pi}{4}. \tag{16}$$

The value of r^* may be different from the one given in Theorems 1 and 1′, but can be chosen to depend only on r_0 and satisfy $r_0 < r^* < 1$.

12. Now let u be a harmonic function on $\Omega = \{z \mid r_0 < |z| < 1\}$ such that

$$[a, \infty) \subset u(\Omega) \tag{17}$$

for some a. Let b be a number such that $b > a$ and $b > u(z)$ for every z with $|z| = r^*$, where r^* is as in Theorem 1. Then for every z with $u(z) > b$, $|z| > r^*$, we have

$$\mu(a, u(z)) = \mu(a, b) + \int_b^{u(z)} \frac{dc}{\Theta(c)} \leqslant \mu(a, b) + \frac{1}{\pi} \log \frac{1-r^*}{1-|z|} + \frac{1}{\pi} \log 2 + \frac{\pi}{4}.$$

If $a < u(z) \leqslant b$, this estimate is trivially true. Since $\mu(a, b) < \infty$ by the assumption (17), we conclude that

$$\mu(a, u(z)) \leqslant \frac{1}{\pi} \log \frac{1}{1-|z|} + O(1) \tag{18}$$

uniformly as $|z| \to 1$, $z \in \Omega$, $u(z) > a$.

Similarly, for u satisfying $[a, \infty) \subset u(S)$, we have

$$\mu_S(a, u(re^{i\varphi})) \leqslant \frac{1}{\pi} \log \frac{1}{1-r} + O(1) \tag{19}$$

as $r \to 1$, $u(re^{i\varphi}) > a$ for $S = S(\varphi, \delta)$ or $S = \Omega$.

In particular, if $D_\Omega[u] < \infty$ then

$$u(z) \leqslant \left(\frac{1}{\pi} D_\Omega[u]\right)^{\frac{1}{2}} \left(\log \frac{1}{1-|z|}\right)^{\frac{1}{2}} + O(1)$$

and, if (3) is satisfied then

$$u(z) \leqslant \frac{\sigma}{\pi} \log \frac{1}{1-|z|} + O(1).$$

§ 3. Regularity of growth

13. The estimate (19) shows that

$$\overline{\lim_{r\to 1}} \left(\mu_S(a, u(re^{i\varphi})) - \frac{1}{\pi} \log \frac{1}{1-r}\right) < \infty.$$

The limiting value (possibly $-\infty$) will be shown to exist if u is a slowly increasing unbounded function (in the sense of 1°) satisfying further $\lim_{r\to 1} u(re^{i\theta}) = \infty$. This result was obtained by Hayman [3, Theorem 5.5] for $u = \log|f|$ with f a circumferentially mean p-valent function, and by Eke [1, Theorem 3] for $u = \log|f|$ with f a regular function with some restrictions. In our proof the method of the extremal metric will be used at various steps.

THEOREM 2. *Suppose a harmonic function u on $\Omega = \{z \,|\, r_0 < |z| < 1\}$ satisfies*

$$\mu_S(a, \infty) = \infty, \quad \overline{\lim_{r\to 1}} \, u(re^{i\theta}) = \infty$$

for some $a \in u(S)$ and φ; here $S = S(\varphi, \delta)$ or $S = \Omega$. Then, as $z \to e^{i\varphi}$ in a Stolz domain, the uniform limit

$$\lim \left(\mu_S(a, u(z)) - \frac{1}{\pi} \log \frac{1}{|z - e^{i\varphi}|}\right) = \alpha,$$

$-\infty \leqslant \alpha < \infty$, *exists.*

Notice that the above u satisfies $[a, \infty) \subset u(S)$, so that $\int_a^b \Theta_S(c)^{-1} dc < \infty$ for every finite $b > a$ and $\int_a^\infty \Theta_S(c)^{-1} dc = \infty$; namely u is a "most slowly growing unbounded harmonic function".

As before, by means of the conformal mapping (13) with $\theta = \varphi$, the proof of Theorem 2 is reduced to the following proposition:

If a harmonic function $u(\zeta)$ on $D = \{\zeta \,|\, \xi^* < \operatorname{Re} \zeta, \ |\operatorname{Im} \zeta| < \pi/2\}$ satisfies

$$\mu(a, \infty) = \infty, \qquad \overline{\lim_{\xi\to\infty}} \, u(\xi) = \infty \quad (\xi\text{: real}) \tag{20}$$

for some $a \in u(D)$, then, as $\operatorname{Re} \zeta \to +\infty$, $|\operatorname{Im} \zeta| \leqslant \pi/2 - \delta$, $0 < \delta < \pi/2$, the uniform limit

$$\lim \left(\mu(a, u(\zeta)) - \frac{1}{\pi} \operatorname{Re} \zeta\right) = \alpha, \tag{21}$$

$-\infty \leqslant \alpha < \infty$, *exists.*

14. For the proof we need some preparation. Take a point $\xi_0 > \xi^*$ on the real axis and fix it once for all. For every $c > u(\xi_0)$, let $D(c)$ be the component of the open set

$\{\zeta \in D \mid u(\zeta) < c\}$ containing ξ_0. It is a simply-connected domain, and every component of $D \cap (\partial D(c))$ is an arc which is piecewise analytic and is contained in $l(c)$. Evidently $D(c_1) \subset D(c_2)$ if $c_1 < c_2$.

If $D(c)$ is bounded to the right, i.e., if the set $\{\operatorname{Re} \zeta \mid \zeta \in D(c)\}$ is bounded above, then the unbounded component $B(c)$ of $D - \operatorname{Cl} D(c)$ is determined uniquely. Clearly $B(c_1) \supset B(c_2)$ if $c_1 < c_2$.

LEMMA 2. *There exists an $a_0 > a$ such that $D(c)$ is bounded to the right for every $c > a_0$. Furthermore* $\lim_{c\to\infty} (\inf \{\operatorname{Re} \zeta \mid \zeta \in B(c)\}) = \infty$.

Proof. It suffices to show the existence of $c_1 < c_2 < \ldots \to \infty$ for which $D(c_n)$ are bounded to the right and $\lim_{n\to\infty}(\inf \{\operatorname{Re} \zeta \mid \zeta \in B(c_n)\}) = \infty$. To prove this, it is sufficient to show that, for every a_1 and ξ_1, there exists $c > a_1$ such that $D(c)$ is bounded to the right and $\xi_1 < \inf \{\operatorname{Re} \zeta \mid \zeta \in B(c)\}$. We may assume $\xi_1 > \xi_0$.

Prepare a ξ_2 with $\xi_2 > \xi_1$ and fix it. Then take $a_2 > \max (a, a_1, \max_{\xi_0 \leq \xi \leq \xi_2} u(\xi))$, and fix it.

Now take an arbitrary b with $b > a_2$. By (20), there exists a real ξ such that $u(\xi) > b$. For every c with $a_2 < c < b$, $D(c)$ does not contain ξ, thus a component $\tilde{\gamma}(c)$ of $D \cap \partial D(c)$ separates ξ from ξ_0. It is a simple arc contained in $l(c)$, meets the interval $[\xi_2, \xi]$, and never meets $[\xi_0, \xi_1]$. The maximum principle shows that $\tilde{\gamma}(c)$ is not a closed curve. The module $\mu(\Gamma)$ of the family $\Gamma = \{\tilde{\gamma}(c) \mid a_2 < c < b\}$ satisfies

$$\mu(a_2, b) \leq \mu(\Gamma).$$

Let Γ_0 be the subfamily of Γ consisting of those $\tilde{\gamma}(c)$ at least one of whose tails T_0, T_1 contains ∞ or else consists of more than one point. As is well known $\mu(\Gamma_0) = 0$.

The rest $\Gamma - \Gamma_0$ consists of those $\tilde{\gamma}(c)$ with finite end points. We let Γ_1 be the subfamily of $\Gamma - \Gamma_0$ consisting of those $\tilde{\gamma}(c)$ which traverse the vertical strip $\xi_1 < \operatorname{Re} \zeta < \xi_2$. It is seen easily that $\mu(\Gamma_1) \leq \pi(\xi_2 - \xi_1)^{-1}$.

