$B(\mathcal{H})$ DOES NOT HAVE THE APPROXIMATION PROPERTY

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In this paper we prove that $B(\mathcal{H})$, the space of all bounded linear operators on a Hilbert space, does not have the approximation property (abbreviated throughout AP).

The first example of a Banach space which does not have AP, was given by P. Enflo [2]. Following the work of Enflo, several other counterexamples to the AP have been constructed.

$B(\mathcal{H})$ is the first Banach space appearing naturally in analysis which is proved to fail AP. $B(\mathcal{H})$ is also the first known example of a C^*-algebra without AP. Our result implies, of course, the existence of a separable C^*-algebra without AP (cf. Corollary on p. 92). Approximation problems in the context of C^*-algebra theory have been considered by several authors (cf. [1], [4], [5], [8], [9]). Let us mention two of these results:

In [4], U. Haagerup proved that the C^*-algebra generated by the left regular representation of the free group on two generators, does have the AP. For some time this C^*-algebra was a candidate for a "natural counterexample" to the AP.

In [9], S. Wasserman proved that $B(\mathcal{H})$ is not nuclear, thus failing the "completely positive approximation property". The latter property, much stronger than AP, is a C^*-algebra analogue of the AP.

Let us now briefly describe the contents of the present paper. It is divided into 5 sections.

In Section 1 we present a criterion for a Banach space not to have the AP. This criterion is a modified version of Enflo's original one. We show how it is related to the ideas of Grothendieck [3], using the tensor product notation, which was originally used in [3] for the purpose of AP but has been neglected since. It seems to the author that the use of this notation makes an essential simplification in several computations.

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The proof of our main result requires a rather complicated construction which is gradually presented in Sections 2, 3, 4, 5. In our presentation we apply a "gliding hump" approach of repeated reduction of the main problem at hand to a number of "technical lemmas" which are proved later on. The whole construction is geared specifically to $B(\mathcal{H})$.

A preliminary exposition of our result appeared in [7]. The presentation of [7] is perhaps more heuristic than the present one.

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Notation. Given a set A, 1_A denotes the indicator function of A, $|A|$ denotes the number of elements of A (if A is finite); $|z|$ denotes also the absolute value of a complex number z.

A *partition* of A is a family of pairwise disjoint sets, which cover A.

If $\mathcal{H}$ is a Hilbert space, then $B(\mathcal{H})$ denotes the space of (bounded, linear) operators from $\mathcal{H}$ to $\mathcal{H}$; otherwise, the space of bounded linear operators from a Banach space X to a Banach space Y will be denoted by $L(X, Y)$.

Given a sequence $X_1, X_2, \ldots$ of Banach spaces, $(X_1 \oplus X_2 \oplus \ldots)_p$ denotes their l_p-sum (for the notation on Banach spaces, cf. [6]).

We shall also use the following tensor production notation:

Let X, Y be Banach spaces, $X \otimes Y$ denotes the algebraic tensor product of X and Y; $X \hat{\otimes} Y$ denotes the projective tensor product of X and Y, i.e., the completion of $X \otimes Y$ in the norm

$$\|\xi\|_\wedge \overset{\text{def}}{=} \inf \{\textstyle\sum \|x_n\|\,\|y_n\| : \xi = \sum x_n \otimes y_n\}.$$

For a bilinear form ξ on $X \otimes Y$ we denote

$$\|\xi\|_\vee = \sup \{|\xi(x, y)| : x \in X, y \in Y, \|x\| \leqslant 1, \|y\| \leqslant 1\}.$$

To every $\xi \in Y^* \hat{\otimes} X$ we assign a functional ξ_* on $L(X, Y)$ defined by

$$\xi_*(T) = \textstyle\sum y_n^*(Tx_n) \quad \text{for } T \in L(X, Y) \tag{0.1}$$

where $\xi = \sum y_n^* \otimes x_n$.

Section 1

A Banach space X is said to have the approximation property (abbreviated AP) if the identity operator on X, denoted I_X, can be approximated uniformly on every compact subset of X by finite rank operators.

Let us recall briefly the approach of Grothendieck [3] to the AP. We find it convenient to use the tensor product notation, as used in [3]; otherwise our presentation follows closely that of Lindenstrauss and Tzafriri ([6], Chapter 1.e).

Let X, Y be Banach spaces. Let us denote $L=L(X, Y)$. In L we have the locally convex topology τ, generated by the seminorms

$$\|T\|_K = \sup \{\|Tx\|: x\in K\},$$

where K ranges over all compact subsets of X. Grothendieck discovered that the dual space $(L,\tau)'$ can be identified with $Y^*\hat{\otimes} X$ by the natural isomorphism $\xi\in Y^*\hat{\otimes} X\to\xi_*\in (L,\tau)'$ where ξ_* is defined by (0.1) (for the proof, see [6], p. 31).

Now, let $X=Y$; for $\beta\in X^*\hat{\otimes} X$ let us denote

$$\operatorname{tr}\beta=\beta_*(I_X) \quad (=\textstyle\sum \varphi_a(x_a) \text{ if } \beta=\sum \varphi_a\otimes x_a).$$

By the Hahn–Banach theorem, X does not have AP if and only if there exists $\beta\in(L,\tau)'=X^*\hat{\otimes} X$ such that

$$\operatorname{tr}\beta=1, \tag{1.1}$$

$$\beta_*(T)=0 \quad \text{if} \quad \operatorname{rk} T=1. \tag{1.2}$$

Since every one dimensional operator on X is of the form $x^*\otimes x$ with $x^*\in X^*$, $x\in X$, (1.2) is clearly equivalent to

$$\sup\{\beta_*(x^*\otimes x): \|x^*\|\leqslant 1, \|x\|\leqslant 1\}=0.$$

It is easy to see that the last supremum is equal to $\|\beta\|_\vee$. Therefore, (1.2) is equivalent to

$$\|\beta\|_\vee=0. \tag{1.3}$$

In other words, X has the AP if and only if the formal identity map from $X^*\hat{\otimes} X$ into $X^*\check{\otimes} X$ is one-to-one.

Remark 1. A $u: X\to Y$ is called *approximable* if it can be approximated uniformly on every compact subset of X by finite dimensional operators. The above argument shows that u is not approximable if and only if there exists a $\beta\in Y^*\hat{\otimes} X$ such that

$$\beta(u)=1 \quad \text{and} \quad \|\beta\|_\vee=0. \tag{1.4}$$

Remark 2. Suppose that $\beta \in X^* \hat{\otimes} X$ satisfies (1.1) and (1.3) and let $\beta = \sum \varphi_n \otimes x_n$ be a "good" representation of β, i.e., $\sum \|\varphi_n\| \, \|x_n\| < \infty$. Then every subspace of X which contains all x_n's fails (obviously) the AP.

COROLLARY (of our main result). *There exists a separable C^*-algebra without* AP: *take the C^*-algebra generated by the corresponding x_n's from our construction for $X = B(\mathcal{H})$.*

Let us now present Enflo's idea leading to his construction of a space without AP. It can be seen as a development of Grothendieck's idea (although, as is apparent from [2], Enflo discovered his new approach to the AP independently of [3]).

