ON THE LINEAR AND ANGULAR CLUSTER SETS OF FUNCTIONS MEROMORPHIC IN THE UNIT CIRCLE

BY

E. F. COLLINGWOOD

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I. Introduction

1. Let the function $w=f(z)$ be meromorphic in the unit circle $|z|<1$ and denote by $C_\varrho = C_\varrho(f, e^{i\theta})$ *the radial cluster set of* $f(z)$ *at the point* $z=e^{i\theta}$ which is defined as follows: $a \in C_\varrho(f, e^{i\theta})$ if there is a sequence $r_1<r_2<\cdots<r_n<\cdots$, $\lim_{n\to\infty} r_n = 1$, such that $\lim_{n\to\infty} f(r_n e^{i\theta}) = a$. Evidently $C_\varrho(f, e^{i\theta})$ is a closed non-empty set and is either a single point or a continuum. In the former case we say that $f(z)$ has a radial limit at the point $z=e^{i\theta}$.

If we donote by $\varrho(\varphi)$ the chord of the unit circle through a point $z=e^{i\theta}$ and inclined at the angle φ, $-\frac{\pi}{2}<\varphi<\frac{\pi}{2}$, to the radius through $e^{i\theta}$, positive angles being measured to the right of the radius and negative angles to the left, we define *the chordal cluster set* $C_{\varrho(\varphi)}(f, e^{i\theta})$ in a similar way. We say that $a \in C_{\varrho(\varphi)}(f, e^{i\theta})$ if there is a sequence $t_1>t_2>\cdots>t_n>\cdots$, $\lim_{n\to\infty} t_n = 0$, such that $\lim_{n\to\infty} f(e^{i\theta}(1-t_n e^{i\varphi})) = a$. Again, $C_{\varrho(\varphi)}(f, e^{i\theta})$ is either a single point or a continuum, and $C_{\varrho(0)}(f, e^{i\theta})$ is the radial cluster set $C_\varrho(f, e^{i\theta})$.

Similarly, let Δ be an angle in the unit circle formed by two chords passing through $e^{i\theta}$, and define the corresponding *angular cluster set* $C_\Delta(f, e^{i\theta})$. We say that $a \in C_\Delta(f, e^{i\theta})$ if there is a sequence of points $\{z_n\}$ contained in Δ such that $\lim_{n\to\infty} z_n = e^{i\theta}$ and $\lim_{n\to\infty} f(z_n) = a$. Again, $C_\Delta(f, e^{i\theta})$ is a closed non-empty set and is either a single point or a continuum.[1] Further, if $\Delta_1 \subseteq \Delta_2$ then plainly $C_{\Delta_1}(f, e^{i\theta}) \subseteq C_{\Delta_2}(f, e^{i\theta})$.

2. We now define the *outer angular cluster set*[2] of $f(z)$ at the point $z=e^{i\theta}$ as the union

$$C_A(f, e^{i\theta}) = \bigcup_\Delta C_\Delta(f, e^{i\theta})$$

[1] This is the notation introduced in COLLINGWOOD and CARTWRIGHT [5], p. 139. Generally, the notation we use is either taken from that paper, which will be cited hereafter as C-C, or is derived from it in an obvious way. A different convention has been adopted by Japanese authors. (Numbers in brackets refer to the list of references at the end of the present paper.)

The connectivity of C_Δ and other cluster sets on locally connected sets can be established by ad hoc arguments (cf. C-C, pp. 90–92). The following general topological method, which puts the matter in a few lines, I owe to Professor WILFRED KAPLAN. Let D_t be the set of values taken by $f(z)$ in the intersection of Δ with the disc $|e^{i\theta}-z|<t$. Then, writing $\bar{D}_t$ for the closure of D_t, an equivalent definition of $C_\Delta(f, e^{i\theta})$ is $C_\Delta(f, e^{i\theta}) = \bigcap_n \bar{D}_{t_n}$, where $t_n \to 0$ as $n\to\infty$. Since D_t is connected for all t it follows by a well-known theorem (see, for example, HAUSDORFF, Mengenlehre, p. 163, XVIII) that $C_\Delta(f\, e^{i\theta})$ is connected.

[2] GROSS [6] used the term *innere Häufungsbereich* for this set.

taken over all angles Δ between pairs of chords through $z=e^{i\theta}$. $C_A(f, e^{i\theta})$ is a non-empty F_σ.

We can clearly define the cluster set of $f(z)$ at $z=e^{i\theta}$ with respect to any curve in $|z|<1$ ending at this point, or any continuum or any sequence having this point as a frontier or limit point. But these definitions will be introduced as they are required.

The *cluster set* $C(f, e^{i\theta})$ of $f(z)$ at the point $z=e^{i\theta}$ is a familiar notion[1] defined as follows: $a\in C(f, e^{i\theta})$ if there is a sequence of points $\{z_n\}$ contained in the unit circle such that $\lim\limits_{n\to\infty} z_n=e^{i\theta}$ and $\lim\limits_{n\to\infty} f(z_n)=a$. $C(f, e^{i\theta})$ is either a single point or a closed continuum.

We shall say that a cluster set or union of cluster sets is *degenerate* if it consists of a single point; otherwise it is *non-degenerate.*[2] We denote the complements of cluster sets with respect to the closed plane or the sphere on which the plane is projected stereographically by $\complement C_\varrho$, $\complement C_\Delta$, $\complement C_A$, $\cdots$ etc. A cluster set or union of cluster sets whose complement is empty, so that it covers the whole w-plane or the whole w-sphere, we shall call *total*; and one whose complement is not empty we shall call *sub-total,*[3] the degenerate cluster sets being a sub-class of the sub-total cluster sets.

3. The classical Theorem of Fatou[4] states that *if $f(z)$ is regular and bounded in $|z|<1$, then* $\lim f(z)$ *exists uniformly as $z\to e^{i\theta}$ in the angle $|\arg(1-ze^{-i\theta})|\leq\frac{\pi}{2}-\delta$ for all $\delta>0$ and p.p. on the circumference $|z|=1$.* That is to say, $f(z)$ has an (outer) angular limit, and therefore also a radial limit p.p. It was subsequently proved by R. Nevanlinna that the same property holds in the more general case of a meromorphic function $f(z)$ whose characteristic $T(r, f)$ is bounded.

The companion *Theorem of F and M. Riesz* states that *if $f(z)$ is regular and bounded in $|z|<1$ and has the same radial limit a, that is to say, if $C_\varrho(f, e^{i\theta})=a$, for a set of points $e^{i\theta}$ of positive measure on $|z|=1$, then $f(z)\equiv a$.* This theorem was also extended by R. Nevanlinna to meromorphic functions of bounded characteristic.[5]

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[1] Notation of C–C, p. 120.

[2] Cf. WHYBURN, *Analytic Topology*, p. 30.

[3] These notions, like that of the cluster set itself, go back to PAINLEVÉ, *C. R.* 131 (1900), pp. 487–492, who spoke of singularities of a function as being points of complete or incomplete indetermination. We avoid the word "complete" because of its other uses in set theory.

[4] We use the notation *p.p.* to denote *almost every-where*; and mE or $m(E)$ to denote the measure of a set E.

[5] R. NEVANLINNA, *Eindeutige analytische Funktionen* (1936), pp. 190–197. Cited hereafter as E.A.F.

