ON COMPACT KÄHLER MANIFOLDS OF NONNEGATIVE BISECTIONAL CURVATURE, I

BY

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This is the first of two papers devoted to the study of compact Kähler manifolds of nonnegative bisectional curvature (see also [Wu2]). The main result of this paper is the following theorem; together with its corollaries below, this theorem shows that such manifolds possess a rigid internal structure. For their statements, recall from [Wu1] that the Ricci curvature is *quasi-positive* iff it is everywhere nonnegative and is positive in all directions at a point; an equivalent definition is that the Ricci tensor Ric is everywhere positive semi-definite and is positive definite at a point.

THEOREM. *Let M be an n-dimensional compact Kähler manifold with nonnegative bisectional curvature and let the maximum rank of* Ric *on M be $n-k$ $(0 \leqslant k \leqslant n)$. Then:*

(A) *The universal covering of M is holomorphically isometric to a direct product $M' \times \mathbf{C}^k$, where M' is an $(n-k)$-dimensional compact Kähler manifold with quasi-positive Ricci curvature and $\mathbf{C}^k$ is equipped with the flat metric.*

(B) *M' is algebraic, possesses no nonzero holomorphic q-forms for $q \geqslant 1$, and is holomorphically isometric to a direct product of compact Kähler manifolds $M_1 \times \ldots \times M_s$, where each M_i has quasi-positive Ricci curvature and satisfies $H^2(M_i, \mathbf{Z}) \cong \mathbf{Z}$.*

(C) *There is a flat, compact complex manifold B and a holomorphic, locally isometrically trivial fibration $p: M \to B$ whose fibre is M'.*

(D) *There exists a compact Kähler manifold M^*, a flat complex torus T, and a commutative diagram:*

$$\begin{array}{ccc} M^* & \longrightarrow & T \\ \downarrow & & \downarrow \\ M & \longrightarrow & B \end{array}$$

(1) Research partially supported by the National Science Foundation.

where the horizontal maps are holomorphic, locally isometrically trivial fibrations with fibre M', and the vertical maps are finite coverings. Furthermore, M^* is globally diffeomorphic to $M' \times T$.

In particular, $\pi_1(M)$ is either trivial or an infinite crystallographic group.

COROLLARY 1. *Let M be a compact Kähler manifold of nonnegative bisectional curvature. Then the following are equivalent:*

(A) *M is simply connected.*

(B) *The first Betti number is zero.*

(C) *M has quasi-positive Ricci curvature.*

COROLLARY 2. *A simply connected compact Kähler manifold of nonnegative bisectional curvature is irreducible (in the sense of the de Rham decomposition theorem) iff its second Betti number is one.*

A theorem slightly weaker than the one above was first announced without proof by the first two authors at the end of [HS] in 1971. Unaware of this result in [HS], but motivated by his attempt to extend the argument of [SY] from positive bisectional curvature to nonnegative bisectional curvature, the third author independently arrived at a theorem also slightly weaker than the one above. The present paper is roughly patterned after the arguments of the third author which rely on the structure theorem of Cheeger–Gromoll ([CG1], [CG2]) on compact Riemannian manifolds of nonnegative Ricci curvature.

Section 1

We summarize the preliminary material in this section. First recall the structure theorem of Cheeger–Gromoll ([CG1], [CG2]) specialized to the Kählerian case. Let M be a compact Kähler manifold with nonnegative Ricci curvature. Then its universal covering manifold is holomorphically isometric to a direct product $M' \times \mathbf{C}^k$, where M' is a compact Kähler manifold and both the flat metric on $\mathbf{C}^k$ and the product metric on $M' \times \mathbf{C}^k$ are understood (here the Kählerian deRham decomposition theorem is needed as well; see [KN], p. 171). Moreover, there is a finite covering M^* of M such that M^* is diffeomorphic to $M_\# \times T^k$, where T^k is a complex k-dimensional torus and $M_\#$ is a compact Kähler manifold covered by M'. This implies that $\pi_1(M_\#)$ is finite. As a consequence, if the Ricci curvature of M is quasi-positive, so is that of $M' \times \mathbf{C}^k$ and hence $k=0$ and $\pi_1(M)$ is itself finite (cf. the comments in [Wu1] on this fact).

Next we review the basic Bochner technique needed for the purpose at hand (cf. [GK] or [L], pp. 3–6). Let ξ be a real (1,1)-form on a compact Kähler manifold and let

R_{AB}, R_{ABCD} be respectively the components of the Ricci and Riemannian curvature tensors (sign convention: R_{ABAB} is a positive multiple of the sectional curvature). Define

$$F(\xi) = 2R_{AB}\xi^{AC}\xi^{B}_{C} - R_{ABCD}\xi^{AB}\xi^{CD}.$$

If ξ is harmonic and $F(\xi) \geqslant 0$, then $F(\xi) = 0$ and ξ is parallel. *This is the basic observation.* Let $\{X_1, ..., X_n, JX_1, ..., JX_n\}$ be a local frame field that diagonalizes ξ, i.e. $\xi_{ii^*} \equiv \xi(X_i, JX_i)$ are the only nonzero components, then

$$F(\xi) = 2\sum_{i,j} R_{ii^*jj^*}(\xi_{ii^*} - \xi_{jj^*})^2 \quad \text{where} \quad R_{ii^*jj^*} \equiv \langle R_{X_iJX_i}X_j, JX_j\rangle. \qquad (*)$$

