SOLUTIONS OF DIFFERENTIAL EQUATIONS AS ANALYTIC FUNCTIONALS OF THE COEFFICIENT FUNCTIONS

BY

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Introduction

In a series of papers during the last several years, ([1], [2], and references therein), one of us has developed a theory of the solutions of linear differential equations as analytic functionals of the coefficient functions. In the present paper, we consider a more general situation in which the differential equation is not restricted to be linear and use different methods. Even in the linear case, the results are a little different.

The method is to establish an implicit function theorem for analytic functions on one complex Banach space to another, and then apply this theorem to the differential equation.

Implicit functional equations in abstract spaces have been studied by various authors[1], and from various points of view. Since we restrict ourselves to the analytic case, it seemed appropriate to develop a theorem by generalizing the classical method of series expansions and dominating functions. A result similar to our theorem of Section 1 was given without proof by Michal and Clifford [3].

In the second section the implicit function theorem is used to study the solution of the differential equation $dy/d\tau = f(\tau, y)$ as a functional of the function f. Here τ is a real variable while y may range over a subset of a complex Banach space.

In particular the theory will include systems of ordinary differential equations and certain types of partial differential equations.

1. Implicit Functions

In the present section we shall make use of the abstract differential calculus and of the theory of analytic functions in complex Banach spaces.[2] In particular

[1] See, e. g. HILDEBRANDT and GRAVES [11].

[2] For a summary of these theories in complex Banach spaces see HILLE [6], chap. 4. For real as well as complex Banach spaces see MICHAL [1, 8]. MICHAL and MARTIN [9], MARTIN [10].

we use the notation $df(x_0; h)$ for the Gateaux differential $\lim_{\lambda\to 0} \lambda^{-1}\{f(x_0+\lambda h)-f(x_0)\}$ of the function $f(x)$ on one complex Banach space E_1 to another E_2 and the notation $d^n f(x_0; h_1, \dots, h_n)$ for the Gateaux differential of the nth order at $x=x_0$ with increments $h_1, \dots, h_n$. It is of importance to observe that *these* "Gateaux differentials" are indeed Fréchet differentials[3] for the analytic $f(x)$ and that the homogeneity of degree one in each increment is with respect to complex number multipliers.

Lemma 1. Let E_1, E_2 be complex Banach spaces and let $f(x)$ be analytic for the sphere $\|x\| \leqq \varrho$ of E_1 with values in E_2 and suppose that $\|f(x)\| \leqq M$ for $\|x\| \leqq \varrho$. Then, given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\left\| \frac{d^n f(\theta; x_1, x_1, \dots, x_n)}{n!} \right\| \leq \frac{(M+\varepsilon)e^n}{\varrho^n \sqrt{2\pi n}} \|x_1\| \dots \|x_n\|$$

for all $n > n_0$ and all $x_1, x_2, \dots, x_n$ in E_1. Hence, if m_n is the modulus of $\frac{1}{n!} d^n f(\theta; x_1, x_2, \dots, x_n)$, the series $\sum_{n=1}^{\infty} m_n \lambda^n$ converges for $0 < \lambda < \varrho/e$.

Proof. Put

$$\varphi(\zeta_1, \zeta_2) = f(\zeta_1 x_1 + \zeta_2 x_2),$$

and in general,

$$\varphi(\zeta_1, \zeta_2, \dots \zeta_n) = f(\zeta_1 x_1 + \zeta_2 x_2 + \dots + \zeta_n x_n).$$

Then

$$d^n f(\theta; x_1 x_2, \dots x_n) = \left.\frac{\partial^n \varphi(\zeta_1, \dots \zeta_n)}{\partial \zeta_1 \partial \zeta_2 \dots \partial \zeta_n}\right|_{\zeta_k = 0}, \quad k = 1, \dots n,$$

and $\varphi(\zeta_1, \dots \zeta_n)$ is analytic for $\|x_k\| \leq \frac{\varrho}{n}$ and $|\zeta_k| \leq 1$. Let Γ denote the unit circle in the complex plane. Then

$$\left(\frac{\partial \varphi(\zeta_1, \zeta_2, \dots \zeta_n)}{\partial \zeta_1}\right)_{\zeta_1 = 0} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\tau_1, \zeta_2, \dots \zeta_n)}{\tau_1^2} d\tau_1,$$

$$\left(\frac{\partial^2 \varphi(\zeta_1, \zeta_2, \dots \zeta_n)}{\partial \zeta_2 \partial \zeta_1}\right)_{\zeta_1 = 0,\, \zeta_2 = 0} = \frac{1}{(2\pi i)^2} \int_{\Gamma}\int_{\Gamma} \frac{\varphi(\tau_1, \tau_2, \zeta_3, \dots \zeta_n)\, d\tau_1\, d\tau_2}{\tau_1^2 \cdot \tau_2^2},$$

and, by induction,

$$d^n f(\theta; x_1, x_2, \dots x_n) = \frac{1}{(2\pi i)^n} \overbrace{\int_{\Gamma} \dots \int_{\Gamma}}^{n} \frac{\varphi(\tau_1, \tau_2, \dots, \tau_n)\, d\tau_1\, d\tau_2 \dots d\tau_n}{\tau_1^2 \cdot \tau_2^2 \dots \tau_n^2}.$$

