MULTIPLE-POINT FORMULAS I: ITERATION

BY

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An *r-fold point* of a map $f: X \to Y$ is a point x of X such that there exist $r-1$ other points of X and each has the same image under f as x. All r points must be "distinct", but some may lie "infinitely close" to others; that is, the infinitely close points determine tangent directions along the fiber $f^{-1}f(x)$. An *r-fold-point formula* is a polynomial expression in the invariants of f that gives, under appropriate hypotheses, the number of r-fold points or the class m_r of a natural positive cycle enumerating the r-fold points. One method for obtaining an r-fold-point formula is the method of iteration, the subject of this article. The setting will be algebraic geometry, but the method and the formulas have a universal character.

Multiple-point theory had its beginnings around 1850 and has attracted attention on and off ever since. About 1973 the field became highly active and has remained so. A survey is found in [10] Chapter V; it includes an introduction to the method of iteration, which at the time was beginning to blossom. Another survey, [11], concentrates on the results of this article and its sequel, [12].

The sequel will present another method for obtaining multiple-point formulas. Based on the Hilbert-scheme, it yields a deeper understanding of the theory and more refined formulas. The method also lends itself better to the study of an important special case, central projections.

The first general double-point formula was obtained in rational equivalence by Todd (1940); he derived it along with a residual-intersection formula, one from the other by induction on the dimension, but his reasoning is specious. Independently, Whitney (1941) gave a double-point formula for an immersion of differentiable manifolds. In [19] Ronga, inspired by Whitney, obtained the double-point formula in ordinary cohomology for a generic map with ramification in both the differential-geometric and complex-analytic

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cases. Central to Ronga's work is a modification R of the set of double points. If f is generic and if $b: B \to X \times X$ denotes the blowing-up along the diagonal Δ, then R is equal to the closure of the inverse image $b^{-1}(X \times_Y X - \Delta)$, so R is the blowup of $X \times_Y X$ along Δ.

Laksov, [13], made a major breakthrough for the algebro-geometric theory. He realized that a residual-intersection formula could be derived on its own and applied to a scheme like Ronga's R. Laksov's R is defined by the relation of ideals,

$$I(R).I(E) = I(b^{-1}X \times_Y X),$$

where E is the exceptional divisor. Pushing the formula for R down to X, Laksov got the double-point formula.

Because of technical difficulties, Laksov had to assume that no component of R lies entirely in E. The difficulties were overcome by Fulton, [4], and the treatment was cleaned up further by Fulton and Laksov, [5], using new-begotten advances in intersection theory, as that theory entered a period of total reconstruction with Fulton and MacPherson as principal architects.

A powerful new intersection theory with a great generality and a fresh point of view has now been constructed. It is a theory of rational-equivalence groups and operators on them. The intersection product on a smooth variety now occupies a secondary position; indeed, it is defined via pullback along the diagonal map, which is a regular embedding. In this article, the product is completely irrelevant. The new "intersection" theory is outlined in [6] § 9. However, there it is set over a field; only a brief footnote is addressed to the generality appropriate to multiple-point theory. Moreover, the theory is presented as one of many bivariant theories; while this point of view certainly has its merits, it leads to notation that seems contrived in the simple context of rational equivalence. To highlight the differentiae and to fix notation, a resumé of operational rational-equivalence theory in the appropriate generality is presented in § 1.6.

One of the principal topics of this article is residual-intersection theory. Section 2 is devoted to general properties of the residual scheme; Section 3, to the setup and derivation of a new residual-intersection formula of wide applicability and a corollary.

The residual scheme R of a closed subscheme W of a scheme Z is defined as the scheme $\mathbf{P}(I)$ where I denotes the ideal of W in Z. The blowup of Z along W is canonically embedded in R, and the two schemes are equal when their algebras, the Rees algebra and the symmetric algebra on I, are equal. The two algebras are equal when W is regularly embedded in Z by the theorem announced in [16], 1. (Other cases in which equality holds are discussed in [8].) Micali's theorem also brings greater simplicity to the theory of blowups along regularly embedded centers (see the proof of Lemma 1.5.1), but it is woefully little known among algebraic geometers. A short direct proof is presented in Proposition 1.4; it is included because of the importance of the result and the interest of this proof.

Suppose that Z is embedded as a closed subscheme of a scheme A in such a way that W is regularly embedded in A. Let $b: B \to A$ denote the blowing-up along W, and E the exceptional divisor. Then (see Proposition 2.3.1) there is a canonical closed embedding of R in B, and the ideals of R, E and $b^{-1}Z$ are related by the formula,

$$I(R).I(E) = I(b^{-1}Z).$$

In particular, therefore, Laksov's modification R of the set of double-points of a map $f: X \to Y$ is equal to the residual scheme of the diagonal in the fibered product $X \times_Y X$. The residual-intersection theorem (3.6) deals with a diagram,

$$\begin{tikzcd} & R \arrow[d, "p"] & \\ X \arrow[d, "f"'] & Z \arrow[l] \arrow[d, "p_2"] \arrow[l, phantom, shift right=6ex, "\square"] & W \arrow[l, "q"'] \\ Y & H \arrow[l, "h"'] & \end{tikzcd}$$

in which the square is cartesian, q is a closed embedding, and R is the residual scheme of W in Z and in which f and p_2p and p_2q are local complete intersections (abbreviation: lci) with f and p_2p of the same codimension. The codimensions need not be constant, in which case "same" means the one function induces the other. The theorem and a corollary (3.7) (of its proof) assert the formulas,

$$f^*|H = p_*(p_2p)^* + q_*c_n(f/p_2q)(p_2q)^*,$$

$$p_*c_1(O_R(1))^k(p_2p)^* = -q_*c_{n+k}(f/p_2q)(p_2q)^* \quad \text{for } k \geqslant 1,$$

where n denotes the excess in the codimension of f over that of p_2q (if the codimensions are not constant, then n is the appropriate function on W) and where $c_j(f/p_2q)$ denotes the jth Chern operator of the difference of the virtual normal bundle of f over that of p_2q. (Thus $c_n(f/p_2q)$ might be termed the *Euler operator of the pair* of maps.)

Additional conditions must be met before the terms in these formulas are defined and the proofs valid. The pullback operator and the virtual normal bundle are not defined for every lci; the map must also factor through an embedding of the source in a smooth scheme P. Usually in practice, P has the form $\mathbf{P}(E)$ with E a locally free sheaf of finite rank, and if so, the map is called *strongly quasi-projective* (a term used in [1]). If the map is only quasi-projective and the target only noetherian, then the source can be embedded in a scheme $\mathbf{P}(E)$ but E need only be coherent. However, if the target is also divisorial (a mild restriction, see § 1.6), then every coherent sheaf is a quotient of a locally free one, and so the map is strongly quasi-projective.

The residual intersection formula is established by more or less following Laksov's original proof. By factoring f, one reduces to the case that f is an embedding and, by blowing-up along W, to the case that W is a divisor and p an embedding. This case is discussed in [6] 9.2.3, and will be assumed here. (It is derived by blowing-up along R and then checking the formula against the modern definitions.) While Laksov used Grothendieck's "key" formula, that formula is no longer needed and, indeed, is now a corollary.

The two formulas are valid, therefore, when in addition to the substantive hypotheses made at the start, the following technical conditions obtain: f is strongly quasi-projective and H is noetherian and divisorial. The condition on H not only guarantees that p_2p and p_2q are strongly quasi-projective but also that the blowing-ups used in the course of the proof are too. Until intersection theory and multiple-point theory become more settled, it is best to work in this reasonably great generality while noting where each hypothesis is used.

The generality of the setup of these formulas affords greater simplicity and clarity to multiple-point theory. Indeed, the author found the setup (March 1979) while seeking these desirables and, in turn, the new generality prompted (in May 1979) the advances in [12].

The setup used in the method of iteration (see § 5.1) has $f\colon X\to Y$ for $h\colon H\to Y$ and the diagonal for W. For convenience, we consider the involution i of R covering the "switch" involution of $P=X\times_Y X$ (it is treated in Proposition 4.2(i), and we set

$$f_1=p_2p,\quad c_k=c_k(\nu_f),\quad t=c_1(O_R(1)).$$

Applying f_{1*} on the right to the residual-intersection formula and its corollary and identifying the terms (see Lemma 5.5 for details), we get

$$f_{1*}i_*f_i^*=f^*f_*-c_n$$
$$f_{1*}i_*t^kf_1^*=-c_{n+k}\quad\text{for } k\geqslant 1,$$

where n denotes the codimension of f. The formulas were obtained by the author in March 1977 by developing ideas in [10] Chapter V, Section D; independently, but about this same way, Roberts in [18] (6.3.1), (5.6.2), obtained a version of the first and nearly one of the second.

Applying the first of these formulas to the fundamental class m_1 of X (see Theorem 5.6 for details), we get the double-point formula,

$$m_2 = f^* f_* m_1 - c_n m_1,$$

where m_2 denotes the pushout under f_1 of the fundamental class m_1' of R.

A triple-point formula for f can be obtained formally from the double-point formula for f_1, then a quadruple-point formula for f from the triple-point formula for f_1, etc.; this procedure is the method of *iteration*. The method occurred to Salomonsen, who told the author about it (30 July 1976). The method is based on the observation that a point z of R is an r-fold point of f_1 if and only if $f_1(z)$ is an $(r+1)$-fold point of f. This observation is evident when f is an immersion (that is, unramified), for then R is equal to the complement in $X \times_Y X$ of the diagonal Δ since Δ is open and closed. When ramification is present, the "observation" becomes more of a definition, but the method still works remarkably well.

Thus, the cycle m_{r+1} enumerating the $(r+1)$-fold points of f is just the pushout under f_{1*} of the cycle m_r' enumerating the r-fold points of f_1. So, by pushing out a formula for m_r', we will get a formula for m_{r+1}. On working it out, we find we need to know that i_* preserves m_r'. For $r=1$, this fact was needed above but was evident because m_1' is the fundamental class. For arbitrary r, the fact is established in § 5.1 on the basis of Proposition 4.2(ii).

To proceed, we also need to know the virtual normal bundle of f_1; so a formula for it is obtained in Proposition 4.7. (The formula is a special case of one in Proposition 3.5, which obtains in the context of the residual-intersection theorem.) Appropriate cases of the formula were obtained before. One was obtained by the author in March 1977 and, independently, by Roberts [18] Section 5. One is implicit in [19], proof of 2.7, according to [17], comment before 3.4. In each derivation, the procedure is basically the same, involving an explicit determination of the virtual normal bundle of a blowing-up along a regularly embedded center. The general determination is included in § 1.5, because the available literature is inadequate.

Multiple-point formulas of all orders can now be obtained in a purely mechanical fashion. Certain ones are obtained explicitly in Theorems 5.8–5.11. While the expression for m_3 in Theorem 5.9 is general in n, unfortunately no such general expression is available for m_r when $r \geqslant 4$.

The formulas in Theorems 5.9–5.11 each begin with the formula in Theorem 5.8, which treats the case of an immersion; the remaining terms correct for the presence of ramification and vanish when none is present. A version of Theorem 5.8 for a generic immersion of differentiable manifolds was obtained by Herbert, [7] Theorem p. 7, as a corollary of a more refined formula. Herbert thus achieved the goals of Lashoff and Smale, who in 1959 erred in an attempt to generalize Whitney's work on the double-point formula. Independent of Herbert, Ronga (using iteration) nearly obtained the unrefined formula; an error appears in his preprint (which finally reached the author 12/79). Ronga chose not to publish (but the basic ideas can be found in [17]). Recently, however, Ronga, [20], gave a short proof of Herbert's refined formula; it is based on a generalization of a result of Quillen (1971), which is a version (with n not constant) of the excess-intersection formula, the case of the residual intersection formula with q an isomorphism and R empty.