We decompose $\Gamma - (\Gamma_0 \cup \Gamma_1)$ into Γ_2, Γ_3, and Γ_4 as follows: Γ_2 is the family of $\tilde{\gamma}(c)$ with both initial and terminal points on the upper edge L^+ of the strip $S = \{\zeta \mid |\operatorname{Im} \zeta| < \pi/2\}$; Γ_3 is that of $\tilde{\gamma}(c)$ with both initial and terminal points on the lower edge L^- of S; Γ_4 is that of $\tilde{\gamma}(c)$ with one of the end points on L^+ and the other on L^-. The module $\mu(\Gamma_2)$ is dominated by the module μ_0 of the triad (S, ξ, L^+), which is finite and independent of ξ. The module $\mu(\Gamma_3)$ is dominated by the module of the triad (S, ξ, L^-), which is equal to μ_0.

If $\Gamma_4 = \emptyset$, then
$$\mu(\Gamma) \leq \frac{\pi}{\xi_2 - \xi_1} + 2\,\mu_0,$$

therefore
$$\mu(a, b) \leqslant \mu(a, a_2) + \frac{\pi}{\xi_2 - \xi_1} + 2\mu_0.$$

Since b is arbitrary, this contradicts (20); accordingly $\Gamma_4 \neq \varnothing$. For a c with $\tilde{\gamma}(c) \in \Gamma_4$ it is clear that $D(c)$ has the desired property. The proof of Lemma 2 is hereby complete.

15. For every $c > a_0$, the set

$$\gamma(c) = D \cap \mathrm{Cl}\, D(c) \cap \mathrm{Cl}\, B(c)$$

is an arc which is piecewise analytic and is contained in $l(c)$. It joins the upper and lower edges of the strip $|\mathrm{Im}\,\zeta| < \pi/2$ within it. The initial point and the terminal point may not exist. On setting

$$\xi'(c) = \inf \{\mathrm{Re}\,\zeta \,|\, \zeta \in \gamma(c)\}, \quad \xi''(c) = \sup \{\mathrm{Re}\,\zeta \,|\, \zeta \in \gamma(c)\},$$

we have
$$\lim_{c \to \infty} \xi'(c) = \infty.$$

Incidentally $\gamma(c) = \tilde{\gamma}(c)$ for $c > a_2$.

We quote here a lemma from Jenkins [5]. Let $\gamma_0, \gamma_1, \ldots, \gamma_n$ be Jordan arcs joining the upper and lower edges of the strip $S = \{\zeta \,|\, |\mathrm{Im}\,\zeta| < \pi/2\}$ within it. Let $\xi_j' = \inf \{\mathrm{Re}\,\zeta \,|\, \zeta \in \gamma_j\}$ and $\xi_j'' = \sup \{\mathrm{Re}\,\zeta \,|\, \zeta \in \gamma_j\}$, and assume $\xi_{j-1}'' < \xi_j'$, $j = 1, \ldots, n$. Let μ_j be the module of the family of arcs joining the upper and lower edges of S within the subdomain of S bounded by γ_{j-1} and γ_j, $j = 1, \ldots, n$.

LEMMA 3.
$$\sum_{j=1}^{n} \mu_j \leqslant \frac{\xi_n' - \xi_0''}{\pi} + 2 - \sum_{j=1}^{n-1} f(\xi_j'' - \xi_j') \tag{22}$$

$$\sum_{j=1}^{n} \mu_j \leqslant \frac{\xi_n'' - \xi_0'}{\pi} - \sum_{j=1}^{n-1} f(\xi_j'' - \xi_j'), \tag{23}$$

where
$$f(x) = \frac{x^3}{9\left(1 + \dfrac{x^2}{3}\right)},$$

a strictly monotone increasing function.

The original lemma in Jenkins [5] contains only (22). The proof of (23) is completely similar if we replace γ_0 and γ_n by vertical segments on $\mathrm{Re}\,\zeta = \xi_0'$ and $\mathrm{Re}\,\zeta = \xi_n''$, respectively.

16. Set
$$\varliminf_{b \to \infty} \left(\mu(a_0, b) - \frac{1}{\pi}\,\xi'(b)\right) = \alpha_0.$$

It will be finally shown that the α in (21) is equal to $\alpha_0 + \mu(a, a_0)$.

First observe that

$$-\infty \leqslant \alpha_0 < \infty.$$

In fact, on denoting by $\bar{\mu}(a, b)$ the module of the family $\{\gamma(c) \mid a < c < b\}$, $a_0 < a < b$, we have $\mu(a, b) \leqslant \bar{\mu}(a, b)$. If b is so large that $\xi'(b) > \xi''(a_0)$, then Lemma 3 for $n=1$ is applied to $\gamma(a_0)$ and $\gamma(b)$ to get $\mu(a_0, b) \leqslant \pi^{-1}(\xi'(b) - \xi''(a_0)) + 2$. Accordingly $\alpha_0 < \infty$.

LEMMA 4. *If* $\alpha_0 > -\infty$, *then*

$$\lim_{c\to\infty} (\xi''(c) - \xi'(c)) = 0. \tag{24}$$

Furthermore, for every $\eta > 0$, *there exists a* ξ_1 *such that, for every* $\zeta \in D$ *with* $\operatorname{Re} \zeta > \xi_1$, *there exists a* c *satisfying*

$$\zeta \in B(c), \quad \operatorname{Re} \zeta - \eta < \xi'(c), \quad \xi''(c) < \operatorname{Re} \zeta + \eta$$

as well as a c' *satisfying*

$$\zeta \notin B(c'), \quad \operatorname{Re} \zeta - \eta < \xi'(c'), \quad \xi''(c') < \operatorname{Re} \zeta + \eta.$$

Proof. If (24) does not hold, there exist $\delta > 0$ and $a_0 = c_0 < c_1 < c_2 < \ldots \to \infty$ such that $\xi''(c_n) - \xi'(c_n) \geqslant \delta$. We may assume $\xi''(c_{n-1}) < \xi'(c_n)$. Apply Lemma 3 to $\gamma(a_0), \gamma(c_1), \ldots, \gamma(c_n)$ to get

$$\sum_{j=1}^{n} \bar{\mu}(c_{j-1}, c_j) \leqslant \frac{1}{\pi}\left(\xi'(c_n) - \xi''(a_0)\right) + 2 - \sum_{j=1}^{n-1} f\left(\xi''(c_j) - \xi'(c_j)\right),$$

which implies

$$\mu(a_0, c_n) - \frac{1}{\pi}\,\xi'(c_n) \leqslant \frac{-1}{\pi}\xi''(a_0) + 2 - (n-1)\,f(\delta) \to -\infty$$

as $n \to \infty$, contrary to the assumption $\alpha_0 > -\infty$. Thus we obtain (24).

Let c^* be a number such that $a_0 < c^*$ and $\xi''(c) - \xi'(c) < \eta/3$ for every $c > c^*$.

If the second assertion of Lemma 4 is not true, there exists a sequence $\{\zeta_n\}$ such that $\operatorname{Re} \zeta_n \to \infty$ and either

$$\gamma(c) \not\subset R_n \text{ for every } c \in C_n = \{c \mid \zeta_n \in B(c)\} \tag{25}$$

or

$$\gamma(c) \not\subset R_n \text{ for every } c \in C_n' = \{c \mid \zeta_n \notin B(c)\}; \tag{26}$$

here $R_n = \{\zeta \in D \mid \operatorname{Re} \zeta_n - \eta < \operatorname{Re} \zeta < \operatorname{Re} \zeta_n + \eta\}$. We may assume that $\xi''(c^*) + \eta < \operatorname{Re} \zeta_n$ and $\operatorname{Re} \zeta_n + 2\eta < \operatorname{Re} \zeta_{n+1}$. If (25) occurs, it is not difficult to see that $\xi''(c) < \operatorname{Re} \zeta_n - 2\eta/3$ for every $c \in C_n$ and $\operatorname{Re} \zeta_n - \eta/3 < \xi'(c)$ for every $c \in C_n'$. Accordingly the set $\bigcup_c \gamma(c)$ is disjoint from the rectangle $R_n' = \{\zeta \in D \mid \operatorname{Re} \zeta_n - 2\eta/3 < \operatorname{Re} \zeta < \operatorname{Re} \zeta_n - \eta/3\}$. If (26) occurs, we get a similar conclusion for the rectangle $R_n'' = \{\zeta \in D \mid \operatorname{Re} \zeta_n + \eta/3 < \operatorname{Re} \zeta < \operatorname{Re} \zeta_n + 2\eta/3\}$.