[3] See Fréchet [7].

Now for $\|x_k\| \leq \varrho/n$, $k = 1, 2, \ldots n$ and $|\tau_k| = 1$, $k = 1, \ldots n$ we have $\|\sum_{k=1}^{n} \tau_k x_k\| \leq \leq \sum_{k=1}^{n} \|x_k\| \leq \varrho$. Hence $\|\varphi(\tau_1, \tau_2, \ldots \tau_n)\| \leq M$. It follows that $\|d^n f(\theta; x_1, x_2, \ldots x_n)\| \leq M$ when $\|x_k\| \leq \varrho/n$, $k = 1, \ldots, n$. Now, for any $x_1, \ldots x_n$ in E_1 with $\|x_k\| \neq 0$, put $x_k' = \frac{\varrho}{n\|x_k\|} \cdot x_k$, so that $\|x_k'\| = \varrho/n$. We have

$$\|d^n f(\theta; x_1', \ldots x_n')\| \leq M,$$

and hence

$$\|d^n f(\theta; x_1, \ldots, x_n)\| \leq \frac{M n^n}{\varrho^n} \|x_1\| \cdot \|x_2\| \ldots \|x_n\|, \quad \text{since } d^n f$$

is homogeneous in each x_k. Now by Stirling's formula,

$$n! = (1 + \varepsilon_n) \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}, \quad \text{where } \varepsilon_n \to 0 \text{ as } n \to \infty.$$

Hence, given $\varepsilon > 0$ there exists a positive integer n_0 such that

$$\left\| \frac{d^n f(\theta; x_1, x_2, \ldots x_n)}{n!} \right\| \leq \frac{(M + \varepsilon)}{\sqrt{2\pi n}} \cdot \left(\frac{e}{\varrho}\right)^n \|x_1\| \ldots \|x_n\|$$

for $n > n_0$ and all $x_1, x_2, \ldots x_n$ in E_1.

If m_n is the modulus of $\frac{1}{n!} d^n f(\theta; x_1, x_2, \ldots x_n)$ then clearly $m_n \leq \frac{M + \varepsilon}{\sqrt{2\pi n}} \left(\frac{e}{\varrho}\right)^n$ for $n > n_0$. Hence if $0 < \lambda < \varrho/e$,

$$m_n \lambda^n \leq \frac{M + \varepsilon}{\sqrt{2\pi n}} \cdot \left(\frac{\lambda e}{\varrho}\right)^n, \quad \text{where } 0 < \frac{\lambda e}{\varrho} < 1.$$

Hence $\sum m_n \lambda^n$ converges.

Lemma 2. Let E_1, E_2 and E_3 be complex Banach spaces and let $x \in E_1$, $y \in E_2$, Let $f(x, y)$ be analytic for $\|x\| \leq \varrho$, $\|y\| \leq \varrho$, with values in E_3, and suppose that $\|f(x, y)\|$ is bounded for $\|x\| \leq \varrho$, $\|y\| \leq \varrho$. Then if m_{jk} is the modulus of the multilinear function

$$\frac{1}{(j+k)!} d^{j+k} f(\theta, \theta; x_1, x_2, \ldots x_j, y_1, y_2, \ldots y_k),$$

the series $\sum_{j,k=0}^{\infty} m_{jk} \lambda^j \mu^k$ converges for $|\lambda| < \varrho/e$, $|\mu| < \varrho/e$.

