On elliptic systems in $\mathbf{R}^n$

by

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1. Statement of results

This paper studies elliptic $k \times k$ systems of partial differential operators in $\mathbf{R}^n$ which may be written in the form

$$A = A_\infty + Q \tag{1.1}$$

where A_∞ is an elliptic system of constant coefficient operators and Q is a variable coefficient perturbation with certain decay properties at $|x| = \infty$.

For the case $k=1$ such operators were studied in [6], [7] and [8] under the conditions

$$\begin{gathered} A_\infty \text{ is an elliptic constant coefficient} \\ \text{operator which is homogeneous of degree } m \end{gathered} \tag{1.2}$$

and the coefficients of

$$Q = \sum_{|\alpha| \leq m} q_\alpha(x)\, \partial^\alpha$$

satisfy $q_\alpha \in C^l(\mathbf{R}^n)$ and

$$\overline{\lim_{|x| \to \infty}} \left| \langle x \rangle^{m-|\alpha|+|\beta|} \partial^\beta q_\alpha(x) \right| = C_{\alpha,\beta} < \infty \tag{1.3}$$

for all $|\beta| \leq l \in \mathbf{N}$. (Here and throughout this paper we let $\mathbf{Z}$ denote the integers, $\mathbf{N}$ denote the nonnegative integers, $\langle x \rangle = (1+|x|^2)^{1/2}$, $p' = p/(p-1)$, and use standard conventions for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ and $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$.)

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Such operators are bounded on certain weighted Sobolev space defined as follows: for $1<p<\infty$, $l\in\mathbf{N}$, and $\delta\in\mathbf{R}$ let $W^p_{l,\delta}$ denote the closure of $C_0^\infty(\mathbf{R}^n)$ in the norm

$$\|u\|_{W^p_{l,\delta}}=\sum_{|\alpha|\leq l}\|\langle x\rangle^{\delta+|\alpha|}\partial^\alpha u\|_{L^p}.$$

(We should mention that these spaces were denoted $M^p_{l,\delta}$ in [2], [3], [7], and [8], and $H^p_{l,\delta}$ in [4] and [6].) Clearly (1.2) and (1.3) imply that

$$A_\infty\colon\ W^p_{l+m,\delta}\to W^p_{l,\delta+m} \qquad (\dagger)_\infty$$

$$A\colon\ W^p_{l+m,\delta}\to W^p_{l,\delta+m} \qquad (\dagger)$$

are bounded operators. In fact, if we let Poly(δ) denote the space of polynomials in $x_1,\dots,x_n$ of degree $\leq\delta$ and $d_P(\delta)$ its dimension (note that Poly$(\delta)=\{0\}$ if $\delta<0$) then the following theorems were proved in [6] and [8]:

THEOREM 1. *If* (1.2) *holds then* $(\dagger)_\infty$ *is Fredholm if and only if*

$$\begin{aligned}&-\delta-\frac{n}{p}\notin\mathbf{N}\quad\text{if }\delta\leq-\frac{n}{p}\\&\delta+m-\frac{n}{p'}\notin\mathbf{N}\quad\text{if }\delta>-\frac{n}{p}.\end{aligned}\tag{1.4}$$

Furthermore, the nullspace and cokernel of $(\dagger)_\infty$ *consist of polynomials, and are of dimension*

$$d_P\left(-\delta-\frac{n}{p}\right)-d_P\left(-\delta-m-\frac{n}{p}\right)\tag{1.5}$$

$$d_P\left(\delta+m-\frac{n}{p'}\right)-d_P\left(\delta-\frac{n}{p'}\right)\tag{1.6}$$

respectively.

THEOREM 2. *If* (1.2) *and* (1.3) *hold with* $C_{\alpha\beta}=0$ *for all* $|\alpha|\leq m$ *and* $|\beta|\leq l$, *then* $(\dagger)$ *is Fredholm if and only if* (1.4) *holds, and the Fredholm index of* $(\dagger)$ *agrees with that of* $(\dagger)_\infty$.