For every n, take a b with $\xi'(b) > \operatorname{Re} \zeta_n + \eta$. Let γ_j be a diagonal of the rectangle R_j', $j=1, \ldots, n$, and let μ_j be the module of the family consisting of those $\gamma(c)$ contained between γ_{j-1} and γ_j. Apply Lemma 3, (22) to $\gamma(c^*), \gamma_1, \ldots, \gamma_n, \gamma_{n+1} = \gamma(b)$ to obtain

$$\bar{\mu}(c^*, b) = \sum_{j=1}^{n+1} \mu_j \leqslant \frac{1}{\pi}\,(\xi'(b) - \xi''(c^*)) + 2 - \sum_{j=1}^{n} f(\xi_j'' - \xi_j')$$

which implies

$$\mu(a_0, b) - \frac{1}{\pi}\,\xi'(b) \leqslant \mu(a_0, c^*) - \frac{1}{\pi}\,\xi''(c^*) + 2 - nf\left(\frac{\eta}{3}\right),$$

contrary to the assumption $\alpha_0 > -\infty$.

17. We now prove the existence and the equality of the following limits:

$$\lim_{b\to\infty}\left(\mu(a_0, b) - \frac{1}{\pi}\,\xi'(b)\right) = \lim_{b\to\infty}\left(\mu(a_0, b) - \frac{1}{\pi}\,\xi''(b)\right). \tag{27}$$

The limit will be equal to α_0 defined in 16°.

For $b < b'$, we have trivially $\mu(a_0, b') - \mu(a_0, b) = \mu(b, b') \leqslant \bar{\mu}(b, b') \leqslant \pi^{-1}(\xi''(b') - \xi'(b))$. Therefore $\mu(a_0, b') - \pi^{-1}\xi''(b') \leqslant \mu(a_0, b) - \pi^{-1}\xi'(b)$, so that

$$\overline{\lim_{b\to\infty}}\left(\mu(a_0, b) - \frac{1}{\pi}\,\xi''(b)\right) \leqslant \underline{\lim_{b\to\infty}}\left(\mu(a_0, b) - \frac{1}{\pi}\,\xi'(b)\right).$$

If b and b' satisfy $\xi''(b) < \xi'(b)$, apply Lemma 3, (22) for $n=1$ to $\gamma(b)$ and $\gamma(b')$. We obtain $\mu(b, b') \leqslant \bar{\mu}(b, b') \leqslant \pi^{-1}(\xi'(b') - \xi''(b)) + 2$ and, therefore

$$\overline{\lim_{b\to\infty}}\left(\mu(a_0, b) - \frac{1}{\pi}\,\xi'(b) - 2\right) \leqslant \underline{\lim_{b\to\infty}}\left(\mu(a_0, b) - \frac{1}{\pi}\,\xi''(b)\right).$$

If $\alpha_0 = -\infty$, this much is sufficient to justify the validity of (27) with the limit $-\infty$.

If $\alpha_0 > -\infty$, we have (24), so that

$$\overline{\lim_{b\to\infty}}\left(\mu(a_0, b) - \frac{1}{\pi}\,\xi''(b)\right) = \overline{\lim_{b\to\infty}}\left(\mu(a_0, b) - \frac{1}{\pi}\,\xi'(b)\right)$$

$$\underline{\lim_{b\to\infty}}\left(\mu(a_0, b) - \frac{1}{\pi}\,\xi''(b)\right) = \underline{\lim_{b\to\infty}}\left(\mu(a_0, b) - \frac{1}{\pi}\,\xi'(b)\right),$$

which show (27) including the existence of the limits.

18. *Proof of (21) for the case* $\alpha_0 > -\infty$. It suffices to show that, for every $\varepsilon > 0$, there exists a ξ_2 such that $\xi_2 > \xi^*$ and

$$\left|\int_{a_0}^{u(\zeta)} \frac{dc}{\Theta(c)} - \frac{1}{\pi} \operatorname{Re} \zeta - \alpha_0 \right| < \varepsilon$$

for every ζ with $\operatorname{Re} \zeta > \xi_2$ and $|\operatorname{Im} \zeta| \leqslant \pi/2 - \delta$.

Take b_0 such that every $b \geqslant b_0$ satisfies

$$\left|\int_{a_0}^{b} \frac{dc}{\Theta(c)} - \frac{1}{\pi} \xi'(b) - \alpha_0 \right| < \frac{\varepsilon}{3}.$$

Take ξ_1 of Lemma 4 for $\eta = \varepsilon\delta/12$. We shall show $\xi_2 = \max(\xi''(b_0) + \eta, \xi_1)$ is the required value. For this purpose, it suffices to show the existence of b such that $b \geqslant b_0$,

$$\frac{1}{\pi} |\xi'(b) - \operatorname{Re} \zeta| \leqslant \frac{\varepsilon}{3} \tag{28}$$

and

$$\left|\int_{b}^{u(\zeta)} \frac{dc}{\Theta(c)}\right| \leqslant \frac{\varepsilon}{3}. \tag{29}$$

Let ζ be such that $\operatorname{Re} \zeta > \xi_2$ and $|\operatorname{Im} \zeta| \leqslant \pi/2 - \delta$. Lemma 4 shows the existence of c_1 with $\zeta \in B(c_1)$, $\operatorname{Re} \zeta - \eta < \xi'(c_1)$, $\xi''(c_1) < \operatorname{Re} \zeta + \eta$, and c_2 with $\zeta \notin B(c_2)$, $\operatorname{Re} \zeta - \eta < \xi'(c_2)$, $\xi''(c_2) < \operatorname{Re} \zeta + \eta$. Clearly $b = c_1$ and $b = c_2$ satisfy $b \geqslant b_0$ and (28). Therefore it remains to show that either $b = c_1$ or $b = c_2$ satisfies (29). There are four possible cases: $u(\zeta) < c_1 < c_2$, $c_1 = u(\zeta) < c_2$, $c_1 < u(\zeta) \leqslant c_2$, and $c_1 < c_2 < u(\zeta)$. In the first case, set $b = c_1$. For every c with $u(\zeta) < c < b = c_1$ the level $l(c)$ contains an arc separating ζ from $\gamma(b)$ contained in the rectangle $R = \{x + iy \mid \operatorname{Re} \zeta - \eta < x < \operatorname{Re} \zeta + \eta,\ |y| < \pi/2\}$. This arc traverses at least one of the rectangles $\{x + iy \in R \mid \pi/2 - \delta < y < \pi/2\}$ and $\{x + iy \in R \mid -\pi/2 < y < -\pi/2 + \delta\}$. Accordingly the module $\mu(u(\zeta), b)$ is dominated by $4\eta\delta/\delta^2 = \varepsilon/3$, namely (29) holds. The reasoning for the other cases is similar; take $b = c_2$, $b = c_1$, $b = c_2$ for the second, third, and the fourth cases, respectively.

19. *Proof of (21) for the case* $\alpha_0 = -\infty$. It suffices to show that, for every M, there exists a ξ_2 such that $\xi_2 > \xi^*$ and

$$\int_{a}^{u(\zeta)} \frac{dc}{\Theta(c)} - \frac{1}{\pi} \operatorname{Re} \zeta < -M$$

for every ζ with $\operatorname{Re} \zeta > \xi_2$ and $|\operatorname{Im} \zeta| \leqslant \pi/2 - \delta$.

For such a ζ we consider the triads (S, ζ, L^+) and (S, ζ, L^-), where L^+ and L^- are respectively the upper and lower edges of the strip $S = \{\zeta \mid |\operatorname{Im} \zeta| < \pi/2\}$. Their modules are readily seen to be bounded by a number μ^* depending only on δ.