Proof. Put

$$z = (x, y), \quad \|z\| = \sqrt{\|x\|^2 + \|y\|^2}, \quad \varphi(z) = f(x, y).$$

Then $\varphi(z)$ is analytic in the sphere $\|z\| \leq \varrho$ of the space $E_1 E_2$. By Lemma 1 we have, for $n > n_0$,

$$\left\| \frac{d^n \varphi(\theta; z_1, z_2, \ldots, z_n)}{n!} \right\| \leq \frac{(M+\varepsilon)\, e^n \, \|z_1\| \ldots \|z_n\|}{\varrho^n \sqrt{2\pi n}}. \tag{1.1}$$

Now,

$$d\varphi(0; z_1) = d f_x(0, 0; x_1) + d f_y(0, 0; y_1)$$

$$d^2 \varphi(0; z_1, z_2) = d^2 f_{xx}(0, 0; x_1, x_2) + d^2 f_{xy}(0, 0; x_1, y_2)$$
$$+ d^2 f_{xy}(0, 0; x_2, y_1) + d^2 f_{yy}(0, 0; y_1, y_2)$$

$$d^3 \varphi(0; z_1, z_2, z_3) = d^3 f_{xxx}(0, 0; x_1, x_2, x_3) + d^3 f_{xxy}(0, 0; x_1, x_2, y_3)$$
$$+ d^3 f_{xxy}(0, 0; x_1, x_3, y_2) + d^3 f_{xyy}(0, 0; x_1, y_2, y_3)$$
$$+ d^3 f_{xxy}(0, 0; x_2, x_3, y_1) + d^3 f_{xyy}(0, 0; x_2, y_1, y_3)$$
$$+ d^3 f_{xyy}(0, 0; x_3, y_1, y_2) + d^3 f_{yyy}(0, 0; y_1, y_2, y_3)$$

$$d^n \varphi(0; z_1, \ldots z_n) = \sum_{i+j=n} \sum d^n_{ix, jy} f(0, 0; x_{k_1}, \ldots x_{k_i}, y_{l_1}, \ldots, y_{l_j})$$

where $k_1, \ldots, k_i$ ranges over all subsets of i members of the integers $1, 2, 3, \ldots, n$, while $l_1, \ldots, l_j$ is the complementary set to $k_1, \ldots, k_i$.

Recalling that $\|z\| = \sqrt{\|x\|^2 + \|y\|^2}$, using the inequality (1.1) and putting $x_{i+1} = x_{i+2} = \cdots = x_n = 0$ and $y_1 = y_2 = \cdots = y_i = 0$ we have

$$\left\| \frac{1}{n!} d^n_{ix, jy} f(0, 0; x_1, x_2, \ldots, x_i;\ y_{i+1}, y_{i+2}, \ldots, y_n) \right\|$$
$$\leq \frac{(M+\varepsilon)\, e^n}{\varrho^n \sqrt{2\pi n}} \|x_1\| \cdot \|x_1\| \ldots \|x_i\| \cdot \|y_{i+1}\| \ldots \|y_n\|$$

for $n > n_0$, so that

$$m_{ij} \leq \frac{(M+\varepsilon)\, e^n}{\varrho^n \sqrt{2\pi n}}, \qquad i + j = n.$$

Hence if $0 < \lambda < \varrho/e$ and $0 < \mu < \varrho/e$,

$$m_{ij} \lambda^i \mu^j \leq \frac{M+\varepsilon}{\sqrt{2\pi(i+j)}} \cdot \left(\frac{\lambda e}{\varrho}\right)^i \left(\frac{\mu e}{\varrho}\right)^j$$

and $\sum m_{ij} \lambda^i \mu^j$ converges, since $0 < \dfrac{\lambda e}{\varrho} < 1$ and $0 < \dfrac{\mu e}{\varrho} < 1$.

Lemma 3. Let E_1 and E_2 be (real or complex) Banach spaces. If $F(x, y) = \sum_{j,k=0}^{\infty} h_{jk}(x, \dots x, y, \dots y)$, where $h_{jk}(x_1, \dots x_j, y_1, \dots y_k)$ is a multilinear function, symmetric in the x's and in the y's on $E_1^j E_2^k$ to E_2 and $h_{01}(y) = 0$ and if the moduli m_{jk} of h_{jk} satisfy $\sum m_{jk} r^j \varrho^k < \infty$ for some $r > 0$ and $\varrho > 0$, then there exists a unique analytic solution of the equation $y = F(x, y)$ in the neighborhood of $x = 0$, such that $y = 0$ when $x = 0$.