The r-fold-point formula obtained by the method of iteration is valid when $f\colon X\to Y$ is projective and *r-generic* and Y is noetherian, divisorial and universally catenary. The condition "universally catenary" is a technical requirement of intersection theory, see § 1.6. The term "r-generic" means that each of the following r maps is an lci of the same codimension: f itself, the derived map f_1, the map f_2 derived from f_1 the way f_1 was from f, ... the map f_{r-1} derived from f_{r-2} the way f_1 was from f. The derived maps are the subject of Section 4.

For example, $f\colon X\to Y$ is r-generic if X and Y are smooth over a base S and f is an *r-fold self-transverse* immersion, see Proposition 4.6. Intuitively, "r-fold self-transverse" means that any r branches are transverse; a precise definition is found in § 4.5. This condition, in an appropriate form, was used by Lashoff-Smale, Herbert, and Ronga. Note, however, that f may be an r-generic immersion without being r-fold self-transverse; consider, for instance, the difference between a tacnode and a simple node.

Suppose that f is an lci and that X or Y (and so X) is Cohen-Macaulay. Then a necessary and sufficient condition for f to be r-generic is that f be *dimensionally r-generic*, that is, that each of the derived maps $f_1, \ldots, f_{r-1}$ simply have the same codimension as f, see Proposition 4.4. (The notion of the codimension of a map is intuitive and convenient; it is developed in § 1.2 and Proposition 1.3, apparently for the first time.) Moreover, obviously, R is Cohen-Macaulay if f is 2-generic.

If X and Y are smooth over some base S, then of course f is an lci, but it is possible that X and Y are smooth and f_1 has the same codimension as f while R is not smooth. Thus one is led (as the author was in March 1977) to abandon smooth schemes (and for that matter, base schemes too) in favor of lci maps. This generalization to singular varieties differs from Johnson's, [9], already when X is a curve in 3-space and f is a general projection to a plane, see [12]. However (see [18] (0.1), (0.2) and [10] Chapter V, Section D and [12]) if f is a general central projection of a smooth variety X over an algebraically closed field of characteristic 0, then R is smooth of the appropriate dimension; a similar result holds in characteristic p but first X may have to be reembedded. Moreover ([14] Proposition 17) in this setup, m_2 has no multiple components if $\operatorname{cod}(f)\geqslant 1$.

For the validity of the r-fold-point formula, the condition that f be r-generic is really too strong. So it is weakened a little in § 5.1 to the condition that f be *practically r-generic*, that is, that f be an lci and its restriction over the complement of an appropriately small closed subset S of Y be r-generic. Nevertheless, see Proposition 5.2, if f is practically r-generic, if f the codimension n of f is constant in case $r\geqslant 4$, and if f has an $\bar{S}_2$-singularity, then r and n are limited to the following range: $r=2$ or 3 and n arbitrary, $r=4$ and $n=1$, 2, 3, and $r=5$ or 6 and $n=1$. The multiple-point formulas in this range are just the ones given in Theorem 5.6, 5.9–5.11. Outside this range of course, the multiple-point formulas obtained by iteration are valid for a practically r-generic map with no $\bar{S}_2$-singularity, just as the Herbert-Ronga formula (5.8) is valid for a practically r-generic immersion, a map with no $\bar{S}_1$-singularity.

A striking feature of the formula for m_r obtained by iteration is that the coefficient of m_s contains the factor,

$$(-1)^{r-1}(r-1)(r-2)\ldots s,$$

except for the first term $f^*f_*m_{r-1}$. During the course of the derivation, it is not at all evident that the coefficient will contain the factor (except for the sign), and so its would-be presence serves as a useful check on the computation.

The presence of the factor is a defect of the method of iteration, however. The defect arises because, for each r-fold point, the $r-1$ other points with the same image are enumerated with a specific order in each of the $(r-1)!$ possible ways. This defect is especially serious in differential topology because, as Herbert, [7] Remark p. 9, points out, unless X and Y are oriented and n is even, $\mathbf{Z}_2$-coefficients are necessary and so m_r vanishes for $r\geqslant 3$. Herbert overcame the difficulty with a more refined theory. In algebraic geometry, the author in May 1979 found that a version of Herbert's refined formula for an immersion and a corresponding refined triple-point formula could be obtained by using the Hilbert scheme for H in the residual-intersection theorem; a resumé of this work will be found in [11], a detailed account in [12]. This work raises hope not only of proving the existence of refined versions of all the formulas obtainable by iteration but also of finding explicit, general closed forms. However, at present the only way of obtaining even in principle an r-fold-point formula for $r\geqslant 4$ for a map with ramification is the method of iteration.

1. Assorted preliminaries

1.1. plci's (*pseudo-local complete intersections*). A map of schemes, $h: S\to T$, will be called a *plci of codimension at most n*, a locally constant function on S, if each point s of S has a neighborhood U such that there exists a closed embedding over T of U into a T-scheme

P which is smooth of constant dimension, say d, and U is defined in P by the vanishing of m global functions, with $m = n(s) + d$.

Thus h is an lci (local complete intersection) of codimension n if, furthermore, for each s, the m functions form a regular sequence. (The converse holds by [21] VIII, 1.2, p. 466.)

1.2. *The codimension of S/T.* Let $h: S \to T$ be a map locally of finite type between locally noetherian schemes. By the *codimension of S over T* (resp. of h) *at a point s* of S will be meant the number defined below. It will be denoted by $\operatorname{cod}_s(S, T)$ or $\operatorname{cod}(S, T)$ (resp. by $\operatorname{cod}_s(h)$ or $\operatorname{cod}(h)$).

Let U be a neighborhood of s and $\alpha: U \to P$ an embedding over T into a flat T-scheme P of finite type. Notice that, since $h: S \to T$ is locally of finite type, s has a neighborhood U that is isomorphic to a closed subscheme of an affine space over a neighborhood of $h(s)$. Now define the codimension at s by the formula,

$$\operatorname{cod}_s(S, T) = \operatorname{cod}_s(U, P) - \operatorname{rel.dim}_s(P, T).$$

The value of the right hand side is the same for a second choice V, Q of U, P, as will now be shown.

We may, obviously, replace V, Q by $U \cap V$, $P \times_T Q$ and so assume that V is contained in U and that there is a flat map $\pi: Q \to P$ compatible with the embeddings of V in Q and U in P. Now, let t be a point of S whose closure contains s, and view t as a point of P, as one of Q, and as one of the fiber $\pi^{-1}t$, as well as one of S, and V. Then we have

$$\dim(O_{Q,t}) = \dim(O_{P,t}) + \dim(O_{\pi^{-1}t,t})$$

because π is flat, by [3] IV_2, 6.1.1, p. 135. Note that t is a closed point of $\pi^{-1}t$ because of the compatibility of the embeddings. Hence we have

$$\dim(O_{\pi^{-1}t,t}) = \operatorname{rel.dim}_t(Q, P).$$

Since the closure of t contains s, we have

$$\operatorname{rel.dim}_t(Q, P) = \operatorname{rel.dim}_s(Q, P).$$

Putting together the above equations and using the additivity of relative dimension we get

$$\dim(O_{Q,t}) - \operatorname{rel.dim}_s(Q, T) = \dim(O_{P,t}) - \operatorname{rel.dim}_s(P, T).$$

Taking the infimum over t, we get

$$\operatorname{cod}_s(V, Q) - \operatorname{rel.dim}_s(Q, T) = \operatorname{cod}_s(U, P) - \operatorname{rel.dim}_s(P, T).$$

Thus $\operatorname{cod}_s(S, T)$ is well-defined.

It is evident that, if $h: S \to T$ is an embedding, then this notion of codimension agrees with the usual one and that, if $h: S \to T$ is flat, then its codimension is equal to the negative of its relative dimension. It is also evident that cod (h) is a locally constant function on S.

PROPOSITION 1.3. *Let $h: S \to T$ be a map locally of finite type between locally noetherian schemes.*

(i) *If $h: S \to T$ is a* plci *of codimension at most n, then (at every point of S) we have*

$$\operatorname{cod}(S, T) \leqslant n,$$

and equality holds if h is an lci *of codimension n. Conversely, if equality holds (at every point), if h is a* plci *of codimension at most n, and if T is Cohen-Macaulay, then h is an* lci *of codimension n and S is Cohen-Macaulay.*

(ii) (additivity) *Let $g: T \to U$ be another map locally of finite type, with U locally noetherian, and let s be a point of S. Then we have*

$$\operatorname{cod}_s(S, T) + \operatorname{cod}_{h(s)}(T, U) \leqslant \operatorname{cod}_s(S, U),$$

with equality if either U is universally catenary or if $g: T \to U$ is an lci *or is flat.*

(iii) *Let s be a closed point of S such that $h(s)$ is a closed point of T. Then we have*

$$\operatorname{cod}_s(S, T) \leqslant \dim_{h(s)}(T) - \dim_s(S),$$

with equality if T is universally catenary.

Proof. (i) The assertion is obvious, because a scheme that is smooth over a Cohen-Macaulay scheme is Cohen-Macaulay also.

(ii) Replacing T by a suitable neighborhood of $h(s)$ and S by a suitable neighborhood of S, we may assume that T is a subscheme of a flat scheme Q over U, say of relative dimension e, and that S is a subscheme of an affine space P over T, say of relative dimension d. Let P' denote the affine space over Q of dimension d. Then we have a commutative diagram

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[dr, "h"'] & P \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & P' \arrow[d] \\
 & T \arrow[r, hook] \arrow[dr, "g"'] & Q \arrow[d] \\
 & & U
\end{tikzcd}$$

in which the square is cartesian.

Since P'/Q is flat, it follows that

$$\mathrm{cod}_s\,(P, P') = \mathrm{cod}_{h(s)}\,(T, Q).$$

On the other hand, we obviously have

$$\mathrm{cod}_s\,(S, P) + \mathrm{cod}_s\,(P, P') \leqslant \mathrm{cod}_s\,(S, P'),$$

with equality if P' is catenary. The first two assertions follow directly.

Suppose now that g is an lci of codimension n. Then T is regularly embedded in Q with codimension $n+e$. Hence P is regularly embedded in P' with codimension $n+e$, because P'/Q is flat and the square is cartesian. Therefore we have

$$\mathrm{cod}_s\,(S, P') = \mathrm{cod}_s\,(S, P) + (n+e).$$

The asserted equality obviously holds now.

If g is flat, then we may take $Q = T$ and the preceding reasoning simplifies greatly; in fact, the asserted equality is clear directly from the definition.

(iii) Since s and $h(s)$ are closed, we have

$$\dim_s\,(S) = \dim\,(O_{S,s})$$
$$\dim_{h(s)}\,(S) = \dim\,(O_{T,h(s)}).$$

Replacing T by a neighborhood of $h(t)$ and S by one of s, we may assume that S is a *closed* subscheme of an affine space over T, say P of dimension d. Then we have

$$\mathrm{cod}_s\,(S, P) \leqslant \dim\,(O_{P,s}) - \dim\,(O_{S,s}),$$

with equality if P is catenary. Moreover, letting F denote the fiber of P over $h(s)$, we have

$$\dim\,(O_{P,s}) = \dim\,(O_{T,h(s)}) + \dim\,(O_{F,s})$$

because P/T is flat. Now, s is closed in S, and S is closed in P; hence, s is closed in F. Therefore $O_{F,s}$ has dimension d. Putting it all together, we get the assertion.

PROPOSITION 1.4 ([16], 1). *Let A be a commutative ring with 1, and let J be an ideal of A. Assume that J is generated by a sequence of elements whose Koszul complex is exact. Then the natural map from the symmetric algebra to the Rees algebra,*

$$S(J) \to \bigoplus_{n\geqslant 0} J^n,$$

is an isomorphism.