Without loss of generality we may assume $M \geqslant 4\mu^*$. Take b_0 sufficiently large so that

$$\int_{b_0}^{b} \frac{dc}{\Theta(c)} - \frac{1}{\pi}\xi'(b) < -2M$$

for every $b \geqslant b_0$. We shall show that $\xi_2 = \xi''(b_0)$ is what we wish to obtain. For ζ with $\operatorname{Re}\zeta > \xi_2$ and $|\operatorname{Im}\zeta| \leqslant \pi/2 - \delta$, it suffices to show the existence of b such that $b > b_0$, $\xi'(b) \leqslant \operatorname{Re}\zeta$, and

$$\int_{b}^{u(\zeta)} \frac{dc}{\Theta(c)} < M. \tag{30}$$

The set $C = \{c \mid \zeta \in B(c)\}$ contains b_0. If the supremum c_0 of C is not less than $u(\zeta)$, we take a $b \in C$ with $b_0 \leqslant b$ and $\int_b^{c_0} \Theta(c)^{-1} dc < M$. It clearly satisfies $\xi'(b) \leqslant \operatorname{Re}\zeta$ and (30). If $c_0 < u(\zeta)$, we consider every c with $c_0 < c < u(\zeta)$. Since $\zeta \notin B(c)$, ζ belongs to a bounded component of $D - \operatorname{Cl} D(c)$. The part of the boundary of this component common to $D(c)$ is an arc, which separates ζ from L^+ or L^- and both of whose tails belong to L^- or L^+, respectively. Accordingly $\mu(c_0, u(\zeta)) \leqslant 2\mu^* \leqslant M/2$. A $b \in C$ with $b_0 \leqslant b$ and $\int_b^{c_0} \Theta(c)^{-1} dc < M/2$ satisfies $\xi'(b) < \operatorname{Re}\zeta$ and (30).

The proof of Theorem 2 is hereby complete.

§ 4. Estimate of the limiting value

20. For a certain class of functions u we can consider the reduced module $\tilde{\mu}(u(z))$ rather than the module $\mu(a, u(z))$. We have not only the regularity of growth analogous to Theorem 2, but also an estimate of the limiting value. The result for $u = \log |f|$ for a mean p-valent function f is found in Eke [2, Theorem 5].

THEOREM 3. *Suppose u is a harmonic function on the punctured disc $0 < |z| < 1$ having the following singularity at the origin: $u(z) - \lambda \log |z|$ is harmonic at the origin, $\lambda > 0$. If there are φ and a such that*

$$\overline{\lim_{r\to 1}}\, u(re^{i\varphi}) = \infty, \quad \mu(a, \infty) = \infty,$$

then the uniform limit in a Stolz domain

$$\tilde{\alpha} = \lim_{z \to e^{i\varphi}} \left(\tilde{\mu}(u(z)) - \frac{1}{\pi}\log\frac{1}{|z - e^{i\varphi}|}\right), \quad -\infty \leqslant \tilde{\alpha} < \infty,$$

exists and satisfies

$$\tilde{\alpha} \leqslant \frac{u_0}{2\pi\lambda}, \quad u_0 = \lim_{z\to 0}\,(u(z) - \lambda \log |z|).$$

The equality is realized if and only if

$$u(z) = \lambda \log \frac{|z|}{|z-e^{i\varphi}|^2} + \text{const.}$$

Proof. Since u satisfies the assumption of Theorem 2 for $r_0 = 0$ and $S = \Omega$, the existence of $\tilde{\alpha}$ is evident as follows:

$$\lim \left(\tilde{\mu}(u(z)) - \frac{1}{\pi} \log \frac{1}{|z-e^{i\varphi}|}\right) = \frac{a}{2\pi\lambda} + \lim \left(\mu(a, u(z)) - \frac{1}{\pi} \log \frac{1}{|z-e^{i\varphi}|}\right).$$

In order to estimate $\tilde{\alpha}$, we use the conformal mapping (15) with $\theta = \varphi$. For the transformed function $u(\zeta) = u(z(\zeta))$ on the strip $\{\zeta \mid |\operatorname{Im} \zeta| < \pi/2\}$, the level $l(a)$ consists of a simple arc joining the upper and lower edges of the strip, provided a is sufficiently small. Take c and b with $a < c < b$ sufficiently large so that $\gamma(c)$ and $\gamma(b)$ exist (cf. Lemma 2). Apply Lemma 3, (23) to $l(a)$, $\gamma(c)$, and $\gamma(b)$ to obtain

$$\mu(a, b) \leqslant \frac{1}{\pi} (\xi''(b) - \xi'(a)) - f(\xi''(c) - \xi'(c)),$$

where $\xi'(a) = \inf \{\operatorname{Re} \zeta \mid \zeta \in l(a)\}$. Thus

$$\frac{a}{2\pi\lambda} + \mu(a, b) - \frac{1}{\pi} \xi''(b) \leqslant \frac{a}{2\pi\lambda} - \frac{1}{\pi} \xi'(a) - f(\xi''(c) - \xi'(c)).$$

Keep a and c fixed and let $b \to \infty$. We know from the proof of Theorem 2 (cf. 16°, 17°) that the limiting value of the left-hand side coincides with $\tilde{\alpha}$. Accordingly

$$\tilde{\alpha} \leqslant \frac{a}{2\pi\lambda} - \frac{1}{2\eta} \inf \left\{\log \frac{|z|}{|z-e^{i\varphi}|^2} \;\middle|\; u(z) = a\right\} - f(\xi''(c) - \xi'(c))$$

$$= \frac{a}{2\pi\lambda} - \frac{a-u_0}{2\pi\lambda} + o(1) - f(\xi''(c) - \xi'(c))$$

as $a \to -\infty$. We conclude

$$\tilde{\alpha} \leqslant \frac{u_0}{2\pi\lambda} - f(\xi''(c) - \xi'(c))$$

for all c sufficiently large. This shows $\tilde{\alpha} \leqslant u_0/2\pi\lambda$ and, in addition, if the equality is realized, $\xi''(c) = \xi'(c)$ for all c sufficiently large, which implies that $u(z(\zeta))$ is a linear transformation of ζ: $u(z(\zeta)) = \lambda \operatorname{Re} \zeta + \text{const.}$

§ 5. Rapid growth in one direction

21. If a function u in Theorem 2 satisfies $\alpha > -\infty$, then the growth of u in the direction $e^{i\varphi}$ may be regarded as being rapid. In fact, even though an upper bound for α is not attained numerically (as in the case of Theorem 3), an upper bound furnished by (19) is essentially attained. It is of course to be noted that, given $e^{i\varphi}$, the condition $\alpha > -\infty$ depends on S, but is independent of $a \in u(S)$.

It will be shown in Theorems 4 and 5 that, if growth in the direction $e^{i\varphi}$ is rapid in this sense, then growth in other directions in S becomes slower. Compare these theorems with Hayman [3, Theorem 2.9] and Eke [1, Theorems 5 and 6].

THEOREM 4. *Suppose a harmonic function u on $\Omega = \{z \mid r_0 < |z| < 1\}$ satisfies $[a, \infty) \subset u(S)$ and*

$$\overline{\lim_{r\to 1}} \left(\mu_S(a, u(re^{i\varphi})) - \frac{1}{\pi} \log \frac{1}{1-r}\right) > -\infty$$

for some a and φ: here $S = S(\varphi, \delta)$ or $S = \Omega$. Then

$$\mu_S(a, u(z)) = o\left(\left(\log \frac{1}{1-|z|}\right)^{\frac{1}{2}}\right)$$

uniformly as $|z| \to 1$, $u(z) > a$ on any sector $S(\theta, \sigma)$ whose closure relative to Ω is contained in $S - \{z \mid \arg z = \varphi\}$.

Observe that a function u satisfies the above assumption if and only if it satisfies that of Theorem 2 and $\alpha > -\infty$.

Proof of Theorem 4. Take a positive $\delta_0 < \delta$ small enough so that the closure of $S_0 = S(\varphi, \delta_0)$ is disjoint from the given sector $\Sigma = S(\theta, \sigma)$. Since $\overline{\lim}\, u(re^{i\varphi}) = \infty$ is implied by the assumption, there exists an a' with $[a', \infty) \subset u(S_0)$. By (19)

$$\mu_{S_0}(a', u(re^{i\varphi})) \leqslant \frac{1}{\pi} \log \frac{1}{1-r} + O(1),$$

thus

$$\mu_{S_0}(a', u(re^{i\varphi})) - \mu_S(a', u(re^{i\varphi})) = O(1).$$

By relation (6) for $G = S_0$ and $\Omega = S$, we have

$$\|\varrho_{S_0} - \varrho_S\|_{\Omega(a', \infty)} < \infty.$$

Since $\mu_S(a, u(z)) = \mu_S(a, a') + \mu_S(a', u(z))$, the conclusion of the theorem is immediately derived from the lemma below.

