ANALYTIC HYPOELLIPTICITY OF THE $\bar{\partial}$-NEUMANN PROBLEM AND EXTENDABILITY OF HOLOMORPHIC MAPPINGS

BY

STEVEN R. BELL(1)

Princeton University, New Jersey, U.S.A.

1. Introduction

In the theory of functions of one complex variable, the proof of the theorem which states that a proper holomorphic mapping between domains with real analytic boundaries extends holomorphically past the boundary consists of two relatively simple steps: first prove that such mappings extend continuously to the boundary; then apply the classical Schwarz reflection principle. Attempts to generalize these techniques to mappings in several complex variables have not been entirely successful. The principle reasons for this are: (1) there is not a satisfactory reflection principle for weakly pseudoconvex hypersurfaces in $\mathbf{C}^n$, and (2) proper maps in $\mathbf{C}^n$ may branch at boundary points. In this paper, we attempt to expose the connection between the problem of extending proper holomorphic mappings and the real analytic hypoellipticity of the $\bar{\partial}$-Neumann problem. To be precise, we prove that if D_1 and D_2 are bounded domains with real analytic boundaries, and if the $\bar{\partial}$-Neumann problem for D_1 is globally real analytic hypoelliptic, then any proper holomorphic mapping f of D_1 onto D_2 extends holomorphically to a neighborhood of $\overline{D_1}$. This result allows us to prove that there can be no proper holomorphic mapping of a bounded domain with real analytic boundary which is strictly pseudoconvex onto such a domain which is weakly pseudoconvex. When our techniques are localized, we are able to prove that if $f\colon D_1 \to D_2$ is a proper holomorphic mapping between bounded pseudoconvex domains with real analytic boundaries, then f maps the set Γ of strictly pseudoconvex boundary points of D_1 into the set of strictly pseudoconvex boundary points of D_2. Furthermore, f extends holomorphically past Γ and is unbranched on Γ.

It should be pointed out that the general problem of proving the global analytic hypoellipticity of the $\bar{\partial}$-Neumann problem in a weakly pseudoconvex domain with real

(1) Research supported by NSF grant no. MCS 80-17205.

analytic boundary is currently a leading open question in the theory of functions of several complex variables. Hence, it might appear that our main theorem is not entirely worthwhile. However, there are many known examples of weakly pseudoconvex domains for which global analytic hypoellipticity is known to hold. Furthermore, if a counterexample to the problem of extending holomorphic mappings between real analytic domains could be found, our theorem would yield a counterexample to the problem of analytic hypoellipticity of the $\bar{\partial}$-Neumann problem.

2. The Bergman projection

The Bergman projection P associated to a bounded domain D contained in $\mathbb{C}^n$ is the orthogonal projection of $L^2(D)$ onto its closed subspace $H(D)$ consisting of L^2 holomorphic functions. The $\bar{\partial}$-Neumann problem and the Bergman projection for a smooth bounded pseudoconvex domain D are fundamentally related via Kohn's formula: $P = I - \bar{\partial}^* N \bar{\partial}$. Here, N is the $\bar{\partial}$-Neumann operator mapping $L^2_{0,1}(D)$ to $L^2_{0,1}(D)$ and $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$ (see Kohn [11]). The operator $\bar{\partial}^*$ is defined via $\bar{\partial}^* (\sum v_i d\bar{z}_i) = -\sum \partial v_i / \partial z_i$.

If D has a real analytic boundary, we say that N is globally real analytic hypoelliptic if whenever α is a $\bar{\partial}$-closed (0, 1)-form whose coefficients extend to be real analytic in a neighborhood of bD, then $N\alpha$ is a (0, 1)-form whose coefficients also extend to be real analytic in a neighborhood of bD. Kohn's formula reveals that whenever N is globally analytic hypoelliptic, then P is also. It is this property of P which is crucial to our arguments in this paper. We shall see momentarily that global analytic hypoellipticity of the Bergman projection associated to a domain with real analytic boundary is equivalent to the apparently weaker condition,

Condition Q. A bounded domain D will be said to satisfy condition Q if $P\varphi$ extends holomorphically to a neighborhood of $\bar{D}$ whenever $\varphi \in C_0^\infty(D)$.