Proof. We may assume that we have

$$A = \mathbf{Z}[T_1, \ldots, T_m] \quad \text{and} \quad J = (T_1, \ldots, T_m),$$

where the T's are indeterminates, because the formation of the map in question commutes with base change. (The formation of J^n does in view of the hypothesis on J by [21] VII, 1.2.3, p. 416.)

The map will be an isomorphism if, given any n-linear symmetric map u on J, there exists a linear map v on J^n satisfying the relation,

$$u(P_1, \ldots, P_n) = v(P_1 \ldots P_n), \quad P_i \in J.$$

It suffices by linearity to define v on the monomials M of degree n or more. Given M factor it in any way as $M = P \cdot P_1 \ldots P_n$ with deg $(P_i) \geq 1$, and define $v(M)$ as $Pu(P_1, \ldots, P_n)$. Note that $v(M)$ is independent of the choice of factorization because u is symmetric and linear. It is obvious that v is linear and satisfies the required relation.

1.5. *Blowups and differentials.* Consider a blowup diagram of schemes,

$$\begin{array}{ccc} W' & \xhookrightarrow{e} & B = \mathrm{Bl}(W, A) \\ \downarrow w & \square & \downarrow b \\ W & \xrightarrow{a} & A \end{array}$$

and assume that a is a regular embedding.

LEMMA 1.5.1. (i)[1] *The natural map* (*whose adjoint is the more usual map*) *is an isomorphism,*

$$\Omega^1_{B/A} \xrightarrow{\sim} e_* \Omega^1_{W'/W}.$$

(ii) *The* T_1-*functor of deformation theory vanishes on* O_B,

$$T_1(B/A, O_B) = 0.$$

Proof. We may check the assertions locally on A and so assume that A is affine and that the ideal J of W in A is generated by a sequence of elements $t_0, \ldots, t_m$ whose Koszul complex is exact. Let $T_0, \ldots, T_m$ be indeterminates, V the free A-module they generate, and v: $V \to A$ the map carrying T_i to t_i. Since the Koszul complex is exact, ker (v) is generated by the various differences, $t_i T_j - t_j T_i$.

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(1) I. Vainsencher points out that (i) is true and easy to prove without assuming a is a regular embedding.

Proposition 1.4 implies that B is equal to $\mathbf{P}(J)$. Hence $v\colon V \to A$ defines an embedding of B in $\mathbf{P}(V)$, whose homogeneous ideal is generated by ker (V), and so by the elements $f_i T_j - f_j T_i$. (Note in passing the improvement that Proposition 1.4 brings to the usual argument, [21] VII, 1.8(ii), p. 425.) Let I denote the ideal of B. It is now straightforward to check that the natural map,

$$I/I^2 \to \Omega^1_{\mathbf{P}(V)/A} \mid B,$$

is injective and that the image is equal to the product of the target with J. The first assertion follows because the map's cokernel is equal to $\Omega^1_{B/A}$ and because W' is the fiber of $\mathbf{P}(V)$ over W. The second assertion follows because the T_1 in question is equal to the map's kernel ([15] 3.1.2, p. 53).

PROPOSITION 1.5.2. *Assume that B is embeddable over A in a smooth scheme P. For any scheme C, let $K(C)$ denote the Grothendieck group of locally finitely presentable sheaves with finite projective dimensions.*

(i) *In $K(B)$, the virtual tangent bundle of B/A is equal to the dual of the class of the direct image of the sheaf of differentials of W'/W,*

$$\tau_{B/A} = (e_* \Omega^1_{W'/W})^*.$$

(ii) *If the conormal bundle $N(W, A)$ is equal in $K(W)$ to the restriction of some virtual bundle N on A, then in $K(B)$ we have*

$$\tau_{B/A} = O_B(-1) - O_B + (b^*N)^*(1) - (b^*N)^*.$$

Proof. (i) Let I denote the ideal of B in P. There is a 4-term exact sequence,

$$T_1(B/A,\, O_B) \to I/I^2 \to \Omega^1_{P/A} \mid B \to \Omega^1_{B/A} \to 0,$$

and the T_1 vanishes by Lemma 1.5.1(ii). Hence the assertion results from 1.5.1(i).

(ii) Since W' is equal to $\mathbf{P}(N(W, A))$, we have the standard exact sequence,

$$0 \to \Omega^1_{W'/W} \to (w^*N(W, A))(-1) \to O_{W'} \to 0.$$

Since $w^*N(W, A)$ is equal in $K(W')$ to the restriction of b^*N, the preceding exact sequence yields the following equation in $K(B)$:

$$e_* \Omega^1_{W'/A} = (b^*N)(-1) \otimes e_* O_{W'} - e_* O_{W'}.$$

Hence (i) and the standard exact sequence ([3] II, 8.1.8.1, p. 156),

$$0 \to O_B(1) \to O_B \to e_* O_{W'} \to 0,$$

yield the asserted formula.

1.6. *Operational rational equivalence.* Let X and Y be noetherian schemes and $f\colon X \to Y$ a map of finite type.

A *cycle* on X is a formal (finite) linear combination with integer coefficients of the closed integral subschemes W. The *components* of a cycle are those W appearing in it with a nonzero coefficient (or *mutliplicity* or *weight*). Associated to any closed subscheme Z of X is its *fundamental cycle* $[Z]$, which is the sum of the components of Z weighted by the lengths of the stalks at the generic points. Similarly, associated to any (locally principal) divisor D on any closed subscheme Z of X is a cycle $[D]$. A cycle is said to be *rationally equivalent* to 0 if it is a linear combination of cycles associated to rational functions on closed integral subschemes. The group of cycles modulo those rationally equivalent to 0 is denoted $A.X$. The group of cycles is naturally *graded by codimension* and, if X is catenary, then this grading passes on to $A.X$.

Let $f\colon X \to Y$ be flat. A *pullback operator* (preserving degrees) is defined by sending a closed integral subscheme of Y to the fundamental cycle of its scheme-theoretic inverse image. The operator preserves rational equivalence, inducing an operator $f^*\colon A.Y \to A.X$. If $X = V(E)$ for some locally free sheaf E on Y, then f^* is an isomorphism.

Let $f\colon X \to Y$ be proper. A pushout operator f_* is defined by sending a closed integral subscheme to its image weighted by the degree of the extension of function fields if finite and by 0 if not. If Y is universally catenary, then the operator preserves rational equivalence, inducing an operator $f_*\colon A.X \to A.Y$.

Let E be a locally free coherent sheaf on a universally catenary X. Chern operators $c_i(E)$ on $A.X$ exist possessing all the standard properties. If E has rank 1, then $c_1(E)$ takes a closed integral subscheme W to the class of any divisor on W associated to the restriction $E \,|\, W$. In general, $c_i(E)$ is defined via $c_1(O(1))$ on $A.\mathbf{P}(E)$ á la Segre theory. Moreover, i need be only an integer-valued locally constant function.

Let $f\colon X \to Y$ be a closed embedding and C its affine normal cone. Then a map $A.Y \to A.C$ is defined by sending a closed integral subscheme V of Y to the fundamental class of the cone of $f^{-1}V$ in V. If f is a regular embedding, then C is equal to $V(N_f)$ where N_f is the conormal sheaf (the restriction of the ideal), whence $A.C$ is isomorphic to $A.Y$ and we get a pullback operator $f^*\colon A.Y \to A.X$.

Consider a cartesian square with Y' noetherian and universally catenary,

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X' \arrow[l, "g'"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "g"] \end{tikzcd} \qquad (1.6\,a)$$

If f is an embedding, f' is too, so there is a map $A.Y' \to A.C'$, where C' is the cone of f'. If f is a regular embedding, then there is a closed embedding of C' in $C \times_Y Y'$, whence an operator $f^*: A.Y' \to A.X'$ (which often differs from f'^* when f'^* is defined). If f is flat, then f' is too, so again f defines an operator $f^*: A.Y' \to A.X'$. Similarly, if f is proper, then f defines an operator $f_*: A.X' \to A.Y'$. Similarly, a locally free sheaf E on X defines (via g'^*E and ig') operators $c_i(E)$ on $A.X'$. Thus we have extended the meaning of the symbols f^*, f_*, $c_i(E)$; they now denote operators with a whole family of components, one component for each map $g: Y' \to Y$ in which Y' is a universally catenary noetherian scheme. There is no longer any need, in general, for Y and X to be noetherian or universally catenary.

The set (group) of *contravariant* linear operators with components, $A.Y' \to A.X'$, is denoted by $A^{\bullet}f$, that of *covariant* linear operators with components, $A.X' \to A.Y'$, by $A.f$. There are evident restriction maps, $A^{\bullet}f \to A^{\bullet}f'$ and $A.f \to A.f'$; the notation $f^*|Y$, etc., will be used to denote the restriction of f^*, etc. Whenever the terms are defined, the following basic compatibility (or commutativity) relations hold:

$$(f^*|Y)g^* = (g^*|X)f^*, \quad (g_*|X)(f^*|Y') = f^*g_*,$$
$$f_*(g_*|X) = g_*(f_*|Y'), \quad c_i(E)c_j(E) = c_j(E)c_i(E),$$
$$(g^*|X)c_i(E) = (c_i(E)|X')(g^*|X), \quad g'_*(c_i(E)|X') = c_i(E)g'_*.$$

The last relation, when applied to an element of $A.X'$, looks like a projection formula.

Suppose that $f: X \to Y$ can be factored as a composition $hg: X \to P \to Y$ in which g is a regular embedding and h is smooth. Then the composition of operators g^*h^* is the same for every choice of g and h. (Is h flat sufficient? It is if Y is smooth over S and if either X is flat over S or S is a field or a Dedekind domain.) Set $f^* = g^*h^*$.

The case that f factors as above is important for another reason, too. Then f possesses a virtual normal bundle, which is given by

$$\nu_f = \nu_g - g^*\tau_h,$$

where ν_g is the normal sheaf (dual to the restriction of the ideal) and τ_h is the tangent sheaf (dual to Ω_h^1). In the Grothendieck group, ν_f is independent of the choice of g and h ([21] VIII, S.2, p. 476). Hence, the Chern operators $c_i(\nu_f)$ are well-defined; for short, they will be denoted by $c_i(f)$.

If $f: X \to Y$ is (quasi-) projective, then X is Y-isomorphic to a (locally) closed subscheme of a scheme $P = \mathbf{P}(E)$ for some coherent sheaf E on Y. If E can be taken locally free, then f will be called *strongly (quasi-) projective*. If f is a strongly quasi-projective lci, then P is smooth and so ([21] VIII, 1.2, p. 466) X is regularly embedded in P; thus, f can be factored as above.

If Y is divisorial (that is, its topology is generated by the complements of its divisors, see [21] II, 2,2, p. 167), then every coherent sheaf is a quotient of a locally free one. Thus, if Y is divisorial and f is (quasi-)projective, it is strongly (quasi-)projective. The family of divisorial schemes is remarkably large; it contains the affine schemes, the schemes with an ample sheaf, the separated regular schemes, the subschemes of a divisorial scheme, the product over any base of two divisorial schemes, and the source of any quasi-projective map whose target is divisorial.

In diagram (1.6a), suppose that f and f' can be factored into a regular embedding followed by a smooth map, for example, that f and f' are strongly projective lci's. Then, whether or not the diagram is cartesian, set

$$c(f'/f) = c(f)/c(f') \ \ (= c(\nu_f - \nu_{f'})).$$

Thus, for example, the excess-intersection formula ([6] 7.2.1), which holds when the diagram is cartesian, becomes

$$f^* \,|\, Y' = c_n(f/f')f'^*, \quad \text{with } n = \operatorname{cod}(f) - \operatorname{cod}(f'),$$

(where, if the codimensions are not constant, cod (f) is to be interpreted as the function induced on X'). This lovely formula yields as special cases rather strong versions of the self-intersection formula and Grothendieck's "key" formula ([21] XIV, pp. 676–677). On the other hand, the excess-intersection formula is a special case of the residual-intersection formula (3.6), the case with q an isomorphism, although additional technical conditions are required at present in the proof of the residual-intersection formula because of the use of blowups.

