ONE-ONE MEASURABLE TRANSFORMATIONS.

By

CASPER GOFFMAN.

1. Introduction. The literature on the theory of functions of a real variable contains a variety of results which show that measurable functions, and even arbitrary functions, have certain continuity properties. As examples, I mention the well known theorems of Vitali-Carathéodory [1], Saks-Sierpinski [2], Lusin [3], and the theorem of Blumberg [4] which asserts that for every real function $f(x)$ defined on the closed interval $[0,1]$ there is a set D which is dense in the interval such that $f(x)$ is continuous on D relative to D.

The related topic of measurable and arbitrary one-one transformations has been given little attention. I know only of Rademacher's work [5] on measurability preserving transformations and my short paper [6] on the approximation of arbitrary one-one transformations.

My purpose here is to fill this void partially by obtaining for one-one measurable transformations an analog of Lusin's theorem on measurable functions. The form of Lusin's theorm I have in mind is that [7] for every measurable real function $f(x)$ defined on the closed interval $[0,1]$ there is, for every $\varepsilon > 0$, a continuous $g(x)$ defined on $[0,1]$ such that $f(x) = g(x)$ on a set of measure greater than $1 - \varepsilon$. The analogous statement for one-one transformations between $[0,1]$ and itself is that for every such one-one measurable $f(x)$ with measurable inverse $f^{-1}(x)$ there is, for every $\varepsilon > 0$, a homeomorphism $g(x)$ with inverse $g^{-1}(x)$ between $[0,1]$ and itself such that $f(x) = g(x)$ and $f^{-1}(x) = g^{-1}(x)$ on sets of measure greater than $1 - \varepsilon$. I shall show that this statement is false but that similar statements are true for one-one transformations between higher dimensional cubes.

I shall designate a one-one transformation by $(f(x), f^{-1}(y))$, where the functions $f(x)$ and $f^{-1}(y)$ are the direct and inverse functions of the transformation. I shall say that a one-one transformation $(f(x), f^{-1}(y))$ between n and m dimensional unit cubes I_n and I_m is measurable if the functions $f(x)$ and $f^{-1}(y)$ are both measurable,

17—533805. *Acta mathematica*. 89. Imprimé le 6 août 1953.

and that $(f(x), f^{-1}(y))$ is absolutely measurable[1] if, for all measurable sets $S\subset I_n$, $T\subset I_m$, the sets $f(S)$ and $f^{-1}(T)$ are measurable, where $f(S)$ is the set of all $y\in I_m$ for which there is an $x\in S$ such that $y=f(x)$, and $f^{-1}(T)$ is defined similarly. It is well known [8] that a measurable transformation $(f(x), f^{-1}(y))$ is absolutely measurable if and only if, for all sets $S\subset I_n$ and $T\subset I_m$, of measure zero, the sets $f(S)$ and $f^{-1}(T)$ are also of measure zero.

I show that if $n=m\geqq 2$, and $(f(x), f^{-1}(y))$ is a one-one measurable transformation between unit n cubes I_n and I_m then for every $\varepsilon>0$, there is a homeomorphism $(g(x), g^{-1}(y))$ between I_n and I_m such that $f(x)=g(x)$ and $f^{-1}(y)=g^{-1}(y)$ on sets whose n dimensional measures both exceed $1-\varepsilon$. This result does not hold if $n=m=1$. I then show that if $1\leqq n<m$ and $(f(x), f^{-1}(y))$ is a one-one measurable transformation between unit cubes I_n and I_m, whose dimensions are n and m, respectively, then for every $\varepsilon>0$, there is a homeomorphism $(g(x), g^{-1}(y))$ between I_n and a subset of I_m whose m dimensional measure exceeds $1-\varepsilon$, such that $f(x)=g(x)$ and $f^{-1}(y)=g^{-1}(y)$ on sets whose n and m dimensional measures exceed $1-\varepsilon$, respectively.

For the case $n=m$, the proof depends on the possibility of extending a homeomorphism between certain zero dimensional closed subsets of the interiors of I_n and I_m to a homeomorphism between I_n and I_m. It has been known since the work of Antoine [9] that such extensions are always possible only if $n=m=2$. However, it is adequate for my needs that such extensions be possible for homeomorphisms between special kinds of zero dimensional closed sets which I call sectional. In § 2, I show that if $n=m\geqq 2$, then every homeomorphism between sectionally zero dimensional closed subsets of the interiors of I_n and I_m may be extended to a homeomorphism between I_n and I_m. For the case $1\leqq n<m$, I show that every homeomorphism between sectionally zero dimensional subsets of the interiors of I_n and I_m may be extended to a homeomorphism between I_n and a subset of I_m. In § 3, I show that for every one-one measurable $(f(x), f^{-1}(y))$ between I_n and I_m, where $n\geqq 1$ and $m\geqq 1$, there are, for every $\varepsilon>0$, closed sets $E_n\subset I_n$ and $E_m\subset I_m$, whose n and m dimensional measures, respectively, exceed $1-\varepsilon$, such that $(f(x), f^{-1}(y))$ is a homeomorphism between E_n and E_m. I then show that the closed sets E_n and E_m may be taken to be sectionally zero dimensional. These facts, when combined with the results of § 2, yield the main results of the paper which were mentioned above. § 4 is concerned whith related matters. I show that for every one-one measurable $(f(x), f^{-1}(y))$ between unit intervals $I=[0,1]$ and $J=[0,1]$ there is a one-one $(g(x), g^{-1}(y))$ between I and J

[1] The transformations which I call absolutely measurable are customarily called measurable. The terms used here seem to conform more nearly to standard real variable terminology.

such that $g(x)$ and $g^{-1}(y)$ are of at most Baire class 2, and $g(x)=f(x)$, $g^{-1}(y)=f^{-1}(y)$ almost everywhere. I have not been able to answer the analogous question for transformations between higher dimensional cubes. Finally, I show that for every one-one measurable transformation $(f(x), f^{-1}(y))$ between I_n and I_m there are decompositions $I_n=S_1 \cup S_2 \cup S_3$ and $I_m=f(S_1)\cup f(S_2)\cup f(S_3)$ into disjoint measurable sets, some of which could be empty, such that S_1 is of n dimensional measure zero, $f(S_2)$ is of m dimensional measure zero, and $(f(x), f^{-1}(y))$ is an absolutely measurable transformation between S_3 and $f(S_3)$.

2. Extension of homeomorphisms. Let $n \geqq 2$ and let I_n be an n dimensional unit cube. I shall say that a set $E\subset I_n$ is *sectionally zero dimensional* if for every hyperplane π which is parallel to a face of I_n and for every $\varepsilon>0$ there is a hyperplane π' parallel to π whose distance from π is less than ε and which contains no points of E. It is clear that every sectionally zero dimensional set is zero dimensional in the Menger-Urysohn sense [10] but that there are zero dimensional sets which are not sectionally zero dimensional. A set $S\subset I_n$ will be called a *p-set* if it consists of a simply connected region, together with the boundary of the region, for which the boundary consists of a finite number of $n-1$ dimensional parallelopipeds which are parallel to the faces of I_n.

Lemma 1. Every subset of a sectionally zero dimensional set is sectionally zero dimensional.

Proof. The proof is clear.