4. These famous theorems have been deepened and extended in a variety of ways and have given rise to what is now an extensive theory. An important generalisation of Fatou's Theorem for an unrestricted meromorphic function is due to Plessner. To state the theorem, and for subsequent developments, it is convenient to introduce some further definitions and technical terms. We shall call a point $z = e^{i\theta}$ *a Fatou point for* $f(z)$ *if* $C_{\Delta}(f, e^{i\theta})$ *is degenerate and if* $\lim f(z)$ *exists uniformly as* $z \to e^{i\theta}$ *in any angle* Δ *between chords through* $e^{i\theta}$; and the set of Fatou points for $f(z)$ on the circumference $|z| = 1$ we shall denote by $F(f)$.[1] We shall call $z = e^{i\theta}$ *a Plessner point for* $f(z)$ *if* $C_{\Delta}(f, e^{i\theta})$ *is total for every angle* Δ *between pairs of chords through* $e^{i\theta}$ *however small the angle may be*; and the set of Plessner points for $f(z)$ on $|z| = 1$ we shall denote by $I(f)$. Another notion that will be useful to us is that of *a Fatou arc for* $f(z)$. This is defined as an arc of the circumference $|z| = 1$ which is an open arc of the frontier of a simply connected Jordan domain in $|z| < 1$ in which either $f(z)$ or, for some $a \neq \infty$, $1/(f(z) - a)$ is bounded. It follows at once from Fatou's Theorem, by conformal mapping, that $F(f)$ *is p.p. on a Fatou arc.*[2]

The theorem of Plessner referred to above is

Plessner's Theorem A.[3] *If* $f(z)$ *is meromorphic in* $|z| < 1$, *then almost all points of* $|z| = 1$ *belong either to* $F(f)$ *or to* $I(f)$.

We note that $I(f) \subseteq \complement F(f)$.[4] Indeed, a Plessner point is in an obvious sense the antithesis of a Fatou point.

For a bounded function, $I(f)$ is empty and the theorem reduces to Fatou's Theorem.

A second theorem of Plessner generalises the Riesz Theorem in a similar way. This theorem is

Plessner's Theorem B.[5] *If* $f(z)$ *is meromorphic in* $|z| < 1$ *and if* $f(z)$ *has the same outer angular limit* a, *that is if* $C_{\Delta}(f, e^{i\theta}) = a$, *for a set of points* $e^{i\theta}$ *of positive measure, then* $f(z) \equiv a$.

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[1] Notation and terminology of C–C, p. 95.

[2] Cf. C–C, p. 98.

[3] PLESSNER [10], p. 220. Plessner states his theorem in the slightly weaker form with the set $\{e^{i\theta}\}$ for which $C_{\Delta}(f, e^{i\theta})$ is degenerate in place of the set $F(f)$ for which the angular limits $C_{\Delta}(f, e^{i\theta})$ are also uniform. His argument, however, actually proves the theorem as we state it here (and as it is stated in C–C). It will be noted that $C_{\Delta}(f, e^{i\theta})$ degenerate does not necessarily imply that $\lim f(z)$ exists uniformly in every Stolz angle Δ at $e^{i\theta}$ as is the case if $e^{i\theta} \in F(f)$.

[4] For sets on the circumference $|z| = 1$ the notation $\complement$ of course denotes the complements with respect to that space.

[5] PLESSNER [10], p. 224. This theorem was first proved for regular functions by PRIVALOFF (see BIEBERBACH, *Funktionentheorie*, vol. II, 2nd ed., p. 158, or LUSIN and PRIVALOFF [8], p. 164).

It has been shown by counter examples[1] that radial limits cannot be substituted for outer angular limits in this theorem. To prove a uniqueness theorem of this kind for radial limits a more stringent condition must be imposed on the set of points $e^{i\theta}$ at which the limit is attained. Lusin and Privaloff were the first to show the significance of sets of category II[2] in this problem. In order to state their theorem we require the following definition. *A set E is said to be metrically dense in (or on) an interval α if, given any sub-interval $\beta \subset \alpha$, the intersection $E \cap \beta$ is of positive measure.* For an open interval this is equivalent to saying that E is metrically dense at every point of the interval.[3]

We can now state the

Theorem of Lusin and Privaloff.[4] *If $f(z)$ is regular in $|z| < 1$ and has the same radial limit a, i.e., if $C_\varrho(f, e^{i\theta}) = a$ for a set $\mathfrak{M}_a(\theta)$ of points $z = e^{i\theta}$, $\mathfrak{M}_a(\theta)$ being both metrically dense and of category* II *on an arc α of $|z| = 1$, then $f(z) \equiv a$.*

More recently, in 1939, F. Wolf[5] proved a related theorem which we state in language of cluster sets as follows:

Wolf's Theorem. *Suppose that $f(z)$ is regular in $|z| < 1$ and that there is a set $\mathfrak{M}(\theta)$ of points $z = e^{i\theta}$, $\mathfrak{M}(\theta)$ being a G_δ dense on an arc α of $|z| = 1$, such that $\infty \in \complement C_\varrho(f, e^{i\theta})$ for all $e^{i\theta} \in \mathfrak{M}(\theta)$. Then if there is a number $a \neq \infty$ such that $a \in C_\varrho(f, e^{i\theta})$ p.p. on α, we have $f(z) \equiv a$.*

5. By combining the ideas in the proofs of these last two theorems we prove a new theorem, no longer restricted to regular functions, which contains them both. This is Theorem 1. of the present paper. In addition to the known theorems of Lusin and Privaloff and of Wolf, which emerge as corollaries, Theorem 1 leads to new results on the coverage of the radial cluster sets of $f(z)$ on a sufficiently extensive set of points of the circumference $|z| = 1$. The typical result is Theorem 2. This in turn leads to the introduction of a new notion, namely that of *the set, which we denote by $S(f)$, of points $z = e^{i\theta}$ at which the radial cluster set $C_\varrho(f, e^{i\theta})$ is total*, and the investigation of its properties.

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[1] Lusin and Privaloff [8], pp. 183–185.

[2] Sets of the first and second categories are called of category I and of category II respectively.

[3] Cf. Hobson, *Theory of Functions of a Real Variable*, vol. I, 2nd ed., pp. 178–179, for the definition of a set metrically dense at a point. A set metrically dense on an interval is called by Lusin and Privaloff (l.c. p. 187) *réduit* on that interval.

[4] Lusin and Privaloff [8], pp. 187–188.

[5] F. Wolf [13].

This set $S(f)$ is the antithesis of the set at which $f(z)$ has a radial limit, which latter is the set that has hitherto been most closely studied; and the relationship between the two sets may be expected to be analogous to that between $I(f)$ and $F(f)$. There is no simple relation between $S(f)$ and $I(f)$, such as inclusion of one in the other. We prove, however, that $S(f)$ and $I(f)$ are topologically equivalent in the sense that they differ only by a set which is of category I on the circumference $|z|=1$ (Theorem 5) and that if either is dense on an arc α of the circumference then both are residual on α (Theorems 3, 6 and 7). We thus obtain existence theorems for $S(f)$ and $I(f)$ in terms of one another and, in virtue of Plessner's Theorem A, in terms also of the set $F(f)$. We can prove, for example (Theorem 9), that for the modular function $\mu(z)$ defined in the circle $|z|<1$, $S(\mu)$ is residual on $|z|=1$.

Although our results are stated for the most part for radial cluster sets our methods are equally applicable to chordal or more general cluster sets, and the basic lemmas are proved in a form sufficiently general to yield these extensions of the main results, the formulations of which are obvious.

By the same methods we prove a somewhat stronger form of a recent theorem of Meier on the distribution of the chords at a given Plessner point on which the chordal cluster sets are total. This result, which is Theorem 10, does not, however, establish the existence of any point $z=e^{i\theta}$ at which the radial cluster set or a chordal cluster set in a given direction is total. It is Theorem 6 that is the existence theorem for such points.

