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Behavior of the Bergman projection on the Diederich–Fornæss worm

by

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§ 1. Introduction

In this paper we show that the Bergman projection operator for certain smooth bounded pseudoconvex domains does not preserve smoothness as measured by Sobolev norms.

Let Ω be a smooth bounded pseudoconvex domain in $\mathbf{C}^n$ and let $P^{(t)}$ denote the orthogonal projection from $L^2(\Omega)$ onto the Bergman subspace $B(\Omega)=L^2(\Omega)\cap\mathcal{O}(\Omega)$ with respect to the weighted norm $\|f\|^{(t)}=(\int_D|f(z)|^2\,e^{-t\|z\|^2}dV)^{1/2}$. Let $W^k(\Omega)$ denote the Sobolev space consisting of functions whose derivatives of order $\leqslant k$ are in $L^2(\Omega)$. An important result of Kohn [Ko1] implies that $P^{(t)}$ maps $W^k(\Omega)$ to $W^k(\Omega)$ when $t\geqslant t_0(k,\Omega)$. On the other hand, there is a large collection of results implying that for certain types of domains the unweighted Bergman projection $P=P^{(0)}$ preserves W^k for all $k\geqslant 0$. (See [FK] for the strictly pseudoconvex case; for results on weakly pseudoconvex domains the reader may consult [Ko2], [Ca], [Si] and the recent [BSt1], [Ch] as well as the references cited therein. Most of these results are focused on the $\bar\partial$-Neumann operator rather than P; see [BSt2] for the connection. Also, except for [Ko1], positive results in this area are typically valid for any choice of smooth positive weight function on $\bar\Omega$.)

The question of whether or not P is similarly well-behaved for *all* weakly pseudoconvex domains has remained open for many years. In this paper we show that this is not the case; in fact when Ω is the so-called "worm domain" of Diederich and Fornæss then P does not map W^k to W^k when $k\geqslant\pi/(\text{total amount of winding})$. This latter quantity is explained in section 4 below, where the construction of the worm is reviewed and the main result is proved. The proof depends on computations for a piecewise Levi-flat model domain depending in turn on certain one-dimensional computations; these are treated in sections 3 and 2, respectively. Section 5 contains additional remarks and questions.

The papers [Ba1] and [BF] provide examples of smooth bounded nonpseudoconvex domains with badly behaved Bergman projections. In these examples the behavior of the projection operator cannot be improved by switching to any norm equivalent to the standard L^2 norm.

This paper was inspired in part by Kiselman's related paper [Ki].

§ 2. Bergman kernels for strips with weights

For $\beta>0$ let S_β denote the one-dimensional strip

$$\{z=x+iy \in \mathbf{C} : |y|<\beta\}$$

and let $\omega(y)$ be a continuous positive bounded function on the interval $I_\beta := \{y : |y|<\beta\}$. In this section we use the Fourier transform to compute the reproducing kernel $K_\omega(z, w)$ for the weighted Bergman space

$$B_\omega = \left\{ f \text{ holomorphic on } S_\beta : \|f\|_\omega^2 = \int_{S_\beta} |f(z)|^2 \, \omega(y)\, dx\, dy < \infty \right\}.$$

For $f \in B_\omega$ the partial Fourier transform

$$\hat{f}(\xi, y) := \int_{\mathbf{R}} f(x+iy)\, e^{-ix\xi}\, dx$$

satisfies $\hat{f}(\xi, y) = e^{-y\xi}\hat{f}_0(\xi)$, where $\hat{f}_0(\xi) = \hat{f}(\xi, 0)$. Thus

$$\|f\|_\omega^2 = (2\pi)^{-1} \int_{\mathbf{R}\times I_\beta} e^{-2y\xi} |\hat{f}_0(\xi)|^2 \, \omega(y)\, d\xi\, dy$$

and an analogous formula holds for weighted inner products.