2. The residual scheme

2.1. *Basics.* Let W be a closed subscheme of a scheme Z. Denote the ideal by $I(W, Z)$. Then the scheme $R = R(W, Z)$ defined by the formula,

$$R(W, Z) = \mathbf{P}(I(W, Z)),$$

will be called the *residual scheme* of W in Z. The pullback of $I(W, Z)$ has a "tautological" invertible quotient, which will be denoted by $O_R(1)$ or $O(1)$.

Let $p\colon R \to Z$ denote the structure map. It is evident that p is an isomorphism off W; in fact, p is an isomorphism over every point of Z at which $I(W, Z)$ is invertible. It is also evident that the fiber over W is

$$p^{-1}W = \mathbf{P}(N(W, Z))$$

where $N(W, Z)$ denotes the conormal sheaf of W in Z.

It is evident that the image $p(R)$ is equal to the support of $I(W, Z)$,

$$p(R) = \operatorname{Supp}(I(W, Z)).$$

So $p(R)$ is equal to the set of points of Z at which W is not equal to Z. Suppose that $I(W, Z)$ is of finite type. Then, therefore, $p(R)$ is the subset of Z defined by the annihilator of $I(W, Z)$,

$$p(R) = V(\operatorname{Ann}(I(W, Z))); \tag{2.1a}$$

in particular, $p(R)$ is closed. In fact, $p: R \to Z$ is projective, so proper. Moreover, it is not hard to see that the two formulas above for $p(R)$ hold scheme-theoretically (that is, $\operatorname{Supp}(I(W, Z))$ is the smallest closed subscheme of Z through which p factors).

The residual scheme in other contexts has been called the "naive blowup" because it is associated to the symmetric algebra of $I(W, Z)$, whereas the blowup $B = B(W, Z)$ is (as Hironaka has taught us) the scheme associated to the Rees algebra, the direct sum of the (ideal-theoretic) powers of $I(W, Z)$.

There is, however, an evident canonical closed embedding,

$$B(W, Z) \subset R(W, Z),$$

and the tautological sheaf of $R(W, Z)$ restricts to that of $B(W, Z)$.

PROPOSITION 2.2. *If W is a* regularly embedded *closed subscheme of Z, then the blowup is equal to the residual scheme,*

$$B(W, Z) = R(W, Z).$$

Moreover, the structure map of $R(W, Z)$ over Z is an lci *(local complete intersection) of codimension* 0.

Proof. The first assertion is a direct consequence of Proposition 1.4. The second is well-known for the blowup $B(W, Z)$ and is reproved implicitly in the proof of Lemma 1.5.1.

2.3. *Nested ambiant schemes.* Let W be a closed subscheme of Z, and let Z be a closed subscheme of A,

$$W \subset Z \subset A.$$

Denote the residual scheme of W in Z by R, that of W in A by B. Let $b: B \to A$ denote the structure map, and $p: R \to A$ the natural map.

PROPOSITION 2.3.1(i) *There is a canonical closed embedding,*

$$R = R(W, Z) \subset B = R(W, A).$$

(ii) *The tautological sheaf of B restricts to that of R,*

$$O_B(1) \mid R = O_R(1).$$

(iii) *If W is regularly embedded in A, then the ideals of R, $b^{-1}W$ and $b^{-1}Z$ satisfy the relation,*

$$I(R, B).I(b^{-1}W, B) = I(b^{-1}Z, B).$$

Proof. There is a canonical exact sequence of ideals,

$$I(Z, A) \to I(W, A) \to I(W, Z) \to 0.$$

Hence, (i) and (ii) hold. In fact, R is, clearly, equal to the subscheme of B, whose homogeneous ideal is

$$I(Z, A).\operatorname{Sym}(I(W, A))[-1].$$

On the other hand, by Proposition 1.4 the homogeneous ideal of $b^{-1}W$ is

$$I(W, A).\operatorname{Sym}(I(W, A)) = \operatorname{Sym}(I(W, A))[1].$$

Hence, the product of the homogeneous ideal of R and that of $b^{-1}W$ is that of $b^{-1}Z$. Therefore, the corresponding weaker inhomogeneous relation holds also.

PROPOSITION 2.3.2. *Assume that W is regularly embedded in A.*

(i) *If R is regularly embedded in B with codimension m (a locally constant function on R), then $p: R \to A$ is an* lci *(local complete intersection) of codimension m.*

(ii) *Assume also that Z is defined in A, locally along $p(R)$, by the vanishing of m functions (m a locally constant function on $p(R)$). Then R is defined in B locally by the vanishing of m functions, and R is regularly embedded in B with codimension m if, in addition, $p: R \to A$ is an* lci *of codimension m.*

Proof. (i) The assertion holds because $b: B \to A$ is an lci of codimension 0 by Proposition 2.2 and because the composition of lci's of codimensions i, j is one of codimension $i+j$ by [21] VIII, 1.5, p. 471 and VIII, 1.10, p. 474.

(ii) Since the matter is local on R, we may assume that W, Z and A are affine and that Z is defined globally in A by the vanishing of m functions, say $f_1, \ldots, f_m$ (m constant). Fix a point of R. Replacing W, Z, A by open subschemes if necessary, we may assume that the point lies in an open subscheme B' of B such that B' is regularly embedded in an affine n-space C over A for some n and such that the exceptional locus $(b^{-1}W)\cap B'$ is defined globally in B' by the vanishing of a single function, t say. (Note that $b\colon B\to A$ is equal to the blowing-up along W by Proposition 2.2.) Then Proposition 2.3.1(iii) implies that $R\cap B'$ is defined in B' by the vanishing of the m fractions $f_1/t, \dots, f_m/t$, which are (global) functions on B'. Thus the first assertion holds.

The m fractions are the restrictions of m functions on C, say $F_1, \dots, F_m$. Let $G_1, \dots, G_n$ be functions on C defining B'. Then the F's and G's together are $n+m$ functions on A defining $R\cap B'$. Suppose that $p\colon R\to A$ is an lci of codimension n. Then $R\cap B'$ is regularly embedded in C with codimension $n+m$ because C is smooth over A with relative dimension m. Hence the F's and G's form a regular sequence on A. Therefore, the (f/t)'s form a regular sequence on B'. Thus $R\cap B'$ is regularly embedded in B' with codimension n.

PROPOSITION 2.3.3. *Assume that A is Cohen-Macaulay (noetherian), that W is regularly embedded in A, and that Z is defined in A, locally along $p(R)$, by the vanishing of m functions (m a locally constant function on $p(R)$). Then, at each point of $p(R)$, we have*

$$\operatorname{cod}(p(R), A) \leqslant m.$$

Proof. The assertion will be derived from Corollary 2.2, p. 312, of [2] (cf. their p. 319). The corollary is applied, by way of contradiction, at a point where the asserted inequality is assumed to fail. The point is taken, furthermore, to be the generic point of a component of $p(R)$.

Consider the local ring of A at the point in question; let M denote its maximal ideal, I the ideal of Z, and J the ideal of W. The set $p(R)$ is defined by the ideal $\operatorname{Ann}(J/I)$ by (2.1 a). Hence, this ideal is primary for M. Therefore, for any nonmaximal prime P, the localizations J_p and I_p are equal. In particular, they are equal for any prime P with $\operatorname{ht}(P)\leqslant m$, because $\operatorname{ht}(M)>m$ since the asserted inequality fails. Thus hypothesis (ii) of the corollary of Artin-Nagata (which is hypothesis (ii) of the theorem before it) is satisfied (with $Q=J$ and $s=m$).

The maximal ideal M is associated to I, because it is to $\operatorname{Ann}(J/I)$. Hence the depth modulo I is zero. However, conclusion (a) of the corollary of Artin-Nagata asserts that this depth is at least the difference, $\operatorname{ht}(M)-m$. Thus conclusion (a) stands in contradiction to the inequality, $\operatorname{ht}(M)>m$.

It remains to check one final hypothesis of the corollary of Artin-Nagata, namely, the inequality, $\operatorname{ht}(J)<m$. Now, all the primes P of I, except M, are also primes of J, because I_P and J_P are equal, as was noted above. So, $\operatorname{ht}(J)\geqslant m$ implies $\operatorname{ht}(I)\geqslant m$. However, I is

generated by m elements and the ambient ring is Cohen-Macaulay. Hence ht $(I) \geq m$ implies that I is unmixed and ht $(I) = m$. However, M is associated to I and ht $(M) > m$. Thus, ht $(J) \geq m$ is untenable, and ht $(J) < m$ holds as required.

PROPOSITION 2.4. *Let S be a ground scheme, W a closed subscheme of an S-scheme Z. Then the formation of the residual scheme $R(W,Z)$ and its tautological sheaf $O(1)$ commutes with base-change to T/S,*

$$R(W \times T, Z \times T) = R(W, Z) \times T,$$

if either W/S or T/S is flat.

Proof. If either is flat, then obviously the formation of $I(W, Z)$ commutes with the base-change. So the assertion is evident from the definition, see § 2.1.

3. The residual-intersection formula

3.1. *The setup.* Fix a diagram of schemes

$$\begin{tikzcd}
 & R = R(W,Z) \arrow[d, "p"] & \\
X \arrow[d, "f"'] & Z \arrow[l, "p_1"'] \arrow[d, "p_2"] \arrow[dl, phantom, "\square"] & W \arrow[l, "q"'] \\
Y & H \arrow[l, "h"] &
\end{tikzcd}$$

in which the square is cartesian and the map q is a closed embedding.

LEMMA 3.2. *In the setup of § 3.1, if f is quasi-projective and H is noetherian, then p is projective, $p_2 p$ and $p_2 q$ are quasi-projective, and Z, R and W are noetherian. If H is also divisorial, then p is strongly projective, $p_2 p$ and $p_2 q$ are strongly quasi-projective, and Z, R and W are noetherian and divisorial.*

Proof. Clearly Z is noetherian. So the ideal $I(W, Z)$ is coherent. Hence p is projective by definition. The rest is standard, involving [3] II, 5.3.2, 5.3.4, p. 99 and the fact that, on a noetherian and divisorial scheme, every coherent sheaf is a quotient of a locally free sheaf of finite rank.

LEMMA 3.3. *In the setup of § 3.1 assume that $f \colon X \to Y$ is a closed embedding and that $p_2 q \colon W \to H$ (which now is a closed embedding) is a regular embedding. Let $b \colon B \to H$ denote the blowing-up of H along W.*

(i) *There is a canonical embedding $g\colon R\to B$, and the following relations of compatibility are satisfied:*

$$g^*O_B(1)=O_R(1);\quad bg=p_2p.$$

Moreover, B is equal to the residual scheme of W in H.

(ii) *Suppose that X is defined in Y locally by the vanishing of m functions (m a locally constant function on X). Then R is defined in B locally by the vanishing of m functions (m now also denoting the function induced on R). Moreover, R is regularly embedded in B with codimension m if and only if p_2p is an* lci *of codimension m.*

(iii) *If the embeddings $f\colon X\to Y$ and $g\colon R\to B$ are both regular of the same (or compatible) codimensions, then their normal sheaves are related by the formula,*

$$\nu_g=[(p_1p)^*\nu_f](1).$$

Proof. (i) The assertion is a direct consequence of Propositions 2.2, 2.3.1(i) and (ii).