The Enflo's criterion. Suppose that there exist $\beta_n \in X^* \hat{\otimes} X$ for $n = 1, 2, \ldots$ such that

(i) $\operatorname{tr} \beta_n = 1$ *for* $n = 1, 2, \ldots$

(ii) $\|\beta_n\|_\vee \to 0$ *as* $n \to \infty$

(iii) $\sum_{n=1}^{\infty} \|\beta_n - \beta_{n+1}\|_\wedge < \infty$.

Then X does not have the AP.

The proof is immediate: let us define

$$\beta = \beta_1 + \sum_{n=1}^{\infty} (\beta_{n+1} - \beta_n) = \lim_{n\to\infty} \beta_n.$$

Then $\beta \in X^* \hat{\otimes} X$, by the first equality and by (iii) and it satisfies (1.1) and (1.3), by the second equality and by (i), (ii), respectively.

In spite of formal similarity, conditions (ii), (iii) are much easier to handle than the condition (1.3): condition (1.3) is, in a way, an extrinsic condition, depending on the whole space X rather than on β alone. Consequently, it is very difficult to control. The corresponding condition (ii) is usually quite easy to control. To illustrate this let us look at a typical situation where

$$\beta_n = N^{-1} \sum_{j=1}^{N} y_j^* \otimes y_j \quad \text{with} \quad \|y_j^*\| = \|y_j\| = y_j^*(y_j) = 1 \quad \text{for} \quad j = 1, \ldots, N$$

(for some N which depends on n and goes to ∞ with n).

Using a very simple estimate (4.5), p. 103, we see that

$$\|\beta_n\|_\vee \leqslant N^{-1} \max_{|\varepsilon_j|=1} \left\| \sum_{j=1}^{N} \varepsilon_j y_j \right\|$$

Thus, unless $\|\sum_{j=1}^{N} \varepsilon_j y_j\| \sim N$ for some choice of signs ε_j, then $\|\beta_n\|_\vee$ is small. In concrete applications we usually obtain $\|\sum_{j=1}^{N} \varepsilon_j y_j\| = o(N)$ quite automatically.

Therefore, the whole difficulty is usually concentrated in the condition (iii). Here the problem is intrinsic, i.e., it is enough to exhibit a single representation $\beta_n - \beta_{n+1} = \sum \varphi_a \otimes x_a$ which is "good".

The rest of the paper is devoted to the construction of a sequence $\beta_n \in B(\mathcal{H})^* \hat{\otimes} B(\mathcal{H})$, satisfying the conditions (i), (ii), (iii).

Section 2

In this section we shall define a Hilbert space $\mathcal{H}$ and $\beta_n \in B(\mathcal{H})^* \hat{\otimes} B(\mathcal{H})$.

Notation. Let A be a finite set. We define the measure μ_A on A by

$$\mu_A(\{a\}) = |A|^{-1} \quad \text{for every } a \in A.$$

Let us denote $L_2(A) = L_2(\mu_A)$. For $B \subset A$ we denote by p_B the projection in $L_2(A)$ defined by $p_B f = f \cdot 1_B$.

By $M(A)$ we denote the set of all $A \times A$ matrices. For $\alpha, \beta \in A$ we denote $\varepsilon_{\alpha,\beta} = 1_{\{\alpha\}} \otimes 1_{\{\beta\}}$ (i.e., it is the matrix which has 1 on (α, β)-th place and zeroes elsewhere). We denote also $M(q) = M(\{1, ..., q\})$. We identify $M(A)$ with the algebraic tensor product $L_2(A) \otimes L_2(A)$, in the usual way. For $x \in M(A)$ let

$$\|x\|_\infty = \|x\|_{L_2(A) \check{\otimes} L_2(A)}, \quad \|x\|_1 = \|x\|_{L_2(A) \hat{\otimes} L_2(A)}.$$

We shall denote $\underline{M}(A) = L_2(A) \check{\otimes} L_2(A)$, $\underline{\underline{M}}(A) = L_2(A) \hat{\otimes} L_2(A)$; let us recall that $L_2(A) \check{\otimes} L_2(A)$ is naturally isometric to $B(L_2(A))$ and that $L_2(A) \hat{\otimes} L_2(A)$ is naturally isometric to $B(L_2(A))^*$.

For an $x \in M(A)$ we shall denote by $\underline{x}$ and $\underline{\underline{x}}$ the corresponding elements of $B(L_2(A))$ and $B(L_2(A))^*$ respectively.

We shall use the following ad hoc definition.

Definition. Let $x \in M(A)$ and $y \in M(B)$. We shall say that x and y are *strictly equivalent* if one can be obtained from another by applying the following four operations:

(1) permutations of rows and columns,

(2) multiplication of rows and columns by numbers of absolute value one,

(3) deleting some rows and columns consisting entirely of zeroes,

(4) transposition.

Needless to say, if x and y are strictly equivalent, then $\|x\|_p = \|y\|_p$ for $p = 1, \infty$.

Now we pass to our construction.

Let $K_1, K_2, \dots$ be some finite sets (to be specified later on). Let μ denote the product measure $\mu = \otimes_{n=1}^{\infty} \mu_{K_n}$ on $\prod_{n=1}^{\infty} K_n$.

Let us denote

$$\mathbf{B} = B(L_2(\mu)).$$

We define $\mathcal{H}$ as the Hilbert sum of countably many copies of the space $L_2(\mu)$, i.e.,

$$\mathcal{H} = (L_2(\mu) \oplus L_2(\mu) \oplus \dots)_2.$$

The l_∞-sum $(\mathbf{B} \oplus \mathbf{B} \oplus \dots)_\infty$ is embedded in a natural way in $B(\mathcal{H})$. Formally, given a sequence $x^1, x^2, \dots \in \mathbf{B}$ such that $\sup \|x^n\|_{\mathbf{B}} < \infty$, we define $\oplus_{n=1}^{\infty} x^n \in B(\mathcal{H})$ by

$$\left(\bigoplus_{n=1}^{\infty} x^n\right)(f_1, f_2, \dots) = (x^1 f_1, x^2 f_2, \dots).$$

Obviously we have

$$\left\| \bigoplus_{n=1}^{\infty} x^n \right\|_{B(\mathcal{H})} = \sup \|x^n\|_{\mathbf{B}}. \tag{2.0}$$

Moreover, $(\mathbf{B} \oplus \mathbf{B} \oplus \dots)_\infty$ is complemented in $B(\mathcal{H})$ by the natural projection R, the restriction.