Let $w \in S_\beta$ and let $k \in B_\omega$ denote the holomorphic function $k(\cdot) = K_\omega(\cdot, w) = \overline{K_\omega(w, \cdot)}$. Then

$$\begin{aligned}
\int_{\mathbf{R}} \hat{f}_0(\xi)\, e^{iw\xi}\, d\xi &= 2\pi f(w) \\
&= 2\pi \int_{S_\beta} f(z) \overline{k(z)} \omega(y)\, dx\, dy \\
&= \int_{\mathbf{R}\times I_\beta} e^{-2y\xi} \hat{f}_0(\xi) \overline{\hat{k}_0(\xi)} \omega(y)\, d\xi\, dy
\end{aligned}$$

for $f \in B_\omega$. Thus

$$\hat{k}_0(\xi) \int_{I_\beta} e^{-2y\xi} \omega(y)\, dy = e^{-i\bar{w}\xi}$$

and

$$\hat{k}_0(\xi) = \hat{\omega}(-2i\xi)^{-1} e^{-i\bar{w}\xi},$$

where $\hat{\omega}$ denotes the Fourier–Laplace transform of ω. (Here we view ω as an integrable function on $\mathbf{R}$ vanishing outside of I_β.) By Fourier inversion we have

$$K_\omega(z, w) = (2\pi)^{-1} \int_{\mathbf{R}} \hat{\omega}(-2i\xi)^{-1} e^{i(z-\bar{w})\xi}\, d\xi.$$

The simplest case is to take $\omega = \chi_\beta$, where here and in the sequel we let χ_β denote the characteristic function of I_β. We have $\hat{\omega}(-2i\xi) = \xi^{-1} \sinh 2\beta\xi$ so that

$$K_\omega(z, w) = \frac{\pi}{16\beta^2} \left(\cosh \frac{\pi(z-\bar{w})}{4\beta} \right)^{-2}.$$

(This can of course also be computed by conformal mapping to the disk.)

We will be interested in the next section in the piecewise-linear weight

$$\omega = \pi \chi_{\beta-\pi/2} * \chi_{\pi/2},$$

where $*$ denotes convolution. (Here we assume that $\beta > \pi/2$.) Then

$$\hat{\omega}(-2i\xi) = \pi\xi^{-2} \sinh(2\beta-\pi)\,\xi \sinh \pi\xi$$

and

$$K_\omega(z, w) = \frac{1}{2\pi^2} \int_{\mathbf{R}} \frac{\xi^2}{\sinh(2\beta-\pi)\,\xi \sinh \pi\xi} e^{i(z-\bar{w})\xi}\, d\xi. \tag{2.1}$$

Now $\xi^2/\sinh(2\beta-\pi)\,\xi \sinh \pi\xi$ has poles at non-zero integer multiples of $\pi i/(2\beta-\pi)$ and i with good decay as $|\mathrm{Re}\,\xi| \to \infty$ so that standard contour integration arguments furnish asymptotic expansions for $K_\omega(z, w)$. In particular, if $\beta > \pi$ we have

$$K_\omega(z, w) = C_\beta e^{-\nu_\beta(z-\bar{w})} + O(e^{-\mu_\beta(z-\bar{w})})$$

for $\mathrm{Re}(z-\bar{w}) > 0$, where $\nu_\beta = \pi/(2\beta-\pi)$, $\mu_\beta = \min\{2\nu_\beta, 1\} > \nu_\beta$, and $C_\beta = \nu_\beta^3/\pi \sin \nu_\beta \pi$; similarly for $\mathrm{Re}(z-\bar{w})<0$ we have

$$K_{\omega}(z,w)=C_{\beta}\,e^{\nu_{\beta}(z-\bar{w})}+O(e^{\mu_{\beta}(z-\bar{w})}).$$

If $\beta=\pi$ we have the corresponding expansions

$$K_{\omega}(z,w)=\pi^{-2}(z-\bar{w}-2)\,e^{-(z-\bar{w})}+O(e^{-2(z-\bar{w})})$$

for $\mathrm{Re}(z-\bar{w})>0$ and

$$K_{\omega}(z,w)=\pi^{-2}(-z+\bar{w}-2)\,e^{z-\bar{w}}+O(e^{2(z-\bar{w})})$$

for $\mathrm{Re}(z-\bar{w})<0$.

(The above expansions hold uniformly as z and w range over any substrip $S_{\beta-\varepsilon}$.)