We should note that the formulae (1.5) and (1.6) do not appear explicitly in [6] or [8] but follow from an easy analysis similar to that of Section 3 of this paper. We also note that in both [6] and [8] it was assumed that $q_\alpha\in C^\infty(\mathbf{R}_n)$ when $|\alpha|=m$, but this may be weakened by perturbation theory as in the proof of Theorem 4 below. For $|\alpha|<m$ the

hypothesis $q_\alpha \in C^l$ may be weakened slightly to assume only bounded derivatives of order l satisfying (1.3), but we retain the above formulation for convenience. (More general coefficients are used in [4], but only for the special case $p=2$, $m<n$, and $-n/p<\delta<-m+n/p'$.)

Now suppose that (1.1) is a system $A=(A_{ij})$ so Au has components

$$(Au)_i = \sum_{j=1}^{k} A_{ij}\, u_j.$$

We shall use the generalized notion of ellipticity provided by Douglis & Nirenberg [5]:

Definition 1. Two k-tuples, $\mathbf{t}=(t_1, \ldots, t_k)$ and $\mathbf{s}=(s_1, \ldots, s_k)$ of nonnegative integers form a *system of orders* for A if for each $1 \leq i,\, j \leq k$ we have order $(A_{ij}) \leq t_j - s_i$. (If $t_j - s_i < 0$ then $A_{ij}=0$.) The *(*$\mathbf{t}, \mathbf{s}$*)-principal part* of A is obtained by replacing each A_{ij} by its terms which are exactly of order $t_j - s_i$, and the $(\mathbf{t}, \mathbf{s})$*-principal symbol* of A is obtained by replacing each ∂ in the $(\mathbf{t}, \mathbf{s})$-principal part by the vector $\xi \in S^{n-1}$. We say A is *elliptic with respect to* $(\mathbf{t}, \mathbf{s})$ if the $(\mathbf{t}, \mathbf{s})$-principal symbol of A has determinant bounded away from zero for $x \in \mathbf{R}^n$ and $\xi \in S^{n-1}$.

We now must replace (1.2) with the condition

$$\begin{gathered} A_\infty \text{ is elliptic with respect to } (\mathbf{t}, \mathbf{s}) \text{ and each operator} \\ (A_\infty)_{ij} \text{ is either zero or constant coefficient} \\ \text{and homogeneous of degree } t_j - s_i. \end{gathered} \tag{1.7}$$

Similarly we must replace (1.3) with $b^{ij}_\alpha \in C^{s_i}(\mathbf{R}^n)$ and

$$\overline{\lim_{|x| \to \infty}}\, |\langle x \rangle^{t_j - s_i - |\alpha| + |\beta|}\, \partial^\beta q^{ij}_\alpha(x)| = C^{ij}_{\alpha\beta} < \infty \tag{1.8}$$

for all $|\beta| \leq s_i$ where

$$Q_{ij} = \sum_{|\alpha| \leq t_j - s_i} q^{ij}_\alpha(x)\, \partial^\alpha.$$

With these conditions we then have

$$A_\infty\colon\ W^p_{\mathbf{t}, \delta - \mathbf{t}} \to W^p_{\mathbf{s}, \delta - \mathbf{s}} \qquad (\dagger\dagger)_\infty$$

$$A\colon\ W^p_{\mathbf{t}, \delta - \mathbf{t}} \to W^p_{\mathbf{s}, \delta - \mathbf{s}} \qquad (\dagger\dagger)$$

are bounded operators where we have defined

$$W^p_{\mathbf{t},\delta-\mathbf{t}}=\prod_{j=1}^{k} W^p_{t_j,\delta-t_j}$$

and $W^p_{\mathbf{s},\delta-\mathbf{s}}$ similarly. The purpose of this paper is to prove the following generalizations of Theorems 1 and 2:

THEOREM 3. *If* (1.7) *holds then* $(\dagger\dagger)_\infty$ *is Fredholm if and only if* δ *satisfies*

$$\begin{aligned} -\delta+t_j-\frac{n}{p}&\notin \mathbf{N} \quad \text{if } \delta-t_j\leq -\frac{n}{p}\\ \delta-s_j-\frac{n}{p'}&\notin \mathbf{N} \quad \text{if } \delta-t_j> -\frac{n}{p}\end{aligned} \tag{1.9}$$

for every $j=1,\ldots,k$. *In fact,* $(\dagger\dagger)_\infty$ *is injective if* $\delta-t_j>-n/p$ *for all* j, *and has dense range if* $\delta-s_j<n/p'$ *for all* j. *In general, the nullspace and cokernel of* $(\dagger\dagger)_\infty$ *consist of polynomials and are of dimension*

$$\sum_{j=1}^{k} d_P\left(-\delta+t_j-\frac{n}{p}\right)-d_P\left(-\delta+s_j-\frac{n}{p}\right) \tag{1.10}$$

$$\sum_{j=1}^{k} d_P\left(\delta-s_j-\frac{n}{p'}\right)-d_P\left(\delta-t_j-\frac{n}{p'}\right) \tag{1.11}$$

respectively.

THEOREM 4. *If* (1.7) *and* (1.8) *hold with* $C^{ij}_{\alpha\beta}=0$ *for all* $|\alpha|\leq t_j-s_i$, $|\beta|\leq s_i$, *and* $i, j=1,\ldots,k$, *then* $(\dagger\dagger)$ *is Fredholm if and only if* (1.9) *holds, and the Fredholm index of* $(\dagger\dagger)$ *then agrees with that of* $(\dagger\dagger)_\infty$.

As an immediate corollary we obtain the following generalization of the results in [9] on the nullspaces of systems which are "classically elliptic" ($t_j\equiv l+m$, $s_i\equiv l$).

COROLLARY 5. *Under the hypotheses of Theorem* 4, *the nullspace of*

$$A: H^p_{\mathbf{t}}\to H^p_{\mathbf{s}}$$

is finite dimensional, where $H^p_{\mathbf{t}}=\prod_{j=1}^{k} H^p_{t_j}$, $H^p_{t_j}$ *denoting the classical* L^p-*Sobolev space of order* t_j *in* $\mathbf{R}^n$.

2. Lemmas on convolution operators

We consider functions $E_m(x)$ of the form

$$
\begin{aligned}
E_0(x) &= \Omega(x)\,|x|^{-n} \\
E_m(x) &= \Gamma_0(x)+\Gamma_1(x)\log|x|, \quad m\geqslant 1
\end{aligned}
\tag{2.1}
$$

where Ω, Γ_0, and Γ_1 are all in $C^\infty(\mathbf{R}^n\setminus\{0\})$; Ω is homogeneous of degree 0 and has mean value 0 on the unit sphere; Γ_0 is homogeneous of degree $m-n$; and Γ_1 is a homogeneous polynomial of degree $m-n$ if n is even and $m-n\geqslant 0$, otherwise $\Gamma_1=0$. Let T be the convolution operator defined by

$$Tu = E_m * u$$

The following lemma is a special case of Theorem 2.11 in [6]. (We should note here that there is a gap in the proof of that theorem; namely, it does not include the case $\beta>-n/p$ and $\beta+m-n/p\in\mathbf{Z}\setminus\mathbf{N}$. However, this gap can be filled with an easy application of standard interpolation theorems, and so the theorem is true as stated.)

LEMMA 2.1. *If $l\in\mathbf{N}$ and $\delta\in\mathbf{R}$ satisfies $m-n/p<\delta<n/p'$, then*

$$T\colon W^p_{l,\delta}\to W^p_{l+m,\delta-m}$$

is bounded.