Finally, for completeness, and although it does not accord strictly with the title of this paper, we give a theorem on the distribution of the total linear cluster sets of a uniform function at an isolated essential singularity. This is Theorem 11. It is an obvious analogue of Theorem 10 and is proved in the same way.

II. A general theorem on linear cluster sets

6. We begin by proving as a lemma what amounts to a generalisation to meromorphic functions and to linear, but not necessarily radial, cluster sets of the preliminary part of Wolf's theorem. The proof derives partly from Wolf and partly from Lusin and Privaloff. We shall use $\mathfrak{M}(\theta)$ and $\mathfrak{N}(\theta)$ throughout to denote sets of points $e^{i\theta}$ on the circumference $|z|=1$.

Lemma 1. *Suppose that $f(z)$ is meromorphic in $|z|<1$ and that for some fixed φ, $-\frac{\pi}{2}<\varphi<\frac{\pi}{2}$, and some complex number a, finite or infinite, there is a set $\mathfrak{M}(\theta)$ of* *points* $z=e^{i\theta}$ *of category* II *on a certain arc* α *of the circumference* $|z|=1$ *and such that* $a\in \complement C_{\varrho(\varphi)}(f, e^{i\theta})$ *for all* $e^{i\theta}\in \mathfrak{M}(\theta)$. *Then the arc* α *contains an arc* β *such that* (i) β *is a Fatou arc for* $f(z)$ *in the neighbourhood of which either* $f(z)$ *or* $1/(f(z)-a)$ *is uniformly bounded according as* $a=\infty$ *or* $a\neq\infty$, *and* (ii) $\mathfrak{M}(\theta)$ *is dense on* β.

It is sufficient to prove the lemma for the case $a=\infty$ since otherwise we have only to make a linear transformation of $f(z)$ which carries a into ∞.

Denote by $A_\varphi(T, N, \theta)$ the set of points $e^{i\theta}$, $0\leq\theta\leq 2\pi$, such that, for all values of t in $0<t<T$,

$$|f(e^{i\theta}(1-te^{i\varphi}))|<N.$$

Now take

$$T_1>T_2>\cdots>T_\nu>\cdots,\quad \lim_{\nu\to\infty} T_\nu=0.$$

Then

$$A_\varphi(T_\nu, N, \theta)\subseteq A_\varphi(T_{\nu+1}, N, \theta).$$

Also, for any $\eta>0$,

$$A_\varphi(T, N, \theta)\subseteq A_\varphi(T, N+\eta, \theta)$$

so that if we take $N_1<N_2<\cdots<N_\nu<\cdots$, $\lim N_\nu=\infty$, we have

$$A_\nu=A_\varphi(T_\nu, N_\nu, \theta)\subseteq A_\varphi(T_{\nu+1}, N_{\nu+1}, \theta)=A_{\nu+1},$$

while plainly

$$\mathfrak{M}(\theta)=(\bigcup_\nu A_\nu)\cap\alpha. \tag{1}$$

Since, by hypothesis, $\mathfrak{M}(\theta)$ is of category II in α it follows that at least one of the sets $A_\nu\cap\alpha$, say $A_k\cap\alpha$, is of category II in α. There is therefore an arc $\beta\subseteq\alpha$ such that A_k is dense on β; and, since $A_k\subseteq\mathfrak{M}(\theta)$, the set $\mathfrak{M}(\theta)$ is dense on β.

For $e^{i\theta}\in A_k\cap\beta$ and for all $0<t<T_k$ we have

$$|f(e^{i\theta}(1-te^{i\varphi}))|<N_k; \tag{2}$$

and, since A_k is dense on β, it follows from this that the inequality

$$|f(z)|\leq N_k \tag{3}$$

is satisfied throughout the annular quadrilateral B (not containing the origin) defined by the arc β, the two chords at the end points of β inclined at the angle φ to the respective radii at these points, and the circular arc $|z|=1-T_k\cos\varphi$ joining the two chords. For every interior point of B is, by (2) and the fact that A_k is dense on β, a limit of points at which $|f(z)|<N_k$, so that $f(z)$ can have no poles in B and is therefore continuous at every interior point of B, and (3) is therefore satisfied at every such point. This completes the proof of the lemma.

For $\varphi = 0$ the chord $\varrho(\varphi)$ through $e^{i\theta}$ becomes the radius at this point and $C_{\varrho(0)} = C_\varrho(f, e^{i\theta})$ is the radial cluster set at $z = e^{i\theta}$. Since in the lemma radial and chordal cluster sets are on exactly the same footing we shall as a rule limit the later applications of the results of this section to radial cluster sets, except where the chordal cluster sets are necessarily involved in the statement of a theorem, leaving the corresponding generalisation for chordal cluster sets to be understood.

As a first application of Lemma 1 we obtain our general theorem, namely

Theorem 1. *If $f(z)$ is meromorphic in $|z| < 1$ and if, for a constant φ, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, there is a set $\mathfrak{M}(\theta)$ of points $z = e^{i\theta}$ of category* II *on an arc α of the circumference $|z| = 1$ such that the intersection $\bigcap_{e^{i\theta} \in \mathfrak{M}(\theta)} C\, C_{\varrho(\varphi)}(f, e^{i\theta})$ is not empty and if, further, there is a number b, finite or infinite, and a set $\mathfrak{N}(\theta)$ metrically dense on α such that $b \in \bigcap_{e^{i\theta} \in \mathfrak{N}(\theta)} C_{\varrho(\varphi)}(f, e^{i\theta})$, then $f(z) \equiv b$.*

By Lemma 1, α contains an arc β which is a Fatou arc for $f(z)$. Therefore, by Fatou's theorem, $F(f)$ is p.p. on β and so, since $m(\mathfrak{N}(\theta) \cap \beta) > 0$ by hypothesis, it follows that

$$(m(\mathfrak{N}(\theta) \cap F(f)) \geq m(\mathfrak{N}(\theta) \cap \beta \cap F(f)) > 0.$$

Now, for $e^{i\theta} \in F(f)$ we have $C_A(f, e^{i\theta}) = C_{\varrho(\varphi)}(f, e^{i\theta})$ and, by hypothesis, $C_{\varrho(\varphi)}(f, e^{i\theta}) = b$ for $e^{i\theta} \in \mathfrak{N}(\theta) \cap F(f)$. Since $m(\mathfrak{N}(\theta) \cap F(f)) > 0$ it now follows from Plessner's Theorem B that $f(z) \equiv b$ and the theorem is proved.

Corollary 1. *If $f(z)$ is meromorphic in $|z| < 1$ and there is a number b and a set $\mathfrak{M}_b(\theta)$ which is both metrically dense and of category* II *on an arc α of the circumference $|z| = 1$, such that the radial cluster set $C_\varrho(f, e^{i\theta}) = b$ for all $e^{i\theta} \in \mathfrak{M}_b(b)$, then $f(z) \equiv b$.*

This is immediate. We have only to put

$$\mathfrak{M}(\theta) = \mathfrak{N}(\theta) = \mathfrak{M}_b(\theta).$$

This corollary extends to meromorphic functions the theorem of Lusin and Privaloff quoted in the Introduction (§ 4 above). These authors also constructed counter examples which show that their theorem is best possible in the sense that, given the condition that $\mathfrak{M}_b(\theta)$ is of category II on α, the condition that it must also be metrically dense on α cannot be dispensed with, and conversely.[1] Theorem 1 must therefore also be best possible in a similar sense.

[1] LUSIN and PRIVALOFF [8], pp. 185–186.