§ 3. Nonsmooth model domains in $\mathbf{C}^2$

In this section we study the Bergman kernels attached to the nonsmooth unbounded domains

$$D_{\beta}=\{(z_1,z_2)\in\mathbf{C}^2:\mathrm{Re}\,z_1\,e^{-i\log z_2\bar{z}_2}>0,\,|\log z_2\bar{z}_2|<\beta-\pi/2\}.$$

D_{β} is a Hartogs domain invariant under the rotations $\varrho_{\theta}\colon(z_1,z_2)\mapsto(z_1,e^{i\theta}z_2)$. By Fourier expansion the Bergman space $B(D_{\beta})$ admits an orthogonal decomposition

$$B(D_{\beta})=\oplus B_j(D_{\beta}),$$

where $B_j(D_{\beta})$ is the subspace consisting of functions $F\in B(D_{\beta})$ satisfying $F\circ\varrho_{\theta}=e^{ij\theta}F$; the projection Q_j from $B(D_{\beta})$ to $B_j(D_{\beta})$ is given by

$$(Q_jF)(z_1,z_2)=(2\pi)^{-1}\int_{-\pi}^{\pi}F(z_1,e^{i\theta}z_2)\,e^{-ij\theta}\,d\theta. \tag{3.1}$$

The Bergman kernel $K_{D_{\beta}}(z,w)$ for D_{β} satisfies

$$K_{D_{\beta}}(z,w)=\sum_j K_j(z,w),$$

where $K_j(z,w)$ is the reproducing kernel for $B_j(D_{\beta})$.

Our computation will be aided by introducing the domains

$$D'_{\beta}=\{(z_1,z_2)\in\mathbf{C}^2:|\mathrm{Im}\,z_1-\log z_2\bar{z}_2|<\pi/2,\,|\log z_2\bar{z}_2|<\beta-\pi/2\}.$$

D'_β is biholomorphic to D_β via the mapping $\Psi\colon D'_\beta \to D_\beta$, $(z_1, z_2) \mapsto (e^{z_1}, z_2)$. The mapping Ψ induces the transformation law

$$K_{D_\beta}(z, w) = K_{D'_\beta}(\Psi^{-1}z, \Psi^{-1}w)/(z_1 \bar{w}_1).$$

Since Ψ commutes with ϱ_θ we have an analogous transformation law

$$K_j(z, w) = K'_j(z, w)/(z_1 \bar{w}_1);$$

here K'_j is the reproducing kernel for the space $B_j(D'_\beta)$ of square-integrable holomorphic functions F on D'_β satisfying $F(z_1, e^{i\theta} z_2) = e^{ij\theta} F(z_1, z_2)$. Such functions F are necessarily of the form $F(z_1, z_2) = f(z_1)\, z_2^j$, where f is holomorphic on the strip S_β. We have

$$\begin{aligned}
\|F\|^2_{B_j(D'_\beta)} &= \int_{D_\beta} |f(z_1)|^2 \, |z_2|^{2j} \, dx_1\, dy_1\, dx_2\, dy_2 \\
&= 2\pi \int_{|y_1 - 2\log r| < \pi/2,\ |2 \log r| < \beta - \pi/2} |f(z_1)|^2 \, r^{2j+1} \, dx_1\, dy_1\, dr \\
&= \pi \int_{|y_1 - s| < \pi/2,\ |s| < \beta - \pi/2} |f(z_1)|^2 \, e^{(j+1)s} \, dx_1\, dy_1\, ds \\
&= \pi \int |f(z_1)|^2 \, e^{(j+1)s} \chi_{\pi/2}(y_1 - s)\, \chi_{\beta - \pi/2}(s)\, dx_1\, dy_1\, ds \\
&= \int_{S_\beta} |f(z)|^2 \, \omega(y)\, dx\, dy,
\end{aligned}$$

where $\omega = \pi(e^{-(j+1)(\cdot)} \chi_{\beta - \pi/2}) * \chi_{\pi/2}$.

Thus the kernels $K'_j(z, w)$ can be computed by the methods of section 1. In particular, if $\beta > \pi$ we have

$$K'_{-1}(z, w) = C_\beta \, e^{\nu_\beta (z_1 - \bar{w}_1)} \, z_2^{-1} \, \bar{w}_2^{-1} + O(e^{\mu_\beta (z - \bar{w})})$$

for $\operatorname{Re}(z - \bar{w}) < 0$ so that

$$K_{-1}(z, w) = C_\beta \, z_1^{\nu_\beta - 1} \, \bar{w}_1^{-\nu_\beta - 1} \, z_2^{-1} \, \bar{w}_2^{-1} + O((z_1/\bar{w}_1)^{\mu_\beta - 1})$$

for $|z_1| < |w_1|$. (The expansion holds uniformly on subdomains $D_{\beta - \varepsilon}$.) Thus it is easy to check that for w fixed we have