Lemma 2. If $(f(x), f^{-1}(y))$ is a homeomorphism between sectionally zero dimensional closed sets S and T, and $\varepsilon>0$, then S may be decomposed into disjoint sectionally zero dimensional closed sets $S_1, S_2, \ldots, S_m$, and T may be decomposed into disjoint sectionally zero dimensional closed sets $T_1, T_2, \ldots, T_m$, each of diameter less than ε, such that, for every $j=1, 2, \ldots, m$, $(f(x), f^{-1}(y))$ is a homeomorphism between S_j and T_j.

Proof. There is a $\delta>0$, which may be taken to be less than ε, such that every subset of S of diameter less than δ is taken by $f(x)$ into a subset of T of diameter less than ε. Let $S_1, S_2, \ldots, S_m$ be a decomposition of S into disjoint sectionally zero dimensional closed sets each of diameter less than δ. Then the sets $T_1=f(S_1)$, $T_2=f(S_2), \ldots, T_m=f(S_m)$ are sectionally zero dimensional closed subsets of T each of diameter less than ε.

Lemma 3. If F is a sectionally zero dimensional closed set which is contained in the interior of a p-set P then, for every $\varepsilon > 0$, there is a finite number of disjoint p-sets in the interior of P, each of which contains at least one point of F and is of diameter less than ε, such that F is contained in the union of their interiors.

Proof. Since F is sectionally zero dimensional, there is, for every pair of parallel faces of I_n, a finite sequence of parallel hyperplanes such that one of the two given faces of I_n is first in the sequence and the other is last, and such that the distance between successive hyperplanes of the sequence is less than $\varepsilon/\sqrt{n}$. The collection of hyperplanes thus obtained for all pairs of parallel faces of I_n decomposes P into a finite number of p-sets, whose interiors are disjoint, such that F is contained in the union of their interiors. Since F is closed, these p-sets may be shrunk to disjoint p-sets which are such that F is still in the union of their interiors. Select among the latter p-sets those whose intersection with F is not empty. It is clear that these p-sets have all the required properties.

Lemma 4. If $k > 0$, and $F_1, F_2, \ldots, F_m$ is a finite number of disjoint sectionally zero dimensional closed sets in the interior of a p-set P, each of diameter less than k, then there are disjoint p-sets $P_1, P_2, \ldots, P_m$ in the interior of P, each of diameter less than $k\sqrt{n}$, such that F_j is contained in the interior of P_j, for every $j = 1, 2, \ldots, m$.

Proof. Every F_j is evidently contained in the interior of a p-set Q_j which is itself in the interior of P and also in a cube of side k. The set P_j will be a subset of Q_j and so its diameter will be less than $k\sqrt{n}$. Since $F_1, F_2, \ldots, F_m$ are disjoint closed sets, there is a constant $d > 0$ such that the distance between any two of them exceeds d. By Lemma 3, each F_j has an associated finite number of disjoint p-sets, all of which are subsets of Q_j of diameter less than $d/2$, each of which contains at least one point of F_j, and are such that F_j is contained in the union of their interiors. Call these sets $P_{j1}, P_{j2}, \ldots, P_{jn_j}$. If $i \neq j$, then every pair of sets P_{ir}, P_{js} is disjoint, since the distance between F_i and F_j exceeds d. For every $j = 1, 2, \ldots, m$, the set P_{j1} can be connected to P_{j2}, P_{j2} to P_{j3}, and so on until P_j, m_{j-1} is connected to P_{jm_j} by means of parallelopipeds with faces parallel to the faces of I_n, which remain in Q_j and do not intersect each other or any of the sets P_{ir}. The set P_j is the union of $P_{j1}, P_{j2}, \ldots, P_{jm_j}$ and the connecting parallelopipeds. P_j is a subset of Q_j. It is a p-set of diameter less than $k\sqrt{n}$ whose interior contains F_j. Moreover, if $i \neq j$, then the intersection of P_i and P_j is empty.

Lemma 5. If P and Q are p-sets, $P_1, P_2, \ldots, P_m$ and $Q_1, Q_2, \ldots, Q_m$ are disjoint p-sets in the interiors of P and Q, respectively, having $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_m$ as their own interiors, then every homeomorphism $(f(x), f^{-1}(y))$ between the boundaries of P and Q may be extended to a homeomorphism between $P-\bigcup_{j=1}^{m} p_j$ and $Q-\bigcup_{j=1}^{m} q_j$ which takes the boundary of P_j into the boundary of Q_j for every $j=1, 2, \ldots, m$.

Proof. Let R be a p-set contained in the interior of P which has the sets $P_1, P_2, \ldots, P_m$ in its interior and let S be a p-set contained in the interior of Q which has the sets $Q_1, Q_2, \ldots, Q_m$ in its interior. There is a homeomorphism $(\varphi(x), \varphi^{-1}(y))$ between $R-\bigcup_{j=1}^{m} p_j$ and $S-\bigcup_{j=1}^{m} q_j$ which takes the boundary of P_j into the boundary of Q_j for every j. I need only show that there is a homeomorphism between the closed region bounded by P and R and the closed region bounded by Q and S which agrees with $(f(x), f^{-1}(y))$ on the outer boundaries and agrees with $(\varphi(x), \varphi^{-1}(y))$ on the inner boundaries. By taking cross-cuts from the outer to the inner boundaries and extending the homeomorphisms along the cross-cuts, the problem is reduced to the following one: if two regions R_1 and R_2 are both homeomorphic to the closed n dimensional sphere σ_n and if $(f(x), f^{-1}(y))$ is a homeomorphism between the boundaries of R_1 and R_2 then $(f(x), f^{-1}(y))$ may be extended to a homeomorphism between R_1 and R_2. In order to show this, I consider arbitrary homeomorphisms $(g(x), g^{-1}(y))$ and $(h(x), h^{-1}(y))$ between R_1 and σ_n and between R_2 and σ_n. I then consider the following special homeomorphism $(k(x), k^{-1}(y))$ between σ_n and itself: For each ξ on the boundary of σ_n, let

$$k(\xi)=h(f(g^{-1}(\xi))).$$

For each ξ in the interior of σ_n, let $k(\xi)$ be defined by first moving ξ along the radius on which it lies to the point ξ' on the boundary of σ_n which lies on the same radius, then by moving ξ' to the point $k(\xi')$, and finally by moving $k(\xi')$ along the radius of σ_n on which it lies to the point on the same radius whose distance from the center of σ_n is the same as the distance of ξ from the center of σ_n. The transformation $k(\xi)$ which is defined in this way is easily seen to be a homeomorphism between σ_n and itself. The transformation

$$\varphi(x)=h^{-1}(k(g(x))),$$

together with its inverse, constitutes a homeomorphism between R_1 and R_2. This homeomorphism is an extension of $(f(x), f^{-1}(y))$, for if x is on the boundary of R_1, then

$$\varphi(x) = h^{-1}(k(g(x))) = h^{-1}(h(f(g^{-1}(g(x)))))$$
$$= h^{-1}(h(f(x))) = f(x).$$

I am now ready to prove a theorem on the extension of homeomorphisms.

Theorem 1. If P and Q are n-dimensional p-sets for $n \geqq 2$, and S and T are sectionally zero dimensional closed subsets of the interiors of P and Q, respectively, every homeomorphism $(f(x), f^{-1}(y))$ between S and T may be extended to a homeomorphism between P and Q.

