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Analyticity of intersection exponents for planar Brownian motion

by

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1. Introduction

The goal of the present paper is to show that the intersection exponents for planar Brownian motions are analytic. Let $k \geqslant 1$ be a positive integer and let $X^1, ..., X^k$ be independent Brownian motions in the complex plane $\mathbf{C}$ started from 0. Let $Y, Y^1, Y^2, ...$ denote other independent planar Brownian motions started from 1, and let Ξ_t be the random variable (measurable with respect to $X^1, ..., X^k$)

$$\Xi_t = \mathbf{P}[Y[0,t] \cap (X^1[0,t] \cup ... \cup X^k[0,t]) = \varnothing \mid X^1[0,t] \cup ... \cup X^k[0,t]].$$

Note that

$$\mathbf{P}[(X^1[0,t] \cup ... \cup X^k[0,t]) \cap (Y^1[0,t] \cup ... \cup Y^p[0,t]) = \varnothing] = \mathbf{E}[\Xi_t^p].$$

The intersection exponent $\xi(k,\lambda)$ is defined for $\lambda > 0$ by

$$\mathbf{E}[\Xi_t^\lambda] \approx t^{-\xi(k,\lambda)/2}, \quad t \to \infty, \tag{1.1}$$

that is,

$$\xi(k,\lambda) := -2 \lim_{t\to\infty} \frac{\log \mathbf{E}[\Xi_t^\lambda]}{\log t}.$$

The existence of such exponents follows easily from a subadditivity argument. For a more detailed account of the definition and properties of these exponents, we refer the reader to our earlier papers [16], [12], [13]. Let us mention, however, that they are related to other critical exponents arising in statistical physics, including those predicted by theoretical physicists for planar critical percolation and self-avoiding walks (see references in [12]).

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In [13], the value of $\xi(k,\lambda)$ was determined for a large collection of pairs (k,λ). In particular, it was shown that

$$\xi(k,\lambda)=\tfrac{1}{48}\big((\sqrt{24k+1}+\sqrt{24\lambda+1}-2)^2-4\big) \tag{1.2}$$

holds for $k=1$, $\lambda\geqslant\frac{10}{3}$, and for $k=2$, $\lambda\geqslant 2$. In [14], we then showed that (1.2) holds for all integers $k\geqslant 1$ and all real $\lambda\geqslant 1$. The idea in the proofs is to compute the exponents associated to another conformally invariant process (called stochastic Loewner evolution and first introduced in [21]) and to identify them with the Brownian intersection exponents via a universality argument (introduced in [17]). This universality argument is not sufficient to derive the value of $\xi(k,\lambda)$ when $\lambda<1$.

The exponents $\xi(2,\lambda)$ are very closely related to the dimension and properties of the so-called outer boundary of a planar Brownian path. The outer boundary, or frontier, of the Brownian path $B[0,1]$ is the boundary of the unbounded connected component of $\mathbf{C}\setminus B[0,1]$. The disconnection exponents η_k are defined by

$$\mathbf{P}[X^1[0,t]\cup\ldots\cup X^k[0,t] \text{ does not disconnect } 1 \text{ from } \infty]\approx t^{-\eta_k/2}.$$

It is easy to show that $\eta_2=\xi(2,0)$ if we use the convention $0^0=0$ in the definition of $\xi(k,\lambda)$ when $\lambda=0$. In [7] it was proved that the Hausdorff dimension of the frontier of $B[0,1]$ is almost surely equal to $2-\eta_2$. Moreover [8], the multifractal spectrum of the frontier with respect to harmonic measure is given in terms of the Legendre transform of the function $\xi(2,\lambda)$. In [9] (see also [15]), it is also shown that

$$\lim_{\lambda\searrow 0}\xi(k,\lambda)=\xi(k,0)=\eta_k.$$

The main result of the present paper is the following:

THEOREM 1.1. *For all integers $k\geqslant 1$, the function $\lambda\to\xi(k,\lambda)$ is real analytic in* $(0,\infty)$.

This result is in fact a consequence of spectral gap estimates for a Markov process on non-disconnecting k-tuples of Brownian paths. In [6], Lawler had defined a Markov process on non-intersecting pairs of Brownian paths (corresponding to the exponent $\xi(1,1)$). Similarly, each exponent $\xi(2,\lambda)$ is associated to a process on non-disconnecting pairs of Brownian paths, where one introduces a weighting corresponding to the value of λ. When λ varies, the Markov process changes, and one then studies how the associated eigenvalues change. Such a strategy was also used in [9] to show that the map $\lambda\mapsto\xi(k,\lambda)$ is strictly concave. Our theorem has the following consequences:

COROLLARY 1.2. *Formula* (1.2) *is valid for all positive integers* k *and all non-negative real* λ. *In particular,*

$$\eta_k = \tfrac{1}{48}\left(\left(\sqrt{24k+1}-1\right)^2-4\right). \tag{1.3}$$

Proof. For $\lambda>0$, this follows from (1.2) and Theorem 1.1 by analytic continuation. For $\lambda=0$, the result follows by the continuity at 0 proved in [9], [15]. □

COROLLARY 1.3. *The Hausdorff dimension of the outer boundary of a planar Brownian path* $B[0,1]$ *is almost surely* $\frac{4}{3}$.

Proof. The case $k=2$ in (1.3) gives $\eta_2=\frac{2}{3}$. The corollary follows from this and the result from [7] saying that the dimension of the frontier is $2-\eta_2$. □

Note that for this corollary, one does not need [14], since (1.2) appears in [13] for $k=2$ and $\lambda\geqslant 2$. Corollary 1.3 has been conjectured by Mandelbrot [19], based on simulations and the analogous conjecture for self-avoiding walks. Non-rigorous arguments from theoretical physics involving quantum gravity [5] also lead to this conjecture. Before our series of papers [12], [13], [14], it had been proved [1], [3], [22], [7] that the Hausdorff dimension of the outer boundary of a planar Brownian path is in the interval $(1.01, 1.48)$.

The Hausdorff dimension of other exceptional subsets of the planar Brownian curve can be described in terms of disconnection exponents. A point z is a pioneer point of $B[0,1]$ if there is some time $t\in[0,1]$ such that $z=B_t$ is in the outer boundary of $B[0,t]$. It is shown in [10] that the Hausdorff dimension of the set of pioneer points is $2-\eta_1$ almost surely. Consequently, (1.3) gives

COROLLARY 1.4. *The Hausdorff dimension of the set of pioneer points of a planar Brownian path is almost surely* $\frac{7}{4}$. □

In the same way, one gets that the Hausdorff dimension of the set of double points of $B[0,1]$ that are also on the outer boundary of $B[0,1]$ is $2-\eta_4=\frac{1}{24}\left(1+\sqrt{97}\right)$ (which is not a rational number).

To prove Theorem 1.1, we show that for every $\lambda>0$, the function $x\mapsto\xi(k,x)$ can be extended to an analytic function in a neighborhood of λ in the complex plane. For notational ease, we will restrict the proof to the case $k=2$; the proof for other values of k is essentially the same. Our proof has similarities with the proof that the free energy of a one-dimensional Ising model with exponentially decaying interactions is an analytic function (see [20]). We shall show that $e^{-\xi(2,x)}$ is the leading eigenvalue of an operator T_x on a space of functions on pairs of paths. A special norm will be chosen such that on the space of functions with finite norm, the dependence of T_x on x is analytic, and the leading eigenvalue is an isolated simple eigenvalue. It then follows by a standard result from operator theory (see, e.g., [4]) that $\xi(2,x)$ is an analytic function of x.

Let us outline the argument in the paper. The Banach spaces and the operators are defined in §2. The operators act on spaces of functions of pairs of paths from the origin to the unit circle. The function spaces consist of functions with the property that their dependence on the behavior of the paths near the origin decays exponentially. These spaces are reminiscent of the spaces defined in [20], but the precise definition in this paper is new. In §3 we review facts about $\xi(2,\lambda)$ from [9] which are needed in the proof. Analyticity of the operator is proved in §4, using a coupling argument. The existence of the spectral gap is established in §5. The main tool to show that the eigenvalue is isolated is also a coupling—but this time a coupling of weighted Brownian paths. A similar coupling was used in [2] for a one-dimensional Ising-type model, and such a coupling was first used in [11] for weighted Brownian paths.

We end the introduction with a few brief words regarding notation. In this paper, c, c', u, etc., denote constants whose values may change from line to line, while C and c_0, c_0', v_0, etc., will denote constants whose values will not change. The values of these constants will depend on λ. The notation $f(t)\approx g(t)$ means $\log f(t)/\log g(t)\to 1$ as $t\to\infty$, while $f(t)\asymp g(t)$ means that there is a constant $c>0$ such that $c^{-1}\leqslant f/g\leqslant c$. The unit disk $\{z:|z|<1\}$ in $\mathbf{C}$ is denoted by $\mathbf{U}$.