7. We conclude this section with a remark on the extension of Lemma 1 and its consequenses to more general linear cluster sets. Consider a curve λ in $|z|<1$ tending to the circumference $|z|=1$. The "end" of λ may be either a point or an arc of $|z|=1$ or, as in the case of a spiral, the whole circumference. For simplicity we may assume that λ has only one point of intersection with every circle $|z|=r<1$ so that, on λ, $|z|$ is strictly increasing as z tends to the circumference $|z|=1$. To define the orientation of λ we may take the argument θ_0 of any point z_0 on λ other than the origin. By rotation about the origin through an angle $\theta-\theta_0$ we obtain the family $\{\lambda(\theta)\}$ of rotational transforms of $\lambda=\lambda(\theta_0)$, the original curve. Evidently, as θ sweeps out an interval the curve $\lambda(\theta)$ sweeps out a domain in $|z|<1$ whose frontier includes an arc of the circumference $|z|=1$ which may be wholly or partly inaccessible from the domain. The cluster set $C_{\lambda(\theta)}(f)$ of $f(z)$ on a curve $\lambda(\theta)$ is defined as follows: $a\in C_{\lambda(\theta)}(f)$ if there is a sequence of points $\{z_n\}$ on $\lambda(\theta)$ such that $\lim\limits_{n\to\infty}|z_n|=1$ and $\lim\limits_{n\to\infty} f(z_n)=a$. $C_{\lambda(\theta)}(f)$ is again a non-empty closed set which is a continuum if it is non-degenerate. For a given family of curves $\{\lambda(\theta)\}$ the result corresponding to Lemma 1 is

Lemma 1 a. *Suppose that $f(z)$ is meromorphic in $|z|<1$ and that, for a given family of curves $\{\lambda(\theta)\}$, which are rotational transforms of one another and on which $|z|$ is strictly increasing as z tends to the circumference $|z|=1$, there is a complex number a, finite or infinite, and a set $\mathfrak{C}(\theta)$ of points θ of category* II *in an interval $\alpha\subseteq(0,2\pi)$ such that $a\in\complement C_{\lambda(\theta)}(f)$ for all $\theta\in\mathfrak{C}(\theta)$. Then the interval α contains an interval β such that* (i), *according as $a=\infty$ or $a\neq\infty$, either $f(z)$ or $1/(f(z)-a)$ is uniformly bounded in that part of the domain swept out by $\lambda(\theta)$ as θ sweeps out the interval β which lies in a certain annulus $r_0<|z|<1$, and* (ii) *$\mathfrak{C}(\theta)$ is dense in β.*

Again, there is no loss of generality in taking $a=\infty$. For a given value of θ every point of $\lambda(\theta)$ is determined by $|z|$, $z\in\lambda(\theta)$. Denote by $L(T,N,\theta)$ the set of values of θ, $0\leq\theta\leq2\pi$, such that, for all values of $z\in\lambda(\theta)$, where $1-T<|z|<1$, the inequality $|f(z)|<N$ is satisfied. Taking $T_1>T_2>\cdots>T_\nu>\cdots$, $\lim\limits_{\nu\to\infty}T_\nu=0$, and $N_1<N_2<\cdots<N_\nu<\cdots$, $\lim\limits_{\nu\to\infty}N_\nu=\infty$, we have, as before,

$$L_\nu=L(T_\nu,N_\nu,\theta)\subseteq L(T_{\nu+1},N_{\nu+1},\theta)=L_{\nu+1}$$

and

$$\mathfrak{C}(\theta)=(\bigcup_\nu L_\nu)\cap\alpha.$$

Since $\mathfrak{C}(\theta)$ is, by hypothesis, of category II in α, at least one of the sets $L_k\cap\alpha$ is of category II in α and is therefore dense in a sub-interval $\beta \subseteq \alpha$. The proof now proceeds exactly as for Lemma 1 and need not be repeated.

The result can easily be generalized by relaxing the restriction that $|z|$, $z \in \lambda(\theta)$, is one valued for a given θ.

III. A theorem on the coverage of the radial cluster sets

8. As a further consequence of Theorem 1 we prove

Theorem 2. *If $f(z)$ is meromorphic in $|z| < 1$ and if there is a set $\mathfrak{N}(\theta)$ of points $z = e^{i\theta}$ metrically dense on an arc α of the circumference $|z| = 1$ such that $\bigcap_{\mathfrak{N}(\theta)} C_\varrho(f, e^{i\theta})$ is not empty, then either given any set $\mathfrak{M}(\theta)$ of category* II *on α, the union $\bigcup_{\mathfrak{M}(\theta)} C_\varrho(f, e^{i\theta})$ is total, or $f(z)$ is a constant.*

Suppose that $\bigcup_{\mathfrak{M}(\theta)} C_\varrho(f, e^{i\theta})$ is sub-total so that its complement $\bigcap_{\mathfrak{M}(\theta)} \complement C_\varrho(f, e^{i\theta})$ is not empty. Since $\mathfrak{M}(\theta)$ is of category II on α the condition of Theorem 1 is satisfied and it follows that if there is a number $b \in \bigcap_{\mathfrak{N}(\theta)} C_\varrho(f, e^{i\theta})$, then $f(z) \equiv b$. This proves the theorem.

We denote by $W(f)$ the set of (Weierstrass) points $z = e^{i\theta}$ for $f(z)$, i.e. the points $e^{i\theta}$ for which $C(f, e^{i\theta})$ is total.[1] Theorem 2 supplements the following quite trivial observation, namely, that *if, for a set $\mathfrak{N}(\theta)$ metrically dense on α, $\bigcap_{\mathfrak{N}(\theta)} C_\varrho(f, e^{i\theta})$ is not empty, then $\alpha \subseteq W(f)$.* For if $e^{i\theta} \in \complement W(f)$ so that $\complement C(f, e^{i\theta})$ is not empty, then $e^{i\theta}$ is contained in a Fatou arc β; and, for $e^{i\theta} \in \mathfrak{N}(\theta)$, we can take $\beta \subseteq \alpha$. But,by Plessner's Theorem B, we must have $m(\mathfrak{N}(\theta) \cap \beta) = 0$; for otherwise there is a subset of $F(f) \cap \beta$ of positive measure in which the angular limit of $f(z)$ is constant, so that $f(z)$ is a constant. This contradicts the hypothesis that $\mathfrak{N}(\theta)$ is metrically dense on α, and our assertion is proved.

An immediate corollary of Theorem 2 is

Corollary 2. *If $f(z)$ is meromorphic in $|z| < 1$ and if $a \in C_\varrho(f, e^{i\theta})$ for all $e^{i\theta} \in \mathfrak{N}(\theta)$, where $\mathfrak{N}(\theta)$ is metrically dense on an arc α of $|z| = 1$, then either $\complement \mathfrak{N}(\theta)$ is residual on α and $\bigcup_{\complement \mathfrak{N}(\theta)} C_\varrho(f, e^{i\theta})$ is total, or $f(z) \equiv a$.*

For, if $\mathfrak{N}(\theta)$ is of category II on α, then $f(z) \equiv a$, by Corollary 1.

This corollary is illustrated by a number of known examples, as for instance

[1] C–C, p. 137.

Koenigs' function $K(z)$[1], a function $\omega(z)$ constructed by Lusin and Privaloff[2] and a function $f(z)$ recently investigated by Bagemihl, Erdös and Seidel.[3] In all these cases there is a radial limit $a = C_\varrho(f, e^{i\theta})$ in a set $\mathfrak{N}(\theta)$ which is p.p. on the circumference; and it is easily verified that the complementary set $\complement\, \mathfrak{N}(\theta)$ is in each case residual on the circumference.