Proof. By Lemma 2, S and T have decompositions into disjoint closed sets $S_1, S_2, \ldots, S_{m_1}$ and $T_1, T_2, \ldots, T_{m_1}$, all of diameter less than 1, such that $T_{j_1} = f(S_{j_1})$, for every $j_1 = 1, 2, \ldots, m_1$. By Lemma 1, these sets are all sectionally zero dimensional, and so, by Lemma 4, there are disjoint p-sets $P_1, P_2, \ldots, P_{m_1}$ in the interior of P and disjoint p-sets $Q_1, Q_2, \ldots, Q_{m_1}$ in the interior of Q, all of diameter less than $\sqrt{n}$, such that, for every $j_1 = 1, 2, \ldots, m_1$, S_{j_1} is in the interior of P_{j_1} and T_{j_1} is in the interior of Q_{j_1}. For every $j_1 = 1, 2, \ldots, m_1$, the sets S_{j_1} and T_{j_1} have decompositions into disjoint sectionally zero dimensional closed sets $S_{j_1 1}, S_{j_1 2}, \ldots, S_{j_1 m_{j_1}}$ and $T_{j_1 1}, T_{j_1 2}, \ldots, T_{j_1 m_{j_1}}$, all of diameter less than $1/2$, such that $T_{j_1 j_2} = f(S_{j_1 j_2})$, for for every $j_2 = 1, 2, \ldots, m_{j_1}$; and there are disjoint p-sets $P_{j_1 1}, P_{j_1 2}, \ldots, P_{j_1 m_{j_1}}$ in the interior of P_{j_1} and disjoint p-sets $Q_{j_1 1}, Q_{j_1 2}, \ldots, Q_{j_1 m_{j_1}}$ in the interior of Q_{j_1}, all of diameter less than $\sqrt{n}/2$, such that for every $j_2 = 1, 2, \ldots, m_{j_1}$, $S_{j_1 j_2}$ is in the interior of $P_{j_1 j_2}$ and $T_{j_1 j_2}$ is in the interior of $Q_{j_1 j_2}$. By repeated application of the lemmas in this way, the following system of sets is obtained: First, there is a positive integer m_1; for every $j_1 \leqq m_1$, there is a positive integer m_{j_1}; for every $j_1 \leqq m_1$, $j_2 \leqq m_{j_1}$, there is a positive integer $m_{j_1 j_2}$; and, for every positive integer k, having defined the positive integers $m_{j_1 j_2 \ldots j_{k-1}}$, there is for every $j_1 \leqq m_1$, $j_2 \leqq m_{j_1}$, $\ldots$, $j_k \leqq m_{j_1 j_2 \ldots j_{k-1}}$, a positive integer $m_{j_1 j_2 \ldots j_k}$. Now, for every positive integer k, for every $j_1 \leqq m_1$, $j_2 \leqq m_{j_1}$, $\ldots$, $j_k \leqq m_{j_1 j_2 \ldots j_{k-1}}$, there are sets $S_{j_1 j_2 \ldots j_k}$, $T_{j_1 j_2 \ldots j_k}$, $P_{j_1 j_2 \ldots j_k}$, and $Q_{j_1 j_2 \ldots j_k}$. The sets $S_{j_1 j_2 \ldots j_k}$ and $T_{j_1 j_2 \ldots j_k}$ are sectionally zero dimensional subsets of $S_{j_1 j_2 \ldots j_{k-1}}$ and $T_{j_1 j_2 \ldots j_{k-1}}$, respectively, all with diameters less than $1/2^k$, such that $T_{j_1 j_2 \ldots j_k} = f(S_{j_1 j_2 \ldots j_k})$. The set $P_{j_1 j_2 \ldots j_k}$ is a p-set of diameter less than $\sqrt{n}/2^k$ which contains $S_{j_1 j_2 \ldots j_k}$ in its interior and is in the interior of $P_{j_1 j_2 \ldots j_{k-1}}$. and $Q_{j_1 j_2 \ldots j_k}$ is a p-set of diameter less than $n/2^k$ which contains $T_{j_1 j_2 \ldots j_k}$ in its interior and is in the interior of $Q_{j_1 j_2 \ldots j_{k-1}}$. Moreover, for every $j_1 \leqq m_1$, $j_2 \leqq m_{j_1}$, $\ldots$, $j_{k-1} \leqq m_{j_1 j_2 \ldots j_{k-2}}$, the sets $P_{j_1 j_2 \ldots j_k}$, as well as the sets $Q_{j_1 j_2 \ldots j_k}$, are disjoint for $j_k = 1, 2, \ldots, m_{j_1 j_2 \ldots j_{k-1}}$.

The desired extension of the homeomorphism $(f(x), f^{-1}(y))$ between S and T to a homeomorphism between P and Q is now obtained by repeated application of Lemma 5 to the p-sets $P_{j_1 j_2 \dots j_k}$ and $Q_{j_1 j_2 \dots j_k}$. Designate the interiors of $P_{j_1 j_2 \dots j_k}$ and $Q_{j_1 j_2 \dots j_k}$ by $p_{j_1 j_2 \dots j_k}$ and $q_{j_1 j_2 \dots j_k}$, respectively. A homeomorphism $(\varphi(x), \varphi^{-1}(y))$ is first effected between $P - \bigcup_{j_1=1}^{m_1} p_{j_1}$ and $Q - \bigcup_{j_1=1}^{m_1} q_{j_1}$ which takes the boundary of P_{j_1} into the boundary of Q_{j_1}, for every $j_1 = 1, 2, \dots, m_1$. For every $j_1 = 1, 2, \dots, m_1$, this homeomorphism between the boundaries of P_{j_1} and Q_{j_1} may be extended to a homeomorphism $(\varphi(x), \varphi^{-1}(y))$ between $P_{j_1} - \bigcup_{j_2=1}^{m_{j_1}} p_{j_1 j_2}$ and $Q_{j_1} = \bigcup_{j_2=1}^{m_{j_1}} q_{j_1 j_2}$ which takes the boundary of $P_{j_1 j_2}$ into the boundary of $Q_{j_1 j_2}$, for every $j_2 = 1, 2, \dots, m_{j_1}$. For every positive integer k, having defined the homeomorphism $(\varphi(x), \varphi^{-1}(y))$ between $P - \bigcup p_{j_1 j_2 \dots j_{k-1}}$ and $Q - \bigcup q_{j_1 j_2 \dots j_{k-1}}$, where the union is taken over all $j_1 \leqq m_1$, $j_2 \leqq m_{j_1}, \dots, j_{k-1} \leqq m_{j_1 j_2 \dots j_{k-2}}$, the homeomorphism $(\varphi(x), \varphi^{-1}(y))$ between the boundary of $P_{j_1 j_2 \dots j_{k-1}}$ and the boundary of $Q_{j_1 j_2 \dots j_{k-1}}$ may, for every $j_1 \leqq m_1$, $j_2 \leqq m_{j_1}, \dots$, $j_{k-1} \leqq m_{j_1 j_2 \dots j_{k-2}}$, be extended to a homeomorphism between

$$P_{j_1 j_2 \dots j_{k-1}} - \bigcup_{j_k=1}^{m_{j_1 j_2 \dots j_{k-1}}} p_{j_1 j_2 \dots j_k}$$

and

$$Q_{j_1 j_2 \dots j_{k-1}} - \bigcup_{j_k=1}^{m_{j_1 j_2 \dots j_{k-1}}} q_{j_1 j_2 \dots j_k}.$$