Let $\mathbf{N}$ denote the set of natural numbers, let U be a fixed free ultrafilter in $\mathbf{N}$. Given a sequence $\varphi^1, \varphi^2, \dots \in \mathbf{B}^*$, we define $\mathrm{Lim}_n\, \varphi^n \in B(\mathcal{H})^*$ in the following way: let $\varphi \in [(\mathbf{B} \oplus \mathbf{B} \oplus \dots)_\infty]^*$ be defined by

$$\varphi\left(\bigoplus_{n=1}^{\infty} x^n\right) = \lim_{n \in U} \varphi^n(x^n);$$

we set then

$$\mathrm{Lim}_n\, \varphi^n = R^* \varphi.$$

Obviously,

$$\|\mathrm{Lim}_n\, \varphi^n\|_{B(\mathcal{H})^*} \leqslant \limsup_n \|\varphi^n\|_{\mathbf{B}^*}. \tag{2.0}^*$$

Now we proceed to define $\beta_n \in B(\mathcal{H})^* \otimes B(\mathcal{H})$ for $n = 1, 2, \dots$. Let us denote $D_n = K_1 \times \dots \times K_n$. For $a \in D_n$ let us define the projection π_a in $L_2(\mu)$ (or in $L_2(D_k)$ for $k \geqslant n$) by

$$\pi_a = p_{\{i \in \prod_{m=1}^{k} K_m : (i_1, \dots, i_n) = a\}}, \quad \text{for} \quad n \leqslant k \leqslant \infty.$$

Given $a \in D_m$, $b \in D_n$ let us define

$$\varrho_{a,b}\, x = \pi_a x \pi_b \quad \text{for} \quad x \in \mathbf{B},$$

and given $x^1, x^2 \ldots \in \mathbf{B}$ and $\varphi^1, \varphi^2, \ldots \in \mathbf{B}^*$ we set

$$x_{a,b} = \bigoplus_{k=1}^{\infty} \varrho_{a,b} x^k, \quad \varphi_{a,b} = \operatorname{Lim}_k \varrho^*_{a,b} \varphi^k,$$

and

$$\beta_n = \beta_n(x^k, \varphi^k) = \sum_{a,b \in D_n} \varphi_{a,b} \otimes x_{a,b}.$$

We shall also denote

$$x^k_{a,b} = \varrho_{a,b} x^k, \quad \varphi^k_{a,b} = \varrho^*_{a,b} \varphi^k.$$

It still remains to define the x^k, φ^k, which will be used in our construction.

We shall first formulate a lemma which is the main combinatorial ingredient of our construction.

Let q be the square of a natural number, say $q = m^2$. A partition ∇ of the set $\{1, \ldots, q\}$ will be called *regular* if $|\nabla| = m$ and if every member of ∇ has m elements. Let $\$_q$ be a fixed regular partition of $\{1, \ldots, q\}$.

LEMMA 1. *Let q be a number of the form 2^{16p} where p is a natural number.*(1) *For $j = 1, 2, \ldots, q^4$ there exist regular partitions ∇^q_j of $\{1, \ldots, q\}$ and Hadamard matrices $v^q_j \in M(q)$ so that for every $S \in \$_q$,*

$$\|p_S v^q_j p_A\|_1 = q^{1/2} \quad \textit{for every } A \in \nabla^q_j, \tag{2.1}$$

$$\|p_S v^q_j p_A\|_\infty \leq q^{15/32} \quad \textit{for every } A \in \nabla^q_i \textit{ with } i \neq j. \tag{2.2}$$

Moreover,

$$\sum_{b=1}^{q} v^q_j(a, b) = q^{1/2} = \sum_{b=1}^{q} v^q_j(b, a) \quad \textit{for every } a. \tag{2.3}$$

(by an Hadamard matrix we mean a square matrix whose all entries have absolute value one and whose columns are mutually orthogonal).

We postpone the proof of this lemma to Section 5.

Let now q_n be a sequence of natural numbers such that:

$$q_n \quad \text{are of the form } 2^{16p} \text{ where } p \text{ is a natural number,} \tag{2.4}$$

$$q_n \quad \text{goes to } \infty \text{ faster than any power of } n, \tag{2.5}$$

$$q_{n+1} < q_n^2 \quad \text{for every } n. \tag{2.6}$$

We set $K_n = \{1, \ldots, q_n\} \times \{1, \ldots, q_n\}$. By (2.6), $|K_n| < q_n^4$, therefore we can find $|K_n| + 1$ Hadamard matrices $v^{q_n}_0$ and $v^{q_n}_j$ for $j \in K_n$ such that they satisfy the conditions of Lemma 1 for $q := q_n$.

(1) Clearly, this p is unrelated to the p of (2.1).

For $i \in K_n$ we denote its coordinates by i^0 and i^1, respectively.

From now on i_n and j_n will always denote elements of K_n.

Let us set

$$w_n(i_n, i_{n+1}; j_n, j_{n+1}) = q_n^{-1} v_{i_{n+1}}^{q_n}(j_n^1, i_n^0)\, v_{j_{n+1}}^{q_n}(i_n^1, j_n^0),$$

$$y_n(i_n, j_n) = q_n^{-1} v_0^{q_n}(j_n^1, i_n^0)\, v_0^{q_n}(i_n^1, j_n^0).$$

Notice that

$$\|y_n\|_\infty = 1 \quad \text{for every } n.$$

For $n \leqslant m$ let us define $u_{n,m} \in M(K_m)$ by

$$u_{n,m}(i, j) = \prod_{k=1}^{n-1} w_k(i_k, i_{k+1}; j_k, j_{k+1}) \prod_{k=n}^{m} y_k(i_k, j_k).$$

By μ^m let us denote the product measure $\mu^m = \otimes_{n=m+1}^\infty \mu_{K_n}$ on $\prod_{n=m+1}^\infty K_n$.

Clearly, y_n is an isometry of $L_2(K_n)$ onto itself. Moreover, by (2.3), $y_n(1) = 1$ (here 1 denotes the function constant 1), consequently, the infinite tensor product $\otimes_{k=m}^\infty y_k$ is well defined and thus defines an element of $B(L_2(\mu^m))$:

$$y^m \overset{\text{def}}{=} \bigotimes_{k=m+1}^{\infty} y_k.$$

We have, obviously,

$$\|y^m\|_{B(L_2(\mu^m))} = 1. \tag{2.7}$$

Let us pick for $m = 1, 2, \ldots$ a $\xi^m \in B(L_2(\mu^m))^*$ which is a Hahn–Banach functional of y^m, i.e.,

$$\xi^m(y^m) = 1, \quad \|\xi^m\|_{B(L_2(\mu^m))^*} = 1. \tag{2.8}$$

Now we define the desired $x^1, x^2, \ldots$ and $\varphi^1, \varphi^2, \ldots$ by

$$x^m = \underline{u}_{m,m} \check{\otimes} y^m, \quad \varphi^m = |D_m|^{-1} \underline{\underline{u}}_{m,m} \hat{\otimes} \xi^m,$$

where we make the natural identifications

$$\mathbf{B} = \underline{M}(D_m) \check{\otimes} B(L_2(\mu^m)),$$

$$\mathbf{B}^* = \underline{\underline{M}}(D_m) \hat{\otimes} B(L_2(\mu^m))^*.$$

Let us notice that we have, for every $n \geqslant m$,

$$x^m = \underline{u}_{m,n} \otimes y^n. \tag{2.9}$$

As follows from the proposition on p. 102, $\|u_{m,n}\|_\infty = 1$, $\|u_{m,n}\|_1 = |D_k|$ and therefore $\oplus_{m=1}^\infty x^m$ and $\operatorname{Lim}_m \varphi^m$ will be well defined.

Section 3

In this section we shall concentrate on the crucial condition (iii). We want to prove that, for every n,

$$\|\beta_n - \beta_{n+1}\|_\wedge \leqslant 2q_n^{-1/32}, \tag{3.0}$$

which, in view of (2.5), clearly implies (iii).

From now on n is fixed.