IV. A property of the set $I(f)$

9. We shall prove that if the set $I(f)$ of Plessner points of $f(z)$ is dense on an arc α then it is also residual on that arc. To do this we require two further lemmas to supplement Lemma 1. First we prove

Lemma 2. *If $f(z)$ is meromorphic in $|z| < 1$ and if, for some fixed φ, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, there is a set $\mathfrak{M}(\theta)$ of category* II *on an arc α of the circumference $|z| = 1$ such that $C_{\varrho(\varphi)}(f, e^{i\theta})$ is sub-total for all $e^{i\theta} \in \mathfrak{M}(\theta)$, then there is a sub-set $\mathfrak{M}_0(\theta) \subseteq \mathfrak{M}(\theta)$, also of category* II *on α, such that $\bigcap_{\mathfrak{M}_0\theta} \complement\, C_{\varrho(\varphi)}(f, e^{i\theta})$ is not empty.*

If $C_{\varrho(\varphi)}(f, e^{i\theta})$ is sub-total we can find a circle contained in $\complement\, C_{\varrho(\varphi)}(f, e^{i\theta})$, since this is an open set.

Now suppose the w-plane to be projected stereographically onto the unit w-sphere so that a circle in the plane is projected onto a circle on the sphere. Adapting Plessner's method, we construct on the sphere a sequence of finite triangular lattices $l_1, l_2, \dots l_n, \dots$ etc., each lattice being a sub-division of its predecessor and such that the length of the longest side of the lattice l_n is less than 2^{-n}, say. We denote the individual triangles in the lattice l_n by $d_{n,1}, d_{n,2}, \dots d_{n,m(n)}$. Denote by $\Gamma_n(\theta)$ the set of points $e^{i\theta} \in \mathfrak{M}(\theta)$ such that n is the smallest number for which $\complement\, C_{\varrho(\varphi)}(f, e^{i\theta})$ contains the whole of at least one of the triangles $d_{n,\nu}$, $1 \le \nu \le m(n)$. Then evidently $\Gamma_1(\theta) \subseteq \Gamma_2(\theta) \subseteq \cdots \subseteq \Gamma_n(\theta) \subseteq \cdots$ and

$$\mathfrak{M}(\theta) = \bigcup_n \Gamma_n(\theta).$$

We now sub-divide the sets $\Gamma_n(\theta)$ as follows. We assign to each triangle

$$d_{n,1}, d_{n,2}, \dots d_{n,m(n)}$$

all those values of $e^{i\theta} \in \Gamma_n(\theta)$ for which $d_{n,\nu}$, $1 \le \nu \le m(n)$, is contained in $\complement\, C_{\varrho(\varphi)}(f, e^{i\theta})$,

[1] C–C, p. 94.

[2] Lusin and Privaloff [8], p. 189.

[3] Bagemihl, Erdös and Seidel [1], p. 139; also J. Wolff [14] and [15].

and we denote the corresponding sub-sets of $\Gamma_n(\theta)$ by $\mathfrak{D}_{n,\nu}(\theta)$, $1 \le \nu \le m(n)$. All but one of these sub-sets $\mathfrak{D}_{n,\nu}(\theta)$ may be empty, but at least one of them must be non-empty, and any two of them which are non-empty may have common points. With these definitions we evidently have

$$\mathfrak{M}(\theta) = \bigcup_n \bigcup_{1 \le \nu \le m(n)} \mathfrak{D}_{n,\nu}(\theta). \tag{4}$$

Since $\mathfrak{M}(\theta)$ is of category II on α one at least of the enumerable set of sets $\mathfrak{D}_{n,\nu}(\theta)$, $n < \infty$, $\nu \le m(n)$, say $\mathfrak{D}_{j,k}(\theta)$, must be of category II on α. But, for all $e^{i\theta} \in \mathfrak{D}_{j,k}(\theta)$, the triangle $d_{j,k}$ is contained in $\complement C_{\varrho(\varphi)}(f, e^{i\theta})$ and so, putting $\mathfrak{M}_0(\theta) = \mathfrak{D}_{j,k}(\theta)$, the lemma is proved.

10. Secondly, we prove

Lemma 3. *If $f(z)$ is meromorphic in $|z| < 1$ and if there is a set $\mathfrak{M}(\theta)$ of points $z = e^{i\theta}$ satisfying the conditions* (a) *that $\mathfrak{M}(\theta) \subseteq \complement I(f)$, and* (b) *that $\mathfrak{M}(\theta)$ is of category* II *on an arc α of the circumference $|z| = 1$, then there is an arc $\beta \subseteq \alpha$ such that* (i) *β is a Fatou arc for $f(z)$, and* (ii) *$\mathfrak{M}(\theta)$ is dense on β.*

Every point $e^{i\theta} \in \complement I(f)$ is the vertex of an angle $\Delta(\theta)$ in $|z| < 1$ in which the angular cluster set $C_{\Delta(\theta)}(f, e^{i\theta})$ is sub-total. Now denote by E_1 the subset of $\mathfrak{M}(\theta)$ at each point $e^{i\theta}$ of which there is a $\Delta(\theta)$ of magnitude greater than $\pi/2$ such that $C_{\Delta(\theta)}(f, e^{i\theta})$ is sub-total, by E_2 the subset of $\mathfrak{M}(\theta)$ at which there is a $\Delta(\theta)$ of magnitude greater than $\pi/4$, by E_n the subset of $\mathfrak{M}(\theta)$ at which there is a $\Delta(\theta)$ of magnitude greater than $\pi/2^n$ in which $C_{\Delta(\theta)}(f, e^{i\theta})$ is sub-total, and so on for indefinitely increasing n. Evidently $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \cdots$, and

$$\mathfrak{M}(\theta) = \bigcup_n E_n.$$

We now proceed to sub-divide each set E_n into a finite set of subsets as follows. Divide the angle $-\frac{\pi}{2}(1 - 2^{-n}) < \varphi < \frac{\pi}{2}(1 - 2^{-n})$ on the interior normal side of the tangent to the unit circle at $e^{i\theta}$ into $N = 2^n - 1$ equal parts of magnitude $\pi/2^n$ by drawing the chords $\varrho_1 = \varrho(\varphi_1)$, $\varrho_2 = \varrho(\varphi_2)$, $\dots \varrho_N = \varrho(\varphi_N)$ through $e^{i\theta}$. Then every angle $\Delta(\theta)$ of magnitude greater than $\pi/2^n$ and contained in $|z| < 1$ must contain at least one of the chords $\varrho_1, \varrho_2, \dots \varrho_N$. For otherwise it must contain one of the half-tangents at $e^{i\theta}$, contrary to the definition of $\Delta(\theta)$. We denote by $E_{n,\nu}$, $1 \le \nu \le N$, the subset of E_n for which there is a $\Delta(\theta)$ of magnitude greater than $\pi/2^n$ such that $C_{\Delta(\theta)}(f, e^{i\theta})$ is sub-total and which contains ϱ_ν. Clearly, a point $e^{i\theta}$ may belong to more than one of the sets $E_{n,\nu}$. With these definitions we have

$$\mathfrak{M}(\theta) = \bigcup_n \bigcup_{1 \leq \nu \leq N} E_{n,\nu}. \tag{5}$$

Since $\mathfrak{M}(\theta)$ is of category II on α one at least of the sets $E_{n,\nu}$, say $E_{j,k}$, must be of category II on α. Consequently, for all $e^{i\theta} \in E_{j,k}$, $C_{\varrho_k}(f, e^{i\theta})$, which is contained in a sub-total set $C_{\Delta(\theta)}(f, e^{i\theta})$, is sub-total; and it follows from Lemma 2 that there is a subset $\mathfrak{M}_0(\theta) \subseteq E_{j,k} \subseteq \mathfrak{M}(\theta)$, also of category II on α, such that $\bigcap_{\mathfrak{M}_0(\theta)} \complement C_{\varrho_k}(f, e^{i\theta})$ is not empty. It now follows from Lemma 1 that there is an arc $\beta \subseteq \alpha$ such that (i) β is a Fatou arc for $f(z)$, and (ii) $\mathfrak{M}_0(\theta)$, and therefore also $\mathfrak{M}(\theta)$, is dense on β. The lemma is therefore proved.