For convenience, we also define

Local condition Q. If z is a boundary point of a domain D, we say that D satisfies condition Q at z if $P\varphi$ extends to be holomorphic in a neighborhood of z whenever $\varphi \in C_0^\infty(D)$.

Smooth bounded strictly pseudoconvex domains with real analytic boundaries satisfy condition Q because the $\bar{\partial}$-Neumann problem is globally real analytic hypoelliptic for such domains (Tartakoff [13], Komatsu [12], Derridj and Tartakoff [5]). Furthermore, a domain D satisfies condition Q whenever the Bergman kernel function $K(z, w)$ associated to it satisfies the condition that for each compact subset E of D, there is an open set G_E containing $\bar{D}$ such that $K(z, w)$ extends holomorphically to G_E as a function of z for each $w \in E$. Hence, for example, bounded complete Reinhardt domains satisfy condition Q.

The $\bar{\partial}$-Neumann problem is locally real analytic hypoelliptic at strictly pseudoconvex boundary points of pseudoconvex domains with real analytic boundaries (Trèves [16], Tartakoff [15]). Hence, pseudoconvex domains with real analytic boundaries satisfy local condition Q at their strictly pseudoconvex boundary points.

With these preliminaries behind us, we can now state our principal results.

3. Results

Our main result is

THEOREM 1. *Suppose that D_1 and D_2 are smooth bounded domains contained in $\mathbf{C}^n$, that D_1 satisfies condition Q, and that D_2 has a real analytic boundary. If f is a proper holomorphic mapping of D_1 onto D_2, then f extends to be holomorphic in a neighborhood of $\overline{D_1}$.*

Remarks made in section 2, together with Theorem 1, yield

COROLLARY 1. *If D_1 and D_2 are smooth bounded pseudoconvex domains contained in $\mathbf{C}^n$ with real analytic boundaries, and if the $\bar{\partial}$-Neumann problem for D_1 is globally real analytic hypoelliptic, then a proper holomorphic mapping f of D_1 onto D_2 extends to be holomorphic in a neighborhood of $\overline{D_1}$.*

When the techniques used in the proof of Theorem 1 are localized, we obtain

THEOREM 2. *Suppose that f: $D_1 \to D_2$ is a proper holomorphic mapping between smooth bounded pseudoconvex domains with real analytic boundaries contained in $\mathbf{C}^n$. Let Γ denote the open subset of bD_1 consisting of strictly pseudoconvex boundary points. Then f extends holomorphically past Γ and is unbranched on Γ. Hence, f maps Γ into the set of strictly pseudoconvex boundary points of D_2.*

We shall now prove that Theorem 2 implies

COROLLARY 2. *There does not exist a proper holomorphic mapping of a smooth bounded domain with real analytic boundary which is strictly pseudoconvex onto such a domain which is weakly pseudoconvex.*

Proof of Corollary 2. Let us assume that Theorem 2 is true, and suppose that f: $D_1 \to D_2$ is a proper mapping which violates the statement of Corollary 2. Let $\{x_k\}$ be a sequence of points in D_1 such that $\{f(x_k)\}$ converges to a weakly pseudoconvex boundary point w_0 of D_2. By passing to a subsequence, if necessary, we may assume that $\{x_k\}$ converges to a point $x_0 \in bD_1$. Then f maps x_0 to w_0, and this contradicts Theorem 2.

The following lemma is crucial to the proofs of all of the results above.

LEMMA 1. *If D is a smooth bounded domain with real analytic boundary, and h is a function on D which extends to be holomorphic in a neighborhood of $\bar{D}$, then there is a function $\varphi \in C_0^\infty(D)$ such that $h = P\varphi$ on D. (Here, P is the Bergman projection associated to D.)*

We shall also require a lemma which is proved in [2] and [3]. The proof of this lemma is so short and simple that we include it in section 6.