2. The operator

Let Γ_0 denote the set of all continuous paths $\alpha:[0,1]\to\overline{\mathbf{U}}$ such that $\alpha(0)=0$, $|\alpha(1)|=1$ and $0<|\alpha(t)|<1$ for $t\in(0,1)$. We identify two paths if one can be obtained from the other by increasing reparameterization, and endow Γ_0 with the metric

$$d(\alpha,\beta):=\inf_{\phi}\sup_{t\in[0,1]}|\alpha(t)-\beta(\phi(t))|,$$

where ϕ ranges over all increasing homeomorphisms $\phi:[0,1]\to[0,1]$. Let $\Gamma\subset\Gamma_0\times\Gamma_0$ denote the set of $\gamma=(\alpha,\beta)$ such that there exists a unique connected component $O=O(\gamma)$ of $\mathbf{U}\setminus(\alpha\cup\beta)$ with $0\in\partial O$ and $\partial\mathbf{U}\cap\partial O\neq\varnothing$. Let $\mathcal{A}$ denote the set of bounded measurable functions $f:\Gamma\to\mathbf{C}$ and $\|f\|=\sup_{\gamma\in\Gamma}|f(\gamma)|$. Here, measurability is with respect to the Borel sets from the metric on Γ (and this is the sole use of this metric in this paper).

We are interested in functions $f\in\mathcal{A}$ that depend little on the part of γ near the origin. For $\alpha\in\Gamma_0$ and $m\geqslant 0$, let α_m be the arc of α after its first point in the circle $e^{-m}\partial\mathbf{U}$ of radius e^{-m} around 0, and for $\gamma=(\alpha,\beta)\in\Gamma$ set $\gamma_m=(\alpha_m,\beta_m)$. If $k<j$, we say that the path $\alpha\in\Gamma_0$ has no downcrossing from e^{-k} to e^{-j} if $\alpha_k\cap e^{-j}\mathbf{U}=\varnothing$, in other

words, if α does not touch $e^{-j}\mathbf{U}$ after its first visit to $e^{-k}\partial\mathbf{U}$. We say that $\gamma=(\alpha,\beta)$ has no downcrossing if both α and β have no downcrossing. Let $\mathcal{Y}_m$ be the set of all $\gamma\in\Gamma$ such that for all $k\in\left[0,\frac{11}{12}m\right]$, γ contains no downcrossing from e^{-k} to $e^{-k-m/12}$. Let $\mathcal{X}_m$ be the set of $(\gamma,\gamma')\in\Gamma^2$ such that

(1) $(\gamma,\gamma')\in\mathcal{Y}_m\times\mathcal{Y}_m$,

(2) $\gamma_m=\gamma'_m$, and

(3) $O(\gamma)\cap\partial\mathbf{U}=O(\gamma')\cap\partial\mathbf{U}$.

For all $f\in\mathcal{A}$ and $u>0$, let

$$\|f\|_u:=\max\{\|f\|,\sup\{e^{mu}|f(\gamma)-f(\gamma')|:m=1,2,...,(\gamma,\gamma')\in\mathcal{X}_m\}\}.$$

This is a norm on the Banach space $\mathcal{A}_u:=\{f\in\mathcal{A}:\|f\|_u<\infty\}$. Let $\mathcal{L}_u$ denote the Banach space of continuous linear operators from $\mathcal{A}_u$ to $\mathcal{A}_u$ with the operator norm

$$N_u(T):=\sup_{\|f\|_u=1}\|T(f)\|_u.$$

For every $\gamma\in\Gamma$, let $\widehat{\alpha}$ and $\hat{\beta}$ be independent planar Brownian motions started, respectively, from the endpoints of α and β on the unit circle. Let $\widehat{\alpha}^n$ denote the path $\widehat{\alpha}$ stopped when it hits $e^n\partial\mathbf{U}$; let $\widetilde{\alpha}^n$ be the path from 0 to $e^n\partial\mathbf{U}$ obtained by concatenating α and $\widehat{\alpha}^n$; and let $\bar{\alpha}^n:=e^{-n}\widetilde{\alpha}^n$. Define $\hat{\beta}^n$, $\tilde{\beta}^n$ and $\bar{\beta}^n$ similarly, and let $\widehat{\gamma}^n:=(\widehat{\alpha}^n,\hat{\beta}^n)$, $\widetilde{\gamma}^n:=(\widetilde{\alpha}^n,\tilde{\beta}^n)$, $\bar{\gamma}^n:=(\bar{\alpha}^n,\bar{\beta}^n)$. We will often omit the superscript n when $n=1$.

Define the event $\mathcal{E}_n:=\{\bar{\gamma}^n\in\Gamma\}$. Note that almost surely this event is satisfied as long as $\widetilde{\gamma}^n$ does not disconnect 0 from ∞ (since $0\notin\widehat{\gamma}^n$ almost surely).

For every $\gamma\in\Gamma$, consider the h-process B started at 0 and conditioned to first leave $O=O(\gamma)$ in $\partial\mathbf{U}\cap\partial O$. Let us say a few words about how this process is defined: it can be viewed as the limit as $z\in O$ tends to 0 of Brownian motion starting from z conditioned to leave O in $\partial\mathbf{U}\cap\partial O$. It is well-defined since 0 is a simple boundary point of O for $\gamma\in\Gamma$. For instance, if we map conformally the strip $(0,1)\times\mathbf{R}$ onto $O(\gamma)$ taking $\{1\}\times\mathbf{R}$ onto $\partial\mathbf{U}\cap\partial O$ and the origin to the origin, then the h-process in $O(\gamma)$ is obtained (after time change, but we will actually only use the paths of the h-processes) as the image under the conformal map of the process $X+iY$ in the strip, where X is a three-dimensional Bessel process started from 0 and stopped when it hits 1 (i.e., the limit when $\varepsilon\to0$ of one-dimensional Brownian motion started from $\varepsilon>0$ conditioned to hit 1 before 0), and Y is an independent one-dimensional Brownian motion started from 0 (stopped at the same time).

Attach to the endpoint of B on $\partial\mathbf{U}$ an independent Brownian motion $\widehat{B}$, and define the paths $\widehat{B}^n$ and $\widetilde{B}^n$ as before. The path $\widetilde{B}^n$ consists of two parts: the h-process (up to its hitting time of $\partial\mathbf{U}$), and the (non-conditioned) Brownian motion $\widehat{B}^n$. Let

$$Z_n=Z_n(\widetilde{\gamma}^n):=\mathbf{P}[\widetilde{B}^n\cap\widetilde{\gamma}^n=\varnothing\mid\widetilde{\gamma}^n]$$

and $Z=Z_1$. Note that although $B\cap\gamma=\varnothing$ almost surely, it can happen that $B\cap\widehat{\gamma}^n\neq\varnothing$ with positive probability. That is, $\widetilde{B}^n\cap\widetilde{\gamma}^n=\varnothing$ can fail in two ways: it may happen that $\widehat{B}^n\cap\widetilde{\gamma}^n\neq\varnothing$, and it may also happen that $B\cap\widehat{\gamma}^n\neq\varnothing$. Note also that $Z_n\neq 0$ if and only if $\mathcal{E}_n$ occurs. We define $\psi_n=-\log Z_n$ and $\psi=-\log Z$ (with $-\log 0=\infty$).

For $n,\lambda>0$ define the linear operator $T_\lambda\colon\mathcal{A}\to\mathcal{A}$ by

$$T_\lambda^n f(\gamma):=\mathbf{E}[f(\bar{\gamma}^n)\exp(-\lambda\psi_n)]=\mathbf{E}[f(\bar{\gamma}^n)Z_n^\lambda],$$

and let $T_\lambda=T_\lambda^1$. The expectation is over the randomness in $\widetilde{\gamma}^n$. Note that $T_\lambda^{n+m}=T_\lambda^n T_\lambda^m$, so the notation is appropriate. Also, there is no need to restrict to real λ; this defines T_z^n for complex z with $\operatorname{Re}(z)>0$. We will prove the following:

PROPOSITION 2.1. (i) *For all real $\lambda>0$, there exist an $\varepsilon>0$ and a $v=v(\lambda)>0$ such that for all $u\in(0,v)$, $z\mapsto T_z$ is an analytic function from the disk $\{z:|z-\lambda|<\varepsilon\}$ into $\mathcal{L}_u$.*

(ii) *For all real $\lambda>0$, there exist an $\varepsilon'>0$ and a $u\in(0,v(\lambda))$ such that the spectrum of T_λ in $\mathcal{L}_u$ is the union of the simple eigenvalue $e^{-\xi(2,\lambda)}$ and a subset of the disk $(1-\varepsilon')e^{-\xi(2,\lambda)}\mathbf{U}$.*

Proof of Theorem 1.1 (assuming Proposition 2.1). The proposition implies that $e^{-\xi(2,\lambda)}$ is an isolated simple eigenvalue of T_λ, for all $\lambda>0$. By Theorem VII.6.9 in [4], it follows that for all $\lambda>0$, there exists $\varepsilon>0$ such that $x\mapsto\xi(2,x)$ can be extended analytically to the disk $\{z:|z-\lambda|<\varepsilon\}$. Hence, there exists a neighborhood $\mathcal{N}$ of the half-line $(0,\infty)$ such that $z\mapsto\xi(2,z)$ is analytic in $\mathcal{N}$, proving the theorem in the case $k=2$. The proof for other k is the same. □

The proofs of Proposition 2.1 (i) and (ii) both rely on coupling arguments. The proof of (i) in §4 uses a coupling of the h-processes B and B' associated to two pairs of paths γ and γ' (when $(\gamma,\gamma')\in\mathcal{X}_m$). In the proof of (ii) (§5), we couple the extensions $\widehat{\gamma}^n$ and $\widehat{\gamma}'^n$ associated to γ and γ' defined under a weighted probability measure.