Since $S = \bigcap_{k=1}^{\infty} (\bigcup P_{j_1 j_2 \dots j_k})$ and $T = \bigcap_{k=1}^{\infty} (\bigcup Q_{j_1 j_2 \dots j_k})$, where the union is taken over all $j_1 \leqq m_1$, $j_2 \leqq m_{j_1}, \dots, j_k \leqq m_{j_1 j_2 \dots j_{k-1}}$, $(\varphi(x), \varphi^{-1}(y))$ is a one-one transformation between $P - S$ and $Q - T$. By letting $\varphi(x) = f(x)$ for every $x \in S$, $(\varphi(x), \varphi^{-1}(y))$ becomes a one-one transformation between P and Q which is an extension of the homeomorphism $(f(x), f^{-1}(y))$ between S and T. For every $x \in S$ and $\varepsilon > 0$, there are $p_{j_1 j_2 \dots j_k}$, and $q_{j_1 j_2 \dots j_k}$ of diameters less than ε, such that $x \in p_{j_1 j_2 \dots j_k}$, $\varphi(x) \in q_{j_1 j_2 \dots j_k}$, and $q_{j_1 j_2 \dots j_k} = \varphi(p_{j_1 j_2 \dots j_k})$, Accordingly, $\varphi(x)$ is continuous at x. For every $x \in P - S$, there is a k such that $x \notin \bigcup P_{j_1 j_2 \dots j_k}$, where the union is taken over all $j_1 \leqq m_1$, $j_2 \leqq m_{j_1}, \dots, j_k \leqq m_{j_1 j_2 \dots j_{k-1}}$, so that it follows from the above construction that $\varphi(x)$ is continuous at x. Hence, $\varphi(x)$ is continuous on P. Similarly, $\varphi^{-1}(y)$ is continuous on Q. This shows that $(\varphi(x), \varphi^{-1}(y))$ is a homeomorphism between P and Q which is an extension of the homeomorphism $(f(x), f^{-1}(y))$ between S and T.

A result similar to that of Theorem 1 holds even if $n \neq m$. Of course, a given homeomorphism between sectionally zero dimensional closed subsets of an n dimensional p-set P and an m dimensional p-set Q, $n < m$, cannot now be extended to a homeomorphism between P and Q. However, it can be extended to a homeomorphism between P and a proper subset of Q. Constructions similar to the one which will be given here have been used by Nöbeling [11] and Besicovitch [12], in their work on surface area.

Theorem 2. If $1 \leqq n < m$, P is an n dimensional p-set and Q is an m dimensional p-set, and S and T are sectionally zero dimensional closed subsets of the interiors of P and Q, respectively, then every homeomorphism $(f(x), f^{-1}(y))$ between S and T may be extended to a homeomorphism between P and a subset of Q.

Proof. I shall dwell only upon those points at which the proof differs from that of Theorem 1. Lemmas 1, 2, and 4 remain valid for $1 \leqq n < m$. The families $S_{j_1 j_2 \dots j_k}$ and $T_{j_1 j_2 \dots j_k}$ of sectionally zero dimensional closed sets, $P_{j_1 j_2 \dots j_k}$ of n dimensional p-sets, and $Q_{j_1 j_2 \dots j_k}$ of m dimensional p-sets, for $k = 1, 2, \dots, j_1 \leqq m_1$, $j_2 \leqq m_{j_1}, \dots, j_k \leqq m_{j_1 j_2 \dots j_{k-1}}, \dots$, may, accordingly, be constructed just as for the case $n = m \geqq 2$. Let R be an n dimensional closed parallelopiped contained in the boundary of Q. Let $R_1, R_2, \dots, R_{m_1}$ be disjoint n dimensional closed parallelopipeds contained in the interior of R, and for every $j_1 \leqq m_1$, let U_{j_1} be an n dimensional closed parallelopiped contained in the boundary of Q_{j_1}. Now, for every $j_1 \leqq m_1$, the boundary of R_{j_1} may be connected to the boundary of U_{j_1} by means of a pipe lying in the interior of Q, whose surface Z_{j_1} is an n dimensional closed polyhedron such that if $j_1 \neq j_1'$ then Z_{j_1}, $Z_{j'_1}$ are disjoint. There is a homeomorphism $(\varphi(x), \varphi^{-1}(y))$ between $P - \bigcup_{j_1=1}^{m_1} p_{j_1}$ and $(R - \bigcup_{j_1=1}^{m_1} r_{j_1}) \cup (\bigcup_{j_1=1}^{m_1} Z_{j_1})$ which takes the boundary of P_{j_1} into the boundary of U_{j_1}, for every $j_1 \leqq m_1$. For every $j_1 \leqq m_1$, let $R_{j_1 1}, R_{j_1 2}, \dots, R_{j_1 m_{j_1}}$ be disjoint n dimensional closed parallelopipeds in the interior of U_{j_1} and, for every $j_2 \leqq m_{j_1}$, let $U_{j_1 j_2}$ be an n dimensional closed parallelopiped contained in the boundary of $Q_{j_1 j_2}$. For every $j_2 \leqq m_{j_1}$, the boundary of $R_{j_1 j_2}$ may be connected to the boundary of $U_{j_1 j_2}$ by means of a pipe, lying in the interior of Q_j, whose surface $Z_{j_1 j_2}$ is an n dimensional polyhedron such that if $j_2 \neq j_2'$ then $Z_{j_1 j_2}$, $Z_{j_1 j'_2}$ are disjoint. The homeomorphism $(\varphi(x), \varphi^{-1}(y))$ between the boundaries of P_{j_1} and U_{j_1} may be extended to a homeomorphism $(\varphi(x), \varphi^{-1}(y))$ between $P_{j_1} - \bigcup_{j_2=1}^{m_{j_1}} p_{j_1 j_2}$ and $(U_{j_1} - \bigcup_{j_2=1}^{m_{j_1}} r_{j_1 j_2}) \cup (\bigcup_{j_2=1}^{m_{j_1}} Z_{j_1 j_2})$ which takes the boundary of $P_{j_1 j_2}$ into the boundary of $U_{j_1 j_2}$, for every $j_2 \leqq m_{j_1}$. By repeating the extension of the homeomorphism for all $k=1, 2, \ldots$, as in the proof of Theorem 1, a homeomorphism is obtained between $P-S$ and a subset of $Q-T$. That this homeomorphism may be extended to one between P and a subset of Q which contains T and is such that $\varphi(x)=f(x)$, for every $x \in S$, follows by a slight modification of the argument used in the proof of Theorem 1.

For the case $n=m=1$, one can easily find one-one transformations between finite sets in I_n and I_m which cannot be extended to homeomorphisms between I_n and I_m. But every one-one transformation between finite sets is a homeomorphism, and every finite set is a sectionally zero dimensional closed set, so that Theorem 1 does not hold for this case.

3. Application to one-one measurable transformations. As stated in the introduction, a one-one measurable transformation, $(f(x), f^{-1}(y))$, between an n dimensional open cube I_n and an m dimensional open cube I_m is one for which $f(x)$ and $f^{-1}(y)$ are both measurable functions. That is to say, for all Borel sets $T \subset I_m$ and $S \subset I_n$, the sets $f^{-1}(T) \subset I_n$ and $f(S) \subset I_m$ are measurable.