LEMMA 2. *Suppose that f: $D_1 \to D_2$ is a proper holomorphic mapping between bounded domains contained in $\mathbf{C}^n$. Let P_i denote the Bergman projection associated to D_i, $i=1, 2$, and let $u = \operatorname{Det}[f']$. Then*

$$P_1(u \cdot (\varphi \circ f)) = u \cdot ((P_2 \varphi) \circ f)$$

for all $\varphi \in L^2(D_2)$.

The proof of Lemma 1 will contain a proof of

COROLLARY 3. *A smooth bounded domain with real analytic boundary satisfies condition Q if and only if the Bergman projection associated to D is globally real analytic hypoelliptic.*

We will prove the theorems, assuming the truth of the lemmas, in section 5, and we will prove the lemmas in section 6.

4. Some remarks

(A) Theorem 1 is well known in the case that both D_1 and D_2 are strictly pseudoconvex domains with real analytic boundaries (Burns and Shnider [4]).

(B) Let $\{k_i\}_{i=1}^n$ be a set of positive integers with at least one k_i greater than one. The weakly pseudoconvex real analytic "ellipsoid" $\{z \in \mathbf{C}^n: \sum_{i=1}^n |z_i|^{2k_i} < 1\}$ satisfies condition Q because it is a complete Reinhardt domain. Hence, if D_1 is one of these ellipsoids and D_2 is a smooth bounded pseudoconvex domain with real analytic boundary and f is a proper mapping of D_1 onto D_2, then f extends to be holomorphic in a neighborhood of $\overline{D_1}$. This is an example of a situation in which mappings extend in the absence of any suitable reflection principle.

(C) Derridj and Tartakoff [5] state sufficient conditions for the $\bar{\partial}$-Neumann problem associated to a weakly pseudoconvex domain with real analytic boundary to satisfy global real analytic hypoellipticity. See also [17].

(D) It should be mentioned that the techniques of this paper can be generalized in a straightforward way to obtain analogous results for domains which are relatively compact inside Stein manifolds (see, for example, Diederich and Fornaess [7]).

(E) Let $B(R)$ denote the ball of radius R in $\mathbf{C}^n$, and let P denote the Bergman projection associated to $B(1)$. It is a simple exercise in the use of power series to prove that a holomorphic function h on $B(1)$ extends to be holomorphic on $B(R)$ for $R>1$ if and only if there is a function $\varphi \in L^2(B(1))$ supported on $B(1/R)$ such that $h=P\varphi$ on $B(1)$. Corollary 3 yields a similar result for an arbitrary strictly pseudoconvex domain D with real analytic boundary. Namely, a holomorphic function on D extends to be holomorphic in a neighborhood of $\bar{D}$ if and only if it is the Bergman projection of a function in $C_0^\infty(D)$.

(F) It will become apparent during the course of the proofs of Theorems 1 and 2 that the following theorem is true.

THEOREM 3. *Suppose that D_1 and D_2 are smooth bounded pseudoconvex domains contained in $\mathbf{C}^n$ and that D_2 has a real analytic boundary. Let Γ denote the set of boundary points of D_1 which satisfy local condition* Q. *If f: $D_1 \to D_2$ is a proper holomorphic mapping, then f extends holomorphically past Γ.*

(G) Combining techniques used in [2] with techniques of the present work, we are able to prove the following theorem of dubious merit:

Suppose that f: $D_1 \to D_2$ is a proper holomorphic mapping of a pseudoconvex domain D_1 with real analytic boundary onto a domain D_2 which satisfies condition Q. It is well known that f is a branched cover of some finite order m. Let $F_1, F_2, \ldots, F_m$ denote the m local inverses of f which are defined locally on D_2 minus the image of the branch locus of f. If h is a function which is holomorphic in a neighborhood of $\overline{D_1}$, then $\sum U_k(h \circ F_k)$ extends to be holomorphic in a neighborhood of $\overline{D_2}$, where $U_k = \operatorname{Det}[F_k']$.