We shall also require the following generalization.

LEMMA 2.2. *For $\alpha\in\mathbf{N}^n$, $l\in\mathbf{N}$, and $\gamma\in\mathbf{R}$ let $r=m-|\alpha|$ and suppose* (i) $|\alpha|>0$, (ii) $l+r\geqslant 0$, *and* (iii) $r-n/p<\gamma<n/p'$. *Then*

$$\partial^\alpha T\colon W^p_{l,\gamma}\to W^p_{l+r,\gamma-r}$$

is bounded.

Proof. If $r\geqslant 0$ then $\partial^\alpha Tu=E'_r * u$ where $E'_r=\partial^\alpha E_m$ is of the form (2.1), so Lemma 2.1 may be applied. If $r<0$ write $\partial^\alpha T=\partial^{\tau_1}\partial^\beta T\partial^{\tau_2}$ where $\tau_i\in\mathbf{N}^n$ satisfy $|\tau_1|+|\tau_2|=-r$ and $-n/p<\gamma+|\tau_2|<n/p'$. Then $|\beta|=m$ and by the $r=0$ case, $\partial^\beta T\colon W^p_{l-|\tau_2|,\gamma+|\tau_2|}\to W^p_{l-|\tau_2|,\gamma+|\tau_2|}$ is bounded, so obviously $\partial^{\tau_1}\partial^\beta T\partial^{\tau_2}\colon W^p_{l,\gamma}\to W^p_{l+r,\gamma-r}$ is bounded.

3. Proof of Theorem 3.

Let $m=\Sigma_{j=l}^{k} t_j-s_j$ and $\tilde{A}_\infty=\det(A_\infty)$ which is an elliptic constant coefficient differential operator, homogenous of degree m. Let ${}^{co}A_\infty$ be the matrix formed by the cofactors of A_∞ so that

$$ {}^{co}A_\infty \cdot A_\infty = A_\infty \cdot {}^{co}A_\infty = \tilde{A}_\infty I $$

where I is the identity matrix. Note that $({}^{co}A_\infty)_{ji}$ is either zero or homogeneous of order $m-t_j+s_j$.

Now if $u=(u_1,\ldots,u_k)$ is in the nullspace of $(\dagger\dagger)_\infty$ then $\tilde{A}_\infty Iu={}^{co}A_\infty\cdot A_\infty u=0$ so $\tilde{A}_\infty u_j=0$ for each j. Since $W^p_{t_j,\delta-t_j}\subset\mathcal{S}'$ the space of "tempered distributions," the Schwartz theory of distributions implies that u_j is a polynomial which must be of degree $<-\delta+t_j-n/p$ in order to be in $W^p_{t_j,\delta-t_j}$. Hence the nullspace of $(\dagger\dagger)_\infty$ is contained in

$$ \prod_{j=1}^{k} \text{Poly}\left(-\delta+t_j-\frac{n}{p}\right) $$

and so is finite dimensional. In particular, if $\delta-t_j>-n/p$ for all j then $(\dagger\dagger)_\infty$ is injective.

Similarly, the dual map to $(\dagger\dagger)_\infty$ is

$$ A^*_\infty\colon W^{p'}_{-\mathbf{s},-\delta+\mathbf{s}} \to W^{p'}_{-\mathbf{t},-\delta+\mathbf{t}} \qquad (\dagger\dagger)^*_\infty $$

where $W^{p'}_{-\mathbf{s},-\delta+\mathbf{s}}$ and $W^{p'}_{-\mathbf{t},-\delta+\mathbf{t}}$ denote the dual spaces of $W^p_{\mathbf{s},\delta-\mathbf{s}}$ and $W^p_{\mathbf{t},\delta-\mathbf{t}}$ respectively, and A^*_∞ is a system of operators satisfying (1.7) for some system of orders $(\mathbf{t}^*, \mathbf{s}^*)$. By duality, $W^{p'}_{-s_i,-\delta+s_i}\subset\mathcal{S}'$. Thus the argument above shows that if $u=(u_1,\ldots,u_k)$ is in the nullspace of $(\dagger\dagger)^*_\infty$ then each u_j is a polynomial of degree $<\delta-s_i-n/p'$. Hence the nullspace of $(\dagger\dagger)^*_\infty$ is contained in