(ii) If X is defined in Y at a point x by the vanishing of m functions, then their pullbacks obviously define Z in H along $p_1^{-1}x$. Hence the assertion follows from Proposition 2.3.2(i) and (ii).

(iii) There is an obvious surjection relating the ideals,

$$(hb)^*I(X,\,Y)\to I(b^{-1}Z,\,B).$$

It and the expression for $I(b^{-1}Z,\,B)$ in Proposition 1.4.1(iii) yield a surjection relating the conormal bundles,

$$(hb)^*N_f\to N_g(1).$$

This surjection is an isomorphism because, under the hypotheses, the source and target are locally free sheaves of the same rank. Taking the dual isomorphism and twisting by $O(1)$ yields the asserted formula.

PROPOSITION 3.4. *In the setup of § 3.1, assume that $f\colon X\to Y$ is a* plci *of codimension at most n(see § 1.1) and let n also denote the induced functions on Z and R. Assume that $p_2q\colon W\to H$ is an* lci.

(i) *The maps $p_2\colon Z\to H$ and $p_2q\colon R\to H$ are* plci's *of codimension at most n.*

(ii) *If H is Cohen-Macaulay, then, at each point of $p(R)$, we have* cod $(p(R),\,H)\leqslant n$.

(iii) *If H is Cohen-Macaulay and if $p_2p\colon R\to H$ has codimension n, then p_2p is an* lci *of codimension n and R is Cohen-Macaulay.*

Proof. The matter is local on X. So we may replace all the schemes by suitable open subschemes and thus assume that X is a closed subscheme of a smooth Y-scheme P with a constant dimension d and that X is defined in P by the vanishing of m functions, $m = n + d$.

Set $A = P \times_Y H$. Then Z may be viewed as the closed subscheme of A defined by the vanishing of the pullbacks of the m functions defining X in P. Now, A/H is smooth of relative dimension d. Hence $p_2: Z \to H$ is a plci of codimension at most n. Moreover, W is regularly embedded in A because $p_2 q: W \to H$ is an lci.

Consider the residual scheme B of W in Z. The structure map $b: B \to A$ is an lci of codimension 0 by Proposition 2.2. By Proposition 2.3.1(i) there is a canonical closed embedding of R in B and, by Proposition 2.3.2(ii), R is defined in B locally by the vanishing of m functions. It follows that $p_2 p: R \to H$ is a plci of codimension at most n. Thus (i) holds.

Suppose H is Cohen-Macaulay. Then A is Cohen-Macaulay because A/H is smooth. Hence, at each point of $p(R)$, we have

$$\operatorname{cod}(p(R), A) \leqslant m$$

by Proposition 2.3.3. Since A/H is flat of relative dimension d, (i) follows. Finally, (iii) follows from (i) and Proposition 1.3(i).

PROPOSITION 3.5. *In the setup of* § 3.1, *assume that f is strongly quasi-projective and that H is noetherian and divisorial. Assume that f and $p_2 p$ and $p_2 q$ are* lci's *with f and $p_2 p$ of the same codimension and that the virtual normal bundle of $p_2 q$ is equal to the pullback of a virtual bundle ν on H. Then the virtual normal bundles of f and $p_2 p$ are related by the formula,*

$$\nu_{p_2 p} = [(p_1 p)^* \nu_f](1) + [O_R - O_R(-1)] + [(p_2 p)^* \nu - ((p_2 p)^* \nu)(1)].$$

Proof. The virtual normal bundles of $p_2 p$ and $p_2 q$ are defined, because these maps are not only lci's but also strongly quasi-projective by Lemma 3.2.

If f is an embedding, then the assertion follows immediately from Lemma 3.3(i), (ii), (iii) and Proposition 1.5.2(ii).

In general, since f is strongly quasi-projective, X is, by definition, isomorphic to a closed subscheme of an open subscheme P of a scheme $\mathbf{P}(E)$ for some locally free sheaf E on Y of finite rank. Set $A = P \times_Y H$. Since P/Y is quasi-projective and since H is noetherian and divisorial, A is noetherian and divisorial. Since P/Y is smooth, the remaining hypotheses for the case in which Y and H are replaced by P and A clearly follow from their counterparts. Since X is embedded and closed in P, the corresponding formula holds; this is the case of an embedding, treated above. Since P/Y is smooth, this formula obviously implies the general one.

THEOREM 3.6 (The residual-intersection theorem). *In the setup of* § 3.1, *assume that* f *is strongly quasi-projective and that* H *is noetherian and divisorial. Assume that* f *and* p_2p *and* p_2q *are* lci's *with* f *and* p_2p *of the same codimension. Let* n *denote the difference between the codimension of* f *and that of* p_2p. *Then the following "residual-intersection" formula holds in* $A^{\bullet}p_2$:

$$f^*|H = p_*(p_2p)^* + q_*c_n(f/p_2q)(p_2q)^*.$$

Proof. First note that the questionable terms in the formula are, in fact, defined. By Lemma 3.2, p is projective, so p_* is defined. By Lemma 3.2 again, p_2p and p_2q are strongly quasi-projective. Since they are lci's, $(p_2p)^*$ and $(p_2q)^*$ are, therefore, defined. Moreover, p_2q has, therefore, a well-defined virtual normal bundle; hence, since f has one, the nth Chern operator of the difference $c_n(f/p_2q)$ is defined.

Since f is strongly quasi-projective, X is isomorphic to a closed subscheme of a smooth Y-scheme P. Set $A = P \times_Y H$. Since P/Y is quasi-projective and since H is noetherian and divisorial, A is noetherian and divisorial. Since P/Y is smooth, the remaining hypotheses of the theorem for the case in which Y and H are replaced by P and A obviously follow from their counterparts, and the formula in that case obviously implies the stated formula. Thus we may assume that f is a closed embedding.

Let $b\colon B \to H$ denote the blowing-up of H along W. Set $m = \mathrm{cod}(f)$. By Proposition 3.3(i) and (ii), there is a canonical closed and regular embedding (as H-schemes) of R in B with codimension m. Set $B' = b^{-1}Y'$ and $W' = b^{-1}W$ and label the various induced maps as indicated in the following diagram:

$$\begin{tikzcd}
 & & R \arrow[d, "p'"] & \\
X \arrow[d, "f"] \arrow[dr, phantom, "\square"] & Z \arrow[l, "p_1"'] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & B \arrow[l, "b'"'] \arrow[d, "e"] & W \arrow[l, "q'"'] \\
Y & H \arrow[l, "h"] & B \arrow[l, "b"] &
\end{tikzcd}$$

Since H is noetherian, b is projective; hence, B is noetherian and divisorial as H is so. Moreover, W' is a divisior in B (in fact, the exceptional divisor). Thus the hypotheses of the theorem hold when H and W are replaced by B and W'. The corresponding formula is

$$f^*|B = p'_*(ep')^* + q'_*c_{m-1}(f/eq')(eq')^* \quad \text{in } A^{\bullet}e. \tag{3.6a}$$

This case of the theorem is treated in [6], 9.2.3, and will be assumed here. (Note that R is the "residual scheme to W' in B" in the sense used by Fulton-MacPherson because of the relation of ideals in Proposition 2.3.1(iii).) The desired formula will now be derived from this one.

Note the relation, $b_*b^*=1$. It is treated in [6], 9.2.2, and holds because b is the blowing-up of a divisorial, noetherian scheme along a closed and regularly embedded, nowhere dense center. To ensure the center's being nowhere dense, it may be necessary, back at the beginning, to replace P by a larger scheme, for example, $P\times_Y P$.

The relation $b_*b^*=1$ and the general commutativity of pushout and pullback yield

$$f^*|H=(f^*|H)b_*b^*=b'_*(f^*|B)b^*.$$

On the other hand, we obviously have

$$b'_*p'_*(ep')^*b^*=p_*(p_2p)^*.$$

Let $w\colon W'\to W$ denote the restriction of b. Then we obviously have

$$b'_*q'_*c_{m-1}(f/eq')(eq')^*b^*=q_*w_*c_{m-1}(f/eq')w^*(p_2q)^*.$$

Finally, the usual sort of reasoning with the projection formula gives

$$w_*c_{m-1}(f/eq')w^*=c_n(f/p_2q);$$

indeed, we have

$$\begin{aligned}w_*c(f/eq')w^*c(p_2q)&=(c(f)\,|\,W)w_*c(eq')^{-1}(c(p_2q)\,|\,W')w^*\\&=(c(f)\,|\,W)w_*c(w^*\nu_{p_2q}/O_{W'}(-1))w^*=c(f)\,|\,W.\end{aligned}$$

Putting together all the above relations, we see that the assumed formula (3.6a) implies the desired one.

COROLLARY 3.7. *Under the conditions of Theorem* 3.6, *the following formula holds in* $A^{\bullet}p_2$:

$$p_*c_1(O_R(1))^k(p_2p)^*=-q_*c_{n+k}(f/p_2q)(p_2q)^*\quad\text{for } k\geqslant 1.$$

Proof. The notation of the proof of Theorem 3.6 will be used. Reasoning again as there, we see that we may assume that f is an embedding and that it suffices to establish the following relation:

$$p_*c_1(O_R(1))^k(ep')^*=-q_*c_{m-1-k}(f/eq')(eq')^*. \tag{3.7a}$$

This relation will also be derived from (3.6a), which is being asumed.

The sheaf $O_R(1)$ is, by Lemma 3.3(i), equal to the pullback of $O_B(1)$ or, what is the same, of $O_B(-W')$. Hence, using the "projection formula" and then (3.6a), we get

$$\begin{aligned} p'_* c_1(O_R(1))^k (ep')^* &= (-1)^k c_1(O_B(W) \mid B')^k p_*(ep')^* \\ &= (-1)^k c_1(O_B(W) \mid B')^k [(f^* \mid B) - q'_* c_{m-1}(f/eq_t)(eq')^*]. \end{aligned} \tag{3.7b}$$

Next we analyze the effect of the term $f^* \mid B$.

Consider the following diagram:

$$\begin{tikzcd} B' \arrow[d, "e"'] & W' \arrow[l, "q'"'] \arrow[d, "1"] \\ B & W' \arrow[l, "eq'"] \end{tikzcd}$$

It is obviously cartesian. With it in mind, we compute

$$\begin{aligned} c_1(O_B(W') \mid B')(f^* \mid B) &= (f^* \mid B) c_1(O_B(W')) = (f^* B)(eq')_*(eq')^* \\ &= q'_*(f^* \mid W')(eq')^* = q'_*(c_m(f) \mid W')(eq')^*. \end{aligned}$$

Here the first and third inequalities hold by general commutativity, and the second and fourth equalities are very easily checked from the definitions. Using this computation and general commutativity, we get

$$c_1(O_B(W') \mid B')^k (f^* \mid B) = q'_*(c_m(f) \mid W') c_1(eq')^{k-1} (eq')^* \tag{3.7c}$$

because $O_B(W') \mid W'$ is equal to $\nu_{eq'}$.

Finally, we obviously have

$$c_1(O_B(W') \mid B')^k q'_* c_{m-1}(f/eq')(eq')^* = q'_* \sum_{i=0}^{m-1} (-1)^i (c_{m-1-i}(f) \mid W') c_1(eq')^{i+k} (eq')^*.$$

Since f has codimension m, we obviously have

$$c_{m-1-k}(f/eq') = \sum_{i=-1}^{m-1} (-1)^{i+k} (c_{m-1-i}(f) \mid W') c_1(eq')^{i+k}.$$

Putting these two equations together with (3.7b) and (3.7c), we find (3.7a).