$$|\operatorname{Re} z_1 e^{-i \log z_2 \bar{z}_2}|^s \left(\frac{\partial}{\partial z_1} \right)^m K_{-1}(z, w) \notin L^2(D_\beta) \quad \text{when} \quad s \leqslant m - \nu_\beta;$$

since

$$\left(\frac{\partial}{\partial z_1}\right)^m K_{-1}(z, w) = Q_{-1}\left(\frac{\partial}{\partial z_1}\right)^m K_{D_\beta}(z, w)$$

it follows from (3.1) that

$$|\operatorname{Re} z_1 e^{-i\log z_2\bar{z}_2}|^s \left(\frac{\partial}{\partial z_1}\right)^m K_{D_\beta}(z, w) \notin L^2(D_\beta) \quad \text{when} \quad s \leq m - \nu_\beta. \tag{3.2}$$

If $\beta=\pi$ then similarly

$$K_{-1}(z, w) = \pi^{-2}(-\log(z_1/\bar{w}_1)-2)\,\bar{w}_1^{-2}\,z_2^{-1}\,\bar{w}_2^{-1} + O((z_1/\bar{w}_1)^2)$$

for $|z_1|<|w_1|$ so that (3.2) follows as before. For $\pi/2<\beta<\pi$ (3.2) can again be verified by examination of higher order terms in the asymptotic expansion of (2.1).

The corresponding calculations for $j\neq-1$ lead to similar formulas containing factors of $(z_1/\bar{w}_1)^{-i(j+1)/2}$.

§4. Worms

We turn our attention finally to the so-called "worm domain(s)" of Diederich and Fornæss. We shall write these domains in the form

$$\Omega = \{(z_1, z_2): |z_1 + e^{i\log z_2\bar{z}_2}|^2 < 1 - \phi(\log z_2\bar{z}_2)\}$$

where ϕ is a smooth nonnegative even function which is chosen so that Ω is smooth, bounded, connected, and pseudoconvex, and moreover $\phi^{-1}(0)$ is an interval which we will take to be $I_{\beta-\pi/2}$ ([DF], see also [Ki]). Thus the fiber of Ω over each z_2 is a disk, and the center of the disk winds by a total of $2\beta-\pi$ radians as $\log z_2\bar{z}_2$ varies over $I_{\beta-\pi/2}$.

THEOREM 1. *The Bergman projection operator attached to Ω does not map $W^k(\Omega)$ into $W^k(\Omega)$ when $k\geq\pi/(2\beta-\pi)$.*

COROLLARY. (See [BSt2].) *The $\bar{\partial}$-Neumann operator for Ω does not map $W^k(\Omega)$ into $W^k(\Omega)$ when $k\geq\pi/(2\beta-\pi)$.*

Proof of Theorem 1. Let $k\geq\pi/(2\beta-\pi)$ and suppose on the contrary that the projection operator P attached to Ω maps $W^k(\Omega)$ to $W^k(\Omega)$ so that we have an estimate

$$\|Pf\|_{W^k(\Omega)} \leq C\|f\|_{W^k(\Omega)}$$

for $f \in W^k(\Omega)$.

For $\lambda \geq 1$ let τ_λ denote the dilation $(z_1, z_2) \mapsto (\lambda z_1, z_2)$, let $\Omega_\lambda = \tau_\lambda(\Omega)$, and let T_λ denote the operator $L^2(\Omega_\lambda) \to L^2(\Omega), f \mapsto f \circ \tau_\lambda$. We have

$$\left\| \left(\frac{\partial}{\partial z}\right)^\alpha \left(\frac{\partial}{\partial \bar{z}}\right)^\beta T_\lambda f \right\|_{L^2(\Omega)} = \lambda^{\alpha_1+\beta_1-1} \left\| \left(\frac{\partial}{\partial z}\right)^\alpha \left(\frac{\partial}{\partial \bar{z}}\right)^\beta f \right\|_{L^2(\Omega_\lambda)}$$

so that

$$\|T_\lambda f\|_{W^k(\Omega)} \leq \lambda^{k-1} \|f\|_{W^k(\Omega_\lambda)}$$

when k is a nonnegative integer; by interpolation, the same estimates hold for all $k \geq 0$.