$$ \prod_{j=1}^{k} \text{Poly}\left(\delta-s_i-\frac{n}{p'}\right) $$

and so $(\dagger\dagger)_\infty$ has dense range if $\delta-s_j<n/p'$ for all j.

Now to show $(\dagger\dagger)_\infty$ has closed range we may assume that the t_j and s_i are arranged so that $s_1\leqslant\ldots\leqslant s_k$ and $t_1\leqslant\ldots\leqslant t_k$. Ellipticity of A_∞ then implies $t_j\geqslant s_j$ for every j. Hence we find that

$$ m+s_j\geqslant t_j \text{ for all } j. \qquad (3.1) $$

We first control the range of $(\dagger\dagger)_\infty$ in the case of

$$\begin{aligned} -\delta+s_i+m-\frac{n}{p} &\notin \mathbf{N} \quad \text{if } \delta-s_i-m \leqslant -\frac{n}{p} \\ \delta-s_i-\frac{n}{p'} &\notin \mathbf{N} \quad \text{if } \delta-s_i-m > -\frac{n}{p}. \end{aligned} \tag{3.2}$$

By Theorem 1

$$\tilde{A}_\infty : W^p_{s_i+m,\,\delta-s_i-m} \to W^p_{s_i,\,\delta-s_i} \tag{3.3}$$

is Fredholm if and only if (3.2) holds, so let us fix δ satisfying (3.2) for all i. Let T_i be a Fredholm inverse for (3.3), and T the diagonal matrix with entries T_i. Then $A_\infty \cdot {}^{co}A_\infty \cdot T = \tilde{A}_\infty I \cdot T = I + P$ where P is a projection of $W^p_{\mathbf{s},\delta-\mathbf{s}}$ onto a complement of the range of $\tilde{A}_\infty I$ in $W^p_{\mathbf{s},\delta-\mathbf{s}}$. Hence the range of $(\dagger\dagger)_\infty$ is closed and we have proven

LEMMA 3.1. *If δ satisfies* (3.2) *for all i, then* $(\dagger\dagger)_\infty$ *is Fredholm.*

In comparing (3.2) with (1.9), note that if for some j we have $\delta-t_j \leqslant -n/p$ and $-\delta+t_j-n/p \notin \mathbf{N}$, then $-\delta+t_j-n/p$ cannot be an integer so (3.2) will be satisfied for *all* i. Similarly, the first line of (3.2) holding for some i implies (1.9) for all j. On the other hand, if $\delta-s_i-m > -n/p$ and $\delta-s_i-n/p' \notin \mathbf{N}$, then by (3.1) we have $\delta-t_i > -n/p$ so we have proved

LEMMA 3.2. *If δ satisfies* (3.2) *for all i, then it satisfies* (1.9) *for all j.*

By the above remarks, δ can satisfy (1.9) for all j but *not* (3.2) only if for *all j*

$$\delta-t_j > -\frac{n}{p} \text{ and } \delta-s_j-\frac{n}{p'} \notin \mathbf{N} \tag{3.4}$$

and for *some i*

$$\delta-s_i-m \leqslant -\frac{n}{p} \text{ and } -\delta+s_i-m-\frac{n}{p} \in \mathbf{N} \tag{3.5}$$