Since $R_{ii^*jj^*}$ is the bisectional curvature defined by $\text{span}_{\mathbf{R}}\{X_i, JX_i\}$ and $\text{span}_{\mathbf{R}}\{X_j, JX_j\}$, we have from $(*)$:

LEMMA 1. *If a compact Kähler manifold M has nonnegative bisectional curvature, then all harmonic forms of type* (1,1) *are parallel.*

From now on assume M is compact Kähler with nonnegative bisectional curvature. Let ω be the Kähler form of M. If ξ is a harmonic (1,1)-form distinct from ω, define a tensor field S, one-fold contravariant and one-fold covariant, by the equation $\xi(X, Y) = (\omega(X), Y)$ for all vector fields X and Y. To be more precise, let ξ' and G be respectively the 2-fold covariant Hermitian tensor fields associated with ξ and ω (i.e., G is the Kähler metric); we then define S by

$$\xi'(X, Y) = G(S(X), Y), \quad \forall X, Y.$$

It is clear that S is self-adjoint relative to the Kähler metric G; consequently S defines a diagonalizable linear transformation at each tangent space M_x of M. On the other hand, since ξ' and G are both parallel tensor fields (Lemma 1), a straightforward reasoning shows that S is also parallel. Thus the linear transformation $S_x: M_x \to M_x$ has the same set of eigenvalues $\{\alpha_1, ..., \alpha_k\}$ ($\alpha_i \in \mathbf{R}$, $\{\alpha_i\}$ distinct) for all $x \in M$ and moreover, if $V_1(x), ..., V_k(x)$ are the corresponding eigenspaces at M_x, then the $\{V_i(x)\}$ are mutually orthogonal complex subspaces of M_x and the distribution $x \mapsto V_i(x)$ is a parallel distribution on M for each i. Since we assume that $\xi \neq \omega$, $k \geqslant 2$. Thus invoking the deRham decomposition theorem for Kähler manifolds, we have proved:

LEMMA 2. *Let M be a simply connected compact Kähler manifold of nonnegative bisectional curvature. If $h^{1,1}(M) > 1$, then M splits holomorphically and isometrically into a direct* *product of compact Kähler manifolds* $M_1 \times M_2$, *where* $\dim M_i \geq 1$ *for* $i=1, 2$ *(we have used the standard notation:* $h^{p,q}(M) \equiv$ *the dimension of the space of harmonic forms of type* (p, q)*).*

Section 2

We now prove the theorem. Thus suppose M is an n-dimensional compact Kähler manifold of nonnegative bisectional curvature such that the maximum rank of Ric on M is $(n-k)$. Since M has nonnegative Ricci curvature, the theorem of Cheeger–Gromoll states that the universal covering of M is holomorphically isometric to $M' \times \mathbf{C}^l$, where $0 \leq l \leq n$ and M' is a simply connected compact Kähler manifold of nonnegative bisectional curvature. We first prove that $l=k$. Let $h^{1,1}(M')=s$. By repeated applications of Lemma 2, M' is holomorphically isometric to a direct product of compact Kähler manifolds $M_1 \times \ldots \times M_s$, where $h^{1,1}(M_i)=1$ for each i. We claim that each M_i must have quasi-positive Ricci curvature. To prove this claim, we need the following lemmas.

LEMMA 3. *Let M be a compact Kähler manifold. If ξ is a positive semi-definite form of type* (1,1) *on M such that its harmonic component $H\xi$ is parallel, then the rank of $H\xi$ equals the maximum rank of ξ.*

Proof. Let ω be the Kähler form of M and let $\dim M = n$. Then for any k,

$$\int_M \xi^k \wedge \omega^{n-k} = \int_M (H\xi)^k \wedge \omega^{n-k}. \qquad (**)$$

Let $r = \max \operatorname{rank} \xi$ and let $t = \operatorname{rank} H\xi$. The positive semi-definiteness of ξ implies that the left side of $(**)$ is positive when $k=r$. Thus $(H\xi)^r \neq 0$, thereby proving $t \geq r$. On the other hand, since $H\xi$ and ω are both parallel 2-forms, $(H\xi)^k \wedge \omega^{n-k}$ is also parallel and hence equals a constant multiple of the volume form of M. Since, by the definition of t, $(H\xi)^t \wedge \omega^{n-t}$ is nonzero at each point of M, it follows that the right side of $(**)$ is nonzero when $k=t$. Thus ξ^t is not identically zero and $t \leq r$. Q.E.D.