3. Previous results on $\xi(2,\lambda)$

In this section, we very briefly review some important facts about the intersection exponent $\xi(2,\lambda)$ that will be useful. The results here were derived in [9] and apply to dimensions 2 and 3. Since some of the arguments are simpler when one considers only the plane, we rewrote detailed self-contained proofs of all these facts ((3.2), (3.3) and (3.5)) in [15], which can be viewed as a preparation paper for the present paper.

3.1. Estimates up to constants

Let $x, y \in \partial \mathbf{U}$, and let $W=(X,Y)$ denote a pair of planar Brownian motions started at x, y, respectively. Let B denote another independent planar Brownian motion started from $b \in \partial \mathbf{U}$. Let B^n be the path B until it reaches the circle $e^n \partial \mathbf{U}$ for the first time, and similarly define X^n and Y^n. Let $W^n = X^n \cup Y^n$,

$$\widehat{Z}_n = \widehat{Z}_n(W^n) := \sup_{b \in \partial \mathbf{U}} \mathbf{P}[B^n \cap W^n = \varnothing \mid W^n]$$

and

$$q_n = q_n(\lambda) := \sup_{x,y \in \partial \mathbf{U}} \mathbf{E}[\widehat{Z}_n^\lambda].$$

Then $q_{n+m} \leqslant q_n q_m$ holds, by the strong Markov property for X, Y and B. Hence, the limit $\xi := -\lim_n \log(q_n)/n = -\inf_n \log(q_n)/n$ exists by subadditivity and

$$q_n \geqslant e^{-\xi n}. \tag{3.1}$$

It is not difficult to verify that, in fact, $\xi = \xi(2, \lambda)$, as defined in the introduction.

An opposite estimate also holds; that is, there exists a constant $c_1 = c_1(\lambda) > 0$ such that

$$q_n \leqslant c_1 e^{-\xi n}. \tag{3.2}$$

This result is a variant of Theorem 2.1 in [9]. The fact that $q_n \asymp e^{-\xi n}$ (which is a more precise statement than the definition of ξ, $q_n \approx e^{-\xi n}$) has been instrumental in showing that $2-\xi$ correspond to Hausdorff dimensions of various subsets of the planar Brownian curve [8].

3.2. Separation and the functions R_n

Let Γ^+ be the set of $\gamma = (\alpha, \beta) \in \Gamma$ such that

$$(\alpha_{1/2}, \beta_{1/2}) \subset \left\{ e^{\varrho + i\theta} : \varrho \in [-1, 0], \theta \in \left(\tfrac{1}{2}\pi, \tfrac{3}{2}\pi\right) \right\}$$

and

$$\left\{ e^{\varrho + i\theta} : \varrho \in \left(-\tfrac{1}{2}, 0\right), \theta \in \left(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right) \right\} \subset O(\gamma).$$

The Separation Lemma [9, Lemma 4.2] states that there exists a c_2 such that for all $\gamma \in \Gamma$,

$$\mathbf{E}[\exp(-\lambda\psi) 1_{\{\bar{\gamma} \in \Gamma^+\}}] \geqslant c_2 \mathbf{E}[\exp(-\lambda\psi)]. \tag{3.3}$$

It is a kind of boundary Harnack principle for the operator T. This type of result was important in the derivation of (3.2).

Define for all $n\geqslant 1$, $\lambda>0$ and $\gamma\in\Gamma$,

$$R_n(\gamma)=R_{n,\lambda}(\gamma):=e^{n\xi}\mathbf{E}[e^{-\lambda\psi_n}]=e^{n\xi}(T_\lambda^n 1)(\gamma).$$

By (3.2), R_n is uniformly bounded in n and γ. Note also that, by the strong Markov property and (3.2),

$$R_n(\gamma)=e^{n\xi}(T_\lambda^n 1)(\gamma)\leqslant e^{-n\xi}q_{n-1}T_\lambda 1(\gamma)\leqslant cR_1(\gamma). \tag{3.4}$$

On the other hand, one can prove that

$$\tilde{c}:=\inf_{n\geqslant 0}\inf_{\gamma\in\Gamma^+}R_n(\gamma)>0. \tag{3.5}$$

This is, loosely speaking, due to the fact that a positive fraction of the extensions $\widehat{B}^n$ and $\widehat{\gamma}^n$ leave quickly the neighborhood of $\mathbf{U}$, so that they do not really feel the influence of γ and B. As $\gamma\in\Gamma^+$, the starting points of γ^n and a positive fraction of the starting points of $\widehat{B}^n$ are not so far from being optimal, so that q_n is within a constant multiple of $\mathbf{E}[e^{-\lambda\psi_n}]$.

Using the strong Markov property and (3.3), then applying (3.5) to $R_{n-1}(\bar{\gamma})$, gives for all $\gamma\in\Gamma$ and $n\geqslant 1$,

$$R_n(\gamma)\geqslant e^{n\xi}\mathbf{E}[e^{-\lambda\psi_n}1_{\{\bar{\gamma}\in\Gamma^+\}}]\geqslant e^{n\xi}\tilde{c}e^{-(n-1)\xi}\mathbf{E}[e^{-\lambda\psi}1_{\{\bar{\gamma}\in\Gamma^+\}}]\geqslant\tilde{c}c_2R_1(\gamma).$$

Combining this with (3.4) shows that there is a c_0 such for all $n,n'\geqslant 1$ and $\gamma\in\Gamma$,

$$R_n(\gamma)\leqslant c_0R_{n'}(\gamma). \tag{3.6}$$

In [9] it was shown that the limit $R(\gamma)=\lim_{n\to\infty}R_n(\gamma)$ exists. We will rederive this result in this paper and simultaneously improve the rate of convergence to the limit.

4. Analyticity

The goal of this section is to prove Proposition 2.1 (i).

4.1. Coupling B and B'

Let $\gamma,\gamma'\in\Gamma$. Let B be the h-process associated with γ, and let B' be the h-process associated with γ'. In this section, we show that there is fast decay for the dependence of B on γ. More precisely, we prove the following proposition.

PROPOSITION 4.1. *There exist a $c>0$ and a $v_0>0$ such that for all $m\geqslant 1$, if $(\gamma,\gamma')\in\mathcal{X}_m$, then one can couple B and B' in such a way that*

$$\mathbf{P}[B\setminus e^{-m/2}\mathbf{U}\neq B'\setminus e^{-m/2}\mathbf{U}]\leqslant ce^{-v_0 m}$$

(here B and B' denote the curves of the processes B and B').

By coupling B and B', we mean that it is possible to define B and B' on the same probability space in such a way that the law of B (resp. B') is that of the h-process associated with γ (resp. γ'). A reference for facts about the coupling method is [18].

This result actually holds for all $(\gamma,\gamma')\in\Gamma$ with

$$\gamma_m=\gamma'_m \quad \text{and} \quad O(\gamma)\setminus e^{-m}\mathbf{U}=O(\gamma')\setminus e^{-m}\mathbf{U},$$

but the proof is more complicated. Proposition 4.1 will be sufficient for our purposes.

It is easy to verify that the processes B and B' satisfy the strong Markov property. This will be very useful in the following.

In preparation for the proof of Proposition 4.1, we first focus on conditioned Brownian motions in the half-infinite rectangle

$$J=\{x+iy:0<x<\pi,\,0<y<\infty\}.$$

There is a conformal transformation taking $O(\gamma)$ to J which takes the origin to infinity and $O(\gamma)\cap\partial\mathbf{U}$ to $[0,\pi]$, so that conditioned Brownian motions in J can be conformally transformed (up to a time change) to conditioned Brownian motions in $O(\gamma)$. Hence results for conditioned Brownian motions in J imply results for conditioned Brownian motions in $O(\gamma)$.