In proving (3.0), it will be convenient to introduce an intermediate step. Let

$$\gamma = \sum_{a \in K_n, c \in K_{n+1}} \varphi_{a,c} \otimes x_{a,c}.$$

We shall prove that

$$\|\beta_n - \gamma\|_\wedge \leqslant q_n^{-1/32} \quad \text{and} \quad \|\gamma - \beta_{n+1}\|_\wedge \leqslant q_n^{-1/32}. \tag{3.1}$$

Let us first notice that, for $a, b \in D_n$, $c \in D_{n+1}$,

$$\varphi_{a;b} = \sum_{h \in K_{n+1}} \varphi_{a;b,h}, \quad \varphi_{a;c} = \sum_{h \in K_{n+1}} \varphi_{a,h;c}$$

(here and everywhere else b, h denotes the element of D_{n+1} whose coordinates are $b_1, \ldots, b_n, h$. The same about a, h etc.).

Therefore, if we denote for $a, b \in D_n$, $c \in D_{n+1}$, $g \in K_{n+1}$,

$$y_{a;b,g} = \sum_{h \neq g, h \in K_{n+1}} x_{a;b,h}, \quad y_{a,g;c} = \sum_{h \neq g, h \in K_{n+1}} x_{a,h;c},$$

we obtain

$$\beta_n - \gamma = \sum_{a \in D_n, c \in D_{n+1}} \varphi_{a;c} \otimes y_{a,c},$$

$$\gamma - \beta_{n+1} = \sum_{a, c \in D_{n+1}} \varphi_{a;c} \otimes y_{a;c}.$$

Now we shall make an appropriate grouping in these sums. In the sequel let:

$$g \in K_{n+1}, \; 1 \leqslant e, f \leqslant q_n, \quad A, S \subset \{1, \ldots, q_n\}, \quad \alpha, \beta \in D_{n-1}$$

and let us denote

$$\delta = \delta_{g,e,f,A,S,\alpha,\beta} = \sum_{(a,b) \in H} \varphi_{a,b} \otimes y_{a,b},$$

$$\delta' = \delta'_{g,e,f,A,S,\alpha,\beta} = \sum_{(a,b) \in H'} \varphi_{a,b} \otimes y_{a,b},$$

where

$$H = H_{g,e,f,A,S,\alpha,\beta} \overset{\text{def}}{=} \{(a, b) \in D_n \times D_{n+1}: (a_1, \dots, a_{n-1}) = \alpha, (b_1, \dots, b_{n-1}) = \beta,$$
$$a_n^0 = e, b_n^1 = f, a_n^1 \in S, b_n^0 \in A, b_{n+1} = g\},$$

$$H' = H'_{g,e,f,A,S,\alpha,\beta} \overset{\text{def}}{=} \{(a, b) \in D_{n+1} \times D_{n+1}: (b_1, \dots, b_{n-1}) = \alpha, (a_1, \dots, a_{n-1}) = \beta,$$
$$b_n^0 = e, a_n^1 = f, b_n^1 \in S, a_n^0 \in A, a_{n+1} = g\}.$$

The proof of (3.1) will be based on the following two lemmas.

LEMMA 2. *We have*

$$\|\delta\|_\wedge \leqslant \Big\| \sum_{(a,b)\in H} \varphi_{a;b} \Big\|_{B(\mathcal{H})^*} \Big\| \sum_{(a,b)\in H} y_{a;b} \Big\|_{B(\mathcal{H})},$$

$$\|\delta'\|_\wedge \leqslant \Big\| \sum_{(a,b)\in H'} \varphi_{a;b} \Big\|_{B(\mathcal{H})^*} \Big\| \sum_{(a,b)\in H} y_{a;b} \Big\|_{B(\mathcal{H})}.$$

Let us denote

$$\Phi_m = \sum_{(a,b)\in H} \varphi^m_{a,b}, \quad \Phi'_m = \sum_{(a,b)\in H'} \varphi^m_{a,b},$$

$$Y_m = \sum_{(a,b)\in H} y^m_{a,b}, \quad Y'_m = \sum_{(a,b)\in H'} y^m_{a,b}.$$

LEMMA 3. *We have*

$$\|\Phi_m\|_{\mathbf{B}^*} = \|\Phi'_m\|_{\mathbf{B}^*} \leqslant (q_1 \dots q_n)^{-3} q_{n+1}^{-2} \|p_S v_g^{q_n} p_A\|_1 \quad \textit{if } m > n, \tag{3.2}$$

$$\|Y_m\|_{\mathbf{B}} = \|Y'_m\|_{\mathbf{B}} = \begin{cases} (q_1 \cdots q_n)^{-1} \max\limits_{h\neq g} \|p_S v_h^{q_n} p_A\|_\infty & \textit{if } m > n, \\ (q_1 \cdots q_n)^{-1} \|p_S v_0^{q_n} p_A\|_\infty & \textit{if } m \leqslant n. \end{cases} \tag{3.3}$$

With these estimates we easily obtain (3.1) and hence (3.0): by (2.1) and (2.2), if $A \in \nabla_g^{q_n}$ and $S \in \$_{q_n}$, then

$$\|\Phi_m\|_{\mathbf{B}^*} = \|\Phi'_m\|_{\mathbf{B}^*} \leqslant (q_1 \dots q_n)^{-3} q_{n+1}^{-2} q_n^{1/2} \quad \text{for } m > n$$

$$\|Y_m\|_{\mathbf{B}} = \|Y'_m\|_{\mathbf{B}} \leqslant (q_1 \dots q_n)^{-1} q_n^{15/32} \quad \text{for all } m.$$

Since

$$\sum_{(a,b)\in H} \varphi_{a,b} = \mathrm{Lim}_m\, \Phi_m, \qquad \sum_{(a,b)\in H'} \varphi_{a,b} = \mathrm{Lim}_m\, \Phi'_m,$$

$$\sum_{(a,b)\in H} y_{a,b} = \bigoplus_{m=1}^{\infty} Y_m, \qquad \sum_{(a,b)\in H'} y_{a,b} = \bigoplus_{m=1}^{\infty} Y'_m,$$

Lemma 2 and (2.0), (2.0)* imply that for every tuple $(g, e, f, A, S, \alpha, \beta)$ such that

$$g \in K_{n+1}, \quad 1 \leqslant e, f \leqslant q_n, \quad A \in \nabla_g^{q_n}, \quad S \in \$_{q_n}, \quad \alpha, \beta \in D_{n-1}, \tag{3.4}$$

we have

$$\left.\begin{array}{l}\|\delta_{g,e,f,A,S,\alpha,\beta}\|_\wedge \leqslant \\ \|\delta'_{g,e,f,A,S,\alpha,\beta}\|_\wedge \leqslant\end{array}\right\} (q_1 \dots q_n)^{-4} \cdot q_n^{1-1/32} \cdot q_{n+1}^{-2}. \tag{3.5}$$

Let us now observe that

$$\beta_n - \gamma = \sum \delta_{g,e,f,A,S,\alpha,\beta}, \quad \gamma = \beta_{n+1} = \sum \delta'_{g,e,f,A,S,\alpha,\beta}$$

where the summations range over all tuples satisfying (3.4). The number of such tuples is obviously equal to

$$|K_{n+1}| \cdot q_\iota \cdot q_n \cdot q_n^{1/2} \cdot q_n^{1/2} |D_{n-1}| \cdot |D_{n-1}| = q_{n+1}^2 \cdot q_n^3 \cdot (q_1 \dots q_{n-1})^4.$$

A glance at (3.5) convinces us that (3.1) holds.