Proof. First we exhibit a formal solution, and then prove convergence. In the equation

$$
\begin{aligned}
y = F(x, y) &= h_{10}(x) + h_{20}(x, x) + h_{11}(x, y) + h_{02}(y, y) \\
&\quad + h_{30}(x, x, x) + h_{21}(x, x, y) + h_{12}(x, y, y) + h_{03}(y, y, y) \\
&\quad + \cdots,
\end{aligned}
\tag{1.2}
$$

substitute the generalized power series $y = \sum_{n=1}^{\infty} k_n(x)$, where $k_n(x)$ is a homogeneous polynomial on E_1 to E_2 which remains to be determined, and equate the resulting homogeneous polynomials of like degree. We have

$$
\left\{
\begin{aligned}
k_1(x) &= h_{10}(x), \\
k_2(x) &= h_{20}(x, x) + h_{11}(x, k_1(x)) + h_{02}(k_1(x), k_1(x)), \\
k_3(x) &= h_{30}(x, x, x) + h_{21}(x, x, k_1(x)) + h_{12}(x, k_1(x), k_1(x)) \\
&\quad + h_{03}(k_1(x), k_1(x), k_1(x)) + h_{11}(x, k_2(x)) + 2 h_{02}(k_1(x), k_2(x)), \\
k_4(x) &= h_{40}(x, x, x, x) + h_{31}(x, x, x, k_1(x)) + h_{22}(x, x, k_1(x), k_1(x)) \\
&\quad + h_{13}(x, k_1(x), k_1(x), k_1(x)) + h_{04}(k_1(x), k_1(x), k_1(x), k_1(x)) \\
&\quad + h_{21}(x, x, k_2(x)) + 2 h_{12}(x, k_1(x), k_2(x)) + 3 h_{03}(k_1(x), k_1(x), k_2(x)) \\
&\quad + h_{11}(x, k_3(x)) + 2 h_{02}(k_1(x), k_3(x)) + h_{02}(k_2(x), k_2(x)), \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\right.
\tag{1.3}
$$

Symbolically, we may write relations (1.3) as

$$
\left\{
\begin{aligned}
k_1 &= h_{10}, \\
k_2 &= h_{20} + h_{11} k_1 + h_{02} k_1^2, \\
k_3 &= h_{30} + h_{21} k_1 + h_{12} k_1^2 + h_{03} k_1^3 + h_{11} k_2 + 2 h_{02} k_1 k_2, \\
k_4 &= h_{40} + h_{31} k_1 + h_{22} k_1^2 + h_{13} k_1^3 + h_{04} k_1^4 + h_{21} k_2 \\
&\quad + 2 h_{12} k_1 k_2 + 3 h_{03} k_1^2 k_2 + h_{11} k_3 + 2 h_{02} k_1 k_3 \\
&\quad + h_{02} k_2^2, \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \text{ etc.}
\end{aligned}
\right.
\tag{1.4}
$$

where $(h_{mn} k_{\alpha_1} h_{\alpha_2} k_{\alpha_3} \ldots k_{\alpha_n}) x = h_{mn} (\overbrace{x, x, \ldots x}^{m}, k_{\alpha_1}(x), k_{\alpha_2}(x), \ldots k_{\alpha_m}(x))$ is a homogeneous polynomial of degree $N = m + \alpha_1 + \cdots + \alpha_n$ in x. Putting $\xi = \| x \|$, $\eta = \| y \|$, we see that the series

$$\begin{aligned} \varphi(\xi, \eta) = & \, m_{10} \xi + m_{20} \xi^2 + m_{11} \xi \eta + m_{02} \eta^2 \\ & + m_{30} \xi^3 + m_{21} \xi^2 \eta + m_{12} \xi \eta^2 + m_{03} \eta^3 \\ & + \cdots \end{aligned} \tag{1.5}$$

dominates (1.2). By hypothesis, this series converges for $0 \leq \xi \leq r$, $0 \leq \eta \leq \varrho$. Eq. (1.5) has a unique analytic solution for η in terms of ξ, and the coefficients in the series $\eta = \sum_{n=1}^{\infty} c_n \xi^n$ for this solution may be determined by substituting the series in Eq. (1.5).

These coefficients are determined successively by the equations

$$\begin{cases} c_1 = m_{10}, \\ c_2 = m_{20} + m_{11} c_1 + m_{02} c_1^2, \\ c_3 = m_{30} + m_{21} c_1 + m_{12} c_1^2 + m_{03} c_1^3 + m_{11} c_2 + 2 m_{02} c_1 c_2, \\ c_4 = m_{40} + m_{31} c_1 + m_{22} c_1^2 + m_{13} c_1^3 + m_{04} c_1^4 + m_{21} c_2 + 2 m_{12} c_1 c_2 \\ \quad + 3 m_{03} c_1^2 c_2 + m_{11} c_3 + 2 m_{02} c_1 c_3 + m_{02} c_2^2, \\ \quad \cdots\cdots\cdots\cdots \end{cases} \tag{1.6}$$

which are of the same form as Eq. (1.4).