But (3.4) and (3.5) imply $\delta-s_j-n/p' \in \mathbf{Z} \setminus \mathbf{N}$ and in particular

$$\delta-s_j < \frac{n}{p'} \tag{3.6}$$

for all j. By monotonicity of the s_i we can find i_0 such that (3.5) holds for all $i \geq i_0$. In fact, together with (3.4) we find

$$\begin{aligned} &t_j < s_i + m \text{ for all } i \geq i_0 \text{ and all } j \\ &\delta - s_i - m > -\frac{n}{p} \text{ for all } i < i_0. \end{aligned} \tag{3.7}$$

Now let $Tu = E_m * u$ where E_m is the fundamental solution of $\tilde{A}_\infty$ of the form (2.1). The operator ${}^{co}A_\infty \cdot TI$ is then a fundamental solution for A_∞. In fact, we claim that ${}^{co}A_\infty \cdot TI$ is the inverse for $(\dagger\dagger)_\infty$ when δ satisfies (3.4) and (3.5). We need only show that for every i and j

$$({}^{co}A_\infty)_{ji} T \colon W^p_{s_i, \delta - s_i} \to W^p_{t_j, \delta - t_j} \tag{3.8}$$

is bounded. If $i < i_0$ then (3.6) and (3.7) imply $m - n/p < \delta - s_i < n/p'$, so by Lemma 2.1 $T\colon W^p_{s_i, \delta - s_i} \to W^p_{s_i + m, \delta - s_i - m}$ is bounded which obviously implies that (3.8) is bounded. On the other hand if $i \geq i_0$ then $|\alpha| = m - t_j + s_i$, $l = s_i$, and $\gamma = \delta - s_i$ satisfy the hypotheses of Lemma 2.2, so (3.8) is bounded. Thus we have proved

LEMMA 3.3. *If δ satisfies* (1.9) *for all j but not* (3.2) *for some i, then $(\dagger\dagger)_\infty$ is an isomorphism.*

We conclude, therefore, that (1.9) is sufficient for $(\dagger\dagger)_\infty$ to be Fredholm.

Next we suppose δ satisfies (1.9) and compute the nullity of $(\dagger\dagger)_\infty$. Note that

$$A_\infty \colon \prod_{j=1}^{k} \operatorname{Poly}\left(-\delta + t_j - \frac{n}{p}\right) \to \prod_{i=1}^{k} \operatorname{Poly}\left(-\delta + s_i - \frac{n}{p}\right). \tag{3.9}$$

We claim that (3.9) is surjective. Indeed, if $v = (v_1, \ldots, v_k) \in \Pi_{i=1}^k \operatorname{Poly}(-\delta + s_i - n/p)$ then v is in the range of $(\dagger\dagger)_\infty$ if and only if $\Sigma_{i=1}^k \int w_i v_i \, dx = 0$ for all $w = (w_1, \ldots, w_k)$ in the nullspace of $(\dagger\dagger)^*_\infty$. If $v_i \neq 0$ then $\delta - s_i < -n/p$, so Poly $(\delta - s_i - n/p') = \{0\}$ implying $w_i = 0$. Thus we can always solve $A_\infty u = v$ for $u \in W^p_{t, \delta - t}$. For $\alpha \in \mathbf{N}^n$ with each α_j sufficiently large, $(\partial^\alpha I) \cdot A_\infty u = (\partial^\alpha I) v = 0$ so u is a polynomial. Thus $u \in \Pi_{j=1}^k \operatorname{Poly}(-\delta + t_j - n/p)$ proving that (3.9) is surjective. Since we have already observed that the nullspace of $(\dagger\dagger)_\infty$ is contained in $\Pi_{j=1}^k \operatorname{Poly}(-\delta + t_j - n/p)$ this proves (1.10).