Proof of Lemma 2. Let E denote the set of all functions from D_n into $\{-1, 1\}$ and let F denote the set of all functions from D_{n+1} into $\{-1, 1\}$. We have the following identities:

$$\delta = |E|^{-1}|E|^{-1} \sum_{\varepsilon \in E} \sum_{\eta \in E} [(\sum_H \varepsilon(a)\eta(b)\varphi_{a;b,g}) \otimes (\sum_H \varepsilon(a)\eta(b) y_{a;b,g})] \tag{3.6}$$

$$\delta' = |E|^{-1}|F|^{-1} \sum_{\varepsilon \in E} \sum_{\eta \in F} [(\sum_{H'} \varepsilon(a)\eta(b)\varphi_{a,g;b}) \otimes (\sum_{H'} \varepsilon(a)\eta(b) y_{a,g;b})] \tag{3.7}$$

(we adopt here the following notational convention: we write

$$\sum_H \text{ instead of } \sum_{\{(a,b):(a;b,g)\in H\}}; \quad \sum_{H'} \text{ instead of } \sum_{\{(a,b):(a,g;b)\in H'\}}).$$

These formulas are simple applications of the invariance of trace. For example

$$\sum_{\varepsilon \in E} \sum_{\eta \in F} [(\sum_{H'} \varepsilon(a)\eta(b)\varphi_{a,g;b}) \otimes (\sum_{H'} \varepsilon(c)\eta(d) y_{c,g;d})] = \sum_{H'} \sum_{H'} (\sum_{\varepsilon,\eta} \varepsilon(a)\varepsilon(c)\eta(b)\eta(d))\varphi_{a,g;b} \otimes y_{c,g;d}. \tag{3.8}$$

Notice now that

$$\sum_{\varepsilon \in E} \varepsilon(a)\varepsilon(c) = \begin{cases} |E| & \text{if } a = c, \\ 0 & \text{if } a \neq c, \end{cases} \qquad \sum_{\eta \in F} \eta(b)\eta(d) = \begin{cases} |F| & \text{if } b = d, \\ 0 & \text{if } b \neq d. \end{cases}$$

Therefore our sum in (3.8) equals

$$|E||F| \sum_{H'} \varphi_{a,g;b} \otimes y_{a,g;b},$$

which gives (3.7). The proof of (3.6) is completely analoguous.

To complete the proof we shall show that, for all $\varepsilon \in E$, $\eta \in F$,

$$\Big\|\sum_{H'} \varepsilon(a)\,\eta(b)\,\varphi_{a,g;b}\Big\|_{B(\mathcal{H})^*} = \Big\|\sum_{H'} \varphi_{a,g;b}\Big\|_{B(\mathcal{H})^*}$$
$$\Big\|\sum_{H'} \varepsilon(a)\,\eta(b)\,y_{a,g;b}\Big\|_{B(\mathcal{H})} = \Big\|\sum_{H'} y_{a,g;b}\Big\|_{B(\mathcal{H})} \tag{3.9}$$

and that analoguous formulas hold in the case of H.

This is true because of "strict equivalence". More precisely, let us define operators T_1: $\mathbf{B}\to\mathbf{B}$, T_2: $(\mathbf{B}\oplus\mathbf{B}\oplus\ldots)_\infty \to (\mathbf{B}\oplus\mathbf{B}\oplus\ldots)_\infty$ and T: $B(\mathcal{H})\to B(\mathcal{H})$ by

$$T_1 x = \Big(\sum_{a\in D_n} \varepsilon(a)\pi_a\Big)\circ x \circ \Big(\sum_{b\in D_{n+1}} \eta(b)\pi_b\Big),$$

$$T_2(x^1, x^2, \ldots) = (T_1 x^1, T_1 x^2, \ldots), \quad T = T_2\circ R.$$

Obviously $\|T\| = 1$ and we have

$$T\Big(\sum_{H'} y_{a,g;b}\Big) = \sum_{H'} \varepsilon(a)\,\eta(b)\,y_{a,g;b} \quad \text{and vice versa,}$$

$$T^*\Big(\sum_{H'} \varphi_{a,g;b}\Big) = \sum_{H'} \varepsilon(a)\,\eta(b)\,\varphi_{a,g;b} \quad \text{and vice versa.}$$

This gives (3.9). The case of H is completely analoguous.

Section 4

Proof of Lemma 3. Throughout this section let

$$\varkappa = \max(m, n+1).$$

For $h\in K_{n+1}$ let us define sets E, $F\subset D_\varkappa$ by

$$E = \{a\in D_\varkappa\colon (a_1, \ldots, a_{n-1}) = \alpha,\ a_n^0 = e,\ a_n^1 \in S\},$$

$$F_h = \{b\in D_\varkappa\colon (b_1, \ldots, b_{n-1}) = \beta,\ b_n^1 = f,\ b_n^0\in A,\ b_{n+1} = h\},$$

and let us put

$$\omega_h = p_E u_{m,\varkappa}\, p_{F_h}, \quad \omega_h' = p_{F_h} u_{m,\varkappa}\, p_E$$

(recall that $u_{m,\varkappa} = w_1\cdot\ldots\cdot w_{m-1}\cdot y_m\cdot\ldots\cdot y_\varkappa$ — coordinatewise multiplication of matrices).

It should be clear that

$$\Phi_m = |D_m|^{-1}\underline{\underline{\omega}}_g\otimes\xi^m, \quad \Phi_m' = |D_m|^{-1}\underline{\underline{\omega}}_g'\otimes\xi^m, \quad \text{for } m > n,$$

$$Y_m = \Big(\sum_{h\neq g} \underline{\omega}_h\Big)\otimes y^\varkappa, \quad Y_m' = \Big(\sum_{h\neq g} \underline{\omega}_h'\Big)\otimes y^\varkappa, \quad \text{for all } m.$$

The following sublemma will be proved in the end of this section.

SUBLEMMA 1. $\|\sum_{h\neq g} \omega_h\|_\infty \leqslant \max_{h\neq g} \|\omega_h\|_\infty$, $\|\sum_{h\neq g} \omega'_h\|_\infty \leqslant \max_{h\neq g} \|\omega'_h\|_\infty$

(*actually, equalities hold*).

Let us notice that $\omega'_h = (\omega_h)^t$ (transposed). We also have $\|\xi^m\|_{B(L_2(\mu^m))^*} = 1$ and $\|y^\varkappa\|_{B(L_2(\mu^\varkappa))} = 1$. Thus we get

$$\|\Phi_m\|_{\mathbf{B}^*} = \|\Phi'_m\|_{\mathbf{B}^*} = |D_m|^{-1} \|\omega_g\|_1,$$

$$\|Y_m\|_{\mathbf{B}} = \|Y'_m\|_{\mathbf{B}} = \max_{h\neq g} \|\omega_h\|_\infty.$$

Now we proceed to compute the norms of ω_h's. It will be convenient to consider three cases:

Case 1°. $m > n$. Let us denote for $l \leqslant m$

$$O_l^m(i_l, \ldots, i_m; j_l, \ldots, j_m) = \prod_{k=l}^{m-1} w_k(i_k, i_{k+1}; j_k, j_{k+1})\, y_m(i_m, j_m). \tag{4.1}$$