LEMMA 4. *Let M be a simply connected compact Kähler manifold of nonnegative bisectional curvature. If φ is its Ricci form, then φ has a nonzero harmonic component $H\varphi$.*

Proof. First observe that φ is not identically zero. Otherwise the bisectional curvature, being nonnegative, would be identically zero and hence the curvature tensor is itself identically zero. By the assumption of simple connectivity, M would then be isometric to complex euclidean space. This contradicts compactness. If $n = \dim M$, then $\varphi \wedge \omega^{n-1}$ (ω = Kähler form of M) is everywhere nonnegative and is positive somewhere. Hence

$$0 < \int_M \varphi \wedge \omega^{n-1} = \int_M (H\varphi) \wedge \varphi^{n-1},$$

thereby proving that $H\varphi$ is not zero. Q.E.D.

We now return to the proof of the theorem. Let φ_i, ω_i be respectively the Ricci form and Kähler form of M_i. Since $h^{1,1}(M_i) = 1$, the harmonic component of φ_i is equal to $c\omega_i$ for some $c \in \mathbf{R}$. Since M_i has nonnegative bisectional curvature, Lemma 4 implies $c \neq 0$. By Lemma 1 and Lemma 3, the maximum rank of φ_i equals the rank of $c\omega_i$, which is equal to $\dim M_i$. Thus M_i has quasi-positive Ricci curvature for each $i = 1, ..., s$. It follows that M' itself has quasi-positive Ricci curvature. Thus $n - l$ ($= \dim M'$) is equal to the maximum rank of the Ricci tensor of M', which equals that of $M' \times \mathbf{C}^l$ (because $\mathbf{C}^l$ has the flat metric), which in turn equals that of M (because $M' \times \mathbf{C}^l \to M$ is a local isometry); hence $n - l = n - k$, i.e., $l = k$. This proves part (A) of the theorem.

To prove part (B), the fact that M' has no holomorphic q-forms for all $q \geqslant 1$ follows from the quasi-positivity of the Ricci curvature and a simple generalization of the Kodaira vanishing theorem (Theorem 6 of [R]; see also Theorem B of [Wu2]). Now we also know that M' is holomorphically isometric to $M_1 \times ... \times M_s$, where $h^{1,1}(M_i) = 1$ for each i. By Kodaira's embedding theorem, each M_i is therefore algebraic and hence so is M itself. Finally to prove $H^2(M_i, \mathbf{Z}) \cong \mathbf{Z}$, observe that $h^{2,0}(M_i) = 0$ so that $H^2(M_i, \mathbf{R}) \cong \mathbf{R}$. Since M_i is simply connected, the universal coefficient theorem for cohomology now gives $H^2(M_i, \mathbf{Z}) \cong \mathbf{Z}$.

To prove (C), let π be the fundamental group of M; π consists of isometries acting freely on $M' \times \mathbf{C}^k$. Introduce the notation: $I(N)$ denotes the group of isometries of any Riemannian manifold N. Consider the natural projection $\varphi: I(M') \times I(\mathbf{C}^k) \to I(\mathbf{C}^k)$. Since $I(M')$ is a compact Lie group, the kernel of the restriction of φ to π is a finite group to be denoted by $\ker \varphi$. The quotient space $M'/\ker \varphi$ is a compact Kähler manifold with quasi-positive Ricci curvature and hence, by part (A) of Theorem B of [Wu2], must be itself simply connected. Thus $\ker \varphi$ is trivial, which is equivalent to saying that $\varphi: \pi \to I(\mathbf{C}^k)$ is an isomorphism onto a crystallograph subgroup $\Gamma \subset I(\mathbf{C}^k)$ (cf. [Wo], Chapter 3). Now let $B = \mathbf{C}^k/\Gamma$. Since π acts as holomorphic isometries on $M' \times \mathbf{C}^k$ respecting the product metric, it is straightforward to verify that $p: M = M' \times \mathbf{C}^k/\pi \to B$ which is defined by projecting on the second factor is a locally isometrically trivial holomorphic fibration with fibre M'. This concludes the proof of part (C).

Finally to prove (D), let π^0 be a free abelian subgroup of rank $2k$ with finite index in the fundamental group π. Define $\Gamma^0 \equiv \varphi(\pi^0)$, $T \equiv \mathbf{C}^k/\Gamma^0$, and $M^* \equiv M' \times \mathbf{C}^k/\pi^0$. Then the commutative diagram in (D) immediately follows. The only nontrivial assertion in (D) is that concerning M^* being globally diffeomorphic to $M' \times T$; this involves a careful choice of π^0 in π and has already been done in [CG2], p. 440. Q.E.D.

Both corollaries are straight forward consequences of the theorem except for the implication (B) $\Rightarrow$ (A) in Corollary 1. To prove this, one invokes the Cheeger–Gromoll structure theorem to show that if the first Betti number of M is zero, then $\pi_1(M)$ must be finite (see Theorem A of [Wu2]). By the above theorem, if $\pi_1(M)$ is finite, it is trivial.

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Received November 7, 1979