LEMMA 4.2. (i) *There exists a constant c such that for all z in J such that* $\mathrm{Im}(z)\geqslant 1$ *and for all $y_0\geqslant 1$, if $\widetilde{B}$ denotes a Brownian motion started from z, and conditioned to leave J on the set $[0,\pi]$, then*

$$\mathbf{P}[\widetilde{B}\text{ hits }\{w:\mathrm{Im}(w)\geqslant\mathrm{Im}(z)+y_0\}\text{ before it leaves }J]\leqslant ce^{-2y_0}. \tag{4.1}$$

(ii) *There exists a constant c such that for every $\widetilde{m}\geqslant 0$, every $\tilde{n}\geqslant 1$ and every $z,z'\in J$ with $\widetilde{m}+\tilde{n}\leqslant\mathrm{Im}(z)\leqslant\mathrm{Im}(z')$, one can find a coupling of $\widetilde{B}$ and $\widetilde{B}'$, Brownian motions conditioned to leave J at $[0,\pi]$, starting at z and z' respectively, such that*

$$\mathbf{P}[\widetilde{B}\cap\{w:\mathrm{Im}(w)\leqslant\widetilde{m}\}\neq\widetilde{B}'\cap\{w:\mathrm{Im}(w)\leqslant\widetilde{m}\}]\leqslant ce^{-\tilde{n}}. \tag{4.2}$$

Before proving this lemma, we first recall the following straightforward fact on coupling (see, e.g., [18]) that we will use in this proof and that will also be instrumental later on in this paper: Suppose that ν is a probability measure on a measurable space $(S,\mathcal{S})$, and that ν_1 and ν_2 are two other probability measures on this space such that the Radon–Nikodym derivative $q(\cdot)$ (resp. $q'(\cdot)$) of ν_1 (resp. ν_2) with respect to ν exists. Then, one can find a probability space $(\Omega,\mathcal{F},\mu)$ and measurable functions $\delta:\Omega\to S$ and $\delta':\Omega\to S$ such that the law of δ is ν_1, the law of δ' is ν_2, and

$$\mu[\delta=\delta'] \geqslant \int_S \min(q,q')\,d\nu = 1-\frac{1}{2}\int |q-q'|\,d\nu = 1-\frac{1}{2}\|\nu_1-\nu_2\|. \tag{4.3}$$

Proof of Lemma 4.2. Let $\tilde{\tau}$ denote the first time a Brownian motion $\widetilde{X}$ (started from $\widetilde{X}=z$ under the probability measure $\mathbf{P}^z$) leaves J, and for all $y_0>0$, let $\tilde{\sigma}_{y_0}$ denote the first time at which $\operatorname{Im}(\widetilde{X})$ hits y_0.

Let $h_z(s)$, $z\in J$, $s\in(0,\pi)$, be the density of $\widetilde{X}_{\tilde{\tau}} 1_{\{\tilde{\tau}=\tilde{\sigma}_0\}}$, where $\widetilde{X}_0=z=x+iy$. It is easy to show that

$$h_z(s) = \frac{2}{\pi}\sum_{n\geqslant 1} \sin(nx)\sin(ns)e^{-n^2 y},$$

for instance, by separation of variables. In particular, this readily implies that for all $y\geqslant 1$,

$$h_z(s) = \frac{2}{\pi}\sin(x)\sin(s)e^{-y}(1+O(e^{-y})) \tag{4.4}$$

and

$$u(z) := \mathbf{P}^z[\tilde{\sigma}_0=\tilde{\tau}] = \frac{4}{\pi}\sin(x)e^{-y}(1+O(e^{-y})), \tag{4.5}$$

where the constants implicit in the notation $O(e^{-y})$ do not depend on x, s or y. By reflecting z on the line $\operatorname{Im}(w)=y+y_0$, it follows that

$$\mathbf{P}^z[\tilde{\sigma}_{y+y_0}<\tilde{\tau}=\tilde{\sigma}_0] \leqslant u(z+2iy_0) = (1+O(e^{-y}))\,e^{-2y_0}u(z),$$

and (4.1) follows.

We now use conditioned Brownian motions $\widetilde{B}$ and $\widetilde{B}'$ as in the lemma that are respectively started from z and z'. Let $\tilde{z}=i\tilde{m}+\tilde{s}$ be the first point on the segment $i\tilde{m}+[0,\pi]$ which $\widetilde{B}$ hits, and let $\tilde{z}'=i\tilde{m}+\tilde{s}'$ be the corresponding point of $\widetilde{B}'$. A straightforward consequence of the strong Markov property shows that the density $\tilde{h}_z$ of $\tilde{s}$ is

$$\tilde{h}_z(s) := \frac{h_{z-i\tilde{m}}(s)\,u(s+i\tilde{m})}{u(z)},$$

and a similar expression holds for the density $\tilde{h}_{z'}$ of $\tilde{s}'$. Using (4.4), this gives

$$\frac{\tilde{h}_z(s)}{\tilde{h}_{z'}(s)} = \frac{h_{z-i\tilde{m}}(s)\,u(z')}{h_{z'-i\tilde{m}}(s)\,u(z)} = 1+O(e^{-\tilde{n}}).$$

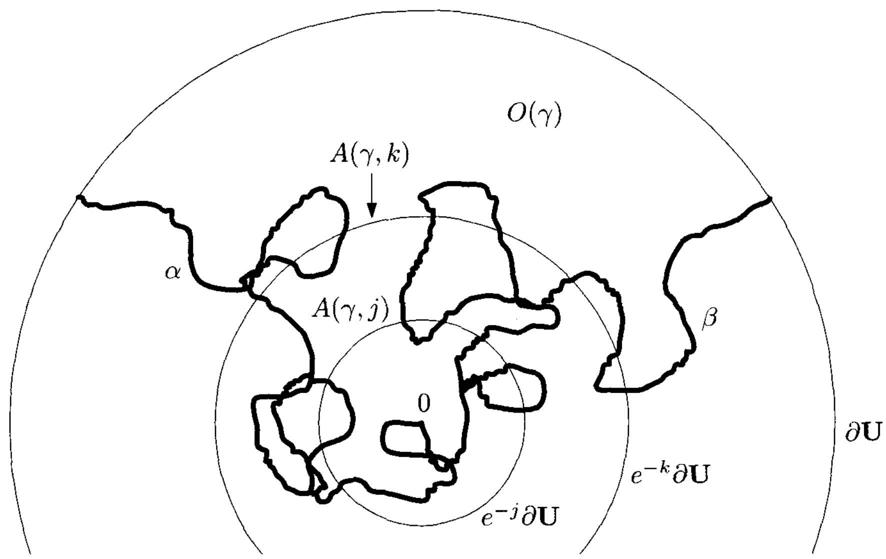

Fig. 4.1. The separating arcs.

Therefore, using (4.3), we may couple the parts of $\widetilde{B}$ and $\widetilde{B}'$ until their first hit of $i\tilde{m}+[0,\pi]$ so that $\mathbf{P}[\tilde{z}\neq\tilde{z}']\leqslant O(e^{-\tilde{n}})$. On the event $\tilde{z}=\tilde{z}'$, we may continue $\widetilde{B}$ and $\widetilde{B}'$ as the same path. This gives (4.2). □

Proof of Proposition 4.1. Prior to the core of the proof, there is a need for some preliminary topological preparations. Suppose that $\gamma\in\Gamma$ and $k>0$. Then, there exists at least one open subarc $A=A(\gamma,k)$ of $e^{-k}\partial\mathbf{U}\cap O(\gamma)$ with the property that the removal of A from $O(\gamma)$ disconnects 0 from $\partial\mathbf{U}$ in $O(\gamma)$. See Figure 4.1. There may be many such arcs; if so, choose $A=A(\gamma,k)$ as being the one closest to the origin, in the sense that every path from the origin to $\partial\mathbf{U}$ in $O(\gamma)$ goes through A before any other such disconnecting arc. The arc A divides $O(\gamma)$ into two components; let $O_+(\gamma,k)$ denote the component whose boundary intersects $\partial\mathbf{U}$, and let $O_-(\gamma,k)$ denote the component whose boundary contains 0.

As in §2, let $\gamma_n=(\alpha_n,\beta_n)$ be the part of γ starting from the first visits to the circle of radius e^{-n}. Note that γ_k contains paths connecting the endpoints of $A(\gamma,k)$ to $\partial\mathbf{U}$. Suppose for a moment that $j>k$ and γ has no downcrossing from e^{-k} to e^{-j}. Let $\widetilde{O}_k(\gamma)$ denote the connected component of $\mathbf{U}\setminus(\gamma_k\cup A(\gamma,k))$ that contains $O_+(\gamma,k)$. Since γ has no downcrossing from e^{-k} to e^{-j}, we have $\partial\widetilde{O}_k(\gamma)\cap e^{-j}\mathbf{U}=\varnothing$, and therefore $\widetilde{O}_k(\gamma)\cap e^{-j}\mathbf{U}=\varnothing$. In particular,

$$O_+(\gamma,k)\cap e^{-j}\mathbf{U}=\varnothing.$$

An entirely similar argument shows that

$$O_-(\gamma, j) \subset e^{-k}\mathbf{U}$$

if γ has no downcrossing from e^{-k} to e^{-j}. In this case, one has to consider the connected component that contains $O_-(\gamma, j)$ of the complement of the union of $A(\gamma, j)$ with the parts of γ until the last visit to the circle $e^{-j}\partial\mathbf{U}$.