22. In general, let Ω be $\{z \mid r_0 < |z| < 1\}$ as before and consider a harmonic function u on a sector $S = S(\varphi, \delta)$. If there exist directions $e^{i\varphi_1}, \ldots, e^{i\varphi_k}$ and mutually disjoint sectors $S(\varphi_j, \delta_j)$, $j = 1, \ldots, k$, such that

$$\|\varrho_{S_1 \cup \ldots \cup S_k} - \varrho_S\|_{\Omega(a, \infty)} < \infty \tag{31}$$

for some a, then the growth of u in directions different from $e^{i\varphi_1}, \ldots, e^{i\varphi_k}$ becomes slower. In fact, (31) shows that, for sufficiently large c, the level $l(c)$ appearing in $S - \bigcup_{j=1}^{k} S_j$ is relatively small. In particular, if u is bounded by b in $S - \bigcup_{j=1}^{k} S_j$ then $\mu_S(b, \infty) = \mu_{S_1 \cup \ldots \cup S_k}(b, \infty)$, so that (31) holds. More explicitly

LEMMA 5. *If* $[a, \infty) \subset u(S_1 \cup \ldots \cup S_k)$ *and* (31) *holds then*

$$\mu_S(a, u(z)) = o\left(\left(\log \frac{1}{1-|z|}\right)^{\frac{1}{2}}\right)$$

uniformly as $|z| \to 1$, $u(z) > a$ *on any sector* Σ *whose closure relative to* Ω *is contained in* $S - \bigcup_{j=1}^{k} \mathrm{Cl}\, S_j$.

Proof. Let $\Sigma = S(\theta, \sigma)$ and take $\eta > 0$ such that $\Sigma' = S(\theta, \sigma + \eta)$ is contained in $S - \bigcup_{j=1}^{k} S_j$. Let r^* be the value in Theorem 1′ with respect to Σ' and η. We may assume that u is not bounded above in $\Sigma^* = \{z \in \Sigma \mid |z| > r^*\}$, since otherwise the conclusion of lemma is trivial. Given $\varepsilon > 0$ take b_0 such that $a < b_0 \in u(\Sigma^*)$, $b_0 > \sup\{u(z) \mid |z| = r^*, z \in \Sigma\}$, and

$$\|\varrho_{S_1 \cup \ldots \cup S_k}\|^2_{\Omega(b_0, \infty)} - \|\varrho_S\|^2_{\Omega(b_0, \infty)} = \|\varrho_{S_1 \cup \ldots \cup S_k} - \varrho_S\|^2_{\Omega(b_0, \infty)} < \varepsilon^2.$$

For a b with $b_0 < b \in u(\Sigma)$ the inequality (11) is read

$$\|\varrho_S\|^2_{\Omega(b_0, b)} \leqslant \|\varrho_{\Sigma'}\|_{\Omega(b_0, b)} \left(\|\varrho_{S_1 \cup \ldots \cup S_k}\|^2_{\Omega(b_0, b)} - \|\varrho_S\|^2_{\Omega(b_0, b)}\right)^{\frac{1}{2}},$$

therefore
$$\mu_S(b_0, b) \leqslant \varepsilon \mu_{\Sigma'}(b_0, b).$$

If $u(z) > b_0$, $z \in \Sigma^*$, then $\mu_S(a, u(z)) = \mu_S(a, b_0) + \mu_S(b_0, u(z))$, $\mu_S(b_0, u(z)) \leqslant \varepsilon \mu_{\Sigma'}(b_0, u(z))$, and an estimate of $\mu_{\Sigma'}(b_0, u(z))$ is given by Theorem 1′ with respect to Σ'. Therefore

$$\mu_S(a, u(z)) \leqslant \mu_S(a, b_0) + \varepsilon \left(\frac{1}{\pi} \log \frac{1}{1-|z|} + A\right)^{\frac{1}{2}},$$

where A is a constant depending only on r^*.

If $a < u(z) \leqslant b_0$, the above estimate is trivial. We conclude that, if $z \in \Sigma$, $u(z) > a$, and $|z|$ is sufficiently close to 1, then

$$\mu_S(a, u(z)) < \varepsilon \left(\log \frac{1}{1-|z|}\right)^{\frac{1}{2}}.$$

The proof of Lemma 5, thus that of Theorem 4, is hereby complete.

23. If u is as in Theorem 4, then Theorem 2 implies the following with respect to an arbitrary Stolz domain Δ with vertex at $e^{i\varphi}$:

$$\lim_{r\to 1} \mu_S(u(re^{i\varphi}), m_\Delta(r)) = 0, \tag{32}$$

where $m_\Delta(r) = \sup\{u(z) \mid |z| = r,\ z \in \Delta\}$. It will be shown that $u(re^{i\varphi})$ is approximately equal to

$$m_S(r) = \sup\{u(z) \mid |z| = r,\ z \in S\}.$$

THEOREM 5. *If u is as in Theorem 4,*

(i) *there exists a Stolz domain Δ with vertex at $e^{i\varphi}$ and a $S^* = S(\varphi, \delta^*) \subset S$ such that*

$$u(re^{i\varphi}) > u(re^{i\theta})$$

for $re^{i\theta} \in S^* - \Delta$ with sufficiently large r.

(ii) $$\lim_{r\to 1} \mu_S(u(re^{i\varphi}), m_S(r)) = 0.$$

Proof. The derivation of (ii) from (i) is immediate, for Theorem 4 shows $u(re^{i\varphi}) > u(re^{i\theta})$ if $re^{i\theta} \in S - S^*$ and r is sufficiently large. Thus $m_S(r) = m_\Delta(r)$ for sufficiently large r, which together with (32) implies (ii).

For the proof of (i), we transform the variable by

$$\zeta = \log \frac{1}{ze^{-i\varphi}}.$$

The image of the sector S is the rectangle

$$R = \left\{\zeta \mid 0 < \operatorname{Re}\zeta < \log\frac{1}{r_0},\quad |\operatorname{Im}\zeta| < \delta\right\},$$

and $z = e^{i\varphi}$ corresponds to $\zeta = 0$. The transformed function $u(\zeta) = u(z(\zeta))$ satisfies

$$\overline{\lim_{\xi\to 0}}\, u(\xi) = \infty \quad (\xi\text{: real}) \tag{33}$$

and

$$\lim_{\zeta\to 0}\left(\mu_R(a, u(\zeta)) - \frac{1}{\pi}\log\frac{1}{|\zeta|}\right) = \alpha. \quad -\infty < \alpha < \infty \tag{34}$$

uniformly as ζ approaches 0 in a Stolz domain. The assertion (i) is reduced to the following: there exist a Stolz domain Δ in R with vertex at 0, a number ξ^* with $0 < \xi^* < \log(r_0^{-1})$, and a number δ^* with $0 < \delta^* < \delta$ such that

$$u(\xi) > u(\xi + i\eta) \tag{35}$$

for every ξ and η satisfying $0 < \xi < \xi^*$, $|\eta| < \delta^*$, $\xi + i\eta \notin \Delta$. We shall prove this by showing

$$\int_{u(\zeta)}^{u(\xi)} \frac{dc}{\Theta_R(c)} > 0, \quad \zeta = \xi + i\eta \tag{36}$$

for these ξ and η.

Consider an auxiliary Stolz domain $\Delta' = \{\zeta = \xi + i\eta \,|\, |\eta| < 2e^{\pi/2}\xi\}$. For a $\zeta = \xi + i\eta \notin \Delta'$ with $|\eta| < \delta_0 = \min(2\delta/3, \frac{1}{2}\log(r_0^{-1}))$, set

$$\xi' = 2^{-1}e^{-\pi/2}|\eta|$$

and express the left-hand side of (36) as follows:

$$\int_{u(\zeta)}^{u(\xi')} \frac{dc}{\Theta_R(c)} + \int_{u(\xi')}^{u(\xi)} \frac{dc}{\Theta_R(c)}.$$

The second integral satisfies

$$\int_{u(\xi')}^{u(\xi)} \frac{dc}{\Theta_R(c)} = \frac{1}{\pi}\log\frac{|\eta|}{\xi} + O(1)$$

as $\zeta \to 0$, $\zeta \notin \Delta'$, which is a direct consequence of (34). For the first integral we need the following:

LEMMA 6. $$\int_{u(\xi')}^{u(\zeta)} \frac{dc}{\Theta_R(c)} \leqslant \left(\frac{1}{2} + o(1)\right)\left(\log\frac{|\eta|}{\xi} + O(1)\right)^{\frac{1}{2}}$$

as $\zeta \to 0$, $\zeta \notin \Delta'$.