Let us now define $x_h \in M(K_{n+1} \times \ldots \times K_m)$, $s_h \in M(K_n)$ and a constant C by

$$x_h(i_{n+1}, \ldots, i_m; j_{n+1}, \ldots, j_m) = \begin{cases} v_{i_{n+1}}^{q_n}(f, e)\, O_{n+1}^m(i_{n+1}, \ldots, i_m; j_{n+1}, \ldots, j_m) & \text{if } j_{n+1} = h \\ 0 & \text{otherwise} \end{cases}$$

$$s_h(i_n, j_n) = \begin{cases} w_{n-1}(\alpha_{n-1}, i_n; \beta_{n-1}, j_n)\, v_h^{q_n}(i_n^1; j_n^0) & \text{if } i_n^0 = e, j_n^1 = f, i_n^1 \in S, j_n^0 \in A \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

$$C = \prod_{k=1}^{n-2} w_k(\alpha_k, \alpha_{k+1}; \beta_k, \beta_{k+1}) \cdot q_n^{-1}.$$

It is easy to see that

$$\omega_h = C\varepsilon_{\alpha,\beta} \otimes s_h \otimes x_h. \tag{4.3}$$

Therefore $\|\omega_h\|_p = |C|\, \|s_h\|_p \|x_h\|_p$ for $p = 1, \infty$. The inequalities (3.2) and (3.3) follow now immediately from the following two sublemmas, proved at the end of this section.

SUBLEMMA 2. $\|x_h\|_\infty = 1$, $\|x_h\|_1 \leqslant (q_{n+2} \cdots q_m)^2$.

SUBLEMMA 3. s_h *is strictly equivalent* (*in the sense of the definition on p.* 93) *to the matrix:*

$$t_h \overset{\text{def}}{=} q_{n-1}^{-1} \cdot p_S v_h^{q_n} p_A.$$

Case 2°. $m=n$. This time we define $\zeta_h \in M(K_{n+1})$ and a constant D by

$$\zeta_h(i_{n+1}, j_{n+1}) = \begin{cases} y_{n+1}(i_{n+1}, j_{n+1}) & \text{if } j_{n+1}=h, \\ 0 & \text{otherwise} \end{cases}$$

$$D = \prod_{k=1}^{n-2} w_k(\alpha_k, \alpha_{k+1}; \beta_k, \beta_{k+1}) \cdot v_0^{q_n}(f, e) \cdot q_n^{-1}$$

We have now (here s_0 is defined by (4.2) with $h=0$)

$$\omega_h = D\varepsilon_{\alpha,\beta} \otimes s_0 \otimes \zeta_h. \tag{4.4}$$

Now we see immediately that $\|\zeta_h\|_\infty = 1$ and (3.3) follows immediately by Sublemma 3 (in the case $h=0$).

Case 3°. $m<n$. Here everything is simpler. Let us define $\sigma \in M(K_n)$ and a constant E by

$$\sigma(i_n; j_n) = \begin{cases} v_0^{q_n}(i_n^1, j_n^0) & \text{if } i_n^0 = e, j_n^1 = f, i_n^1 \in S, j_n^0 \in A \\ 0 & \text{otherwise} \end{cases}$$

$$E = \prod_{k=1}^{m-1} w_k(\alpha_k, \alpha_{k+1}; \beta_k, \beta_{k+1}) \prod_{k=m}^{n-1} y_k(\alpha_k, \beta_k) v_0^{q_n}(f, e) \cdot q_n^{-1}.$$

We have $\omega_h = E\varepsilon_{\alpha,\beta} \otimes \sigma \otimes \zeta_h$ and (3.3) obviously holds.

To complete the proof of Lemma 3, we should prove Sublemmas 1, 2, 3. We shall need the following

PROPOSITION. *The matrices O_l^m, defined by* (4.1), *are orthogonal for every $l \leq m$.*

Proof. For $l \leq n < m$ let us define $\Gamma_n \in M(K_l \times \ldots \times K_m)$ by

$$\Gamma_n(i_l, \ldots, i_m; j_l, \ldots, j_m) = \begin{cases} q_n^{-1/2} v_{i_{n+1}}^{q_n}(j_n^1, i_n^0) & \text{if } i_n^1 = j_n^0 \text{ and } i_k = j_k \text{ for } k \neq n \\ 0 & \text{otherwise.} \end{cases}$$

We also define $T \in M(K_l \times \ldots \times K_m)$ by

$$T(i_l, \ldots, i_m; j_l, \ldots, j_m) = \begin{cases} y_m(i_m, j_m) & \text{if } i_k^0 = j_k^1 \text{ and } j_k^0 = i_k^1 \text{ for } l \leq k < m, \\ 0 & \text{otherwise.} \end{cases}$$

We see that Γ_n is a direct sum of orthogonal matrices $q_n^{-1/2} v_{i_{n+1}}^{q_n}$, therefore Γ_n is orthogonal.

For similar reasons, T is orthogonal. On the other hand, we have the identity

$$O_l^m = \Gamma_l \circ \Gamma_{l+1} \circ \ldots \circ \Gamma_m \circ T \circ \Gamma_m^t \circ \Gamma_{m-1}^t \circ \ldots \circ \Gamma_l^t,$$

therefore O_l^m is also orthogonal.

Proof of Sublemma 1. We shall use the following general fact which is very easy to prove:

Let X, Y *be Banach spaces, let* $x_1, \ldots, x_k \in X$, $y_1, \ldots, y_k \in Y$. *Then*

$$\|\sum x_j \otimes y_j\|_{X \check{\otimes} Y} \leqslant \max_{|\varepsilon_j|=1} \|\sum \varepsilon_j y_j\| \max_j \|x_j\|. \tag{4.5}$$

A glance at the formulas (4.3) and (4.4) convinces us that it suffices to show that

$$\|\sum_{h \neq g} \varepsilon_h x_h\|_\infty = \max_{h \neq g} \|x_h\|_\infty \quad \text{for every } |\varepsilon_h| = 1. \tag{4.6}$$

To prove this, let us notice that $x \overset{\text{def}}{=} \sum_{h \neq g} \varepsilon_h x_h$ is strictly equivalent to the matrix

$$O \overset{\text{def}}{=} O^m_{n+1} \circ p_{\{j \in K_{n+1} \times \ldots \times K_m : j_{n+1} \neq g\}}.$$

Indeed, x is obtained from O by multiplying its ith row by $v^{q_n}_{i_{n+1}}(f, e)$ and its jth column by $\varepsilon_{j_{n+1}}$. Similarly,

$$x_h \text{ is strictly equivalent to } O_h \overset{\text{def}}{=} O^m_{n+1} \circ p_{\{j \in K_{n+1} \times \ldots \times K_m : j_{n+1} = h\}}. \tag{4.7}$$

Now, since O^m_{n+1} is an orthogonal matrix (by the proposition),

$$\|O^m_{n+1}\|_\infty = \|O_h\|_\infty = 1, \tag{4.8}$$

which implies (4.6).

Proof of Sublemma 2. The first equality is contained in (4.8) and (4.7). For the norm $\|x_h\|_1 = \|O_h\|_1$ we use the following obvious estimate:

$$\textit{for every matrix } x,\ \|x\|_1 \leqslant \text{sum of the norms of the columns of } x. \tag{4.9}$$

In our case, the last number is $(q_{n+2} \cdots q_m)^2$.