Now take $(\gamma, \gamma') \in \mathcal{X}_m$, as in the proposition. Let $k \in [0, \frac{11}{12}m]$; so that γ and γ' have no downcrossing from e^{-k} to $e^{-k-m/12}$. Observe that $\widetilde{O}(\gamma, k) = \widetilde{O}(\gamma', k)$, since both can be characterized as the largest domain which does not contain 0, has $\partial O(\gamma) \cap \partial\mathbf{U} = \partial O(\gamma') \cap \partial\mathbf{U}$ on its boundary, and is bounded by $\gamma_k = \gamma'_k$ together with an arc of $e^{-k}\partial\mathbf{U}$. This gives $A(\gamma, e^{-k}) = A(\gamma', e^{-k})$. Set

$$\begin{aligned}
A_1 &:= A(\gamma, e^{-11m/12}), \\
A_2 &:= A(\gamma, e^{-10m/12}), \\
A_3 &:= A(\gamma, e^{-9m/12}), \\
A_\infty &:= \partial O(\gamma) \cap \partial\mathbf{U}, \\
O_+ &:= O_+(\gamma, e^{-11m/12}), \\
O_- &:= O_-(\gamma, e^{-9m/12}).
\end{aligned}$$

Except for O_-, these are the same as the corresponding objects for γ'. The extremal distance from A_1 to A_2 in $O(\gamma)$ is bounded from below by the extremal distance in $\mathbf{C}$ from $e^{-11m/12}\partial\mathbf{U}$ to $e^{-10m/12}\partial\mathbf{U}$, which is $m/(12\cdot 2\cdot \pi)$. Let ϕ be the conformal map from $O(\gamma)$ onto J which takes 0 to ∞ and takes A_∞ onto $[0, \pi]$. Note that $\phi(A_j)$ is a path joining the two lines $\{\mathrm{Re}(w) = 0\}$ and $\{\mathrm{Re}(w) = \pi\}$. By conformal invariance of extremal length, it follows that there is a constant $v > 0$ such that

$$\forall z_1 \in \phi(A_1), \forall z_2 \in \phi(A_2), \quad \mathrm{Im}(z_1) - \mathrm{Im}(z_2) \geqslant vm - 1/v.$$

It follows immediately from Lemma 4.2 (i), the strong Markov property and conformal invariance that the probability that B hits A_1 after its first hit to A_2 is bounded by ce^{-vm}. The same holds for B'. Note that if these processes do not hit A_1 again, then they stay in O_+.

To construct the coupling of B and B', let them evolve independently until they first hit A_2 (at some random points z and z', respectively). The law of B (resp. B') after that time is that of Brownian motion in $O(\gamma)$ (resp. $O(\gamma')$) conditioned to exit $O(\gamma)$ (resp. $O(\gamma')$) in A_∞. Each of these laws are at distance at most ce^{-vm} from the law of Brownian motion starting from z and z', respectively, in O_+ conditioned to exit O_+

in A_∞, where distance is in the sense of the measure norm; that is, L_1-distance. Hence, it suffices to show that for all $z, z' \in A_2$, it is possible to couple two Brownian motions started from z and z', respectively, conditioned to exit $O(\gamma)$ in A_∞, in such a way that they agree after their first hitting of A_3 with probability at least $1-ce^{-v'm}$. (Note that before their first hitting of A_3, they are in $O_- \subset e^{-8m/12}\mathbf{U}$.) By conformal invariance, one can use the coupling described in Lemma 4.2 with

$$\tilde{m} := \sup \operatorname{Im}(\phi(A_3)), \quad \tilde{n} := \inf \operatorname{Im}(\phi(A_2)) - \tilde{m}.$$

By the conformal invariance of extremal length, as explained above, $\tilde{n}/m$ is bounded away from 0, when m is large. Small values of m can be handled by adjusting the constant c in the statement of the proposition. This completes the proof of Proposition 4.1. □

4.2. Exponential decay

If $\gamma \in \Gamma$ and $\gamma' \in \Gamma$ have the same terminal point (which will be the case, in particular, if $\gamma_m = \gamma'_m$ for some $m>0$), then we will attach the same Brownian motions $(\hat{\alpha}, \hat{\beta}) = \hat{\gamma} = \hat{\gamma}'$ to γ and γ' in defining $\tilde{\gamma}^n$ and $\tilde{\gamma}'^n$. In this case, write ψ_n as shorthand for $\psi_n(\tilde{\gamma}^n)$ and ψ'_n as a shorthand for $\psi_n(\tilde{\gamma}'^n)$. Similarly, $Z'_n = Z_n(\tilde{\gamma}'^n)$.

PROPOSITION 4.3. *For every $\lambda>0$, there exist $c>0$ and $v_1>0$ such that if $n \geqslant 0$, $m \geqslant 1$ and $(\gamma, \gamma') \in \mathcal{X}_m$, then*

$$\mathbf{E}[|e^{-\lambda\psi_n} - e^{-\lambda\psi'_n}|] \leqslant ce^{-\xi n}e^{-v_1 m}.$$

The proof of this proposition relies heavily on Proposition 4.1. An easy estimate on the disconnection exponent will also be needed. Let X denote a planar Brownian motion started from the unit circle. As before, for $\varrho>0$, let X^ϱ denote the part of X until its first hit of the circle $e^\varrho \partial\mathbf{U}$. Let $v'_0>0$ be such that the probability that X^1 disconnects the origin from infinity is $1-e^{-v'_0}$. Let $c'_0 := e^{v'_0}$. The strong Markov property immediately implies that for all $\varrho>0$,

$$\mathbf{P}[X^\varrho \text{ does not disconnect the origin from infinity}] \leqslant e^{v'_0(1-\varrho)} = c'_0 e^{-v'_0 \varrho}. \tag{4.6}$$

Proof of Proposition 4.3. Suppose that $m \geqslant 1$ and that $(\gamma, \gamma') \in \mathcal{X}_m$. Couple the h-processes B and B' as in Proposition 4.1. When the coupling succeeds, i.e., when the event $\mathcal{H} := \{B \setminus e^{-m/2}\mathbf{U} = B' \setminus e^{-m/2}\mathbf{U}\}$ holds, then we attach to B and B' the same Brownian path $\hat{B}$.

Step 1. Define the events

$$\mathcal{U}(\hat{\alpha}) := \{\hat{\alpha}^n \cap e^{-m/2}\mathbf{U} = \varnothing\}, \quad \mathcal{U}(\hat{\beta}) := \{\hat{\beta}^n \cap e^{-m/2}\mathbf{U} = \varnothing\}$$

and $\mathcal{U}=\mathcal{U}(\hat{\gamma}):=\mathcal{U}(\hat{\alpha})\cap\mathcal{U}(\hat{\beta})$. When $\mathcal{U}(\hat{\alpha})$ is not satisfied, then $\hat{\alpha}$ hits the circle $e^{-m/2}\partial\mathbf{U}$ before the circle $e^n\partial\mathbf{U}$. Let σ be the first time at which $\hat{\alpha}(t)\in e^{-m/2}\partial\mathbf{U}$, and let τ be the first time $t>\sigma$ such that $\hat{\alpha}(t)\in\partial\mathbf{U}$. Let η^n be the path $\hat{\alpha}$ after τ and until its first hit of the circle $e^n\partial\mathbf{U}$ after time τ. Conditioned on $\neg\mathcal{U}(\hat{\alpha})$ (i.e., the complement of $\mathcal{U}(\hat{\alpha})$), the probability that $\hat{\alpha}[\sigma,\tau]$ does not disconnect 0 from ∞ is bounded by $c_0'e^{-v_0'm/2}$ by (4.6). Hence, we get

$$1_{\neg\mathcal{U}(\hat{\alpha})}Z_n \leqslant c_0'e^{-v_0'm/2}\widehat{Z}_n(\eta^n,\hat{\beta}^n).$$

The same applies to $1_{\neg\mathcal{U}(\hat{\beta})}Z_n$. Therefore, by (3.2),

$$\mathbf{E}[(Z_n^\lambda+Z_n'^\lambda)\,1_{\neg\mathcal{U}}]\leqslant ce^{-v_0'm\lambda/2}e^{-n\xi}.$$

It remains to study

$$\mathbf{E}[\,|Z_n^\lambda-Z_n'^\lambda|\,1_{\mathcal{U}}].$$

Step 2. We now show that

$$1_{\mathcal{U}}|Z_n-Z_n'|\leqslant ce^{-vm}\widehat{Z}_n(\hat{\gamma}^n), \tag{4.7}$$

assuming $v\leqslant\min\{v_0,v_0'\}$. When $\mathcal{U}$ is satisfied, then the contribution to $|Z_n-Z_n'|$ comes from two possible events: the coupling between B and B' does not succeed (this occurs with probability at most ce^{-mv_0}, independently from $\hat{\gamma}$, $\widehat{B}$ and $\widehat{B}'$), or the coupling succeeds, but $\widehat{B}^n$ visits $e^{-m}\mathbf{U}$ and feels the difference between γ and γ'. In the latter case, the Brownian motion $\widehat{B}$ has a probability at most $c_0'e^{-v_0'm}$ not to disconnect the origin from infinity after the first visit to $e^{-m}\partial\mathbf{U}$ and before the next visit to $\partial\mathbf{U}$, and after that, it is again an ordinary Brownian motion up to its hitting time of the circle $e^n\partial\mathbf{U}$. From this, (4.7) follows.