Since $\| h_{ij} \| = m_{ij}$, it follows that $\| k_n \| \leq c_n$, where $\| h_{ij} \|$ is the modulus of $h_{ij}(x, y)$ and $\| k_n \|$ that of $k_n(x)$. Hence, since for some $\alpha > 0$, $\sum_1^{\infty} c_n \xi^n$ converges for $| \xi | < \alpha$, then $\sum_1^{\infty} k_n(x)$ converges for $\| x \| < \alpha$. Thus $y = \sum_1^{\infty} k_n(x)$ is the unique analytic solution of equation $y = F(x, y)$ in the neighborhood of $x = 0$, satisfying the condition $y(0) = 0$.

Theorem 1. Let E_1, E_2 and E_3 be complex Banach spaces and let $f(x, y)$ be analytic in a region $R_1 R_2$ where $R_1 < E_1$, $R_2 < E_2$. If the equation

$$f(x, y) = \theta \tag{1.7}$$

has a solution at $x = x_0$, $y = y_0$ where $x_0 \in R_1$, $y_0 \in R_2$, and if the differential $d_y f(x_0, y_0; \delta y)$ is a solvable linear function of δy then there exists a sphere S around the point x_0 in the space E_1 and an analytic function $\varphi(x)$ on S to E_2 such that $y = \varphi(x)$ is the unique analytic solution of Eq. (1.7) on S such that $y_0 = \varphi(x_0)$. This solution $\varphi(x)$ may be calculated recursively by the method of Formula (1.3) in the proof of Lemma 3.

Proof. For convenience, we may assume that $x_0 = \theta$, $y_0 = \theta$. Expanding the left member of (1.7) we obtain

$$p_{10}(x) + p_{01}(y) + \sum_{k=2}^{\infty} \sum_{i+j=k} p_{ij}(x, y) = \theta, \tag{1.8}$$

where

$$p_{ij}(x, y) = \frac{1}{(i+j)!} d_{x, y}^{i+j} f(\theta, \theta; x, x, \ldots x, y, \ldots y)$$

is a polynomial of degree i in x and j in y, and where in particular $p_{01}(y) = d_y f(\theta, \theta; y)$ is a solvable linear function of y. Hence Eq. (1.8) may be rewritten in the form

$$y = G(x, y) = -p_{01}^{-1}(p_{10}(x)) - \sum_{k=2}^{\infty} \sum_{i+j=k} p_{01}^{-1}(p_{ij}(x, y))$$

or

$$y = h_{10}(x) + \sum_{i+j>1} h_{ij}(x, x, \ldots x, y, \ldots, y), \tag{1.9}$$

where p_{01}^{-1} denotes the inverse of p_{01} and where

$$h_{ij}(x_1, x_2, \ldots x_i, y_1, y_2 \ldots y_j) = p_{01}^{-1}\left[\frac{1}{(i+j)!} d^{i+j} f(\theta, \theta; x_1 \ldots x_i, y_1, \ldots y_j)\right].$$

It is sufficient to show the existence of a unique analytic solution of Eq. (1.9). By hypothesis, the left member of (1.8) and hence the right member of (1.9) is bounded and convergent in the neighborhood of (0, 0), say for $\|x\| \leq \alpha$, $\|y\| \leq \alpha$. By Lemma 2, $\sum m_{ij} \lambda^i \mu^j$ converges, for

$$|\lambda| < \alpha/e, \quad |\mu| < \alpha/e,$$

where m_{ij} is the modulus of $h_{ij}(x_1, \ldots x_i, y_1 \ldots y_j)$.

Hence by Lemma 3, there is a unique analytic solution of Eq. (1.9) which reduces to θ for $x = \theta$, and the theorem is proved.

2. Differential Equations

Consider the differential equation

$$\frac{dy}{d\tau} = f(\tau, y) \tag{2.1}$$

subject to the initial condition $y = y_0$ for $\tau = \tau_0$. Here τ is a real variable on $|\tau - \tau_0| \leq a$, while y and $f(\tau, y)$ are elements of a complex Banach space B. We assume that $f(\tau, y)$ is continuous in the pair (τ, y) for $|\tau - \tau_0| \leq a$ and $\|y - y_0\| \leq b$, that there exists a positive number M such that $\|f(\tau, y)\| \leq M$ for $|\tau - \tau_0| \leq a$,

$\| y - y_0 \| \leq b$, and that for each τ, $f(\tau, y)$ is an analytic function of y for $\| y - y_0 \| < b$. Then if $0 < \beta < b$, $f(\tau, y)$ satisfies a Lipschitz condition

$$\| f(\tau, y_1) - f(\tau, y_2) \| \leq N \| y_1 - y_2 \|$$

for all τ in $|\tau - \tau_0| \leq a$ and all y_1, y_2 in the sphere $\| y - y_0 \| < \beta$.