11. We are now in a position to prove

Theorem 3. *If $f(z)$ is meromorphic in $|z| < 1$ and the set $I(f)$ of Plessner points of $f(z)$ is dense on an arc α of the circumference $|z| = 1$, then $I(f)$ is also residual on α.*

Suppose that $\complement I(f)$ is of category II on α. Then, putting $\mathfrak{M}(\theta) = \complement I(f)$ and applying Lemma 3, it follows that there is a Fatou arc $\beta \subseteq \alpha$. But, since evidently $\beta \subseteq \complement I(f)$, this implies that $I(f)$ is not dense on α, and the theorem is proved.

As a further consequence of the argument we have

Theorem 4. *If $f(z)$ is meromorphic in $|z| < 1$ and if on an arc α of the circumference $|z| = 1$, $m(F(f) \cap \alpha) = 0$, then $F(f)$ is of category I on α.*

For $F(f) \subseteq \complement I(f)$ so that $\complement I(f) \cap \alpha$ is of category II if $F(f) \cap \alpha$ is of category II; and this implies that α contains a Fatou arc β. Since $m(F(f) \cap \alpha) \geq m(F(f) \cap \beta) > 0$ this proves the theorem.

V. The set $S(f)$

12. The set $S(f)$ is defined as the set of points $z = e^{i\theta}$ for each of which the radial cluster set $C_\varrho(f, e^{i\theta})$ of $f(z)$ is total. The definition can obviously be extended to chordal or more general linear cluster sets, as for example the set $S_{\varrho(\varphi)}$ of points $e^{i\theta}$ for each of which $C_{\varrho(\varphi)}(f, e^{i\theta})$ is total, or the set $S_{\lambda(\theta)}(f)$ on which $C_{\lambda(\theta)}(f, e^{i\theta})$ is total. We shall, however, confine the detailed discussion to the radial case $S(f)$ since it will be obvious that our theorems are equally applicable to $S_{\varrho(\varphi)}(f)$ and, under appropriate limitations on $\lambda(\theta)$, to $S_{\lambda(\theta)}(f)$ also. We first prove

Theorem 5. *If $f(z)$ is meromorphic in $|z|<1$, then the sets $S(f)$ and $I(f)$ differ by a set of category* I *on the circumference $|z|=1$.*

We prove first that $S(f) \cap \complement I(f)$ is of category I. To do this, put $\mathfrak{M}(0) = S(f) \cap \complement I(f)$ and apply Lemma 3 under the hypothesis that $\mathfrak{M}(0)$ is of category II. This implies the existence of a Fatou arc β on which $\mathfrak{M}(\theta)$ is dense. But evidently no point of β can belong to $S(f)$ so that $\beta \subseteq \complement S(f)$ and $\mathfrak{M}(0) \cap \beta$ is empty. This contradiction proves our assertion.

Similarly, we prove that $\complement S(f) \cap I(f)$ is of category I. Put $\mathfrak{M}(0) = \complement S(f) \cap I(f)$ and apply Lemma 2 under the hypothesis that $\mathfrak{M}(0)$ is of category II. This implies the existence of a subset $\mathfrak{M}_0(0) \subseteq \mathfrak{M}(0)$ such that $\bigcap_{\mathfrak{M}_0(0)} \complement C_\varrho(f, e^{i\theta})$ is not empty. Now, applying Lemma 1, it follows that there is a Fatou arc β on which $\mathfrak{M}_0(0)$, and therefore also $\mathfrak{M}(0)$, is dense. But evidently no point of β can belong to $I(f)$, so that $\beta \subseteq \complement I(f)$ and we again have a contradiction, since $\mathfrak{M}(0) \cap \beta$ must be empty. The theorem is therefore proved.

13. As an immediate deduction from Theorems 3 and 5 we may state

Theorem 6. *If $f(z)$ is meromorphic in $|z|<1$ and the set $I(f)$ is dense on an arc α of the circumference $|z|=1$, then $S(f) \cap I(f)$ is residual on α.*

Corollary 6.1. *If $m(F(f) \cap \alpha) = 0$, then $S(f)$ is residual on α.*

For, by Plessner's Theorem A, $I(f)$ is dense on α if $m(F(f) \cap \alpha) = 0$.

Corollary 6.2. *If $I(f)$ is dense on α, then $F(f)$ is of category* I *on α.*

For $S(f)$ is residual on α and $S(f) \subseteq \complement F(f)$ so that $\complement F(f)$ is residual on α. This is also a corollary of Theorem 3, since $I(f) \subseteq \complement F(f)$.

14. Theorem 5 shows the sets $S(f)$ and $I(f)$ to be topologically equivalent. They are not, however, metrically equivalent, as can be shown by known examples. For instance, in the case of the function $\omega(z)$ of Lusin and Privaloff referred to in § 8 above, $\omega(z)$ takes the radial limit zero p.p., from which it follows that $\complement S(\omega)$ is p.p. But $m F(\omega) = 0$, for otherwise we should have $C_A(\omega, e^{i\theta}) = 0$ in a set of points $e^{i\theta}$ of positive measure which, by Plessner's Theorem B, would imply $\omega(z) \equiv 0$. Therefore $I(\omega)$ is p.p. and hence, by Theorem 6, $S(\omega)$ is residual and, by Theorem 5, $I(\omega)$ is also residual. Thus in this example $m I(\omega) = 2\pi$ and $m S(\omega) = 0$ while $I(\omega)$ and $S(\omega)$ are both residual sets.

VI. Relations between the sets $F(f)$, $I(f)$ and $S(f)$

15. As counterpart to Theorem 6 we have

Theorem 7. *If $f(z)$ is meromorphic in $|z|<1$ and the set $S(f)$ is dense on an arc α of the circumference $|z|=1$, then $S(f)$ is residual on α and consequently $S(f) \cap I(f)$ is also residual on α.*

For, if $S(f)$ is not residual on α then $\complement S(f) \cap \alpha$ is of category II and it follows from Lemmas 1 and 2 that there is a Fatou arc $\beta \subseteq \alpha$. Since, however, $\beta \subseteq \complement S(f)$ it follows from this that $S(f)$ it not dense and we have a contradiction. That $S(f) \cap I(f)$ is residual follows from Theorem 5.

This argument also proves that *if an arc α is contained in $W(f)$, then $S(f) \cap I(f)$ is residual on α.* For $\beta \subseteq \complement W(f)$ so that $\complement W(f) \cap \alpha$ is not empty if $\complement S(f) \cap \alpha$ is of category II.

Corollary 7. *If $S(f)$ is dense on α, then $F(f)$ is of category* I *on α.*

For $I(f)$ is dense and the conclusion follows immediately from Corollary 6.2.

In the same order of ideas we prove

Theorem 8. *If $f(z)$ is meromorphic in $|z|<1$ and if the set $I(f)$ (or the set $S(f)$) is of category* I *on an arc α, then $F(f)$ is metrically dense on α.*

For, $\complement I(f) \cap \alpha$ is residual and hence, by Lemma 3, every arc $\alpha' \subseteq \alpha$ contains a Fatou arc β'; and $m(F(f) \cap \beta') > 0$. Hence $m(F(f) \cap \alpha') \geq m(F(f) \cap \beta') > 0$, and since α' is an arbitrary arc in α the theorem is proved.