5. Proofs of the theorems

We now prove the theorems, assuming the truth of the lemmas.

Proof of Theorem 1. Let us denote the Bergman projection associated to D_i by P_i, $i=1, 2$. For each monomial z^α, we choose a function $\varphi_\alpha \in C_0^\infty(D_2)$ such that $P_2\varphi_\alpha = z^\alpha$. The existence of such functions is guaranteed by Lemma 1. The transformation rule for the Bergman projections stated in Lemma 2 yields that $u \cdot f^\alpha = u \cdot ((P_2\varphi_\alpha) \circ f) = P_1(u \cdot (\varphi_\alpha \circ f))$. The function $u \cdot (\varphi_\alpha \circ f)$ is in $C_0^\infty(D_1)$ because f is proper. From this, we can conclude that $u \cdot f^\alpha$ extends to be holomorphic in a neighborhood of $\overline{D_1}$ because D_1 satisfies condition Q. Repeat the argument above using a function $\varphi_0 \in C_0^\infty(D_2)$ such that $P_2\varphi_0 = 1$ to conclude that u extends to be holomorphic in a neighborhood of $\overline{D_1}$.

To finish the proof, we must show that u divides $u \cdot f^\alpha$ as a holomorphic function at boundary points where u vanishes. Suppose that $z \in bD_1$ is a point where u vanishes. Let

f_k denote the kth component of f. The ring O_z of germs of holomorphic functions at z is a unique factorization domain. We now factor the functions u and $u\cdot f_k$ which have just been shown to belong to O_z. Suppose that $u=\prod p_i$ and $u\cdot f_k=\prod q_j$ where the p_i's and q_j's are powers of irreducible elements of O_z. The fact that $u\cdot f_k^m$ is an element of O_z for each positive integer m implies that $(\prod p_i)^{m-1}$ divides $(\prod q_j)^m$ for each m. This is only possible if $\prod q_j$ divides $\prod p_i$, i.e., if $f_k=(\prod q_j)/(\prod p_i)$ is actually holomorphic in a neighborhood of z.

Hence, we have shown that f extends holomorphically to a neighborhood of $\overline{D_1}$.

Proof of Theorem 2. The local real analytic hypoellipticity of the $\bar\partial$-Neumann problem at strictly pseudoconvex points of bounded pseudoconvex domains with real analytic boundaries (Trèves [16], Tartakoff [15]) implies that local condition Q holds at all points in Γ. The same procedure used in the proof of Theorem 1 can be applied to yield that f extends holomorphically past Γ. The only thing remaining to be proved is the fact that f is unbranched on Γ, i.e., that $u\neq 0$ on Γ.

A smooth real valued function r on $\mathbf{C}^n$ is called a defining function for a domain D if $D=\{r<0\}$, $bD=\{r=0\}$, and $dr\neq 0$ on bD. Similarly, r is called a local defining function for an open subset Λ of bD if r is smooth near Λ and if these conditions are met locally on Λ.

We shall now employ an argument due to Fornaess [9], used originally in the biholomorphic mapping case. Diederich and Fornaess [8] prove that if D is a smooth bounded pseudoconvex domain, then there is a defining function r for D such that $-(-r)^{2/3}$ is *strictly plurisubharmonic* on D. Let r_2 be such a defining function for D_2. We wish to prove that $r_2\circ f$ is a local defining function for Γ. To do this, we need only show that $d(r_2\circ f)\neq 0$ on Γ. Since $-(-r_2\circ f)^{2/3}$ is a plurisubharmonic function on D_1, we may apply the classical Hopf's lemma to conclude that $-(r_2\circ f)^{2/3}\leqslant -Cd(z)$ where $d(z)$ is equal to the euclidean distance of z to bD_1 and C is a constant independent of z. Hence $(r_2\circ f)(z)\geqslant Cd(z)^{3/2}$. At points near Γ, this can only be true if $d(r_2\circ f)\neq 0$ on Γ.