Proof of Sublemma 3. The matrix $s \in M(K_n)$ defined by

$$s(i_n, j_n) = \begin{cases} v^{q_n}_h(i^1_n, j^0_n) & \text{if } i^0_n = e,\ j^1_n = f,\ i^1_n \in S,\ j^0_n \in A, \\ 0 & \text{otherwise,} \end{cases}$$

can be obtained from t_h by permutations of rows and columns and by adding some zero rows and columns, whence s_h is obtained from s by multiplying its i_nth row by the number $v^{q_{n-1}}_{i_n}(\beta^1_{n-1}, \alpha^0_{n-1})$ and its j_nth column by the number $v^{q_{n-1}}_{j_n}(\alpha^1_{n-1}, \beta^0_{n-1})$, all these numbers having absolute value 1.

Section 5

Now we are going to prove the remaining conditions (i) and (ii) as well as Lemma 1.

Condition (i). First let us notice that, for $a, b \in D_n$,

$$\varphi_{a,b}(x_{c,d}) = 0 \quad \text{unless} \quad (a, b) = (c, d). \tag{5.0}$$

Indeed, we have $\varphi_{a,b}(x_{c,d}) = \lim_{k \in U} \varrho^*_{a,b}\,\varphi^k(\varrho_{c,d}\,x^k)$ and

$$\varrho^*_{a,b}\,\varphi^k(\varrho_{c,d}\,x^k) = \varphi^k(\pi_a \pi_c x^k \pi_d \pi_b) = \begin{cases} \varphi^k(\varrho_{a,b}\,x^k) & \text{if } (a, b) = (c, d) \\ 0 & \text{otherwise.} \end{cases}$$

This gives (5.0). Now we can write

$$\beta_n = \sum_{a,b \in D_n} \varphi_{a,b}(x_{a,b}) = \Big(\sum_{a,b \in D_n} \varphi_{a,b}\Big)\Big(\sum_{c,d \in D_n} x_{c,d}\Big) = \lim_{k \in U} \varphi^k(x^k).$$

By (2.7) and (2.8) we have

$$\varphi^k(x^k) = |D_k|^{-1} \underline{\underline{u}}_{k,k}(\underline{u}_{k,k}) \cdot \xi^k(y^k) = 1,$$

which proves (i).

Condition (ii). We are going to prove that

$$\Big\| \sum_{a,b \in D_n} \varepsilon(a, b)\, \varphi_{a,b} \Big\|_{B(\mathcal{H})^*} \leqslant 1 \quad \text{for every } |\varepsilon(a, b)| = 1, \tag{5.1}$$

$$\|x_{a,b}\|_{B(\mathcal{H})} = (q_1 \dots q_n)^{-1} \quad \text{for every } a, b \in D_n. \tag{5.2}$$

By (4.5), this yields $\|\beta_n\|_{B(\mathcal{H})^* \check{\otimes} B(\mathcal{H})} \leqslant (q_1 \dots q_n)^{-1}$ and obviously implies (ii).

Proof of (5.1). By (2.0)*, it suffices to prove that $\|\sum \varepsilon(a, b) \varphi^k_{a,b}\|_{\mathbf{B}^*} \leqslant 1$ for every $k \geqslant n$. For $a, b \in D_n$ let us denote $W^k_{a,b} = \pi_a u_{k,k} \pi_b$, thus

$$\sum \varepsilon(a, b) \varphi^k_{a,b} = |D_k|^{-1} \Big(\sum \varepsilon(a, b) \underline{\underline{W}}^k_{a,b}\Big) \otimes \xi^k,$$

therefore

$$\Big\|\sum \varepsilon(a, b) \varphi^k_{a,b}\Big\|_{\mathbf{B}^*} = |D_k|^{-1} \Big\|\sum \varepsilon(a, b)\, W^k_{a,b}\Big\|_1.$$

Since all the entries of the matrix $\sum \varepsilon(a, b)\, W^k_{a,b}$ have absolute value $(q_1 \dots q_k)^{-1}$, each column of it has norm 1. Consequently, by (4.9), $\|\sum \varepsilon(a, b)\, W^k_{a,b}\|_1 \leqslant |D_k|$ and this yields (5.1).

Proof of (5.2). We shall prove that for every m,

$$\|x^m_{a,b}\|_{\mathbf{B}} = (q_1 \dots q_n)^{-1}, \tag{5.3}$$

which obviously implies (5.2), by (2.0). We shall use the matrices O^m_l as defined by (4.1).

If $m \leqslant n$, then $x^m_{a,b} = C\varepsilon_{a,b} \otimes y^n$ where

$$C = \prod_{k=1}^{m-1} w_k(a_k, a_{k+1}; b_k, b_{k+1}) \prod_{k=m}^{n} y_k(a_k, b_k)$$

and (5.3) follows, because $|C| = (q_1 \dots q_n)^{-1}$, $\|\varepsilon_{a,b}\|_\infty = \|y^n\|_\infty = 1$.

If $m > n$, then $x^m_{a,b} = D \cdot \varepsilon_{a,b} \otimes \chi_{a,b} \otimes y^m$ where $\chi_{a,b} \in M(K_{n+1} \times \dots \times K_m)$ and the constant D are defined by

$$\chi_{a,b}(i_{n+1}, \dots, i_m; j_{n+1}, \dots, j_m) = w_n(a_n, i_{n+1}; b_n, j_{n+1})\, O^m_{n+1}(i_{n+1}, \dots, i_m; j_{n+1}, \dots, j_m),$$

$$D = \prod_{k=1}^{n-1} w_k(a_k, a_{k+1}; b_k, b_{k+1}).$$

The argument of the proof of Sublemma 3 in Section 4 shows that $\chi_{a,b}$ is strictly equivalent to $q_n^{-1} \cdot O^m_{n+1}$, thus $\|\chi_{a,b}\|_\infty = q_n^{-1}$. Since $|D| = (q_1 \dots q_{n-1})^{-1}$, we obtain (5.3).

Proof of Lemma 1. The proof will be based on the following combinatorial

SUBLEMMA 4. *There exist regular partitions* ∇^q_j, $j = 1, \dots, q^4$ *of* $\{1, \dots, q\}$ *such that*

$$|A \cap B| \leqslant q^{7/16} \quad \textit{for every } A \in \nabla^q_i,\ B \in \nabla^q_j \textit{ with } i \neq j. \tag{5.4}$$

Proof. Let K be the (abelian) field of order 2^p, i.e., $K = GF(2^p)$. We identify $\{1, \dots, q\}$, as a set, with the vector space K^{16}. It is a standard fact that, given a $2P$-dimensional vector space V over a field of order r, there are at least r^{P^2} different P-dimensional subspaces of V. (To see this, let us choose a basis for V, say $e_1, e_2, \dots, e_{2P}$ and to a tuple $j = (j_{\alpha,\beta}: 1 \leqslant \alpha, \beta \leqslant P)$ with $j_{\alpha,\beta} \in K$ let us assign the P-dimensional subspace of V,

$$E_j \overset{\text{def}}{=} \operatorname{span}\left\{\sum_{\beta=1}^{P} j_{\alpha,\beta} e_\beta + e_{P+\alpha}: \alpha = 1, \dots, P\right\}.$$

It should be clear that $E_i = E_j$ only if $i = j$ and there are obviously r^{P^2} different j's like above.)