Similarly, we derive (1.11) from the surjectivity of

$$A^*_\infty \colon \prod_{i=1}^{k} \operatorname{Poly}\left(\delta - s_i - \frac{n}{p'}\right) \to \prod_{j=1}^{k} \operatorname{Poly}\left(\delta - t_j - \frac{n}{p'}\right).$$

To show that (1.9) is necessary for $(\dagger\dagger)_\infty$ to be Fredholm, suppose that for some j we have $-\delta+t_j-n/p\in\mathbf{N}$ or $\delta-s_j-n/p'\in\mathbf{N}$. Consider the one-parameter family of operators

$$A_\infty(\tau)=\langle x\rangle^\tau A_\infty\langle x\rangle^{-\tau}: W^p_{\mathbf{t},\delta-\mathbf{t}}\to W^p_{\mathbf{s},\delta-\mathbf{s}} \tag{3.10}$$

defined for $-\varepsilon\leq\tau\leq\varepsilon$ where $0<\varepsilon<1$. Since $u\to\langle x\rangle^\sigma u$ is an isomorphism of $W^p_{l,\delta+\sigma}$ onto $W^p_{l,\delta}$ we conclude that (3.10) is Fredholm if and only if

$$A_\infty: W^p_{\mathbf{t},\delta+\tau-\mathbf{t}}\to W^p_{\mathbf{s},\delta+\tau-\mathbf{s}} \tag{3.11}$$

is Fredholm, and the index of (3.10) equals that of (3.11). We have seen that $A_\infty(\tau)$ is Fredholm for $\tau\neq0$, and by (1.10) and (1.11) index $[A_\infty(\varepsilon)]<$ index $[A_\infty(-\varepsilon)]$. Hence $A_\infty(0)$ cannot be Fredholm, as to be shown.

4. Proof of Theorem 4.

First note that (1.8) with $C^{ij}_{\alpha\beta}=0$ implies

$$\sum_{|\alpha|<t_j-s_i} q^{ij}_\alpha(x)\,\partial^\alpha: W^p_{t_j,\delta-t_j}\to W^p_{s_i,\delta-s_i}$$

is compact by Theorem 5.2 of [6] or Lemma 4.1 of [8]. Therefore we may assume

$$Q_{ij}=\sum_{|\alpha|=t_j-s_i} q^{ij}_\alpha(x)\,\partial^\alpha.$$

Now let $\varphi\in C^\infty_0(\mathbf{R}^n)$ satisfy $\varphi(x)\equiv1$ for $|x|\leq1$ and $\varphi(x)\equiv0$ for $|x|\geq2$, and define $\varphi_R(x)=\varphi(x/R)$ for $R>1$. From (1.8) with $C^{ij}_{\alpha\beta}=0$ we can find $R>1$ such that for every i,j and $|\beta|\leq s_i$ and $|\alpha|=t_j-s_i$

$$\left|\langle x\rangle^{|\beta|}\partial^\beta q^{ij}_\alpha(x)\right|<\varepsilon$$

whenever $|x|>R$. Thus there is a constant C which depends only on φ, s_i, and n for which

$$\left|\langle x\rangle^{|\beta|}\partial^\beta((1-\varphi_R(x))q^{ij}_\alpha(x))\right|<C\cdot\varepsilon$$

holds for all $x\in\mathbf{R}^n$, $|\beta|\leq s_i$, $|\alpha|=t_j-s_i$, and all i,j. Hence by choosing R sufficiently large, the norm of

$$(1-\varphi_R)\,Q=(1-\varphi_R)\,I\cdot Q: W^p_{\mathbf{t},\delta-\mathbf{t}}\to W^p_{\mathbf{s},\delta-\mathbf{s}}$$

may be made arbitrarily small. Therefore, if δ satsifies (1.9) for all j, then we may choose R_0 so that

$$A'_\infty = A_\infty + (1-\varphi_R)\,Q : W^p_{t,\delta-t} \to W^p_{s,\delta-s}$$

is Fredholm whenever $R \geqslant R_0$.