A remark concerning this definition seems to be appropriate. That the measurability of $f^{-1}(y)$ does not follow from that of $f(x)$ is shown by the following example: Let I and J be open unit intervals (0,1). Let $S \subset I$ be a Borel set of measure zero, but of the same cardinal number c as the continuum, and $T \subset J$ a Borel set of positive measure such that $J-T$ is also of positive measure. Then T contains disjoint non-measurable sets T_1 and T_2, both of cardinal number c, such that $T=T_1 \cup T_2$; and S contains disjoint Borel sets S_1 and S_2, both of cardinal number c, such that $S=S_1 \cup S_2$. Define $(f(x), f^{-1}(y))$ by means of a one-one correspondence between $I-S$ and $J-T$ which takes every Borel set in $I-S$ into a measurable set in $J-T$ and every Borel set in $J-T$ into a measurable set in $I-S$, and by means of arbitrary one-one correspondences between S_1 and T_1 and between S_2 and T_2. The function $f(x)$ is measurable. For, let B be any Borel set in J. Then $B=B_1 \cup B_2$, where $B_1=B \cap (J-T)$ and $B_2=B \cap T$ are also Borel set. But $f^{-1}(B_1)$ is measurable and $f^{-1}(B_2)$ is of measure zero, so that $f^{-1}(B)$ is measurable. The function $f^{-1}(y)$ is non-measurable, since S_1 is a Borel set and $T_1=f(S_1)$ is non-measurable.

On the other hand, if $(f(x), f^{-1}(y))$ is a one-one transformation such that $f(x)$ is measurable and takes all sets of measure zero into sets of measure zero, then $f^{-1}(y)$ is also measurable, and $(f(x), f^{-1}(y))$ is a one-one measurable transformation. For, by the Vitali-Carathéodory theorem, there is a function $g(x)$, of Baire class 2 at most, such that $f(x)=g(x)$, except on a Borel set $Z\subset I$ of measure zero. Now $g(x)$ as a Baire function on an interval I, takes all Borel sets [13] in I into Borel sets in J. Let $B\subset I$ be a Borel set. Then B is the union of Borel sets $B_1\subset I-Z$ and $B_2\subset Z$. Since $f(B_1)=g(B_1)$ is a Borel set and $f(B_2)$ is of measure zero, $f(B)$ is measurable, so that $f^{-1}(y)$ is a measurable function.

The usual form of Lusin's Theorem [14] is that for every measurable real valued function $f(x)$ defined, say, on an open n dimensional unit cube I_n, and for every $\varepsilon>0$, there is a closed set $S\subset I_n$, whose n dimensional measure exceeds $1-\varepsilon$, such that $f(x)$ is continuous on S relative to S. Since every measurable function on I_n with values in an m dimensional cube I_m is given by m measurable real valued functions, and the continuous functions on a set $S\subset I_n$ relative to S, with values in I_m, are those for which the corresponding set of m real functions are all continuous on S relative to S, the theorem is readily seen to hold just as well for functions on I_n with values in I_m. Moreover, the following result is valid for one-one measurable transformations.

Theorem 3. If $(f(x),\ f^{-1}(y))$ is a one-one measurable transformation between open n dimensional and m dimensional unit cubes I_n and I_m, where n and m are any positive integers then, for every $\varepsilon>0$, there is a closed set $S\subset I_n$ of n dimensional measure greater than $1-\varepsilon$ and a closed set $T\subset I_m$ of m dimensional measure greater than $1-\varepsilon$ such that $(f(x),\ f^{-1}(y))$ is a homeomorphism between S and T.

Proof. It is known [15] that if $(\varphi(x),\ \varphi^{-1}(y))$ is a one-one transformation between a closed set $S\subset\mathfrak{S}$ and a set $T\subset\mathfrak{J}$, where $\mathfrak{S}$ and $\mathfrak{J}$ are subsets of compact sets, and if $\varphi(x)$ is continuous, then T is a closed set and $\varphi^{-1}(y)$ is continuous, so that $(\varphi(x),\ \varphi^{-1}(y))$ is a homeomorphism. This assertion holds for the case $\mathfrak{S}=I_n$, $\mathfrak{J}=I_m$, since their closures are compact sets. Since $f(x)$ is measurable, there is a closed set $S\subset I_n$, of n dimensional measure greater than $1-\varepsilon$, such that $f(x)$ is continuous on S relative to S. The set $f(S)$ is a closed subset of I_m, and $f^{-1}(y)$ is continuous on $f(S)$ relative to $f(S)$. The complement, $\complement f(S)$, is measurable, and the function $f^{-1}(y)$ defined on it is measurable. Accordingly, again by Lusin's Theorem, there is a closed subset T of $\complement f(S)$, whose measure exceeds $m(\complement f(S))-\varepsilon$, such that $f^{-1}(y)$ is continuous on T relative to T. The set $f^{-1}(T)$ is closed and $f(x)$ is continuous on $f^{-1}(T)$ relative to $f^{-1}(T)$. Now, the set $S\cup f^{-1}(T)$ is closed and of n dimensional measure greater than $1-\varepsilon$, the set $T\cup f(S)$ is closed and of m dimensional measure greater than $1-\varepsilon$. The transformation $(f(x),\ f^{-1}(y))$ is a homeomorphism between $S\cup f^{-1}(T)$ and $T\cup f(S)$. For, the fact that $f(x)$ is continuous on $S\cup f^{-1}(T)$ relative to $S\cup f^{-1}(T)$

follows from the facts that it is continuous on S relative to S and on $f^{-1}(T)$ relative to $f^{-1}(T)$ and that S and $f^{-1}(T)$, as disjoint closed sets, have positive distance from each other. The function $f^{-1}(y)$ is continuous on $T \cup f(S)$ relative to $T \cup f(S)$ for similar reasons.

Theorem 4. The sets S and T of Theorem 3 may be taken to be sectionally zero dimensional closed sets.

Proof. Let $U \subset I_n$ and $V \subset I_m$ be closed sets, U of n dimensional measure greater than $1-\varepsilon/2$ and V of m dimensional measure greater than $1-\varepsilon/2$, such that $(f(x), f^{-1}(y))$ is a homeomorphism between U and V. For convenience, I shall designate the intersection of a hyperplane π with the open cube I_n by π and shall refer to this intersection as the hyperplane. Among all hyperplanes π which are parallel to faces of I_n, there is only a finite or denumerable number for which the set $f(\pi)$ is of positive m dimensional measure. For, if the set of hyperplanes with this property were non-denumerable, then a non-denumerable number of them would be parallel to one of the faces of I_n. Then, for some positive integer k, an infinite number of these hyperplanes π would be such that the m dimensional measure of $f(\pi)$ exceeds $1/k$. This contradicts the fact that $m(I_m)=1$, where the notation $m(S)$ will henceforth indicate m dimensional measure for subsets of I_m and n dimensional measure for subsets of I_m. It then follows that for every face of I_n, there is a denumerable set of hyperplanes parallel to the face, whose union is dense in I_n, such that $m(f(\pi))=0$ for every hyperplane π in the set. As the union of a finite number of denumerable sets, this totality of hyperplanes is denumerable in number, and so it may be ordered as