Step 3. Suppose first that $\lambda\geqslant1$. Then, since $\max\{Z_n,Z_n'\}\leqslant\widehat{Z}_n(\hat{\gamma}^n)$,

$$\begin{aligned}\mathbf{E}[1_{\mathcal{U}}|Z_n^\lambda-Z_n'^\lambda|] &\leqslant \lambda\mathbf{E}[1_{\mathcal{U}}|Z_n-Z_n'|\,\widehat{Z}_n^{\lambda-1}]\\ &\leqslant c\lambda e^{-vm}\mathbf{E}[\widehat{Z}_n^\lambda] \quad \text{(by (4.7))}\\ &\leqslant c'\lambda e^{-vm}e^{-n\xi} \quad \text{(by (3.2))}.\end{aligned}$$

When $\lambda\leqslant1$,

$$\begin{aligned}\mathbf{E}[1_{\mathcal{U}}|Z_n^\lambda-Z_n'^\lambda|] &\leqslant \mathbf{E}[1_{\mathcal{U}}|Z_n-Z_n'|^\lambda]\\ &\leqslant c^\lambda\mathbf{E}[e^{-\lambda vm}\,\widehat{Z}_n^\lambda]\\ &\leqslant c'e^{-\lambda vm}e^{-n\xi}.\end{aligned}$$

This concludes the proof of Proposition 4.3 with $v_1=\min\{\frac{1}{2}v_0'\lambda, v_0', v_0, v_0\lambda\}$. □

4.3. Proof of Proposition 2.1 (i)

Fix $\lambda>0$. Define for all integers $k\geqslant 0$, and all $f\in\mathcal{A}$, $\gamma\in\Gamma$,

$$U_k f(\gamma)=\mathbf{E}\left[f(\bar{\gamma})\frac{\psi^k}{k!}e^{-\lambda\psi}\right].$$

Taking $y=\lambda\psi$ in the inequality $|y|^k/k!\leqslant e^{|y|}$ gives for all $f\in\mathcal{A}$ and $\gamma\in\Gamma$,

$$|U_k f(\gamma)|\leqslant \mathbf{E}[f(\bar{\gamma})\lambda^{-k}]\leqslant \|f\|\lambda^{-k}. \tag{4.8}$$

Hence, by dominated convergence, for all $z\in\mathbf{C}$ such that $|z|<\lambda$, for all $f\in\mathcal{A}$ and $\gamma\in\Gamma$,

$$T_{\lambda-z}f(\gamma)=\sum_{k=0}^{\infty} z^k U_k f(\gamma).$$

We will show that there are constants $a,c,v>0$ such that $N_u(U_k)\leqslant ca^k$ for all $u\in(0,v)$ and all $k\in\mathbf{N}$. From this, Proposition 2.1 (i) immediately follows.

To find an upper bound for $N_u(U_k)$, first note that $\|U_k\|\leqslant\lambda^{-k}$ by (4.8), and consequently, for all $m<24$, for all $(\gamma,\gamma')\in\mathcal{X}_m$,

$$|U_k f(\gamma)-U_k f(\gamma')|\leqslant 2\|f\|\lambda^{-k}.$$

Suppose that $m\geqslant 24$ and $(\gamma,\gamma')\in\mathcal{X}_m$. As in Proposition 4.3, since γ and γ' have the same endpoints, we can choose $\hat{\gamma}=\hat{\gamma}'$. Define the event $\mathcal{U}'$ that neither $\hat{\alpha}$ nor $\hat{\beta}$ hit the circle $e^{-m/24}\partial\mathbf{U}$ before $e\partial\mathbf{U}$. Note that when $\mathcal{U}'$ is satisfied and $\bar{\gamma}\in\Gamma$, then $(\bar{\gamma},\bar{\gamma}')\in\mathcal{X}_m$ (this is where the assumption $m\geqslant 24$ is needed), so that

$$1_{\mathcal{U}'}|f(\bar{\gamma})-f(\bar{\gamma}')|\leqslant \|f\|_u e^{-um}.$$

An elementary computation shows that if $k\in\mathbf{N}$ and $x\in(0,1)$,

$$\frac{1}{k!}\left|\frac{d}{dx}(x^2(\log x)^k)\right|\leqslant 3, \tag{4.9}$$

so that

$$\frac{1}{k!}\mathbf{E}\left[\left|\left(\tfrac{1}{2}\lambda\psi\right)^k e^{-\lambda\psi}-\left(\tfrac{1}{2}\lambda\psi'\right)^k e^{-\lambda\psi'}\right|\right]\leqslant 3\mathbf{E}[|e^{-\lambda\psi/2}-e^{-\lambda\psi'/2}|].$$

Hence, for all $f\in\mathcal{A}_u$ and for all $(\gamma,\gamma')\in\mathcal{X}_m$ (with $m\geqslant 24$),

$$\begin{aligned}
&|U_k f(\gamma)-U_k f(\gamma')|\\
&\quad\leqslant \mathbf{E}\left[(1_{\mathcal{U}'}+1_{\neg\mathcal{U}'})|f(\bar{\gamma})-f(\bar{\gamma}')|\frac{\psi'^k e^{-\lambda\psi'}}{k!}\right]+\|f\|\frac{1}{k!}\mathbf{E}[|\psi^k e^{-\lambda\psi}-\psi'^k e^{-\lambda\psi'}|]\\
&\quad\leqslant \|f\|_u e^{-um}\lambda^{-k}+2\|f\|\lambda^{-k}\mathbf{E}[1_{\neg\mathcal{U}'}1_{\bar{\gamma}'\in\Gamma}]+\|f\|\cdot 3(2/\lambda)^k\mathbf{E}[|e^{-\lambda\psi/2}-e^{-\lambda\psi'/2}|]\\
&\quad\leqslant c\|f\|_u(2/\lambda)^k e^{-um}
\end{aligned}$$

for all $u\leqslant\min\{v_1(\tfrac{1}{2}\lambda),\tfrac{1}{24}v_0'\}$, by Proposition 4.3 and (4.6).

Finally, combining the above estimates shows that for all u smaller than both $v_1(\tfrac{1}{2}\lambda)$ and $\tfrac{1}{24}v_0'$, there exists $c>0$ such that for all $k\geqslant 0$, $N_u(U_k)\leqslant c(2/\lambda)^k$. This completes the proof of Proposition 2.1 (i). □

5. Spectral gap

We now study the spectrum of T_λ for fixed $\lambda>0$. The proof of the existence of a spectral gap will be based on a coupling argument.

5.1. Coupling the weighted paths

Let $\varrho>0$. For $\hat{\gamma}^\varrho$-measurable events $\mathcal{A}$ define the weighted probability measures

$$\widetilde{\mathbf{P}}^\varrho_\gamma[\mathcal{A}] := \frac{\mathbf{E}[1_\mathcal{A} Z^\lambda_\varrho]}{\mathbf{E}[Z^\lambda_\varrho]}. \tag{5.1}$$

For $0<k\leqslant n$, let $\delta(n,k,\gamma)$ denote a random variable with the same law that $\bar{\gamma}^k$ has under $\widetilde{\mathbf{P}}^n_\gamma$. It is easy to verify that

$$\delta(n-j,k,\delta(n,j,\gamma)) \text{ has the same law as } \delta(n,k+j,\gamma), \tag{5.2}$$

when $j+k\leqslant n$.

The following coupling result will be crucial in our proof:

PROPOSITION 5.1. *There exist constants $v_2, c>0$ such that for all $n\geqslant 1$, for all $\gamma,\gamma'\in\Gamma$, one can define $\delta=\delta(n,n,\gamma)$ and $\delta'=\delta(n,n,\gamma')$ on the same probability space $(\Omega,\mathcal{F},\mu)$ such that*

$$\mu[(\delta,\delta')\notin\mathcal{X}_{n/3}]\leqslant ce^{-v_2 n}.$$

This subsection will be devoted to the proof of this proposition. The rough strategy of the proof is as follows. The first step is to get both paths to be in Γ^+ (recall the definition of Γ^+ from §3.2). The second step is to make the two paths match up and walk together a little while. The third step is to show that if the two paths have walked together for some time, then it is unlikely that they will decouple. If any of these steps fails, the coupling process returns to the beginning. (For technical reasons, the order in which we address these steps is different from the logical order indicated above.)