Consequently the left-hand side of (36) is bounded below by

$$\frac{1}{\pi}\log\frac{|\eta|}{\xi} + O(1) - \left(\log\frac{|\eta|}{\xi} + O(1)\right)^{\frac{1}{2}}$$

as $\zeta \to 0$. We can make this positive on taking $|\zeta|$ sufficiently small and $|\eta|/\xi$ sufficiently large. In other words, if we take a sufficiently wide Stolz domain Δ and sufficiently small ξ^* and δ^*, then (36) is satisfied for $\zeta = \xi + i\eta \notin \Delta$ such that $0 < \xi < \xi^*$ and $|\eta| < \delta^*$. The proof of Theorem 5 will be complete if Lemma 6 is verified.

24. *Proof of Lemma 6.* We need a counterpart of Theorem 1′ for R. On mapping R by $\zeta \to \log((\zeta - i\eta_0)^{-1})$ and applying Lemma 1, we obtain the following: given η_0 with $|\eta_0| < \delta$ and σ with $0 < \sigma < \min(\delta - |\eta_0|, \log(r_0^{-1}))$,

$$\left| \int_{u(\xi_1 + i\eta_0)}^{u(\xi_2 + i\eta_0)} \frac{dc}{\Theta_D(c)} \right| \leqslant \frac{1}{\pi}\log\frac{\xi_1}{\xi_2} + \frac{\pi}{4} \tag{37}$$

for every ξ_1, ξ_2 such that $0 < \xi_2 < \xi_1 \leqslant \sigma e^{-\pi/2}$; here

$$D = D(\eta_0, \sigma) = \{\zeta \,|\, \mathrm{Re}\,\zeta > 0, \ |\zeta - i\eta_0| < \sigma\}.$$

Given a $\zeta \notin \Delta'$, set $\zeta' = \xi' + i\eta$. The relation (34) shows

$$\int_{u(\zeta')}^{u(\xi')} \frac{dc}{\Theta_R(c)} = \frac{1}{\pi} \log \frac{|\zeta'|}{\xi'} + o(1) = \frac{1}{2\pi} \log (1 + 4e^{\pi}) + o(1)$$

as $\xi' \to 0$. By taking ξ' sufficiently small we can make this integral positive, so that

$$u(\xi') > u(\zeta'). \tag{38}$$

To estimate the integral in the lemma, take $\zeta \notin \Delta'$ arbitrarily. We may assume

$$u(\xi') < u(\zeta) \tag{39}$$

since otherwise the result is trivial. Consider the half discs $D_0 = D(0, |\eta|/2)$ and $D_\eta = D(i\eta, |\eta|/2)$. They are disjoint and contained in R, thus by (11) we obtain the following:

$$\|\varrho_R\|^2 \leqslant \|\varrho_{D_\eta}\| (\|\varrho_{D_0}\|^2 - \|\varrho_R\|^2)^{\frac{1}{2}} \tag{40}$$

with norms considered over the domain $\Omega(u(\xi'), u(\zeta))$. Clearly

$$\|\varrho_R\|^2 = \int_{u(\xi')}^{u(\zeta)} \frac{dc}{\Theta_R(c)}.$$

On the other hand, by (38)

$$\|\varrho_{D_\eta}\|^2 = \int_{u(\xi')}^{u(\zeta)} \frac{dc}{\Theta_{D_\eta}(c)} \leqslant \int_{u(\zeta')}^{u(\zeta)} \frac{dc}{\Theta_{D_\eta}(c)}.$$

The estimate (37) is applied to this integral to get

$$\|\varrho_{D_\eta}\|^2 \leqslant \frac{1}{\pi} \log \frac{\xi'}{\xi} + \frac{\pi}{4}. \tag{41}$$

The relations (33) and (39) guarantee the existence of $\tilde{\xi}$ such that $0 < \tilde{\xi} < \xi'$ and $u(\tilde{\xi}) = u(\zeta)$. Then

$$\|\varrho_{D_0}\|^2 = \int_{u(\xi')}^{u(\tilde{\xi})} \frac{dc}{\Theta_{D_0}(c)} \leqslant \frac{1}{\pi} \log \frac{\xi'}{\tilde{\xi}} + \frac{\pi}{4}$$

by (37) and, by (34)

$$\|\varrho_R\|^2 = \int_{u(\xi')}^{u(\tilde{\xi})} \frac{dc}{\Theta_R(c)} = \frac{1}{\pi} \log \frac{\xi'}{\tilde{\xi}} + o(1)$$

as $\xi' \to 0$. Accordingly

$$\|\varrho_{D_0}\|^2 - \|\varrho_R\|^2 \leqslant \frac{\pi}{4} + o(1). \tag{42}$$

On substituting (41) and (42) in (40), we complete the proof of Lemma 6.

25. We shall present here some examples of functions with rapid growth in the direction $e^{i\varphi}$, more specifically functions satisfying the condition of Theorem 4.

Needless to say the function $u(z)=\lambda \log(|z|\cdot|z-e^{i\varphi}|^{-2})+\text{const}$, $\lambda>0$, mentioned in Theorem 3 is such an example.

Another example is as follows: *a harmonic function u on $\Omega=\{z\,|\,r_0<|z|<1\}$ satisfies the condition of Theorem 4 with respect to $a=c_0+2L$ determined below if $u+iu^*$ furnishes a one-to-one conformal mapping of $S=S(\varphi,\delta)$ onto a domain which is contained in $D=\{w\,|\,-\theta_1(\operatorname{Re} w)<\operatorname{Im} w<\theta_2(\operatorname{Re} w),\ -\infty<\operatorname{Re} w<\infty\}$ and contains $D_0=\{w\in D\,|\,\operatorname{Re} w>c_0\}$ such that*

(i) $0<\theta\leqslant\theta_j(u)\leqslant L<\infty$, $j=1, 2$, $c_0\leqslant u<\infty$, *for some θ and L,*

(ii) *the total variations of $\theta_1(u)$ and $\theta_2(u)$ over any closed interval (c_0, ∞) are bounded by $V<\infty$,*

(iii) $z=e^{i\varphi}$ *corresponds to $w=\infty$.*

In fact, $[a, \infty)\subset u(S)$ for $a=c_0+2L$ is trivially satisfied. In order to verify

$$\overline{\lim_{r\to 1}}\left(\mu_S(a, u(re^{i\varphi}))-\frac{1}{\pi}\log\frac{1}{1-r}\right)>-\infty, \tag{43}$$

we transform the independent variable by (13) with $\theta=\varphi$. Observe that, for $c>a=c_0+2L$ the level $l(c)\cap S$ corresponds to the line segment $\{w\,|\,\operatorname{Re} w=c,\ -\theta_1(c)<\operatorname{Im} w<\theta_2(c)\}$. Accordingly, in the ζ-plane, $l(c)\cap S$ coincides with $\gamma(c)$, and $\Theta_S(c)=\theta_1(c)+\theta_2(c)$. For any $b>a$ we apply Ahlfors' Second Inequality (see Jenkins–Oikawa [6]). We obtain

$$\frac{1}{\pi}(\xi''(b)-\xi'(a))\leqslant\int_a^b\frac{dc}{\Theta_S(c)}+\frac{LV}{\theta^2}+\frac{4L}{\theta}$$

and, therefore
$$\underline{\lim_{b\to\infty}}\left(\mu_S(a,b)-\frac{1}{\pi}\xi''(b)\right)>-\infty.$$

Since $\overline{\lim}_{r\to 1}u(re^{i\varphi})=\infty$ and $\mu_S(a, \infty)=\infty$, we conclude (43) via Theorem 2, and see that u satisfies the condition of Theorem 4.