$$\pi_1, \pi_2, \ldots, \pi_k, \ldots.$$

I associate with each π_k an open set G_k, as follows: For every positive integer r, let G_{kr} be the set of all points in I_n whose distance from π_k is less than $1/r$. Since $f(\pi_k)=\bigcap_{r=1}^{\infty} f(G_{kr})$, the sets $f(G_{kr})$ are non-increasing, and $m(f(\pi_k))=0$, there is an r_k for which $m(f(G_{kr_k}))<\eta/2^k$, where $\eta=\varepsilon/4$. Moreover, r_k may be taken so large that $m(G_{kr_k})<\eta/2^k$. Let $G=\bigcup_{k=1}^{\infty} G_{kr_k}$. Then I_n-G is a sectionally zero dimensional closed set of n dimensional measure greater than $1-\eta$ such that $f(I_n-G)$ is of m dimensional measure greater than $1-\eta$. In the same way, there is an $H \subset I_m$ for which I_m-H is a sectionally zero dimensional closed set of m dimensional measure greater than $1-\eta$ such that $f^{-1}(I_m-H)$ is of n dimensional measure greater than $1-\eta$. The set $(I_m - H) \cap V$ is sectionally zero dimensional, closed, and of m dimensional measure greater than $1-(\varepsilon/2+\eta)$; and $f^{-1}[(I_m - H) \cap V]$ is closed and of n dimensional measure greater than $1-(\varepsilon/2+\eta)$. Then, the set $S = f^{-1}[(I_m - H) \cap V] \cap (I_n - G)$ is a closed, sectionally zero dimensional set of n dimensional measure greater than $1-(\varepsilon/2+\eta+\eta) = 1-\varepsilon$, whose image $T = f(S)$ is a closed, sectionally zero dimensional set of m dimensional measure greater than $1-\varepsilon$. Since $S \subset U$, the transformation $(f(x), f^{-1}(y))$ is a homeomorphism between S and T.

The main results of this paper now follow:

Theorem 5. If $n = m \geqq 2$, I_n and I_m are n dimensional open unit cubes, and $(f(x), f^{-1}(y))$ is a one-one measurable transformation between I_n and I_m, then for every $\varepsilon > 0$, there is a homeomorphism $(g(x), g^{-1}(y))$ between I_n and I_m such that $f(x) = g(x)$ and $f^{-1}(y) = g^{-1}(y)$ on sets whose n dimensional measures exceed $1-\varepsilon$.

Proof. By Theorem 4, $(f(x), f^{-1}(y))$ is a homeomorphism between sectionally zero dimensional closed sets $S \subset I_n$ and $T \subset I_m$, both of whose n dimensional measures exceed $1-\varepsilon$. Let $(g(x), g^{-1}(y))$ be the extension of this homeomorphism between S and T to a homeomorphism between I_n and I_m, whose existence is assured by Theorem 1.

That Theorem 5 does not hold for the case $n = m = 1$ is shown by the following one-one measurable transformation between $I_n = (0,1)$ and $I_m = (0,1)$:

$$\begin{aligned} f(x) &= x + 1/2 && 0 < x < 1/2 \\ &= x - 1/2 && 1/2 < x < 1 \\ &= 1/2 && x = 1/2. \end{aligned}$$

Suppose $(g(x), g^{-1}(y))$ is a homeomorphism between I_n and I_m. Then $g(x)$ is either strictly increasing or strictly decreasing on I_n. If $g(x)$ is strictly decreasing, then $f(x) = g(x)$ for at most three values of x. If $g(x)$ is strictly increasing, then if there is a ξ such that $0 < \xi < 1/2$ and $f(\xi) = g(\xi)$, it follows that $f(x) \neq g(x)$ for every x such that $1/2 < x < 1$. In either case, the set on which $f(x) = g(x)$ is of measure not greater than $1/2$.

Theorem 6. If $1 \leqq n < m$, I_n is an n dimensional open unit cube, I_m is an m dimensional open unit cube, and $(f(x), f^{-1}(y))$ is a one-one measurable transformation between I_n and I_m, then for every $\varepsilon > 0$, there is a homeomorphism $(g(x), g^{-1}(y))$ between I_n and a subset of I_m, such that $f(x) = g(x)$ on a set whose n dimensional measure exceeds $1-\varepsilon$ and $f^{-1}(y) = g^{-1}(y)$ on a set whose m dimensional measure exceeds $1-\varepsilon$.

Proof. Just as in the proof of Theorem 5 except that Theorem 2 is needed instead of Theorem 1.

In Theorem 6, the subset of I_m into which I_n is taken by $g(x)$ is of m dimensional measure greater than $1-\varepsilon$. I show now that it cannot be of m dimensional measure 1. For, suppose $(g(x), g^{-1}(y))$ is a homeomorphism between I_n and a subset U of I_m of m dimensional measure 1. Then U is dense in I_m. Let $x \in I_n$ and $y=g(x) \in U$. Let $\{I_{nk}\}$ be the sequence of closed cubes concentric with I_n such that, for every k, the n dimensional measure of I_{nk} is $1-1/k$. The set $g(I_{nk})$ is a closed subset of U which is nowhere dense in I_m since, otherwise, as a closed set, it would contain an m dimensional sphere, making an n dimensional set homeomorphic with an $m>n$ dimensional set. The sphere σ_k of center y and radius $1/k$, accordingly, contains a point $y_k \in U$ such that $y_k \notin g(I_{nk})$. The sequence $\{y_k\}$ converges to y, but the distances from the boundary of I_n of the elements of the sequence $\{g^{-1}(y_k)\}$ converge to zero so that the sequence does not converge to x, and the function $g^{-1}(y)$ is not continuous. This contradicts the assumption that $(g(x), g^{-1}(y))$ is a homeomorphism. The following theorem should be of interest in this connection.

Theorem 7. If $1 \leqq n < m$, I_n is an open n dimensional unit cube, I_m is an open m dimensional unit cube, and $(f(x), f^{-1}(y))$ is a one-one measurable transformation between I_n and I_m, then, for every $\varepsilon>0$, there is a one-one transformation $(g(x), g^{-1}(y))$ between I_n and a subset of I_m of m dimensional measure 1, such that $g(x)$ is continuous, $f(x)=g(x)$ on a set of n dimensional measure greater than $1-\varepsilon$, and $f^{-1}(y)=g^{-1}(y)$ on a set of m dimensional measure greater than $1-\varepsilon$.

Proof. By Theorem 4, there are sectionally zero dimensional sets $S \subset I_n$ and $T \subset I_m$ such that $(f(x), f^{-1}(y))$ is a homeomorphism between S and T, and the n dimensional measure of S and m dimensional measure of T both exceed $1-\varepsilon$. The distance of S from the boundary of I_n is positive, so that there is a closed cube I_{n1} in I_n such that S is contained in the interior of I_{n1}. The homeomorphism $(f(x), f^{-1}(y))$ between S and T may be extended, by Theorem 2, to a homeomorphism $(g_1(x), g_1^{-1}(y))$ between I_{n1} and a subset, E_1, of I_m whose boundary is the boundary of an n dimensional cube. Now, let I_{n1} be the first member of an increasing sequence

$$I_{n1}, I_{n2}, \ldots, I_{nk}, \ldots$$

of closed unit cubes whose union is I_n, each of which is contained in the interior of its immediate successor, and let

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots$$

be a decreasing sequence of positive numbers which converges to zero. The set E_1, as a closed homeomorphic image of an n dimensional set, is nowhere dense in I_m. Let $T_1 \subset I_m - E_1$ be a sectionally zero dimensional closed set such that the m dimensional measure of $I_m - (E_1 \cup T_1)$ is less than ε_2. Now, T_1 may be taken to be the intersection of a decreasing sequence of sets each of which consists of a finite number of disjoint closed m dimensional cubes contained in $I_m - E_1$, so that the homeomorphism $(g_1(x), g_1^{-1}(y))$ between I_{n1} and E_1 may then be extended, in the manner described by Besicovitch [12], to a homeomorphism $(g_2(x), g_2^{-1}(y))$ between I_{n2} and a closed subset $E_2 \supset T_1$ of I_m, of m dimensional measure greater than $1-\varepsilon_2$, whose boundary is the boundary of a n dimensional cube. In this way, the sequence of homeomorphisms $(g_1(x), g_1^{-1}(y))$, $(g_2(x), g_2^{-1}(y))$, . . ., $(g_k(x), g_k^{-1}(y))$, . . ., each of which is an extension of its immediate predecessor, such that, for every k, $(g_k(x), g_k^{-1}(y))$ is a homeomorphism between I_{nk} and a subset E_k of I_m of m dimensional measure greater than $1-\varepsilon_k$, is obtained. The sequence $\{g_k(x)\}$ converges to a function $g(x)$ defined on I_n which has an inverse $g^{-1}(y)$. The one-one transformation $(g(x), g^{-1}(y))$ evidently has the desired properties.