4. The derived maps

4.1. *Basics.* Let $f: X \to Y$ be a separated map of schemes. Define a sequence of satellite separated maps,

$$f_r: X_{r+1} \to X_r \quad \text{for } r \geq 0,$$

inductively as follows. Define $f_0\colon X_1 \to X_0$ to be $f\colon X \to Y$. Now, assume f_{r-1} defined. Consider, the fibered product of X_r with itself over X_{r-1} and consider its diagonal subscheme Δ, which is a closed subscheme because f_{r-1} is separated. Define X_{r+1} to be the corresponding residual scheme, see § 2.1, and define f_r to be the composition of the structure map p and the second projection p_2,

$$\begin{tikzcd} & X_{r+1} = R(\Delta, X_r \times X_r) \arrow[d, "p"] & \\ X_r \arrow[d, "f_{r-1}"'] & X_r \times X_r \arrow[l, "p_1"'] \arrow[d, "p_2"] \arrow[dl, phantom, "\square"] & \Delta \arrow[l, "q"'] \\ X_{r-1} & X_r \arrow[l, "f_{r-1}"] & f_r = p_2 p \end{tikzcd} \tag{4.1 a}$$

The map f_r will be called the *r-th derived map* of f; the scheme X_r, the *r-th derived scheme of X/Y*.

Note that for $r \geqslant 2$ the derived scheme X_r possesses a "tautological" invertible sheaf

$$O(1) = O_{X_r}(1),$$

because it is a residual scheme (and so a projective bundle), and the fiber over the diagonal Δ is

$$p^{-1}\Delta = \mathbf{P}(\Omega_f^1),$$

because the conormal sheaf of Δ is equal to the sheaf of differentials.

The formations of the derived map $f_r\colon X_{r+1} \to X_r$ and for $r \geqslant 2$ of the tautological sheaf $O(1)$ commute with arbitrary base-change of Y by Proposition 2.4.

It is evident that the sth derived map of f_r is equal to f_{r+s} and that for $s \geqslant 1$ the two corresponding tautological sheaves on X_{r+s+1} are equal too.

If f is locally of finite type, of finite type, or proper, then so is each derived map f_r. Indeed, suppose f_{r-1} is so. Then so is the projection p_2 in (4.1 a). Moreover, he ideal I of the diagonal Δ is locally finitely generated ([3] IV_1, 1.4.3.1); hence the structure map p in (4.1 a) is projective by definition. Therefore f_r is locally of finite type, of finite type, or proper.

Suppose f is locally of finite type (resp. of finite type). It is evident that, if Y is locally noetherian (resp. noetherian) or universally catenary, then so is each derived scheme X_r.

It is clear from Lemma 3.2 that, if f is quasi-projective and Y is noetherian and divisorial, then each derived map f_r is strongly projective and each derived scheme X_r is noetherian and divisorial.

PROPOSITION 4.2. *Let $f: X \to Y$ be a separated map of schemes.*

(i) *The "switch" involution of $X \times_Y X$, which interchanges (x, y) and (y, x), has a natural covering, which is an involution i of the derived scheme X_2,*

$$i: X_2 \to X_2.$$

The restriction of i is equal to the identity on the preimage $p^{-1}\Delta$ of the diagonal Δ, although it acts by multiplication by -1 on the normal bundle, and i preserves the tautological sheaf of X_2,

$$i^*\mathcal{O}(1) = \mathcal{O}(1). \tag{4.2a}$$

Moreover, there is a basic relation, involving maps of (4.1a) *with $r=1$,*

$$f_1 i = p_1 p. \tag{4.2b}$$

(ii) *If f is proper and Y is noetherian, then, for any $r \geq 3$, the image on X_2 of the fundamental cycle $[X_r]$ is preserved by i_*,*

$$i_*(f_2 \dots f_{r-1})_*[X_r] = (f_2 \dots f_{r-1})_*[X_r].$$

Proof. (i) Let I denote the ideal of Δ. It is easy to check that the switch involution preserves I and that it induces the operation of multiplication by -1 on I/I^2. The assertion follows immediately.

(ii) Write $[X_r]=v+w$, where w has support in $p^{-1}\Delta$ but no component of v does. Since i induces the identity on $p^{-1}\Delta$ by (i), the image of w is invariant under i_*. It remains to prove that the image of v is invariant under i_*.

Set $U_2=X_2-p^{-1}\Delta$. Set $U_r=(f_2 \dots f_{r-1})^{-1}U_2$. Let $g: U_r \to U_2$ denote the restriction of $f_2 \dots f_{r-1}$ and $j_2: U_2 \to U_2$ the restriction of i. Finally set $u=[U_r]$. Then $v \mid U_r=u$ and every component of v meets U_r. Hence it will suffice to prove $j_{2*}g_*u=g_*u$.

There is an involution $j_r: U_r \to U_r$ such that $j_2 g=gj_r$; it will be constructed in a moment. Now, u is a linear combination of cycles c of the form $c=[C]+j_{r*}[C]$, where C is an irreducible component of U_r. Obviously, $j_{r*}c=c$. So $j_{2*}g_*c=g_*c$. Hence $j_{2*}g_*u=g_*u$.

It remains to construct j_r. Each j_r will cover j_{r-1}, whence $j_2 g=gj_r$. Suppose j_r and j_{r-1} have been constructed. Since j_r covers j_{r-1}, the cartesian product is a well-defined involution (covering j_{r-1}),

$$j_r \times_{j_{r-1}} j_r: U_r \times_{U_{r-1}} U_r \to U_r \times_{U_{r-1}} U_r.$$

Since this involution preserves the diagonal, it is covered by an involution j_{r+1} of U_{r+1}. Obviously, j_{r+1} covers j_r.

It remains to construct j_3. Set $U=(X\times_Y X)-\Delta$. Consider the following two "diagonal" closed subschemes of $X\times_Y U$:

$$\Delta_1=\{(x,x,y)\}$$
$$\Delta_2=\{(x,y,x)\}.$$

Let I_1 and I_2 denote the respective ideals. Consider the composition of canonical maps,

$$X_3\to X_2\times_X X_2\to (X\times_Y X)\times_X(X\times_Y X)\tilde{\to} X\times_Y X\times_Y X$$
$$((x,y),(z,y))\mapsto(x,y,z).$$

It is covered by an isomorphism, $U_3\tilde{\to}\mathbf{P}(I_1I_2)$, which will be constructed next.

The canonical map $U_2\to U$ is an isomorphism, and the canonical isomorphism from $(X\times_Y X)\times_X U$ onto $X\times_Y U$ identifies $\Delta\times_X U$ with Δ_1. Hence the map from $X_2\times_X X_2$ into $X\times_Y X\times_Y X$ is covered by an isomorphism from $U_2\times_X X_2$ onto $\mathbf{P}(I_1)$. This isomorphism identifies the restriction of the diagonal of $X_2\times_X X_2$ with the pullback of Δ_2 because the intersection $\Delta_1\cap\Delta_2$ is empty. Denote the pullback of I_2 by J. Then there is an induced isomorphism, $U_3\tilde{\to}\mathbf{P}(J)$, which covers the map from X_3 into $X\times_Y X\times_Y X$. Since $\Delta_1\cap\Delta_2$ is empty, $\mathbf{P}(J)$ is equal to $\mathbf{P}(I_1I_2)$.

Finally, consider the involution of $X\times_Y X\times_Y X$ that switches the second two factors. It switches I_1 and I_2. Hence it is covered by an involution of $\mathbf{P}(I_1I_2)$. Transport this involution over to U_3. The result is the desired j_3.

4.3. *r-generic maps.* Let $f\colon X\to Y$ be a separated map of schemes. Suppose that f is a plci of codimension at most n (see § 1.1). Let n also denote the locally constant function on X_{r+1} defined inductively as the function induced via $p_1p(=f_1i)$ by the function n on X_r. Then, it is clear from (4.1a) and Proposition 3.4(i) that each derived map f_r is a plci of codimension at most n. Hence, by Proposition 1.3(i) if for some r the derived scheme X_r is Cohen-Macaulay and the derived map f_r has codimension n, then f_r is an lci of codimension n and X_{r+1} is Cohen-Macaulay.

The map f will be called *r-generic* of codimension n if, for each $s=0,\ldots,r-1$, the sth derived map f_s is an lci of codimension n.

PROPOSITION 4.4. *Let $f\colon X\to Y$ be a separated map of schemes. If f is an* lci *of codimension n and if X is Cohen-Macaulay, then a necessary and sufficient condition for f to be r-generic of codimension n is that f be dimensionally r-generic, that is, that for each $s=1,\ldots,r-1$ the derived map f_s have codimension n.*

Proof. The assertion is obvious from § 4.3.

4.5. *Immersions.* Let $f: X \to Y$ be a separated map of schemes. Then f will be called an *immersion* if f is locally of finite type and formally unramified ([3] IV_4, 17.1.1, p. 56).

Suppose f is an immersion. Then each derived map f_r is an immersion too. In fact, it is evident (from the proof of [3] IV_4, 17.4.1 a, p. 63) that the structure map p in (4.1 a) is an open and closed embedding, whose image is the complement of the diagonal Δ, and that f_1 is equal to the restriction of the projection p_2,

$$X_{r+1} = X_r \times X_r - \Delta$$
$$f_r = p_2 \,|\, X_{r+1}.$$

Moreover, it is evident that for $r \geqslant 1$ the tautological sheaf is trivial,

$$O_{X_{r+1}}(1) = O_{X_{r+1}}.$$

Suppose that X and Y are smooth over a ground scheme S and that $f: X \to Y$ is an immersion of S-schemes. Then f will be called *r-fold self-transverse* if, for each geometric point y of Y, say with image geometric point s of S, the tangent spaces of the fiber $X(s)$ at any r points of the fiber $f^{-1}(y)$, when viewed as subspaces of the tangent space of the fiber $Y(s)$ at y, are in general position.

PROPOSITION 4.6. *Let X and Y be smooth schemes over a ground scheme S, and let $f: X \to Y$ be an r-fold self-transverse immersion. Then f is r-generic of codimension n, with*

$$n(x) = \mathrm{rel.dim}_{f(x)}\,(Y, S) - \mathrm{rel.dim}_x\,(X, S),$$

and $X_2, \ldots, X_r$ are smooth over S.

Proof. Note that f is an lci of codimension n; indeed, because Y is smooth, the graph of f is a regular embedding by [3] IV_4, 17.12.3, p. 87. Hence the assertion will follow by induction on r once it is proved that X_2 is smooth over S with relative dimension at z equal to

$$\dim_{f_1(z)}\,(X, S) - nf_1 i(z)$$

and that f_1 is an $(r-1)$-fold self-transverse immersion.

As noted in § 4.5, f_1 is an immersion because f is; in fact, X_2 is equal to $X \times_Y X - \Delta$ and f_1 is equal to the restriction of p_2. Hence it suffices to prove that X_2 is smooth and that, for each geometric point $z = (x_1, x_2)$ of X_2, say with image geometric points y of Y and s of S, the tangent space of the fiber $X_2(s)$ at z is equal to the intersection of the tangent spaces of the fiber $X(s)$ at x_1 and at x_2, when the three spaces are viewed as subspaces of the tangent space of the fiber $Y(s)$ at y.

These assertions are easy to prove in a more or less straightforward fashion using the reasoning of [3] IV_4, 17.13.2, p. 90. There is one hitch, however. The basic result used [3] IV_4, 17.12.1, p. 85, asserts that a subscheme of a smooth scheme is smooth itself if and only if the canonical map from the conormal sheaf into the restriction of the sheaf of differentials is left invertible; it is stated and proved only for an embedded subscheme and not an immersed one. However, the proofs of this and related results carry over to the case of an immersed subscheme with little or no change.