In our case this means that there are at least $2^{64p} = q^4$ different 8-dimensional subspaces of K^{16}, say $E_1, E_2, \dots, E_{q^4}$. Let ∇^q_j be the partition of K^{16} into 8-dimensional hyperplanes parallel to E_j. Then ∇^q_j are, obviously, regular partitions of $K^{16} = \{1, \dots, q\}$. If $A \in \nabla^q_i$ and $B \in \nabla^q_j$, then either $A \cap B = \emptyset$ or $A \cap B = E_i \cap E_j + x$ for some x. In either case we have

$$|A \cap B| \leqslant |E_i \cap E_j|.$$

If $i \neq j$, then $E_i \neq E_j$, and thus $\dim_K (E_i \cap E_j) \leqslant 7$ and therefore $|E_i \cap E_j| \leqslant 2^{7p} = q^{7/16}$. This implies (5.4).

Now we are going to construct a Hadamard matrix $w \in M(q)$ which has the following properties:

$$\begin{gathered}\text{for every } S,\ U \in \$_q, \text{ the matrix } p_S w p_U \text{ has rank 1, i.e., there exists} \\ \text{a vector } \alpha_{S,U} \text{ such that every non-zero column of } p_S w p_U \text{ is of the} \\ \text{form } z \cdot \alpha_{S,U} \text{ where } |z| = 1. \text{ Moreover,} \\ \alpha_{S,U} \perp \alpha_{S,T} \quad \text{if} \quad U \neq T.\end{gathered} \tag{5.5}$$

$$\sum_{b=1}^{q} w(a,b) = 1 = \sum_{b=1}^{q} w(b,a) \quad \text{for every } a. \tag{5.6}$$

Let us first notice that, without loss of generality we can take as $\$_q$ any regular partition of $\{1, ..., q\}$. It will be convenient to regard $\{1, ..., q\}$ as $\{1, ..., m\} \times \{1, ..., m\}$ (here $m = \sqrt{q} = 2^{8p}$) and to let $\$_q$ to be the partition of $\{1, ..., m\} \times \{1, ..., m\}$ into the sets

$$S_j \overset{\text{def}}{=} \{j\} \times \{1, \dots, m\} \quad \text{for } j = 1, \dots, m.$$

To construct w, let us start by defining matrices $U_r \in M(4^r)$:

$$U_1 = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}, \quad U_r = \underbrace{U_1 \otimes U_1 \otimes \dots \otimes U_1}_{r \text{ times}}.$$

The matrix U_{4p} is an $m \times m$-matrix. We set now

$$w(i_1, i_2; j_1, j_2) = U_{4p}(i_1, j_2)\, U_{4p}(i_2, j_1).$$

We see easily that w fulfills (5.5) and (5.6).

We shall also need the following, entirely trivial, remark:

$$\begin{gathered}\text{If } \mathcal{A} \text{ and } \mathcal{Z} \text{ are arbitrary regular partitions of } \{1, ..., q\}, \text{ then there exists} \\ \text{a permutation } \varrho \text{ of } \{1, ..., q\} \text{ which carries } \mathcal{A} \text{ onto } \mathcal{Z}, \\ \text{i.e., such that for every } A \in \mathcal{A},\ \varrho(A) \in \mathcal{Z}.\end{gathered} \tag{5.7}$$

Now we define v_j^q. Let ∇_j^q, $j = 1, ..., q^4$, be the partitions of $\{1, ..., q\}$ from the sublemma and, for $j = 1, ..., q^4$, let ϱ_j be a permutation of $\{1, ..., q\}$ which carries ∇_j^q onto $\$_q$. We define v_j^q by

$$v_j^q(e, f) = w(e, \varrho_j f),$$

i.e., v_j^q is obtained from w by applying ϱ_j^{-1} to its columns. It is evident that (2.3) holds. Let us check (2.1) and (2.2). We shall use the following standard facts: Let $x, y \in M(Z)$, where Z is any finite set. We have:

$$\text{if} \quad \operatorname{rk} x = 1, \quad \text{then} \quad \|x\|_1 = \|x\|_\infty = \Big(\sum_{a,b \in Z} |x(a,b)|^2\Big)^{1/2}, \tag{5.8}$$

if $D(x) \perp D(y)$ and $R(x) \perp R(y)$ (where D, R denote the domain and the range of an operator, respectively), then (5.9)

$$\|x+y\|_\infty = \max(\|x\|_\infty, \|y\|_\infty).$$

Let $S \in \$_q$, let $j = 1, ..., q^4$ be fixed. We see that, for every $B \in \nabla_j^q$, $p_S v_j^q p_B$ is obtained from $p_S w p_{\varrho_j B}$ by a permutation of columns. On the other hand, $\varrho_j(B) \in S_j$. Therefore, by (5.5),

$$\operatorname{rk} p_S v_j^q p_B = 1 \quad \text{for every } B \in \nabla_j^q \tag{5.10}$$

and, moreover

$$R(p_S v_j^q p_B) \perp R(p_S v_j^q p_C) \quad \text{if } B, C \in \nabla_j^q,\ B \neq C. \tag{5.11}$$

Now, (2.1) follows from (5.10) and (5.8).

Let $i = 1, ..., q^4$ be fixed, let $A \in \nabla_i^q$. For every $B \in \nabla_j^q$ let us denote

$$u_B = p_S v_j^q p_{A \cap B}.$$

We have, obviously,

$$p_S v_j^q p_A = \sum_{B \in \nabla_j^q} u_B.$$

By (5.11), $Ru_B \perp Ru_C$ if $B \neq C$. Since, obviously, we also have $Du_B \perp Du_C$ if $B \neq C$, by (5.9) we obtain

$$\|p_S v_j^q p_A\|_\infty = \max_{B \in \nabla_j^q} \|u_B\|_\infty.$$

Clearly, u_B has $q^{1/2} \cdot |A \cap B|$ non zero entries, all of them of absolute value 1. Therefore by (5.10),

$$\|u_B\|_\infty = q^{1/4} |A \cap B|^{1/2}.$$

If now $i \neq j$, then, by (5.4), $|A \cap B| \leqslant q^{7/16}$ and this yields (2.2).

References

[1] Choi, M. & Effros, E., Nuclear C^*-algebras and the approximation property. *Amer. J. Math.*, 100 (1978), 61–79.

[2] Enflo, P., A counterexample to the approximation property in Banach spaces. *Acta Math.*, 130 (1973), 309–317.

[3] GROTHENDIECK, A., Produits tensoriels topologiques et espaces nucleaires. *Memoirs Amer. Math. Soc.*, 16 (1955).

[4] HAAGERUP, U., An example of a non-nuclear C^*-algebra, which has the metric approximation property. *Invent. Math.*, 50 (1979), 279–293.

[5] LANCE, C., On nuclear C^*-algebras. *J. Functional Analysis*, 12 (1973), 157–176.

[6] LINDENSTRAUSS, J. & TZAFRIRI, L., *Classical Banach Spaces*, Vol. 1. Springer-Verlag 1977.

[7] SZANKOWSKI, A., The space of all bounded operators on Hilbert space does not have the approximation property, exposé XIV–XV. *Seminaire d'analyse fonctionnelle*, 1978–79, Ecole Polytechnique.

[8] TAKESAKI, M., On the cross-norm of the direct product of C^*-algebras. *Tohoku Math. J.*, 16 (1964), 111–122.

[9] WASSERMAN, S., On tensor products of certain group C^*-algebras. *J. Functional Analysis*, 23 (1976), 239–254.

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