Theorem 5 has the following interpretation. For any two one-one measurable transformations $\mathfrak{J}_1 : (f_1(x), f_1^{-1}(y))$ and $\mathfrak{J}_2 : (f_2(x), f_2^{-1}(y))$ between a given n dimensional open unit cube I_n, $n \geqq 2$, and itself, let

$$\delta(\mathfrak{J}_1, \mathfrak{J}_2) = m(E) + m(F),$$

where E is the set of points for which $f_1(x) \neq f_2(x)$, F is the set of points for which $f_1^{-1}(y) \neq f_2^{-1}(y)$, and $m(E)$ and $m(F)$ are their n dimensional measures. If $\mathfrak{J}_1$ is equivalent to $\mathfrak{J}_2$ whenever $\delta(\mathfrak{J}_1, \mathfrak{J}_2)=0$, the equivalence classes obtained in the usual way are readily seen to form a metric space. Theorem 5 may now be restated:

Theorem 5'. The set of homeomorphisms is dense in the metric space of all one-one measurable transformations between an n dimensional open cube I_n, $n \geqq 2$, and itself.

A different distance between transformations has been introduced by P. R. Halmos [16] in his work on measure preserving transformations. A metric similar to the one used by Halmos could be introduced here. Theorem 5' could then be stated in terms of this metric δ', since it would follow that $\delta' \leqq \delta$ for every pair of transformations.

4. Related results and questions. The theorem of Vitali-Carathéodory says that for every measurable $f(x)$ on, say, the open interval (0,1) there is a $g(x)$ on (0,1), of Baire class 2 at most, such that $f(x) = g(x)$ almost everywhere. I prove the following analogous theorem for one-one measurable transformations.

Theorem 8. If $(f(x), f^{-1}(y))$ is a one-one measurable transformation between $I=(0,1)$ and $I=(0,1)$ there is a one-one transformation $(g(x), g^{-1}(y))$ between I and J such that $g(x)$ and $g^{-1}(y)$ are of Baire class 2 at most and are such that $f(x)=g(x)$ and $f^{-1}(y)=g^{-1}(y)$ almost everywhere.

Proof. The proof depends upon the following relations between Baire functions and Borel sets (17]. A real function $f(x)$, defined on a set S, is continuous relative to S if and only if, for every k, the set of points for which $f(x)<k$ is open relative to S and the set of points for which $f(x)\leqq k$ is closed relative to S; it is of at most Baire class 1 relative to S if and only if the sets of points for which $f(x)<k$ and $f(x)\leqq k$ are of types F_σ and G_δ relative to S, respectively; it is of at most Baire class 2 relative to S if and only if the sets of points for which $f(x)<k$ and $f(x)\leqq k$ are of types $G_{\delta\sigma}$ and $F_{\sigma\delta}$ relative to S, respectively. Now, by Theorem 4, there are closed sets $S_1\subset I$, and $T_1\subset J$, each of measure greater than $1/2$, such that $(f(x), f^{-1}(y))$ is a homeomorphism between S_1 and T_1. Again, by Theorem 4, there are closed sets $S_2\supset S_1$ and $T_2\supset T_1$, each of measure greater than $3/4$, such that $(f(x), f^{-1}(y))$ is a homeomorphism between S_2 and T_2. In this way, obtain increasing sequences $S_1\subset S_2\subset \ldots \subset S_n\subset \ldots$ and $T_1\subset T_2\subset \ldots \subset T_n\subset \ldots$, such that $S=\lim S_n$ and $T=\lim T_n$ are both of measure 1, $(f(x), f^{-1}(y))$ is a one-one transformation between S and T, and for every n, S_n and T_n are closed sets and $(f(x), f^{-1}(y))$ is a homeomorphism between them. Moreover, the sets S_n and T_n may be taken to be zero dimensional, hence nowhere dense, so that S and T are sets of type F_σ which are of the first category. $f(x)$ is of Baire class 1 on S relative to S. For, by the Tietze extension theorem [18], the continuous function $f(x)$ on S_n relative to S_n may be extended to a continuous function $\varphi_n(x)$ on I. The functions of the sequence $\{\varphi_n(x)\}$ are all continuous on S relative to S and converge to $f(x)$ on S so that $f(x)$ is of at most Baire class 1 on S relative to S. Similarly, $f^{-1}(y)$ is of at most Baire class 1 on T relative to T. Since S and T are of type F_σ, of measure 1, and of the first category, the sets $I-S$ and $J-T$ are of type G_δ, of measure 0, and residual. Since they are of measure 0, they are frontier sets, and since residual they are everywhere dense. By a theorem of Mazurkiewicz [19], they are accordingly homeomorphic to the set of irrationals and hence to each other. Let $(\varphi(x), \varphi^{-1}(y))$ be a homeomorphism between $I-S$ and $J-T$. Let

$$
\begin{aligned}
g(x) &= f(x) & & x\in S \\
&= \varphi(x) & & x\in I-S.
\end{aligned}
$$

Then $(g(x), g^{-1}(y))$ is a one-one transformation between I and J. For every k, the set of points of S for which $f(x)<k$ is of type F_σ relative to the set S of type F_σ, and so is of type F_σ relative to I; and the set of points of $I-S$ for which $\varphi(x)<k$ is open relative to the set $I-S$ of type G_δ, and so is of type G_δ relative to I. Hence, the set of points of I for which $g(x)<k$, as the union of sets of type F_σ and G_δ is of type $G_{\delta\sigma}$ relative to I. In the same way, the set of points of S for which $f(x)\leqq k$ is of type $F_{\sigma\delta}$ relative to I, and the set of points of $I-S$ for which $\varphi(x)\leqq k$ is of type G_δ relative to I, so that the set of points of I for which $g(x)\leqq k$, as the union of sets of type $F_{\sigma\delta}$ and of type G_δ, is of type $F_{\sigma\delta}$ relative to I. Hence, $g(x)$ is of Baire class 2 at most. Similarly, $g^{-1}(y)$ is of Baire class 2 at most.

The method used here does not seem to apply to higher dimensional transformations, and I have not found a way to treat this problem in such cases.

The following converse to Theorem 8 holds.

Theorem 9. There is a one-one measurable transformation $(f(x), f^{-1}(y))$ between open unit intervals $I=(0,1)$ and $J=(0,1)$ such that, for every one-one transformation $(g(x), g^{-1}(y))$ between I and J for which $f(x)=g(x)$ and $f^{-1}(y)=g^{-1}(y)$ almost everywhere, the functions $g(x)$ and $g^{-1}(y)$ are both of Baire class 2 at least.