We will make use of the fact that for $s \geq 0$ the W^{-s} norm of a harmonic function f is comparable to the L^2 norm of $|r|^s f$, where r is a defining function for Ω [Li]. We choose r so that it coincides with $|z_1|^2 + 2\,\mathrm{Re}\, z_1 e^{-i \log z_2 \bar{z}_2}$ when $\log z_2 \bar{z}_2 \in I_{\beta-\pi/2}$. Let $r_\lambda = \lambda r \circ (\tau_\lambda)^{-1}$ so that $r_\lambda \to r_\infty := 2\,\mathrm{Re}\, z_1 e^{-i \log z_2 \bar{z}_2}$ as $\lambda \to \infty$.

Write k as $m-s$, where m is an integer and $s \geq 0$.

Let P_λ denote the Bergman projection for Ω_λ. Then $P_\lambda = T_\lambda^{-1} P T_\lambda$ so that

$$\begin{aligned}
\left\| |r_\lambda|^s \left(\frac{\partial}{\partial z_1}\right)^m P_\lambda f \right\|_{L^2(\Omega_\lambda)} &= \left\| |r_\lambda|^s \left(\frac{\partial}{\partial z_1}\right)^m T_\lambda^{-1} P T_\lambda f \right\|_{L^2(\Omega_\lambda)} \\
&= \lambda^{1-k} \left\| |r|^s \left(\frac{\partial}{\partial z_1}\right)^m P T_\lambda f \right\|_{L^2(\Omega)} \\
&\leq C_1 \lambda^{1-k} \|P T_\lambda f\|_{W^k(\Omega)} \\
&\leq C_2 \lambda^{1-k} \|T_\lambda f\|_{W^k(\Omega)} \\
&\leq C_2 \|f\|_{W^k(\Omega_\lambda)},
\end{aligned}$$

where C_1 and C_2 are independent of λ.

We will prove the following lemma momentarily.

Lemma 1. *If the estimate*

$$\left\| |r_\lambda|^s \left(\frac{\partial}{\partial z_1}\right)^m P_\lambda f \right\|_{L_2(\Omega_\lambda)} \leq C \|f\|_{W^k(\Omega_\lambda)}$$

holds for $f \in W^k(\Omega_\lambda)$ with C independent of $\lambda \geq 1$ then the Bergman projection operator P_∞ for D_β satisfies

$$\left\| |r_\infty|^s \left(\frac{\partial}{\partial z_1}\right)^m P_\infty f \right\|_{L^2(D_\beta)} \leq C \|f\|_{W^k(\mathbf{C}^n)}$$

for $f \in W^k(\mathbf{C}^n)$ with $\mathrm{supp} f \subset \overline{D_\beta}$.

Granting the lemma for now we conclude that since for each $w \in D_\beta$ the Bergman kernel function $K_{D_\beta}(\cdot, w)$ lies in $P_\infty(C_0^\infty(D_\beta))$ [BL] we must have

$$|r_\infty|^s \left(\frac{\partial}{\partial z_1}\right)^m K_{D_\beta}(\cdot, w) \in L^2(D_\beta).$$

But this is false by (3.2). The contradiction proves the theorem. □

The proof of Lemma 1 uses the following approximation result.

LEMMA 2. *For any finite $M>1$ the space $B(D_{M\beta})$ is dense in $B(D_\beta)$.*

Proof of Lemma 2. It suffices to show that each $B_j(D_{M\beta})$ is dense in $B_j(D_\beta)$. But from sections 1 and 2 we find that $B_j(D_\beta)$ is isometric via the Fourier transform to the space of functions on $\mathbf{R}$ which are in L^2 with respect to the weight

$$\frac{\sinh\left((2\beta-\pi)\left(\xi-\frac{j-1}{2}\right)\right)\sinh \pi\xi}{2\left(\xi-\frac{j-1}{2}\right)\xi}.$$

But $C_0^\infty(\mathbf{R})$ is dense in the latter space for any value of β, so the lemma follows.

Proof of Lemma 1. The key geometric facts are:

(1) Each compact subset K of D_β is contained in Ω_λ for all $\lambda \geq \lambda_K$.

(2) Each compact subset K of the complement of $\overline{D_\beta}$ is disjoint from $\overline{\Omega_\lambda}$ for all $\lambda \geq \lambda_K$.