Suppose that $u>0$ is fixed (and small enough), and choose C such that for all $\gamma\in\Gamma^+$, $R_1(\gamma)\geqslant C$. Define for all $k\geqslant 1$,

$$\mathcal{K}_k=\{(\gamma,\gamma')\in\Gamma^2 : R_1(\gamma)\geqslant Ce^{-ku},\ R_1(\gamma')\geqslant Ce^{-ku} \text{ and } (\gamma,\gamma')\in\mathcal{X}_k\}.$$

LEMMA 5.2. *There exist $c,w>0$ such that for all $n\geqslant 1$, for all $k\geqslant 24$, and for all $(\gamma,\gamma')\in\mathcal{K}_k$, one can define $\delta=\delta(n,1,\gamma)$ and $\delta'=\delta(n,1,\gamma')$ on the same probability space $(\Omega,\mathcal{F},\mu)$ such that*

$$\mu[(\delta,\delta')\in\mathcal{K}_{k+1}]\geqslant 1-ce^{-wk}.$$

Proof of Lemma 5.2. In this proof, we will use (4.3). Suppose that $(\gamma,\gamma')\in\mathcal{K}_k$ with $k\geqslant 24$. The law of δ has Radon–Nikodym derivative

$$q:=\frac{e^{\xi-\lambda\psi_1}R_{n-1}(\bar\gamma)}{R_n(\gamma)}$$

with respect to the law of $\bar\gamma$, and similarly for δ':

$$q':=\frac{e^{\xi-\lambda\psi_1'}R_{n-1}(\bar\gamma')}{R_n(\gamma')}.$$

Since $\widehat\gamma^1$ has the same law as $\widehat\gamma'^1$, we may take them to be the same. Our first goal is to estimate $\mathbf{E}[|q-q'|]$.

Recall that $R_{n-1}(\bar\gamma')\leqslant c_1$ by (3.2), and that $R_n(\gamma')\geqslant(C/c_0)e^{-uk}$ by (3.6) and the fact that $(\gamma,\gamma')\in\mathcal{K}_k$. Hence, by Proposition 4.3,

$$\mathbf{E}\left[\frac{R_{n-1}(\bar\gamma')}{R_n(\gamma')}|e^{-\lambda\psi}-e^{-\lambda\psi'}|\right]\leqslant ce^{uk}e^{-v_1k}\leqslant ce^{-uk}$$

when $u\leqslant\frac12 v_1$. On the other hand, using Proposition 4.3 again, it follows that

$$\begin{aligned}\mathbf{E}\left[1_{(\bar\gamma,\bar\gamma')\in\mathcal{X}_k}\frac{|R_{n-1}(\bar\gamma')-R_{n-1}(\bar\gamma)|}{R_n(\gamma')}e^{-\lambda\psi}\right]&\leqslant ce^{uk}\mathbf{E}[1_{(\bar\gamma,\bar\gamma')\in\mathcal{X}_k}|R_{n-1}(\bar\gamma)-R_{n-1}(\bar\gamma')|]\\&\leqslant c'e^{uk}e^{-v_1k}\leqslant c'e^{-uk}\end{aligned}$$

when $u\leqslant\frac12 v_1$. Since $k\geqslant 24$ and $(\gamma,\gamma')\in\mathcal{X}_k$, if $(\bar\gamma,\bar\gamma')\in\Gamma\times\Gamma\setminus\mathcal{X}_{k+1}$, then $\widehat\gamma^1$ has a down-crossing from 1 to $e^{-k/24}$. This shows readily that

$$\mathbf{E}\left[1_{(\bar\gamma,\bar\gamma')\notin\mathcal{X}_k}\frac{R_{n-1}(\bar\gamma')+R_{n-1}(\bar\gamma)}{R_n(\gamma')}\right]\leqslant ce^{uk}e^{-kv_0'/24}\leqslant ce^{-ku}$$

for all $u<\frac{1}{48}v_0'$. Finally,

$$\mathbf{E}[R_{n-1}(\bar\gamma)|R_n(\gamma)^{-1}-R_n(\gamma')^{-1}|e^{-\lambda\psi}]\leqslant ce^{2uk}|R_n(\gamma)-R_n(\gamma')|\leqslant c'e^{2uk}e^{-kv_1}\leqslant c'e^{-uk}$$

for all $u\leqslant\frac13 v_1$.

Putting all the pieces together, we see that if we take $u<\min(\frac{1}{48}v_0',\frac13 v_1)$, then $\mathbf{E}[|q-q'|]\leqslant ce^{-uk}$, and hence there is a coupling of δ and δ' with $\mu[\delta_{k+1}=\delta'_{k+1}]\geqslant 1-ce^{-uk}$.

We now check that $\mu[R_1(\delta)\leqslant Ce^{-u(k+1)}]=\widetilde{\mathbf{P}}^n_\gamma[R_1(\bar\gamma)\leqslant Ce^{-u(k+1)}]$ is also exponentially small in k. Recall that $R_{n-1}(\bar\gamma)\leqslant c_0R_1(\bar\gamma)$, by (3.6), so that

$$\begin{aligned}\widetilde{\mathbf{P}}^n_\gamma[R_1(\bar\gamma)<Ce^{-u(k+1)}]&=\mathbf{E}\left[1_{R_1(\bar\gamma)\leqslant Ce^{-u(k+1)}}\frac{e^{\xi-\lambda\psi}R_{n-1}(\bar\gamma)}{R_n(\gamma)}\right]\\&\leqslant\mathbf{E}\left[\frac{e^{\xi-\lambda\psi}}{R_n(\gamma)}c_0Ce^{-u(k+1)}\right]\\&=c_0Ce^{-u(k+1)}\frac{R_1(\gamma)}{R_n(\gamma)}\leqslant c_0^2Ce^{-uk},\end{aligned}$$

and a similar inequality holds for γ'.

Finally, it remains to bound the probability $\mu[\delta\notin\mathcal{Y}_{k+1}]$. Since $\gamma\in\mathcal{Y}_k$ and $k\geqslant 24$, if $\delta\notin\mathcal{Y}_{k+1}$ then $\widehat{\gamma}^1$ has a downcrossing from 1 to $e^{-k/24}$. Hence,

$$\begin{aligned}\mu[\delta\notin\mathcal{Y}_{k+1}] &= \mathbf{E}\left[1_{\bar\gamma\notin\mathcal{Y}_{k+1}} e^{\xi-\lambda\psi}\frac{R_{n-1}(\bar\gamma)}{R_n(\gamma)}\right]\\ &\leqslant 2c_0' e^{-v_0' k/24}(c_0c_1/C)e^{uk}e^{\xi}\\ &\leqslant ce^{-uk}\end{aligned}$$

when $u<\frac{1}{48}v_0'$. This completes the proof of the lemma. □

We now choose $p\geqslant 25$ so that (for the constants defined in Lemma 5.2) $ce^{-w(p-1)}<\frac{1}{2}$. This is to make sure that the coupling in Lemma 5.2 occurs with positive probability for all $k\geqslant p-1$.

LEMMA 5.3. *There exists a constant $c=c(p)>0$ such that for all $\gamma,\gamma'\in\Gamma^+$, and for all $n\geqslant p$, one can define $\delta=\delta(n,p,\gamma)$ and $\delta'=\delta(n,p,\gamma')$ on the same probability space $(\Omega,\mathcal{F},\mu)$, such that*

$$\mu[(\delta,\delta')\in\mathcal{K}_{p-1}]\geqslant c.$$

Proof. Take $\gamma,\gamma'\in\Gamma^+$. Define the Brownian motions $\widehat{\alpha}$ and $\widehat{\alpha}'$ on the same probability space by mirror coupling. That is, we take $|\widehat{\alpha}(t)|=|\widehat{\alpha}'(t)|$ and keep $\arg\widehat{\alpha}(t)+\arg\widehat{\alpha}'(t)$ constant up to the first time t at which $\widehat{\alpha}(t)=\widehat{\alpha}'(t)$. After they have met, they stay together. Couple $\hat\beta$ and $\hat\beta'$ in the same way.

It is easy to see that there is a $c=c(p)>0$, which does not depend on γ or γ', such that with probability at least c,

(1) $\widehat{\gamma}$ and $\widehat{\gamma}'$ coalesce before they reach $e\partial\mathbf{U}$,

(2) $\bar\gamma^p,\bar\gamma'^p\in\Gamma^+$,

(3) $(\bar\gamma^p,\bar\gamma'^p)\in\mathcal{X}_{p-1}$, and

(4) $\widehat{\gamma}^p\cup\widehat{\gamma}'^p\subset\{re^{i\theta}:r>e^{-1/8},\ \theta\in(\frac{1}{4}\pi,\frac{7}{4}\pi)\}$.