PROPOSITION 4.7. *Let $f\colon X \to Y$ be a quasi-projective map of schemes, with Y noetherian and divisorial. If f is 2-generic, then the virtual normal bundles of f and f_1 are related by the formula,*

$$\nu_{f_1} = [i^* f_1^* \nu_f](1) + [O_{X_2} - O_{X_2}(-1)].$$

Moreover, in case f is an immersion, the above formula reduces to

$$\nu_{f_1} = i^* f_1^* \nu_f.$$

Proof. The first formula holds by Proposition 3.5 in view of § 4.1, (4.2b) and § 4.3 The second formula comes from the first because, when f is an immersion, the tautological sheaf is trivial by § 4.5.

4.8. *Remark.* The second formula of Proposition 4.7 can be obtained simply and directly, without appealing to Proposition 3.5. In fact, because f and f_1 are "regular immersions", their virtual normal bundles are representable by canonical locally free sheaves (because the H^0's of their cotangent complexes vanish, being the sheaves of relative differentials) and the formula comes from a canonical isomorphism of sheaves.

Indeed, identifying X_2 with $X \times_Y X - \Delta$ and f_1 and $f_1 i$ with the restrictions of the projections p_2 and p_1 (see § 4.5 and (4.2b),) it is not hard to construct a canonical surjection of conormal sheaves,

$$i^* f_1^* N_f \to N_{f_1}.$$

This surjection is an isomorphism because, under the hypotheses, the source and target are locally free sheaves of the same rank.

5. The multiple-point formulas

5.1. *The setup.* Fix a map of schemes, $f\colon X \to Y$. Assume that f is a projective lci of codimension $n \geqslant 1$. (Note that n is a locally constant function on X.) Assume that Y is noetherian, divisorial and universally catenary. Recall from § 4.1 that then the derived schemes $X_1(=X)$, X_2, X_3, ... are each noetherian, divisorial and universally catenary and that the derived maps $f_0(=f)$, f_1, f_2, ... are each strongly projective.

For $r \geqslant 1$, the pushout to X of the fundamental cycle of X_r will be denoted by m_r,

$$m_r = (f_1 \dots f_{r-1})_* [X_r], \quad m_1 = [X].$$

The rational-equivalence class of m_r in $A.X$ will be denoted by m_r too.

The definition of m_r did not involve the hypothesis that f is an lci. So it applies to $f_1: X_2 \to X$. The corresponding cycle and its class will both be denoted by m'_r. Thus we have

$$m'_r = (f_2 \dots f_r)_* [X_{r+1}], \quad m'_1 = [X_2], \tag{5.1a}$$

$$f_{1*} m'_r = m_{r+1}, \tag{5.1b}$$

$$i_* m'_r = m'_r, \tag{5.1c}$$

the first two relations being evident, the third holding by Proposition 4.2(i) for $r=1$ and Proposition 4.2(ii) for $r \geqslant 2$. Moreover, if f_1 is an lci (for example, if f is r-generic for an $r \geqslant 2$), then we have

$$f_1^* m_1 = m'_1 \tag{5.1d}$$

because, in general, the pullback of the fundamental class of a target is the fundamental class of the source.

Several Chern operators will be used often, so they will be abbreviated as follows:

$$t = c_1(O_{X_2}(1));$$
$$c_k = c_k(f) = c_k(\nu_f);$$
$$c'_k = c_k(f_1),$$

the latter being defined only when f_1 is an lci.

Recall from § 4.3 that f is called r-generic (of codimension n) if f_s for $s=0, \dots, r-1$ is an lci of codimension n (where n now is a function on X_{s+1} suitably induced by the original n, the codimension of f). Recall from Proposition 4.4 that, if X (or Y) is Cohen-Macaulay, then f is r-generic if and only if $f_1, \dots, f_{r-1}$ have codimension n. Now, f will be called *practically r-generic* if there is a closed subset S of Y such that the restriction of f,

$$(X - f^{-1}S) \to (Y - S),$$

is r-generic and such that, setting

$$n_r(x) = \max \{n(x_2) + \dots + n(x_r) \mid x_2, \dots, x_r \in f^{-1}f(x)\}, \tag{5.1e}$$

we have, for each x in $f^{-1}S$,

$$\operatorname{cod}_x (f^{-1}S, X) > n_r(x).$$

Note that we have

$$n_r(x) \geqslant (r-1)\,n(x),$$

with equality if n is constant along $f^{-1}f(x)$.

Proposition 5.2. *In the setup of § 5.1, if f is practically r-generic, if the codimension n of f is constant in case $r \geqslant 4$, and if the singularity set $\bar{S}_2(f)$ is nonempty, then r and n must lie in the following range:*

$$r = 1,\ 2 \text{ or } 3,\ n \text{ arbitrary};$$

$$r = 4, \quad n = 1, 2, 3;$$

$$r = 5 \text{ or } 6, \quad n = 1.$$

Proof. By definition, $\bar{S}_2(f)$ is the subset of X whose geometric points x satisfy the relation,

$$\dim_{k(x)} \Omega^1_f(x) \geqslant 2.$$

Now, since f is an lci of codimension n, we have

$$\operatorname{cod} (\bar{S}_2(f), X) \leqslant 2(n+2). \tag{5.2a}$$

Indeed, embed X in a smooth Y-scheme P (locally on X is enough). Then the conormal bundle-cotangent bundle sequence,

$$N \to \Omega^1_{P/Y} | X \to \Omega^1_f \to 0,$$

is a presentation of Ω^1_f by locally free sheaves whose ranks differ by n; whence (5.2a) holds.

It is evident from the definitions, see § 4.1, that at each point x of $\bar{S}_2(f)$ the fiber $f_1^{-1}(x)$ has dimension at least 1 and that therefore the dimension of the fiber $\varphi^{-1}(x)$, where $\varphi = f_1 \cdots f_{r-1}$, is at least $r-1$. Thus we have

$$\operatorname{cod} (\varphi^{-1}(\bar{S}_2(f)), \bar{S}_2(f)) \leqslant -(f-1).$$

Hence, by additivity Proposition 1.3(ii) (which holds because Y is universally catenary), (5.2a) yields the second inequality in the expression,

$$\operatorname{cod}(\varphi) \leqslant \operatorname{cod} (\varphi^{-1}(\bar{S}_2(f)), X) \leqslant 2(n+2) - (r-1); \tag{5.2b}$$

the first inequality, which holds along $\varphi^{-1}(\bar{S}_2(f))$, is obviously true.

By hypothesis, there is a subset S of Y such that $f^{-1}S$ has codimension at least $(r-1)n+1$ in X and the restriction of f to the complement of $f^{-1}S$ is r-generic. If $f^{-1}S$ contains $\bar{S}_2(f)$, then (5.2a) yields

$$(r-1)n+1 \leqslant 2(n+2), \tag{5.2c}$$

an inequality that is satsified just for the values of r, n listed in the assertion. On the other hand, if S does not contain $\bar{S}_2(f)$ and if $r \geqslant 4$, then, because the restriction of f is r-generic of constant codimension n, (5.2b) yields

$$(r-1)n \leqslant 2(n+2)-(r-1),$$

an inequality that is satisfied for no more values of r and n than (5.2c) is.

PROPOSITION 5.3. *In the setup of* § 5.1, *for each component M of the cycle m_r and for each $r \geqslant 2$, we have*

$$\mathrm{cod}_x\,(M, X) \leqslant n(x_2)+\ldots+n(x_r) \leqslant n_r(x) \tag{5.3a}$$

for suitable points $x_2, \ldots, x_r$ of the fiber $f^{-1}f(x)$. Moreover, equality holds in the first relation if f is practically r-generic of codimension n.

Proof. Let Z be an irreducible component of X_r and z its generic point. Set $\varphi = f_1 \ldots f_{r-1}$. Since each f_s has codimension at most n (see § 4.3), by additivity we have

$$\mathrm{cod}_z\,(Z, X) \leqslant n(z_2)+\ldots+n(z_r) \tag{5.3b}$$

for certain points $z_2, \ldots, z_r$ of the fiber $f^{-1}f(\varphi(z))$. On the other hand, additivity yields

$$\mathrm{cod}_z\,(Z, X) = \mathrm{cod}_z\,(Z, \varphi(Z)) + \mathrm{cod}_{\varphi(z)}\,(\varphi(Z), X).$$

Since, by definition, the components of m_r are the $\varphi(Z)$ such that cod $(Z, \varphi(Z))$ is 0, the first inequality of (5.3a) follows. The second inequality holds by the definition (5.1e) of n_r.

Suppose that f is practically r-generic of codimension n; that is, there is a closed subset S of Y such that the restriction of f to the complement of $f^{-1}S$ is r-generic of codimension n and the codimension of $f^{-1}S$ in X at x is more than $n_r(x)$. Then, by (5.3a), no component of m_r can lie in $f^{-1}S$. Hence, because the restriction of f is r-generic of codimension n, the derivation of (5.3b) shows that equality holds in it for each Z such that $\varphi(Z)$ is a component of m_r. The second assertion follows.

5.4. *The double-point set.* In the setup of § 5.1, the image $D = f_1(X_2)$ deserves the name, *the double-point set of f*. Indeed, it is evident from the definitions that the geometric points of D are of these two types: (1) the strict double points, those for which there is a second geometric point of X, distinct from first but with the same image under f, and (2) the cuspidal or ramification points, those at which Ω_f^1 is nonzero. It is evident that D is stratified by the singularity sets $\bar{S}_i(f)$, whose geometric points x are those where $\Omega_f^1(x)$ has dimension at least i. It is evident that D, in fact, $\bar{S}_1(f)$ contains the closed set F of points at which the fibers of f are positive dimensional. It is not hard to prove that the complement of D is just the open set on which f is an isomorphism. (None of these observations require f to be an lci or Y to be divisorial or universally catenary.)

Suppose that f is practically 2-generic. Let M be an irreducible component of D and x the generic point of M. The proof of Proposition 5.3 shows that we have

$$\operatorname{cod}_x(M, X) \leqslant n(x_2)$$

for every point x_2 of the fiber $f^{-1}f(x)$, except possibly for $x_2 = x$ in the case that x is not a ramification point, and that equality holds for some x_2 if and only if M is a component of the cycle m_2. Moreover, it is clear that, if M is a component of m_2, then $x \notin \bar{S}_2(f)$ and some $x_2 \notin F$.

Suppose that Y is Cohen-Macaulay and that $f^{-1}f(F)$ is nowhere dense in D. Then we have

$$\operatorname{cod}_x(D, X) \leqslant n(x_2) \tag{5.4a}$$

for every point $x_2 \notin F$ of the fiber $f^{-1}f(x)$, except possibly for $x_2 = x$ in the case that x is not a ramification point, because by Proposition 3.4 we have

$$\operatorname{cod}_{(x_2, x)}(p_2 \,|\, p(X_2)) \leqslant n(x_2).$$

Suppose in addition that $\bar{S}_2(f)$ is nowhere dense in D and that equality holds in (5.4a) for every x and x_2 (except possibly for $x_2 = x$ in the case that x is not a ramification point). Then it is not hard to prove that m_2 has no embedded components, that the support of m_2 is all of D, and that f is practically 2-generic, with S being the union of the images of the components Z of x_2 such that the $f_1(Z)$ are not components of m_2.

LEMMA 5.5. *In the setup of* § 5.1, *if f is 2-generic of codimension n, then*

$$\begin{aligned} f_{1*} i_* f_1^* &= f^* f_* - c_n, \\ f_{1*} i_* t^k f_1^* &= -c_{k+n} \quad \text{for } k \geqslant 1. \end{aligned} \tag{5.5a}$$

Proof. The residual-intersection theorem (3.6) and its corollary (3.7) applied in the setup of (4.1a) with $r = 1$ yield

$$\begin{aligned} f^* \,|\, X &= p_* f_1^* + q_* c_n(f/p_2 q)(p_2 q)^* \\ p_* t^k (p_2 p)^* &= -q_* c_{n+k}(f/p_2 q)(p_2 q)^*, \end{aligned}$$

because $p_2 q$ has codimension 0, being an isomorphism. Applying p_{1*} to both sides of these formulas yields the desired ones as follows.