Let $f \in W^k(\mathbf{C}^n)$ with $\operatorname{supp} f \subset \overline{D_\beta}$. Let Φ_λ denote the characteristic function of Ω_λ and let $h_\lambda \in L^2(\mathbf{C}^2)$ be the function which equals $P_\lambda(\Phi_\lambda f)$ in Ω_λ and vanishes outside of Ω_λ. Since $\|h_\lambda\|_{L^2(\mathbf{C}^2)} \leq \|f\|_{L^2(\mathbf{C}^2)}$ we may choose a sequence $\{\lambda_k\}_k$ tending to ∞ so that h_{λ_k} converges weakly to a function $h \in L^2(\mathbf{C}^2)$. It is clear that h is holomorphic in D_β and vanishes outside of $\overline{D_\beta}$. Since $|r_\infty|^s(\partial/\partial z_1)^m h$ is a weak limit on each compact subset of D_β of a subsequence of $|r_{\lambda_k}|^s(\partial/\partial z_1)^m P_{\lambda_k}(\Phi_{\lambda_k} f)$ it follows that

$$\left\| |r_\infty|^s \left(\frac{\partial}{\partial z_1}\right)^m h \right\|_{L^2(D_\beta)} \leq C\|f\|_{W^k(D_\beta)}.$$

We will be done if we can show that $h = P_\infty f$. We must show that $f-h \in L^2(D_\beta)$ is orthogonal to $B(D_\beta)$; courtesy of Lemma 2 it will suffice to show that $f-h$ is orthogonal to $B(D_{M\beta})$. We choose M so that $\Omega_\lambda \subset D_{M\beta}$ for all λ. Then $\Phi_\lambda f - h_\lambda$ is orthogonal to $B(D_{M\beta})$ so we reach the desired conclusion by passing to the limit. □

§ 5. Remarks

(1) A domain Ω is said to satisfy "condition R" if the Bergman projection operator for Ω maps $C^\infty(\bar{\Omega})$ into $C^\infty(\bar{\Omega})$. (This terminology was introduced in [BL].) It is not clear from what we have done here whether or not the worm domain satisfies condition R.

(2) Boas and Straube have shown [BSt3] that for all $k \geqslant 0$ the counterexample domain in [Ba1] does admit the *a priori* estimate

$$\|Pf\|_{W^k} \leqslant C_k \|f\|_{W^k} \quad \text{for all} \quad f \in W^k \text{ such that } Pf \in W^k.$$

It would be interestering to know whether such estimates hold on the worm domain.

(3) The importance of the unweighed Bergman projection stems in large part from its utility in the study of biholomorphic (and equidimensional proper holomorphic) maps. In particular, it is known that any biholomorphic map between smooth bounded pseudoconvex domains in $\mathbf{C}^n$, one of which satisfies condition R, extends smoothly to the boundary [Be]. If condition R indeed fails on the worm domain it is reasonable to ask if there is a corresponding failure of regularity for biholomorphic maps. In [Ba1] badly behaved biholomorphic maps are constructed between cousins of the worm domain which do not lie in $\mathbf{C}^n$.

(4) The methods of this paper extend easily to the study of weighted Bergman projections on the worm domain taken respect to a positive weight function $\eta \in C^\infty(\bar{\Omega})$ which is invariant under the rotations ϱ_θ. In place of Theorem 1 we have the conclusion that if the weighted projection maps W^k to W^k then the Fourier–Laplace transform $\hat{\omega}(\xi)$ of the function $\omega(s) = \eta(e^{s/2}, 0)\,\chi_{\beta-\pi/2}(s)$ is zero-free in the strip $|\mathrm{Re}\,\xi| \leqslant 2k$.

Since the only compactly supported distributions with zero-free Fourier–Laplace transforms are point masses it follows that it is impossible to choose η so that the projection maps W^k to W^k for all $k \geqslant 0$. On the other hand, it is of course easy to see that for any η the transform $\hat{\omega}(\xi)$ must be zero-free for

$$|\mathrm{Re}\,\xi| \leqslant \frac{\pi}{2\beta - \pi} = \pi/(\text{total amount of winding});$$

weights concentrated near the endpoints can be constructed which have zeros lying just outside this strip. Thus the best possible weight-independent result would be that all the weighted projections map W^k to W^k when $k \leqslant \pi/(2 \times \text{total amount of winding})$. Boas and Straube [BSt3] have in fact shown that the unweighted projection maps W^k to W^k when $k < \pi/(2 \times \text{total amount of winding})$ and k is an integer or $k = 1/2$; their method in fact works for any smooth positive (not necessarily ϱ_θ-invariant) η.

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