Another metrical relation may be noted. If $m(S(f) \cap \alpha) = m\,\alpha$ it follows, since $S(f) \subseteq \complement F(f)$, that $m(F(f) \cap \alpha) = 0$ and hence that $m(I(f) \cap \alpha) = m\,\alpha$ and $I(f)$, and also therefore $S(f)$, is residual on α. On the other hand, as we saw in the example of the function $\omega(z)$, the condition $m(I(f) \cap \alpha) = m\,\alpha$, while it implies that $I(f)$ and $S(f)$ are residual on α does not imply that $m(S(f) \cap \alpha) > 0$. In general terms we may say that although $S(f)$ and $I(f)$ are topologically equivalent sets $S(f)$ may be smaller than $I(f)$ by a set of measure $m\,I(f)$. There is no simple relation, such as inclusion, between the sets $I(f)$ and $S(f)$ and no simple metrical relation between them. But Theorems 3–8 and their corollaries have led to a number of relations involving some metrical element which may be tabulated as follows:

For any function $f(z)$ meromorphic in $|z|<1$ and any arc α of the circumference $|z|=1$,

$$\left.\begin{array}{c} m\,(F\,(f)\cap\alpha)=0 \\ \simeq \\ m\,(I\,(f)\cap\alpha)=m\,\alpha \end{array}\right\} \Rightarrow \left\{\begin{array}{l} F\,(f)\cap\alpha \text{ is of category I} \\ I\,(f)\cap\alpha \text{ is residual} \\ S\,(f)\cap\alpha \text{ is residual} \end{array}\right. \tag{6 a}$$

$$m\,(S\,(f)\cap\alpha)=m\,\alpha \;\Rightarrow\; m\,(I\,(f)\cap\alpha)=m\,\alpha \tag{6 b}$$

$$\left.\begin{array}{c} I\,(f)\cap\alpha \text{ of category I} \\ \simeq \\ S\,(f)\cap\alpha \text{ of category I} \end{array}\right\} \Rightarrow F\,(f) \text{ is metrically dense on } \alpha. \tag{6 c}$$

VII. A theorem on the modular function

16. Since a set of category II is not empty Theorems 6 and 7 are existence theorems for the sets $S\,(f)$ and $I\,(f)$ under appropriate conditions on the set $F\,(f)$. A particular case of interest is that of the modular function $\mu\,(z)$ defined in the unit circle $|z|<1$. It is known that the enumerable set of vertices of the modular figure on the circumference $|z|=1$ are all Fatou points[1]; and it is easily shown that every other point of the circumference is spanned by an infinity of copies of all three sides of the fundamental triangle so that, for these points $e^{i\theta}$, $C_\varrho\,(\mu, e^{i\theta})$ is non-degenerate and hence $e^{i\theta}\in \complement F\,(\mu)$. It follows that $F\,(\mu)$ is enumerable, so that $I\,(\mu)$ is p.p. by Plessner's Theorem A, and hence, by Theorems 5 and 6, both $I\,(\mu)$ and $S\,(\mu)$ are residual on the circumference $|z|=1$. We have thus proved

Theorem 9. *For the modular function $\mu\,(z)$ regular in $|z|<1$ the set $S\,(\mu)$ of points $z=e^{i\theta}$ at which the radial cluster set $C'_\varrho\,(\mu, e^{i\theta})$ is total is a residual set on $|z|=1$; and the set $I\,(\mu)$ of Plessner points, which is p.p. on $|z|=1$, is also residual.*

Recurring to a previous remark (§ 12 above), the theorem is also true for the sets $S_{\varrho(\varphi)}\,(\mu)$, $-\dfrac{\pi}{2}<\varphi<\dfrac{\pi}{2}$, and $S_{\lambda(\theta)}\,(\mu)$ under suitable restrictions on the curve $\lambda\,(\theta)$.[2]

It is known that $F\,(\mu)$ is dense[3] and $I\,(\mu)$ is p.p. on $|z|=1$. We now see that $S\,(\mu)$, being residual, is also dense on $|z|=1$.

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[1] See, for example, CARATHEODORY [4], p. 275.

[2] For example, if λ has an end point on and is non-tangential to $|z|=1$ while $|z|$ is strictly increasing on λ.

[3] C–C, p. 140.

Other special cases for which $S(f)$, and therefore also $I(f)$, are residual sets are those functions for which $F(f)$ is empty, such as the unbounded regular functions which are bounded on a spiral or on a sequence of closed curves enclosing the origin and converging to the circumference $|z|=1$.[1]

VIII. Behaviour in the set of Plessner points

17. We consider the chordal cluster sets $C_{\varrho(\varphi)}(f, e^{i\theta})$ of $f(z)$ at a given point $e^{i\theta}\in I(f)$. It has recently been proved by Meier[2] that the set $S(f, \theta)$ of values of φ for which $C_{\varrho(\varphi)}(f, e^{i\theta})$ is total is of category II in the open interval $-\frac{\pi}{2}<\varphi<\frac{\pi}{2}$. Our method is immediately applicable to this problem and gives the stronger result that $S(f, \theta)$ is in fact a residual set in the interval. We prove

Theorem 10. *If $f(z)$ is meromorphic in $|z|<1$ and $e^{i\theta}\in I(f)$, then the set $S(f, \theta)$ of values of φ for which $C_{\varrho(\varphi)}(f, e^{i\theta})$ is total is a residual set in $-\frac{\pi}{2}<\varphi<\frac{\pi}{2}$.*

If, for a given value of φ, $C_{\varrho(\varphi)}(f, e^{i\theta})$ is sub-total we can find a circle contained in $\complement C_{\varrho(\varphi)}(f, e^{i\theta})$. We again project the w-plane onto the w-sphere and proceed exactly as in the proof of Lemma 2. Using the same sequence of triangular lattices $l_1, l_2, \dots l_n, \dots, n\to\infty$, as in that argument $\Gamma_n(\varphi)$ now denotes the set of values of φ in $-\frac{\pi}{2}<\varphi<\frac{\pi}{2}$ such that n is the smallest number for which $\complement C_{\varrho(\varphi)}(f, e^{i\theta})$ contains the whole of at least one of the triangles $d_{n,\nu}$, $1\le\nu\le m(n)$. We assign to each triangle $d_{n,\nu}$, $1\le\nu\le m(n)$ those values of $\varphi\in\Gamma_n(\varphi)$ for which $d_{n,\nu}\subseteq\complement C_{\varrho(\varphi)}(f, e^{i\theta})$. Again, all but one of these sub-sets of $\Gamma_n(\varphi)$, which we denote by $\mathfrak{D}_{n,\nu}(\varphi)$, may be empty, but at least one of them must be non-empty and any two of them may have common points. We thus obtain the decomposition

$$\complement S(f, \theta)=\bigcup_{n}\bigcup_{1\le\nu\le m(n)}\mathfrak{D}_{n,\nu}(\varphi). \tag{7}$$

If one of the sets $\mathfrak{D}_{n,\nu}(\varphi)$, say $\mathfrak{D}_{j,k}(\varphi)$, is of category II it follows by an argument similar to the proof of Lemma 1, which it is not necessary to repeat, that there is an angle η contained in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $d_{j,k}\subseteq\complement C_\eta(f, e^{i\theta})$ so that $C_\eta(f, e^{i\theta})$

[1] Lusin and Privaloff [8], pp. 147–150; Bagemihl, Erdös and Seidel [1], p. 144; and Valiron [11] and [12], pp. 430–432. Another example is given by Bieberbach, l.c., pp. 152–155.