For, if C is the unit circle in the complex plane, then by Cauchy's integral formula[1]

$$\delta f(\tau, y; h) = \frac{1}{2\pi i} \int_C \frac{f(\tau, y + \sigma h)}{\sigma^2} d\sigma,$$

for $\| y - y_0 \| < \beta$, $\| h \| \leq b - \beta$.
Hence $\| \delta f(\tau, y; h) \| \leq M$ for $|\tau - \tau_0| \leq a$, $\| y - y_0 \| < \beta$, $\| h \| \leq b - \beta$.
Since $\delta f(\tau, y; h)$ is homogeneous in h of degree one, it follows that $\| \delta f(\tau, y; h) \| \leq$ $\leq N \| h \|$ for $|\tau - \tau_0| \leq a$, $\| y - y_0 \| < \beta$, where $N = \dfrac{M}{b - \beta}$.

Now $f(\tau, y_2) - f(\tau, y_1) = \int_0^1 \delta f(\tau, y_1 + \lambda (y_2 - y_1); \; y_2 - y_1) \, d\lambda$ for all y_1 and y_2 in $\| y - y_0 \| < \beta$. It follows that

$$\| f(\tau, y_2) - f(\tau, y_1) \| \leq N \| y_2 - y_1 \|$$

for all y_1, y_2 in the sphere $\| y - y_0 \| < \beta$.

Thus, all the hypotheses of a known existence theorem ([6], p. 95) for differential equations are satisfied. It follows that Eq. (2.1) has a unique continuous solution $y = \bar{y}(\tau)$ satisfying the initial conditions $y = y_0$ for $\tau = \tau_0$, and defined for $|\tau - \tau_0| \leq a$, where $\alpha < \min(a, \beta/M)$.

Also,

$$\| \bar{y}(\tau) - y_0 \| \leq M \alpha < \beta \quad \text{for} \quad |\tau - \tau_0| \leq \alpha. \tag{2.2}$$

We use the following notations: I is the interval $|\tau - \tau_0| \leq \alpha$, S the sphere $\| y - y_0 \| < \beta$ of the Banach space B, while X will denote the space of all functions $x = x(\tau, z)$ on IS to B which are continuous in the pair (τ, z), bounded, and analytic in z for each τ. With the norm defined by

$$\| x \| = \sup \{ \| x(\tau, z) \|_B ; \; \tau \in I, \; z \in S \},$$

X is evidently a normed (complex) vector space. It is also complete, since if $\| x_m - x_n \| \to 0$ as m and n tend to infinity, then the sequence of functions $x_n(\tau, z)$

[1] See [6], p. 74.

converges uniformly to a limiting function $x_0(\tau, z)$ which is continuous and bounded on IS. By the generalized convergence theorem of Weierstrass (see A. E. Taylor, [4]) it follows that $x_0(\tau, z)$ is also analytic in z for each $\tau \in I$, and for $z \in S$. Hence X is a complex Banach space.

Let Y be the complex Banach space of all continuous functions on I to B with the norm $\|y\| = \max\{\|y(\tau)\|;\ \tau \in I\}$.

Putting

$$G(x, y) = y_0 + \int_{\tau_0}^{\tau} x(\sigma, y(\sigma))\, d\sigma, \tag{2.3}$$

we shall be concerned with the equation

$$y = G(x, y), \tag{2.4}$$

where $G(x, y)$ is defined on XK to Y, and where K is the sphere $\|y - y_0\| < \beta$ in the space Y. It follows from inequality (2.2) that $\bar{y} \in K$. It is clear that $G(x, y)$ is single valued and well defined for $x \in X$, $y \in K$, and that $G(x, y)$ is locally bounded in the region XK. For if x_1 is any given element of X and if $\|x - x_1\| < \gamma$ and $y \in K$ then

$$\|G(x, y)\| \le \|y_0\| + (\|x_1\| + \gamma)\,\alpha.$$

$G(x, y)$ is Gateaux differentiable at each point of the region XK. For if (x, y) and $(x + h, y + k)$ are points of this region we have

$$\begin{aligned} \frac{G(x+\lambda h, y+\lambda k) - G(x, y)}{\lambda} &= \int_{\tau_0}^{\tau} \frac{x[\sigma, y(\sigma) + \lambda k(\sigma)] - x[\sigma, y(\sigma)]\, d\sigma}{\lambda} \\ &\quad + \int_{\tau_0}^{\tau} h[\sigma, y(\sigma) + \lambda k(\sigma)]\, d\sigma \\ &= \int_{\tau_0}^{\tau} d\sigma \int_0^1 \delta x[\sigma, y(\sigma) + \lambda \nu k(\sigma);\ k(\sigma)]\, d\nu \\ &\quad + \int_{\tau_0}^{\tau} h[\sigma, y(\sigma) + \lambda k(\sigma)]\, d\sigma. \end{aligned} \tag{2.5}$$