We must now prove that f is unbranched on Γ. To do this, we use an argument due to Kerzman, Kohn, and Nirenberg [10]. For $t>0$, define $\varrho=\exp(tr_2)-1$. The function $\varrho\circ f$ is a local defining function for Γ. Furthermore, for a fixed $z\in\Gamma$, t can be chosen to be sufficiently large so that $\varrho\circ f$ is strictly plurisubharmonic near z (see, for example [10]). Hence

$$\operatorname{Det}\left[\frac{\partial^2(\varrho\circ f)}{\partial z_i\partial\bar z_j}\right]=|\operatorname{Det}[f']|^2\operatorname{Det}\left[\frac{\partial^2\varrho}{\partial w_i\partial\bar w_j}\circ f\right]$$

must be strictly positive on Γ near z. We conclude that $\operatorname{Det}[f']=u$ does not vanish on Γ and that f is unbranched on Γ. Hence, f maps Γ into the set of strictly pseudoconvex boundary points of D_2. This completes the proof of Theorem 2.

6. Proofs of the lemmas

Proof of Lemma 1. Suppose that h is a function in $C^\infty(\bar{D})$ which extends to a neighborhood of $\bar{D}$ in such a way that h is real analytic in a neighborhood of bD. Let v be the solution to the Cauchy problem:

$$\Delta v = h \quad \text{near } bD$$

with

$$v = \frac{\partial v}{\partial \eta} = 0 \quad \text{on } bD.$$

Here, $\partial v/\partial \eta$ is the normal derivative of v on bD. The Cauchy–Kowalewski theorem guarantees that there is an open set U containing bD such that v satisfies the Cauchy problem in U and is real analytic there. Let ψ be a function in $C_0^\infty(U)$ which is equal to one in a neighborhood of bD. Now, a simple integration by parts reveals that $\Delta(\psi v)$ is a function which is orthogonal to holomorphic functions on D. Define $\varphi = h - \Delta(\psi v)$. Notice that φ is a function in $C_0^\infty(D)$ such that $Ph = P(h - \Delta(\psi v)) = P\varphi$. In the event that h extends to be a holomorphic function on a neighborhood of $\bar{D}$, then $h = Ph = P\varphi$, and the proof of Lemma 1 is complete. Note that we have also proved Corollary 3.

Proof of Lemma 2. A classical theorem due to R. Remmert states that f is a branched cover of some finite order m and that the set $V = \{w \in D_2 \colon w = f(z);\ u(z) = 0\}$ is a complex analytic variety in D_2. Let $F_1, F_2, \ldots, F_m$ denote the m inverses to f which are defined locally on $D_2 - V$ and let $U_k = \operatorname{Det}[F_k']$. We shall employ the following well known version of Riemann's removable singularity theorem: if D is a bounded domain contained in $\mathbf{C}^n$ and X is a complex analytic variety contained in D, then every function which is holomorphic on $D - X$ and in $L^2(D)$ is actually holomorphic on all of D. For a simple proof of this theorem, see [3].

The Jacobian determinant of f viewed as a mapping on $\mathbf{R}^{2n} \approx \mathbf{C}^n$ is equal to $|u|^2$. Hence, $\|u \cdot (\varphi \circ f)\|_{L^2(D_1)} = m^{\frac{1}{2}} \|\varphi\|_{L^2(D_2)}$ and the terms in the transformation formula are well defined. The equation

$$P_1(u \cdot (\varphi \circ f)) = u \cdot ((P_2 \varphi) \circ f) \tag{6.1}$$

is certainly true when φ is in $H(D_2)$. We shall now complete the proof of Lemma 2 by showing that (6.1) holds whenever φ is in a certain dense subset of $H(D_2)^\perp$. Let Ω be equal to the linear span of $\{\partial \psi/\partial z_i \colon \psi \in C_0^\infty(D_2 - V) \colon i = 1, \ldots, n\}$. We claim that Ω is a dense subspace of $H(D_2)^\perp$. Indeed, if $v \in H(D_2)^\perp$ is orthogonal to Ω, then v is a distributional solution to $\bar{\partial} v = 0$ on $D_2 - V$. Hence v is a function in $L^2(D_2)$ which is holomorphic on $D_2 - V$ and is therefore in $H(D_2)$ by the removable singularity theorem. Hence, $v = 0$. Now if $\varphi = \partial \psi/\partial z_i$