Proof. I first note that there is a Borel set S such that both S and its complement $I-S$ are of positive measure in every subinterval of I. For, if $S_1, S_2, \ldots, S_n, \ldots$ is a sequence of nowhere dense closed sets, such that S_n has positive measure in each of the intervals

$$I_{n1}=(0,1/n),\ I_{n2}=(1/n,\ 2/n),\ \ldots,\ I_{nn}=(1-1/n,\ 1)$$

and, for every n,

$$m(S_n)=1/3\ \min\ [m(I_{ni}-\bigcup_{j=1}^{n-1} S_j);\ \ i=1,\ 2,\ \ldots,\ n],$$

the set $S=\bigcup_{n=1}^{\infty} S_n$ has this property. Now, let S be a Borel subset of $(0,\ 1/2)$ such that both S and its complement have positive measure in every subinterval of $(0,\ 1/2)$. Let $S+1/2$ be the set obtained by adding $1/2$ to all the points in S. Now, let

$$f(x)=\begin{cases} x & x\in S \\ x+1/2 & x\in I-S \\ x & x\in S+1/2 \\ x-1/2 & x\in(I-S)+1/2 \\ x & x=1/2. \end{cases}$$

The function $f(x)$ has an inverse $f^{-1}(y)$. Suppose $g(x)=f(x)$ almost everywhere. Since every interval contains a set of positive measure on which $f(x)<1/2$ and a set of positive measure on which $f(x)>1/2$, the same holds for $g(x)$. Then $g(x)$ is discontinuous wherever $g(x)\neq\frac{1}{2}$ (i.e., almost everywhere) and so is not of Baire class 1. Similarly, if $g^{-1}(y)=f^{-1}(y)$ almost everywhere, it is not of Baire class 1.

One might ask if whenever one-one measurable transformations are absolutely measurable or measure preserving the approximating homeomorphisms of Theorems 5 and 6 may also be taken to be absolutely measurable or measure preserving. I have not yet considered these matters.

Finally, I obtain a decomposition theorem for one-one measurable transformations analogous to the Hahn decomposition theorem for measures [20]:

Theorem 10. If $(f(x), f^{-1}(y))$ is a one-one measurable transformation between I_n and I_m, $1\leqq n\leqq m$, I_n has a decomposition into three disjoint Borel sets S_1, S_2, and S_3, some of which might be empty, such that S_1 is of n dimensional measure zero, $f(S_2)$ is of m dimensional measure zero, and $(f(x), f^{-1}(y))$ is a one-one absolutely measurable transformation between S_3 and $f(S_3)$.

Proof. Consider the set $\mathfrak{F}_1$ of all closed sets in I_n whose n dimensional measures are positive but which are taken by $f(x)$ into sets of m dimensional measure zero. Let $F_1\in\mathfrak{F}_1$ be such that its measure is not less than half the measure of any set in $\mathfrak{F}_1$. Consider the set $\mathfrak{F}_2$ of all closed sets in I_n-F_1 whose n dimensional measures are positive but which are taken by $f(x)$ into sets of m dimensional measure zero. In this way, obtain a sequence of disjoint closed sets $F_1, F_2, \ldots, F_k, \ldots$ each of positive n dimensional measure, each taken by $f(x)$ into a set of m dimensional measure zero, such that for every k, the n dimensional measure of F_k is more than half the n dimensional measure of any closed subset of $I_n-\bigcup_{j=1}^{k-1}F_j$ which is taken by $f(x)$ into a set of m dimensional measure zero. Let $F=\bigcup_{k=1}^{\infty}F_k$. Obtain an analogous sequence $K_1, K_2, \ldots, K_k, \ldots$ of disjoint closed subsets of $I_m-f(F)$ and let $K=\bigcup_{k=1}^{\infty}K_k$. Now, $f(F)$ is of m dimensional measure zero and $f^{-1}(K)$ is of n dimensional measure zero. Let $S_1=f^{-1}(K)$, $S_2=F$, and $S_3=I_n-(F\cup f^{-1}(K))$. Let $E\subset S_3$ be a measurable set such that $f(E)$ is of m dimensional measure zero. Suppose E is of positive n dimensional measure. Then E contains a closed subset S of positive n dimensional measure. But the measure of S then exceeds twice the measure of F_k, for some k, and so S should appear in the sequence $F_1, F_2, \ldots$ instead of F_k. Hence E must be of n dimensional measure zero. Similarly, every measurable subset of $f(S_3)$ which is taken by $f^{-1}(y)$ into a set of n dimensional measure zero is itself of m dimensional measure zero. The transformation $(f(x), f^{-1}(y))$ between S_3 and $f(S_3)$ is, accordingly, absolutely measurable.

Bibliography.

1. W. SIERPINSKI, *Démonstration de quelques théorèmes fondamentaux sur les fonctions mesurables*, Fund. Math., vol. 3 (1922), p. 319.
2. C. GOFFMAN, *Proof of a Theorem of Saks and Sierpinski*, Bull. Amer. Math. Soc., vol. 54 (1948), pp. 950—952.
3. N. LUSIN, *Sur les propriétés des fonctions mesurables*, C. R. Acad. Sci. Paris, vol. 154 (1912), pp. 1688—1690.
4. H. BLUMBERG, *New Properties of All Real Functions*, Trans. Amer. Math. Soc., vol. 24 (1922), pp. 113—128.
5. H. RADEMACHER, *Eineindeutige Abbildungen und Messbarkeit*, Monatshefte für Mathematik und Physik, vol. 27 (1916), pp. 183—290.
6. C. GOFFMAN, *The approximation of Arbitrary Biunique Transformations*, Duke Math. J., vol. 10 (1943), pp. 1—4.
7. E. J. MCSHANE, *Integration*, Princeton, 1947, p. 237.
8. J. VON NEUMANN, *Functional Operators I*, Princeton, 1950, p. 77.
9. W. A. BLANKINSHIP, *Generalization of a Construction of Antoine*, Annals of Math., vol. 53 (1951), pp. 276.
10. W. HUREWICZ and H. WALLMAN, *Dimension Theory*, Princeton, 1941, Chapter 2.
11. G. NÖBELING, *Über die Flächenmasse im Euklidischen Raum*, Mathematische Annalen, vol. 118 (1941—3), pp. 687—701.
12. A. S. BESICOVITCH, *On the definition and Value of the Area of a Surface*, Oxford Quarterly Journal of Math., vol. 16 (1945), pp. 86—102.
13. W. SIERPINSKI, *Sur l'inversion des fonctions représentables analytiquement*, Fund. Math., vol. 3 (1922), pp. 26—34.
14. L. W. COHEN, *A New Proof of Lusin's Theorem*, Fund Math., vol. 9 (1927), pp. 122—3.
15. C. KURATOWSKI, *Topologie II*, Warsaw, 1950, p. 11.
16. P. R. HALMOS, *Approximation Theories for Measure Preserving Transformations*, Trans. Amer. Math. Soc., vol. 55 (1944), pp. 1—18.
17. F. HAUSDORFF, *Mengenlehre*, Leipzig, 1927.
18. H. TIETZE, *Über Funktionen, die auf einer abgeschlossenen Menge stetig sind*. Journal für die Reine und Angewandete Math., vol. 145 (1915), pp. 9—14.
19. C. KURATOWSKI, *Topologie I*, Warsaw, 1933, p. 225.
20. P. R. HALMOS, *Measure Theory*, New York, 1950, p. 121.

University of Oklahoma Norman, Oklahoma.