Put

$$\Delta = \frac{G(x+\lambda h, y+\lambda k) - G(x, y)}{\lambda} - \int_{\tau_0}^{\tau} \delta x[\sigma, y(\sigma);\ k(\sigma)]\, d\sigma - \int_{\tau_0}^{\tau} h[\sigma, y(\sigma)]\, d\sigma.$$

Then from (2.5), $\Delta = \Delta_1 + \Delta_2$ where

$$\Delta_1 = \int_{\tau_0}^{\tau} d\sigma \int_0^1 \{\delta x[\sigma, y(\sigma) + \lambda \nu k(\sigma); k(\sigma)] - \delta x[\sigma, y(\sigma); k(\sigma)]\} d\nu$$

$$\Delta_2 = \int_{\tau_0}^{\tau} \{h[\sigma, y(\sigma) + \lambda k(\sigma)] - h[\sigma, y(\sigma)]\} d\sigma.$$

We must show that $\Delta \to 0$ in the space Y as $\lambda \to 0$. Now the norm of the integrand of the second integral is a continuous function of (σ, λ) on $I\Lambda$, where Λ is an interval around zero. Hence the integrand of the second integral tends to zero uniformly with respect to σ when $\lambda \to 0$, and the second integral Δ_2 tends to zero in the space Y, as $\lambda \to 0$.

Since the range of the function $y(\sigma)$ is a compact subset of the open sphere $S \subset B$, the distance of the subset from the boundary of S is a positive number δ. Choose ω such that $0 < \omega < \delta / \| k \|$ where $\| k \|$ is the norm in the space Y. Then $y(\sigma) + \varrho k(\sigma) \in S$ for all $\sigma \in I$ and all complex numbers ϱ with $|\varrho| < \omega$.

We may rewrite Δ_1 as

$$\Delta_1 = \lambda \int_{\tau_0}^{\tau} d\sigma \int_0^1 d\nu \int_0^1 \nu\, \delta^2 x[\sigma, y(\sigma) + \lambda \mu \nu k(\sigma); k(\sigma)] d\mu. \tag{2.6}$$

Now by the generalized Cauchy integral formula, if we select C as a circle with center at the origin and radius $\frac{\omega}{2}$, and take $|\lambda| < \frac{\omega}{2}$,

$$\delta^2 x[\sigma, y + \lambda \mu \nu k; k] = \frac{1}{\pi i} \int_C \frac{x[\sigma, y + (\lambda \mu \nu + \zeta) k]}{\zeta^3} d\zeta. \tag{2.7}$$

The integrand is well defined, since $|\lambda \mu \nu + \zeta| \leq |\lambda| + |\zeta| < \omega$ so that $y(\sigma) + (\lambda \mu \nu + \zeta) k(\sigma) \in S$. It follows from (2.7) that

$$\| \delta^2 x[\sigma, y + \lambda \mu \nu k; k] \| \leq 8 \omega^{-2} \| x \|.$$

Hence, by (2.6), $\| \Delta_1 \| \leq 8 \omega^{-2} |\lambda| \alpha \| x \|$, so that $\Delta_1 \to 0$ in the space Y as $\lambda \to 0$. Therefore $G(x, y)$ is Gateaux differentiable and locally bounded, and hence analytic in K, with its differential given by

$$d G(x, y; h, k) = \int_{\tau_0}^{\tau} \delta x[\sigma, y(\sigma); k(\sigma)] d\sigma + \int_{\tau_0} h[\sigma, y(\sigma)] d\sigma. \tag{2.8}$$

Putting $F(x, y) = G(x, y) - y$, Eq. (2.4) may be written in the form $F(x, y) = 0$. The differential $dF_y(x, y; k)$ takes the form

$$\begin{aligned} dF_y &= dG_y(x, y; k) - k \\ &= \int_{\tau_0}^{\tau} \delta x[\sigma, y(\sigma); k(\sigma)]\, d\sigma - k(\tau). \end{aligned}$$

We put $\bar{x} = f(\tau, y)$.

Then since $\bar{y}(\tau)$ is the solution of Eq. (2.1) with the initial value of y_0, we have $F(\bar{x}, \bar{y}) = 0$.