§ 6. Functions with maximum growth

26. Let u be a harmonic function on $\Omega=\{z\,|\,r_0<|z|<1\}$ satisfying

$$[a, \infty)\subset u(\Omega)$$

for some a. The growth of the quantity

$$m(r)=\max_{|z|=r}u(z),\quad r_0<r<1$$

is, by (18), subject to the restriction either $m(r) \leqslant a$ or $\mu(a, m(r)) \leqslant \pi^{-1} \log (1-r)^{-1} + O(1)$. We shall say that u *attains maximum growth* if

$$\overline{\lim_{r\to 1}} \left(\mu(a, m(r)) - \frac{1}{\pi} \log \frac{1}{1-r}\right) > -\infty; \tag{44}$$

the condition is independent of a with $[a, \infty) \subset u(\Omega)$. In this case we have (cf. the paragraph after Theorem 4)

$$\lim_{r\to 1} m(r) = \infty, \quad \mu(a, \infty) = \infty.$$

It is to be noted that (44) is equivalent to the existence of $\{b_n\}$, $\{r_n\}$, $\{\theta_n\}$ such that

$$\begin{cases} a < b_n \leqslant u(r_n e^{i\theta_n}), \quad \lim_{n\to\infty} r_n = 1 \\ \lim_{n\to\infty} \left(\mu(a, b_n) - \frac{1}{\pi} \log \frac{1}{1-r_n}\right) > -\infty. \end{cases} \tag{44'}$$

Concerning the growth in a direction $e^{i\varphi}$ we consider the condition of Theorem 4 with respect to $S = \Omega$. Namely, we shall say that u *attains maximum growth in the direction* $e^{i\varphi}$ if

$$\overline{\lim_{r\to 1}} \left(\mu(a, u(re^{i\varphi})) - \frac{1}{\pi} \log \frac{1}{1-r}\right) > -\infty, \tag{45}$$

the condition is independent of a with $[a, \infty) \subset u(\Omega)$. In this case we have as above

$$\overline{\lim_{r\to 1}}\ u(re^{i\varphi}) = \infty, \quad \mu(a, \infty) = \infty,$$

so that, by Theorem 2, the limiting value of (45) exists.

We remark that (45) implies that u satisfies the condition of Theorem 4 for every $S = S(\varphi, \delta)$, but the validity of the latter for a particular S does not necessarily imply (45).

The first paragraph of the following theorem shows that (44) and (45) are equivalent; cf. Hayman [3, Theorem 2.8]:

THEOREM 6. *A harmonic function u on Ω satisfying $[a, \infty) \subset u(\Omega)$ for some a attains maximum growth if and only if it attains maximum growth in one direction. The direction $e^{i\varphi}$ is determined uniquely, for which the condition of Theorem 4 is satisfied for every $S(\varphi, \delta)$. In particular finite limiting values*

$$\lim_{r\to 1} \left(\mu(a, m(r)) - \frac{1}{\pi} \log \frac{1}{1-r}\right)$$

$$\lim_{r\to 1}\left(\mu(a, u(re^{i\varphi})) - \frac{1}{\pi}\log\frac{1}{1-r}\right)$$

exist and coincide, and the growth in directions different from $e^{i\varphi}$ *is relatively slower in the sense that*

$$\mu(a, u(z)) = o\left(\left(\log\frac{1}{1-|z|}\right)^{\frac{1}{2}}\right)$$

uniformly as $|z|\to 1$, $u(z)>a$ *on every sector whose closure does not meet the ray* $\arg z=\varphi$.

Proof. Only the only-if part of the first paragraph needs proof. If u attains maximum growth, an accumulation point φ of the $\{\theta_n\}$ in (44′) gives a direction $e^{i\varphi}$ in which u attains maximum growth. This is apparent from the lemma below applied for $S=\Omega$ and a subsequence $\{\theta_{n_\nu}\}$ such that $\lim\theta_{n_\nu}=\varphi$.

27. Let u be a harmonic function on $r_0<|z|<1$.

LEMMA 7. *If* $b_n\leqslant u(r_n e^{i\theta_n})$, $n=1, 2, \ldots$, *is valid for sequences* $\{b_n\}$, $\{r_n\}$, $\{\theta_n\}$ *with* $\lim_{n\to\infty} r_n=1$, $\lim\theta_n=\varphi$, *then*

$$\overline{\lim_{n\to\infty}}\left(\int_a^{b_n}\frac{dc}{\Theta_S(c)}-\frac{1}{\pi}\log\frac{1}{1-r_n}\right)\leqslant\int_a^{u(re^{i\varphi})}\frac{dc}{\Theta_S(c)}-\frac{1}{\pi}\log\frac{1}{1-r}+\frac{1}{\pi}\log 2+\frac{\pi}{4}$$

for every $S=S(\varphi,\delta)$, a *with* $[a,\infty)\subset u(S)$, *and* r *sufficiently large.*

Proof. Given $S(\varphi,\delta)$ take η with $0<\eta<\delta$, and let r^* be the value in Theorem 1′. We shall show the above inequality for every $r\geqslant r^*$. Take n sufficiently large so that $r_n>r$ and $|\theta_n-\varphi|<\eta$. Theorem 1′ shows

$$\left|\int_{u(re^{i\theta_n})}^{u(r_n e^{i\theta_n})}\frac{dc}{\Theta_S(c)}\right|\leqslant\frac{1}{\pi}\log\frac{1-r}{1-r_n}+\frac{1}{\pi}\log 2+\frac{\pi}{4}.$$

Since
$$\int_a^{b_n}\frac{dc}{\Theta_S(c)}\leqslant\int_a^{u(re^{i\theta_n})}\frac{dc}{\Theta_S(c)}+\int_{u(re^{i\theta_n})}^{u(r_n e^{i\theta_n})}\frac{dc}{\Theta_S(c)},$$

we get

$$\int_a^{b_n}\frac{dc}{\Theta_S(c)}-\frac{1}{\pi}\log\frac{1}{1-r_n}\leqslant\int_a^{u(re^{i\theta_n})}\frac{dc}{\Theta_S(c)}-\frac{1}{\pi}\log\frac{1}{1-r}+\frac{1}{\pi}\log 2+\frac{\pi}{4}.$$

On fixing r and letting $n\to\infty$ we obtain the desired inequality.

§ 7. Simultaneous growth in more than one direction

28. If a harmonic function on $\Omega=\{z \mid r_0<|z|<1\}$ satisfying $[a, \infty)\subset u(\Omega)$ for some a grows rapidly in more than one direction, the previous argument shows that u never attains maximum growth. A smaller bound than that of (18) should be obtained.

Let $e^{i\varphi_1}, \ldots, e^{i\varphi_k}$ be assigned directions. Suppose there exist $\{b_n\}$, $\{r_n\}$, $\{\theta_{nj}\}$ such that $b_n \leqslant u(r_n e^{i\theta_{nj}})$, $\lim_{n\to\infty} r_n=1$, $\lim_{n\to\infty}\theta_{nj}=\varphi_j$, $j=1, \ldots, k$. Apply the inequality (8′) for mutually disjoint sectors $S_j=S(\varphi_j, \delta_j)$, $j=1, \ldots, k$, and a with $[a, \infty)\subset \bigcap_{j=1}^{k} u(S_j)$:

$$\int_a^{b_n}\frac{dc}{\Theta(c)}\leqslant\frac{1}{k^2}\sum_{j=1}^{k}\int_a^{b_n}\frac{dc}{\Theta_{S_j}(c)}. \tag{46}$$

Application of Lemma 7 to each term in the right-hand side yields

$$\overline{\lim_{n\to\infty}}\left(\int_a^{b_n}\frac{dc}{\Theta(c)}-\frac{1}{\pi k}\log\frac{1}{1-r_n}\right)$$
$$\leqslant\frac{1}{k^2}\sum_{j=1}^{k}\left(\int_a^{u(re^{i\varphi_j})}\frac{dc}{\Theta_{S_j}(c)}-\frac{1}{\pi}\log\frac{1}{1-r}\right)+\frac{1}{k^2}\left(\frac{1}{\pi}\log 2+\frac{\pi}{4}\right). \tag{47}$$

The right-hand side is bounded as (19) shows. Consequently

$$\int_a^{b_n}\frac{dc}{\Theta(c)}\leqslant\frac{1}{\pi k}\log\frac{1}{1-r_n}+O(1) \tag{48}$$

must be satisfied.