[2] Meier [9], p. 241.

is sub-total and consequently $e^{i\theta} \in \complement I(f)$. Therefore, if $e^{i\theta} \in I(f)$ every set $\mathfrak{D}_{n,\nu}(\varphi)$ must be of category I in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and the theorem follows from (7).

18. For any given value of φ, Theorem 10 does not of course establish the existence of any point $z = e^{i\theta}$ at which the corresponding chordal cluster set $C_{\varrho(\varphi)}(f, e^{i\theta})$ of $f(z)$ is total. But we have such an existence theorem in Theorem 6, which, as we have already pointed out, may also be proved for any of the sets $S_{\varrho(\varphi)}(f)$ and certain of the sets $S_{\lambda(\theta)}(f)$. This leads at once to the following generalisation of Theorem 6. Let $\varphi_1, \varphi_2, \dots \varphi_n, \dots,$ be any enumerable set of values in the interval $-\frac{\pi}{2} < \varphi_n < \frac{\pi}{2}$ and denote by $S_{\Sigma\varrho(\varphi_n)}(f)$ the set $\bigcap\limits_n S_{\varrho(\varphi_n)}(f)$ so that $S_{\Sigma\varrho(\varphi_n)}(f)$ is the set of points $z = e^{i\theta}$ at each of which all the chordal cluster sets $C_{\varrho(\varphi_n)}(f, e^{i\theta})$ are total. We then have

Theorem 11. *If $f(z)$ is meromorphic in $|z| < 1$ and if either $I(f)$ or a set $S_{\varrho(\varphi)}(f)$ for some φ in the open interval $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$ is dense on an arc α of the circumference $|z| = 1$, then, given any enumerable set $\varphi_1, \varphi_2, \dots \varphi_n, \dots,$ where $-\frac{\pi}{2} < \varphi_n < \frac{\pi}{2}$, $n = 1, 2, \dots,$ the intersection*

$$S_{\Sigma\varrho(\varphi_n)}(f) \cap I(f) \tag{8}$$

is residual on α.

For by Theorems 6 and 7, both of which are valid with $S_{\varrho(\varphi)}(f)$ in place of $S(f)$, $I(f)$ is residual on α. Hence, by Theorem 6, $S_{\varrho(\varphi_n)}(f)$ is residual on α for every value of n and it follows that the complement $(\bigcup\limits_n \complement S_{\varrho(\varphi_n)}(f)) \cup \complement I(f)$ of (8) being the union of an enumerable set of sets of category I on α, is of category I on α. This proves the theorem.

Some years ago it was proved by Kierst and Szpilrajn[1] for regular functions that in general, i.e. *in a residual sub-space of the space of functions regular in $|z| < 1$, the cluster set $C_{\varrho(\varphi)}(f, e^{i\theta})$ is total for all values of θ and φ.* More generally, these authors proved that for a class of curves $\lambda(\theta)$ the cluster set $C_{\lambda(\theta)}(f)$ is total for all values of θ and in a residual sub-space of the space of regular functions.[2] What Theorem 6

[1] Kierst and Szpilrajn [7], p. 291.

[2] Although Kierst and Szpilrajn only discuss regular functions in the circle $|z| < 1$ it is clear that their methods will in fact prove that in the space of functions meromorphic in $|z| < 1$ the set of functions $f(z)$ for which $C_\lambda(f)$ is total for *all* curves λ on which $|z| \to 1$ is residual.

and its generalisations, of which Theorems 10 and 11 are examples, now show is that assertions in the same direction, although less sweeping, can be made under specified conditions applicable to individual meromorphic functions.

IX. A theorem on isolated essential singularities

19. The method of § 17 is equally applicable to the case of a uniform function $g(z)$ having an isolated essential singularity, which we may assume to be at infinity. We denote by $C_{\varrho(\varphi)}(g)$ the cluster set of $g(z)$ as $z \to \infty$ along the ray $\varrho(\varphi)$ defined by $z = r e^{i\varphi}$, $0 \leq r < \infty$, and by $C_{\Delta}(g)$ the angular cluster set of $g(z)$ as $z \to \infty$ by sequences $\{z_n\}$ contained in Δ. With these definitions we prove

Theorem 12. *Suppose that $g(z)$ is non-rational and meromorphic for $K < |z| < \infty$. The necessary and sufficient condition that, given an angle A defined by $z = r e^{i\varphi}$, $\varphi_1 < \varphi < \varphi_2$, $K < r < \infty$, the angular cluster set $C_{\Delta}(g)$ of $g(z)$ in any $\Delta \subseteq A$ shall be total is that the set of values of φ in $\varphi_1 < \varphi < \varphi_2$ for which the radial cluster set $C_{\varrho(\varphi)}(g)$ is sub-total is of category* I.

That the condition is necessary follows at once by an application to the sets $C_{\varrho(\varphi)}(g)$ and $S(g)$, the set of values of φ for which $C_{\varrho(\varphi)}(g)$ is total, of the argument of § 17, practically without modification. The sequence of triangular lattices $\{l_n\}$ is constructed and the corresponding subsets $\Gamma_n(\varphi)$ of $\complement S(g)$ are defined as in § 17 above and we obtain the decomposition

$$\complement S(g) = \bigcup_{n} \bigcup_{1 \leq \nu \leq m(n)} \mathfrak{D}_{n,\nu}(\varphi), \tag{9}$$

where $\mathfrak{D}_{n,\nu}(\varphi)$ is now the subset of $\Gamma_n(\varphi)$ for which $\complement C_{\varrho(\varphi)}(g)$ contains the triangle $d_{n,\nu}$, $1 \leq \nu \leq m(n)$. If a set $\mathfrak{D}_{j,k}(\varphi)$ is of category II in (φ_1, φ_2) then there is an angle Δ contained in $\varphi_1 < \varphi < \varphi_2$ such that $d_{j,k} \subseteq \complement C_{\Delta}(g)$. The angular cluster set $C_{\Delta}(g)$ is thus sub-total and the necessity of the condition follows. For if every $C_{\Delta}(g)$ is total then all the $\mathfrak{D}_{n,\nu}(\varphi)$ must be of category I and it follows from (9) that $\complement S(g)$ must be of category I in (φ_1, φ_2).

The condition is also sufficient. For if it is satisfied every angle $\Delta \subseteq A$ contains a ray $z = r e^{i\varphi}$, $0 \leq r < \infty$, on which $C_{\varrho(\varphi)}(g)$ is total; and $C_{\Delta}(g)$, which contains $C_{\varrho(\varphi)}(g)$, is therefore total.

The theorem can evidently be generalised for domains swept out by a curve $\Lambda(\varphi)$ rotated about the origin, $\Lambda(\varphi)$ being subject to suitable restrictions, as for example that $|z|$ is strictly increasing on $\Lambda(\varphi)$.

Postscript

It was not until after this paper was finished that I became aware of two very recent papers of Bagemihl and Seidel [2] and [3]. The authors there study problems closely related to those studied here and in [2] they prove some of the same results. Theorem 7 (b) and Corollary 1 of [2] are respectively Theorem 1 and Corollary 1 of the present paper, while Theorems 7 (a) and 7 (b) of [2] are special cases of our Theorem 6, and Theorem 9 of [2] contains our Theorem 10. However, the methods of these two quite independent investigations are sufficiently different to give each of them an independent interest and there are a considerable number of results which are not common to both. In particular, our Theorem 5 and its consequences may be mentioned. It has therefore seemed best, in spite of the overlapping of the results referred to[1], to leave this paper unaltered except for the addition of this brief note.

Lilburn Tower, Alnwick, England

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