The differential $dF_y(x, y; k)$ is a solvable linear function of k if the linear integral equation

$$k(\tau) - \int_{\tau_0}^{\tau} \delta x[\sigma, \bar{y}(\sigma); k(\sigma)]\, d\sigma = z(\tau) \tag{2.9}$$

has a continuous solution $k(\tau)$ defined for $|\tau - \tau_0| \leq \alpha$ for every continuous $z(\tau)$ on I to B. It follows by using the Cauchy integral formula as before that there exists a constant $\mu > 0$ such that

$$\| \delta x[\sigma, \bar{y}(\sigma); k] \| \leq \mu \| k \| \tag{2.10}$$

for all $\sigma \in I$ and all $k \in B$. Hence it is easily shown that the abstract Volterra integral equation (2.9) has a unique continuous solution.

Thus all the hypotheses of the implicit function Theorem 1 are satisfied, and we conclude that Eq. (2.4) has a solution $y = \phi(x)$ analytic in x in the neighborhood of $\bar{x} = f(\tau, y)$. Thus we have proved the following result:

Theorem 2. *Under the restrictions and definitions given above, the solution $y(\tau)$ of the differential equation*

$$\frac{dy}{d\tau} = x(\tau, y)$$

with $y(\tau_0) = y_0$, considered as a function of the right hand side $x(\tau, z) \in X$ with values in Y, is an analytic function of x in the neighborhood of $\bar{x} = f(\tau, z)$.

Corollary. If $f(\tau, y)$ is a polynomial in y, and continuous in τ for $|\tau - \tau_0| \leq a$ then all the hypotheses on $f(\tau, y)$ will be satisfied for an arbitrary sphere $\| y - y_0 \| \leq b$, so Theorem 2 will hold.

Proof. Is follows from a result of Kerner[1] that $f(\tau, y)$ is continuous in the pair (τ, y) and that if τ varies in a sufficiently small neighborhood, $\| f(\tau, y) \|$ is bounded

[1] See [5], p. 548.

for $\| y-y_0 \| < b$ where b is any positive number. By an application of the Heine Borel theorem to the interval $| \tau - \tau_0 | \leq a$, there exists an $M > 0$ such that $\| f(\tau, y \| < M$ for all τ in the interval $| \tau - \tau_0 | \leq a$ and all y in the sphere $\| y - y_0 \| < b$, Thus all the hypotheses are satisfied.

Another important special case of Theorem 2 is that of a system of n numerical differential equations. Since in this case the space B is a finite dimensional space of n complex variables, the hypothesis of the boundedness of the norm $\| f(\tau, y) \|$ is redundant providing the numbers a and b are finite.

By taking B to be a function space, it is also possible to include certain types of partial differential equations under Theorem 2.

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References

[1]. A. D. MICHAL, On a non-linear total differential equation in normed linear spaces, *Acta Mathematica*, vol. 80, 1–21 (1948).

[2]. ———, Solutions of systems of linear differential equations as entire analytic functionals of coefficient functions, *Mathematics Magazine*, vol. 22, 57–66 (1948).

[3]. A. D. MICHAL and A. H. CLIFFORD, Fonctions analytiques implicites dans des espaces vectoriels abstraits, *Comptes Rendus*, Paris, vol. 197, 735–737 (1933).

[4]. A. E. TAYLOR, Analytic functions in general analysis, *Annali della R. Scuola Normale Superiore di Pisa*, Series II, vol. 6, 277–292 (1937). This is a shortened account of TAYLOR's California Institute of Technology Thesis, 1936.

[5]. M. KERNER, Die Differentiale in der Allgemeinen Analysis, *Annals of Math.*, vol. 34,-546–572 (1933).

[6]. E. HILLE, Functional Analysis and Semi-Groups, *American Mathematical Society Colloquium Publication*, vol. 31 (1948).

[7]. M. FRÉCHET, La notion de différentielle dans l'Analyse Générale, *Ann. Ec. Norm. Sup.*, t. XLII, 293–323 (1925). This paper is reproduced in FRÉCHET's recent book: "Pages choisies d'Analyse générale", pp. 180–204, Gauthier-Villars, 1953.

[8]. A. D. MICHAL, Functional analysis in topological group spaces, *Mathematics Magazine*, vol. XXI, 80–90 (1947).

[9]. A. D. MICHAL and R. S. MARTIN, Some expansions in vector space, *Journal de Math. Pures et Appliquées*. vol. 13, 69–91 (1934).

[10]. R. S. MARTIN, "Contributions to the theory of functionals", California Institute of Technology Thesis, 1932 (unpublished).

[11]. T. H. HILDEBRANDT and L. M. GRAVES, Implicit functions and their differentials in general analysis, *Trans. American Math. Soc.*, vol. 29, 127–153 (1927).