We shall say that u with $[a, \infty)\subset u(\Omega)$ *attains maximum simultaneous growth in directions* $e^{i\varphi_1}, \ldots, e^{i\varphi_k}$ if there exist sequences $\{b_n\}$, $\{r_n\}$, $\{\theta_{nj}\}$, such that

$$\begin{gathered} a<b_n\leqslant u(r_n e^{i\theta_{nj}}), \quad n=1,2,\ldots, \\ \lim_{n\to 1} r_n=1, \ \lim_{n\to\infty}\theta_{nj}=\varphi_j, \quad j=1,2,\ldots,k, \\ \lim_{n\to\infty}\left(\mu(a, b_n)-\frac{1}{\pi k}\log\frac{1}{1-r_n}\right)>-\infty. \end{gathered} \tag{49}$$

The condition implies $\lim_{n\to\infty} b_n=\infty$. Accordingly we may make a to satisfy $a\in \bigcap_{j=1}^{k} u(S_j)$ thus $[a, \infty)\subset\bigcap_{j=1}^{k} u(S_j)$. By means of (47), we see that if (49) is satisfied then the condition of Theorem 4 holds for $S=S_j$, $j=1, \ldots, k$, as well as for $S=\Omega$.

29. The inequalities (46), (48), (49) show

$$\frac{1}{k^2}\sum_{j=1}^{k}\mu_{S_j}(a, b_n)-\mu(a, b_n)=O(1).$$

Since $\lim b_n = \infty$ we obtain
$$\|\tilde{\varrho} - \varrho\|_{\Omega(a,\infty)} < \infty \tag{50}$$
with respect to $\tilde{\varrho} = \tilde{\varrho}_{S_1, \dots, S_k}$. We know, by (8),
$$\|\varrho\|_{\Omega(a,b_n)} \leq \|\varrho_{S_1 \cup \dots \cup S_k}\|_{\Omega(a,b_n)} \leq \|\tilde{\varrho}\|_{\Omega(a,b_n)}.$$
Accordingly, by (6) and (10), we see that (50) implies
$$\|\varrho_{S_1 \cup \dots \cup S_k} - \varrho\|_{\Omega(a,\infty)} < \infty \tag{51}$$
and, therefore, by Lemma 5, that the growth of u in directions different from $e^{i\varphi_1}, \dots, e^{i\varphi_k}$ is slower.

The validity of (50) indicates, in turn, simultaneous growth of u in directions $e^{i\varphi_1}, \dots, e^{i\varphi_k}$ in a sense different from (49). Observe, in particular, the validity of (51) and $\Theta_{S_1} = \dots = \Theta_{S_k}$ imply (50).

In general, consider a harmonic function u on $\Omega = \{z \mid r_0 < |z| < 1\}$, directions $e^{i\varphi_j}$, and mutually disjoint sectors $S_j = S_j(\varphi_j, \delta_j)$, $j = 1, \dots, k$. We obtain the following result, which was obtained by Eke [1, Theorems 5, 7] for u with some restrictions:

LEMMA 8. *If u satisfies $[a, \infty) \subset \bigcap_{j=1}^{k} u(S_j)$ and*
$$\|\tilde{\varrho}_{S_1, \dots, S_k} - \varrho\|_{\Omega(a,\infty)} < \infty$$
then the growth of u is

(i) *relatively slow in $\Omega - \bigcup_{j=1}^{k} S_j$ in the sense that*
$$\mu(a, u(z)) = o\left(\left(\log \frac{1}{1-|z|}\right)^{\frac{1}{2}}\right)$$
as $|z| \to 1$, $u(z) > a$ uniformly on any sector whose closure is contained in $\mathrm{Cl}\,\Omega - \bigcup_{j=1}^{k} S_j$, and

(ii) *equally rapid in $S_1, \dots, S_k$ in the sense that, for each $j = 1, \dots, k$,*
$$\mu(a, u(z)) = \frac{1}{k}\,\mu_{S_j}(a, u(z)) + o\left(\left(\log \frac{1}{1-|z|}\right)^{\frac{1}{2}}\right)$$
as $|z| \to 1$, $u(z) > a$, uniformly on Ω.

30. *Proof.* (i) is evident from (51) and Lemma 5. To prove (ii), let r^* be the value in Theorem 1. For a given $\varepsilon > 0$, take b_0 such that $a < b_0$, $m(r^*) \leq b_0$, and
$$\|\tilde{\varrho}\|^2_{\Omega(b_0,\infty)} - \|\varrho\|^2_{\Omega(b_0,\infty)} = \|\tilde{\varrho} - \varrho\|^2_{\Omega(b_0,\infty)} < \left(\frac{\varepsilon}{k}\right)^2,$$

where $\tilde{\varrho} = \tilde{\varrho}_{S_1, \ldots, S_k}$. With respect to an arbitrary $b > b_0$ we obtain the following, where inner products and norms without subscript (with subscript S_j) are those taken on $\Omega(b_0, b)$ (on $\Omega(b_0, b) \cap S_j$, respectively): $\|\varrho_{S_j}\| = k\|\tilde{\varrho}\|_{S_j}$ by definition and $\|\varrho\|^2 = (\varrho, \varrho_{S_j}) = k(\varrho, \tilde{\varrho})_{S_j}$ by (5). Then

$$\frac{1}{k}\left(\mu(b_0, b) - \frac{1}{k}\mu_{S_j}(b_0, b)\right) = \frac{1}{k}\left(\|\varrho\|^2 - \frac{1}{k}\|\varrho_{S_j}\|^2\right) = (\varrho, \tilde{\varrho})_{S_j} - \|\tilde{\varrho}\|^2_{S_j} = -(\varrho, \tilde{\varrho})_{S_j} + \|\varrho\|^2_{S_j} - \|\varrho - \tilde{\varrho}\|^2_{S_j}$$

$$= (\varrho, \varrho - \tilde{\varrho})_{S_j} - \|\varrho - \tilde{\varrho}\|^2_{S_j}.$$

Accordingly

$$\left|\mu(b_0, b) - \frac{1}{k}\mu_{S_j}(b_0, b)\right| \leq k\|\varrho\| \cdot \|\varrho - \tilde{\varrho}\| + k\|\varrho - \tilde{\varrho}\|^2 \leq \varepsilon\left(\|\varrho\| + \frac{\varepsilon}{k}\right).$$

Now if $u(z) > b_0$, we use Theorem 1 to estimate $\|\varrho\|$ for $b = u(z)$, and obtain

$$\left|\mu(a, u(z)) - \frac{1}{k}\mu_{S_j}(a, u(z))\right| \leq \mu(a, b_0) + \frac{1}{k}\mu_{S_j}(a, b_0) + \varepsilon\left(\left(\frac{1}{\pi}\log\frac{1}{1-|z|} + A\right)^{\frac{1}{2}} + \frac{\varepsilon}{k}\right),$$

where A is a constant. If $a < u(z) \leq b_0$, this inequality is trivially satisfied. We conclude that

$$\left|\mu(a, u(z)) - \frac{1}{k}\mu_{S_j}(a, u(z))\right| < \varepsilon\left(\log\frac{1}{1-|z|}\right)^{\frac{1}{2}}$$

if $|z|$ is sufficiently close to 1.

31. On summarizing we obtain

THEOREM 7. *Suppose a harmonic function u on Ω satisfying $[a, \infty) \subset u(\Omega)$ for some a attains maximum simultaneous growth in the directions $e^{i\varphi_1}, \ldots, e^{i\varphi_k}$. Let $S_j = S(\varphi_j, \delta_j)$ be mutually disjoint, and assume $a \in u(S_j)$, $j = 1, \ldots, k$. Then for each j*

(i) $$\lim_{z \to e^{i\varphi_j}}\left(\mu_{S_j}(a, u(z)) - \frac{1}{\pi}\log\frac{1}{|z - e^{i\varphi_j}|}\right) = \alpha_j,$$

$-\infty < \alpha_j < \infty$, uniformly as z approaches $e^{i\varphi_j}$ in a Stolz domain,

(ii) $$\mu(a, u(z)) = \frac{1}{\pi k}\log\frac{1}{|z - e^{i\varphi_j}|} + o\left(\left(\log\frac{1}{1-|z|}\right)^{\frac{1}{2}}\right)$$

uniformly as z approaches $e^{i\varphi_j}$ in a Stolz domain,

(iii) $\mu(a, u(z)) = o\left(\left(\log\frac{1}{1-|z|}\right)^{\frac{1}{2}}\right)$ *uniformly as $|z| \to 1$, $u(z) > a$ in every sector whose closure does not meet the rays* $\arg z = \varphi_j$, $j = 1, \ldots, k$.

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