for $\psi \in C_0^\infty(D_2 - V)$, then $P_2\varphi = 0$ because $\partial\psi/\partial z_i$ is orthogonal to holomorphic functions. Furthermore, for $h \in H(D_1)$, we see that

$$\int_{D_1} h\,\overline{u \cdot (\varphi \circ f)} = \int_{D_2} \left(\sum_{k=1}^{m} U_k \cdot (h \circ F_k) \right) \frac{\partial \bar{\psi}}{\partial \bar{z}_i} = 0$$

via integration by parts. Hence $P_1(u \cdot (\varphi \circ f)) = 0 = u \cdot ((P_2\varphi) \circ f)$ whenever $\varphi \in \Omega$ and the proof of the transformation rule is finished.

References

[1] BELL, S., Biholomorphic mappings and the $\bar{\partial}$-problem. *Ann. of Math.*, 114 (1981), 103–113.

[2] —— Proper holomorphic mappings and the Bergman projection. *Duke Math. J.*, 48 (1981), 167–175.

[3] —— The Bergman kernel function and proper holomorphic mappings. *Trans. Amer. Math. Soc.* In press, 1981.

[4] BURNS, D. & SHNIDER, S., Geometry of hypersurfaces and mapping theorems in $\mathbf{C}^n$. *Comment. Math. Helv.*, 54 (1979), 199–217.

[5] DERRIDJ, M. & TARTAKOFF, D. S., On the global real analyticity of solutions to the $\bar{\partial}$-Neuman problem. *Comm. Partial Differential Equations*, 1 (1976), 401–435.

[6] —— Sur la régularité locale des solutions du problème de Neumann pour $\bar{\partial}$. *Journées sur les fonctions analytiques.* Springer Lecture Notes in Math. 578, 1977, 207–216.

[7] DIEDERICH, K. & FORNAESS, J. E., A remark on a paper of S. R. Bell. *Manuscripta Math.*, 34 (1981), 31–44.

[8] —— Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions. *Invent. Math.*, 39 (1977), 129–141.

[9] FORNAESS, J. E., Biholomorphic mappings between weakly pseudoconvex domains. *Pacific J. Math.*, 74 (1978), 63–65.

[10] KERZMAN, N., A Monge–Ampère equation in complex analysis. *Proc. of Symposia in Pure Math.*, 30 (1977), 161–167.

[11] KOHN, J. J., Harmonic integrals on strongly pseudoconvex manifolds, I and II. *Ann. of Math.*, 78 (1963), 112–148 and 79 (1964), 450–472.

[12] KOMATSU, G., Global analytic-hypoellipticity of the $\bar{\partial}$-Neumann problem. *Tôhoku Math. J. Ser 2*, 28 (1976), 145–156.

[13] TARTAKOFF, D. S., On the global real analyticity of solutions to $\square_b$. *Comm. Partial Differential Equations*, 1 (1976), 283–311.

[14] —— Local analytic hypoellipticity for $\square_b$ on non-degenerate Cauchy–Riemann manifolds. Proc. Nat. Acad. Sci. USA, 75 (1978), 3027–3028.

[15] —— The local real analyticity of solutions to $\square_b$ and the $\bar{\partial}$-Neumann problem. *Acta Math.*, 145 (1980), 177–204.

[16] TRÈVES, F., Analytic hypo-ellipticity of a class of pseudodifferential operators with double characteristics and applications to the $\bar{\partial}$-Neumann problem. *Comm. Partial Differential Equations*, 3 (1978), 475–642.

[17] DERRIDJ, M., Regularité pour $\bar{\partial}$ dans quelques domaines faiblement pseudo-convexes. *J. Diff. Geom.*, 13 (1978), 559–576.

Received April 28, 1981