The commutativity of pushout and pullback yields

$$p_{1*}(f^*|X) = f^* f_*.$$

Next, (4.2b) yields

$$p_{1*} p_* = f_{1*} i_*.$$

Finally, $p_1 q$ and $p_2 q$ are isomorphisms; hence

$$p_{1*} q_* c(f/p_2 q)(p_2 q)^* = c(f).$$

Putting it altogether, we get the desired formula.

THEOREM 5.6 (the double-point theorem). *In the setup of* § 5.1, *if* f *is practically* 2-*generic of codimension* n, *then the following "double-point" formula holds in* $A.X$:

$$m_2 = f^* f_* m_1 - c_n m_1.$$

Proof. We may assume that f is 2-generic of codimension n. Indeed, by hypothesis, there is a closed subset S of Y such that $f^{-1}S$ has codimension in X at x more than $n_2(x)$ and the restriction of f to the complement of $f^{-1}S$ is 2-generic. Each term of the double-point formula has degree at most $n_2(x)$ (in the grading by codimension) by Proposition 5.3 and general properties of the operators involved. Each term's formation clearly commutes with restriction to the complement of $f^{-1}S$. Hence the formula will hold for f if it holds for its restriction, which is 2-generic.

If f is 2-generic, the double-point formula comes from applying (5.5a) to m_1 and using (5.1d), (5.1c) and (5.1b).

LEMMA 5.7. *In the setup of* § 5.1, *if* f *is* 2-*generic of codimension* n, *then for any* k *we have*

$$f_{1*} i_* c'_k = c_k f_{1*} i_* + \sum_{j=0}^{k-1} \left[\sum_{l=0}^{k-1} \binom{n-j}{l} \right] c_j f_{1*} i_* t^{k-j}.$$

Moreover, if f_1 *is an immersion, then the second term on the right vanishes.*

Proof. If f_1 is an immersion, the $O(1)$ is trivial (see § 4.5), so t is 0 and the second term vanishes.

In the general case, Proposition 4.7 yields the relation,

$$c(f_1) = c((i^* f_1^* \nu_f)(1)(1-t)^{-1}.$$

Now, the total Chern operator of the product of a virtual bundle N of rank n and a line bundle L is given by the following formula:

$$c(N \otimes L) = \sum_j c_j(N)(1 + c_1(L))^{n-j}.$$

(This well-known formula is easy to verify using the splitting principle because both sides are additive in N.) Hence a little computation yields the relation,

$$c(f_1) = \sum_j c_j(i^* f_1^* \nu_f) \sum_k \left(\sum_{l=0}^{k} \binom{n-j}{l} \right) t^k.$$

Applying $f_{1*} i_*$, using the projection formula, and rearranging the sums yields the desired relation.

THEOREM 5.8 (the Herbert-Ronga formula). *In the setup of* § 5.1, *if f is practically $(r+1)$-generic of codimension n (for any $r \geq 1$) and if f is an immersion, then the following formula holds in $A.X$:*

$$m_{r+1} = f^* f_* m_r - r c_n m_r.$$

Proof. For $r=1$, the assertion is a special case of the double-point theorem (5.6). Reasoning as in the proof of that theorem, we may assume that f is r-generic of codimension n. Proceeding by induction on r, we may assume the following formula:

$$m_r' = f_1^* f_{1*} m_{r-1}' - (r-1) c_n' m_{r-1}'.$$

Applying $f_{1*} i_*$ and using (5.1c), (5.1b), (5.5a) and Lemma 5.7, we get the desired formula.

THEOREM 5.9 (the triple-point theorem). *In the setup of* § 5.1, *if f is practically* 3-*generic of codimension n, then the following "triple-point" formula holds in $A.X$:*

$$m_3 = f^* f_* m_2 - 2 c_n m_2 + \left(\sum_{j=0}^{n-1} 2^{n-j} c_j c_{2n-j} \right) m_1.$$

Proof. Reasoning as in the proof of the double-point theorem (5.6), we may assume that f is 3-generic of codimension n. Then that theorem applies to f_1 and yields

$$m_2' = f_1^* f_{1*} m_1' - c_n' m_1'.$$

Applying $f_{1*} i_*$ and using (5.1c), (5.1b), (5.5a) and Lemma 5.7, we get

$$m_3 = f^* f_* m_2 - 2 c_n m_2 - \left(\sum_{j=0}^{n-1} \left[\sum_{l=0}^{n-j} \binom{n-j}{l} \right] c_j f_{1*} i_* t^{n-j} \right) m_1'.$$

Finally, (5.1d) and (5.5b) yield the desired formula.

THEOREM 5.10 (the quadruple-point theorem for $n=1, 2, 3$). *In the setup of* § 5.1, *if* f *is practically* 4-*generic of codimension* n *for* $n=1, 2$ *or* 3, *then the following "quadruple-point" formulas hold in* $A.X$:

$n=1$, $$m_4 = f^*f_*m_3 - 3c_1m_3 + 6c_2m_2 - 6(c_1c_2+2c_3)m_1.$$

$n=2$, $$m_4 = f^*f_*m_3 - 3c_2m_3 + 6(c_1c_3+2c_4)m_2 - 6(c_1c_2c_3+2c_1^2c_4+5c_1c_5+3c_2c_4+12c_6+c_3^2)m_1.$$

$n=3$, $$m_4 = f^*f_*m_3 - 3c_3m_3 + 6(c_2c_4+2c_1c_5+4c_6)m_2 - 6(c_2c_3c_4+2c_2^2c_5+10c_1c_2c_6 + 26c_2c_7+3c_1c_3c_5+12c_1^2c_7+60c_1c_8+9c_3c_6+72c_9+9c_4c_5)m_1.$$

Proof. Reasoning as in the proof of Theorem 5.6, we may assume that f is 4-generic of codimension n. Then applying $f_{1*}i_*$ to the triple-point formula for f_1 and using (5.1c), (5.1b) and (5.5a), we get

$$m_4 = f^*f_*m_3 - c_nm_3 - 2f_{1*}i_*c_n'm_2' + \sum_{j=0}^{n-1} 2^{n-j}f_{1*}i_*c_j'c_{2n-j}'m_1'.$$

This expression presents some minor new features.

To facilitate the computation, the work may be organized as follows. First, work out the expression for $f_{1*}i_*c_j'$ for $j=1, ..., 2n$ from Lemma 5.7 (n may be fixed). Next, using this expression and (5.1b), (5.1c), (5.1d) and (5.5b), work out an expression for

$$f_{1*}i_*c_j't^km_1', \quad k\geq 0,\ j=1, ..., 2n;$$

there are basically only two different cases, $k=0$ and $k>0$, and they differ only at the first term. Next, work out an expression for $f_{1*}i_*t^nm_2'$ for $k>0$ by using the double-point formula for f_1 and then (5.1b) and (5.5b) and the expression for $f_{1*}i_*c_n't^km_1'$; the latter may be used because

$$t^kc_j' = c_j't^k$$

as any two Chern operators commute. Finally, the terms

$$f_{1*}i_*c_n'm_2' \quad \text{and} \quad f_{1*}i_*c_j'c_{2n-j}'m_1'$$

appearing in the expression above for m_4 can be conveniently worked out by using the expression for $f_{1*}i_*c_j'$ and then the required, but available expressions.

THEOREM 5.11 (quintuple-point (resp. sextuple-point) theorem for $n=1$). *In the setup of* § 5.1, *if* f *is practically* 5-*generic* (*resp.* 6-*generic*) *of codimension* 1, *then the following "quintuple-point"* (*resp.* "*sextuple-point*") *formula holds in* $A.X$:

$$m_5 = f^*f_*m_4 - 4c_1m_4 + 12c_2m_3 - 24(c_1c_2+2c_3)m_2 + 24(c_1^2c_2+5c_1c_3+6c_4+c_2^2)m_1$$

$$(\text{resp. } m_6 = f^*f_*m_5 - 5c_1m_5 + 20c_2m_4 - 60(c_1c_2+2c_3)m_3 + 120(c_1^2c_2+5c_1c_3+6c_4+c_2^2)m_2 - 120(c_1^3c_2+9c_1^2c_3+26c_1c_4+3c_1c_2^2+8c_2c_3+24c_5)m_1).$$

Proof. The proof is similar in spirit to that of the quadruple-point theorem (5.10). There are no new features. However, the computation is more involved, with fewer fresh starts.

References

[1] ALTMAN, A. & KLEIMAN, S., Compactifying the Picard scheme. *Advances in Math.*, 35: 1 (1980), 50–112.
[2] ARTIN, M. & NAGATA, M., Residual intersections in Cohen–Macauley rings. *J. Math. Kyoto Univ.*, 12 (1972), 307–323.
[3] EGA II, IV_1, IV_2, IV_4, *Eléments de géométrie algébrique*. A. Grothendieck and J. Dieudonné. Inst. Hautes Etudes Sci. Publ. Math., Nos. 8 (1961), 20 (1964), 24 (1965), 32 (1967).
[4] FULTON, W., A note on residual intersections and the double point formula. *Acta math.*, 140 (1978), 93–101.
[5] FULTON, W. & LAKSOV, D., Residual intersections and the double point formula. *Real and complex singularities, Oslo 1976*. P. Holm, ed., pp. 171–177. Sijthoff & Noordhooff (1977).
[6] FULTON, W. & MACPHERSON, *Bivariant theories*. Preprint, Brown Univ., Providence (1980).
[7] HERBERT, R., *Multiple points of immersed manifolds*. Thesis, Univ. of Minnesota (1975).
[8] HUNEKE, C., On the symmetric and Rees algebra of an ideal generated by a d-sequence. To appear in a volume of the *J. Alg.* dedicated to N. Jacobson.
[9] JOHNSON, K., Immersion and embedding of projective varieties. *Acta math.*, 140 (1978), 49–74.
[10] KLEIMAN, S., The enumerative theory of singularities. *Real and complex singularities, Oslo 1976*. P. Holm, ed., pp. 297–396. Sijthoff & Noordhoof (1977).
[11] —— Multiple-point formulas for maps. To appear.
[12] —— Multiple-point formulas II: The Hilbert scheme. In preparation.
[13] LAKSOV, D., Residual intersections and Todd's formula for the double locus of a morphism. *Acta math.*, 140 (1978), 75–92.
[14] —— Secant bundles and Todd's formula for the double points of maps into $\mathbf{P}^n$. *Proc. Lond. Math. Soc.*, 37 (1978), 120–142.
[15] LICHTENBAUM, S. & SCHLESSINGER, M., The cotangent complex of a morphism. *Trans. Amer. Math. Soc.*, 128: 1 (1967), 41–70.
[16] MICALI, A., Algèbra symétrique d'un idéal. *C. R. Acad. Sci. Paris*, 251 (1960), 1954–1956.
[17] PIENE, R. & RONGA, F., A geometric approach to the arithmetic genus of a projective manifold of dimension three. Preprint, July (1979).
[18] ROBERTS, J., Some properties of double point schemes. Preprint May (1979). To appear in *Compositio math.*
[19] RONGA, F., Le calcul des classes duales aux points doubles d'une application. *Compositio math.*, 27: 2 (1973), 223–232.
[20] —— On multiple points of smooth immersions. Preprint March (1980).
[21] SGA6, *Théorie des Intersections et Théorème de Riemann-Roch*. P. Berthelot et al. Lecture Notes in Math. No. 225, Springer (1971).

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