In terms of à priori inequalities this means that

$$|u|_t \leqslant C(|A'_\infty u|_s + |\pi u|_t) \tag{4.1}$$

for $u \in W^p_{t,\delta-t}$, where we have abbreviated the norms in $W^p_{t,\delta-t}$ and $W^p_{s,\delta-s}$ by $|\cdot|_t$ and $|\cdot|_s$ respectively, and where π is a projection of $W^p_{t,\delta-t}$ onto the kernel of A'_∞ and thus is compact. We shall apply (4.1) to $(1-\varphi_{3R})u$ and use $A'_\infty = A$ in the support of $(1-\varphi_{3R})$ to conclude

$$|(1-\varphi_{3R})\,u|_t \leqslant C(|A(1-\varphi_{3R})\,u|_s + |\pi(1-\varphi_{3R})\,u|_t). \tag{4.2}$$

On the other hand, since $\varphi_{3R}\,u$ has compact support, standard elliptic estimates [1] imply

$$|\varphi_{3R}\,u|_t \leqslant C(|A\varphi_{3R}\,u|_s + |\varphi_{3R}\,u|_0). \tag{4.3}$$

Combining (4.2) and (4.3) yields

$$\begin{aligned} |u|_t &\leqslant C(|A(1-\varphi_{3R})\,u|_s + |A\varphi_{3R}u|_s + |\pi(1-\varphi_{3R})\,u|_t + |\varphi_{3R}u|_0) \\ &\leqslant C(|(1-\varphi_{3R})\,Au|_s + |\varphi_{3R}\,Au|_s \\ &\qquad + |[A,(1-\varphi_{3R})]\,u|_s + |[A,\varphi_{3R}]u|_s \\ &\qquad + |\pi(1-\varphi_{3R})\,u|_t + |\varphi_{3R}u|_0) \end{aligned} \tag{4.4}$$

where [,] denotes the commutator. By Rellich's compactness theorem, $[A,(1-\varphi_{3R})]$, $[A,\varphi_{3R}]$: $W^p_{t,\delta-t} \to W^p_{s,\delta-s}$ and $\varphi_{3R}: W^p_{t,\delta-t} \to W^p_{0,\delta}$ are all compact, so the à priori inequality (4.4) shows that $A: W^p_{t,\delta-t} \to W^p_{s,\delta-s}$ has a finite dimensional nullspace and closed range, hence is "semi-Fredholm". Furthermore, we may find R_1 large so that $A_\infty + \varphi_R\,Q$ is an elliptic system which is semi-Fredholm and

$$\text{index}\,(A_\infty + \varphi_R Q) = \text{index}\,(A) \tag{4.5}$$

whenever $R \geqslant R_1$, although we do not as yet know that (4.5) is finite.

Now for $R \geq \max(R_0, R_1)$ and $0 \leq \tau \leq 1$ let $(\varphi_R Q)_\tau$ be the matrix with entries

$$\varphi_R(\tau x) \sum_{|\alpha| = t_j - s_i} q_\alpha^{ij}(\tau x)\, \partial^\alpha.$$

For each τ, $A_\tau = A_\infty + (\varphi_R Q)_\tau$ is an elliptic system of the form (1.1) with coefficients satisfying (1.8) (since A_0 has constant coefficients and A_τ for $\tau > 0$ has coefficients constant for $|x| \geq 2/\tau$). Thus we have a one-parameter family of semi-Fredholm operators, and so

$$\text{index}(A_0) = \text{index}(A_1). \tag{4.6}$$

But $A_1 = A_\infty + \varphi_R Q$ so (4.5) and (4.6) imply that index(A)=index(A_0). However, the index of A_0 is given by Theorem 3: index(A_0)=index(A_∞) is finite. Hence A is indeed Fredholm.

In other words, we have shown that if δ satisfies (1.9) then (††) is Fredholm and has the same index as $(††)_\infty$. Conversely, we can show that (††) is not Fredholm where its index changes (i.e., where (1.9) fails for some j) by the same method as used for $(††)_\infty$ in Section 3.

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