Let $\mathcal{G}$ denote this event. The law of δ has Radon–Nikodym derivative

$$q:=\frac{e^{p\xi-\lambda\psi_p}R_{n-p}(\bar\gamma^p)}{R_n(\gamma)}$$

with respect to the law of $\bar\gamma$, and similarly for δ':

$$q':=\frac{e^{p\xi-\lambda\psi_p'}R_{n-p}(\bar\gamma'^p)}{R_n(\gamma')}.$$

Since $\mathbf{P}[\mathcal{G}]$ is bounded from below, to prove that there is a constant $c>0$ such that for all $n\geqslant p$ and all $\gamma,\bar{\gamma}\in\Gamma^+$ there is a coupling μ with

$$\mu[\{\delta_{p-1}=\delta'_{p-1}\}\cap\{\delta,\delta'\in\Gamma^+\}]\geqslant c,$$

it suffices to show that q and q' are bounded away from 0 on $\mathcal{G}$. This does hold, since R_n is bounded and bounded from 0 on Γ^+, and one can verify directly that ψ_p and ψ'_p are bounded on $\mathcal{G}$. Because $R_1\geqslant C$ on Γ^+, such (δ,δ') are in $\mathcal{K}_{p-1}$, and hence the proof is now complete. □

Proof of Proposition 5.1. Set $p^*=p+1$. Suppose first that $n=mp^*$, where $m\in\mathbf{N}$. Let $\gamma,\gamma'\in\Gamma$. The coupling μ is defined in the following way. Inductively, we construct a sequence $(\delta(j),\delta'(j))$, $j=0,1,...,m$, such that $\delta(0):=\gamma$, $\delta'(0):=\gamma'$, $\delta(j+1)$ has the law of $\delta(n-jp^*,p^*,\delta(j))$, and $\delta'(j+1)$ has the law of $\delta(n-jp^*,p^*,\delta'(j))$. Then, we set $(\delta,\delta'):=(\delta(m),\delta'(m))$. Repeated use of (5.2) shows that δ and δ' have the desired laws.

Let

$$K_j:=\max\{k\geqslant 1:(\delta(j),\delta'(j))\in\mathcal{K}_k\},$$

and let $K_j:=0$ if the set on the right-hand side is empty.

It follows easily from Lemma 5.2 iterated p^* times that it is possible to construct $(\delta(j+1),\delta'(j+1))$ in such a way that

$$\mathbf{P}[K_{j+1}\geqslant K_j+p^*\,|\,\delta(j),\delta'(j)]\geqslant(1-ce^{-wK_j})^{p^*}1_{K_j\geqslant p-1}.$$

For the case where $K_j<p-1$, the construction of $(\delta(j+1),\delta'(j+1))$ proceeds as follows. Note that the strong Markov property, the Separation Lemma (3.3), and (3.5) imply that

$$\inf_{\varrho\geqslant 1}\inf_{\gamma\in\Gamma}\widetilde{\mathbf{P}}^{\varrho}_{\gamma}[\tilde{\gamma}\in\Gamma^+]>0.$$

Therefore, Lemma 5.3 shows that it is possible to construct $(\delta(j+1),\delta'(j+1))$ in such a way that

$$\inf_{n,j,\gamma,\gamma'}\inf\mathbf{P}[K_{j+1}\geqslant p-1\,|\,\delta(j),\delta'(j)]>0.$$

By comparison with a Markov chain on the integers (see, e.g., Proposition 2 (iii) in [2]), this implies readily that

$$\mathbf{P}\big[K_m\geqslant\tfrac{1}{2}p^*m\big]\geqslant 1-c'e^{-mw'}$$

for some constants $c',w'>0$. This proves the proposition when $n/p^*\in\mathbf{N}$. The general case follows easily. For instance, if $n=mp^*+m'$ with $m'\in[0,...,p^*)$, we can apply the result for $n'=mp^*$ to $\delta=\delta(n,m',\gamma)$ and to $\delta'=\delta(n,m',\gamma')$, and note that $\delta(n',n',\delta)=\delta(n,n,\gamma)$ and $\delta(n',n',\delta')=\delta(n,n,\gamma')$. The small values of n can be handled by modifying the constant c in the statement of the lemma. □

5.2. Proof of Proposition 2.1 (ii)

We now conclude the proof of Proposition 2.1 (ii) and thereby the proof of Theorem 1.1.

Step 1. Let $k>0$. Suppose that $\gamma\in\Gamma$ is fixed, and let $\gamma':=\delta(n+k,k,\gamma)$. Set $\delta:=\delta(n,n,\gamma')$ and $\delta':=\delta(n,n,\gamma)$. Then δ has the law of $\delta(n+k,n+k,\gamma)$. By Proposition 5.1, δ and δ' may be defined on the same probability space $(\Omega,\mathcal{F},\mu)$ so that

$$\mu[(\delta,\delta')\notin\mathcal{X}_{n/3}]\leqslant ce^{-nv_2}.$$

Hence, for all $f\in\mathcal{A}_u$,

$$\left|\frac{T^nf(\gamma)}{T^n1(\gamma)}-\frac{T^{n+k}f(\gamma)}{T^{n+k}1(\gamma)}\right|\leqslant\int|f(\delta)-f(\delta')|\,d\mu\leqslant\|f\|_u(2ce^{-nv_2}+e^{-nu/3}). \tag{5.3}$$

It follows that for all $f\in\mathcal{A}_u$, $T^nf(\gamma)/T^n1(\gamma)$ converges when $n\to\infty$ to some limit $h(f,\gamma)$.

The same kind of argument gives for all $\gamma,\gamma'\in\Gamma$,

$$\left|\frac{T^nf(\gamma)}{T^n1(\gamma)}-\frac{T^nf(\gamma')}{T^n1(\gamma')}\right|\leqslant\|f\|_u(2ce^{-nv_2}+e^{-nu/3}),$$

and therefore the limit $h(f,\gamma)=h(f)$ is in fact independent of γ. Clearly, $h:\mathcal{A}\to\mathbf{C}$ is linear and $|h(f)|\leqslant\|f\|$ for all $f\in\mathcal{A}$, so that h is a bounded linear functional on $\mathcal{A}_u$.

Step 2. We are going to find an upper bound for the operator norm N_u of the operator $f\mapsto T^nf-h(f)T^n1$. Inequality (5.3) shows that for all $f\in\mathcal{A}_u$ and $\gamma\in\Gamma$,

$$|T^nf(\gamma)-h(f)T^n1(\gamma)|\leqslant\|f\|_u(2ce^{-nv_2}+e^{-nu/3})T^n1(\gamma)\leqslant c'\|f\|_ue^{-n(\xi+u/3)}$$

for all sufficiently small $u\leqslant 3v_2$. Suppose now that $(\gamma,\gamma')\in\mathcal{X}_m$ and that $m\leqslant\frac{1}{6}n$. Then, the previous estimate gives

$$|T^nf(\gamma)-h(f)T^n(\gamma)-T^nf(\gamma')+h(f)T^n(\gamma')|\leqslant 2c'\|f\|_ue^{-n\xi}e^{-nu/6}e^{-mu}.$$

Assume now that $(\gamma,\gamma')\in\mathcal{X}_m$ and that $m\geqslant\frac{1}{6}n$. Defining $\hat\gamma=\hat\gamma'$, and using Proposition 4.3 and the fact that $|h(f)|\leqslant\|f\|$, we get

$$\begin{aligned}
&|T^nf(\gamma)-T^nf(\gamma')|+|h(f)|\,|T^n1(\gamma)-T^n1(\gamma')|\\
&\quad\leqslant\mathbf{E}[|f(\bar\gamma^n)e^{-\lambda\psi_n}-f(\bar\gamma'^n)e^{-\lambda\psi'_n}|]+\|f\|\,\mathbf{E}[|e^{-\lambda\psi'_n}-e^{-\lambda\psi_n}|]\\
&\quad\leqslant\mathbf{E}[|f(\bar\gamma^n)-f(\bar\gamma'^n)|e^{-\lambda\psi_n}]+2\|f\|\,\mathbf{E}[|e^{-\lambda\psi'_n}-e^{-\lambda\psi_n}|]\\
&\quad\leqslant\mathbf{E}[1_\mathcal{V}|f(\bar\gamma^n)-f(\bar\gamma'^n)|e^{-\lambda\psi_n}]+2\|f\|\,\mathbf{E}[1_{\neg\mathcal{V}}e^{-\lambda\psi_n}]+2\|f\|\,\mathbf{E}[|e^{-\lambda\psi'_n}-e^{-\lambda\psi_n}|],
\end{aligned}$$

where $\mathcal{V}$ is the event that $(\bar{\gamma}^n, \bar{\gamma}'^n) \in \mathcal{X}_{n+m}$. We are going to bound the three terms separately. For the first one, we have

$$\mathbf{E}[1_{\mathcal{V}} |f(\bar{\gamma}^n) - f(\bar{\gamma}'^n)| e^{-\lambda\psi_n}] \leqslant \|f\|_u e^{-(n+m)u} \mathbf{E}[e^{-\lambda\psi_n}] \leqslant c' \|f\|_u e^{-n\xi} e^{-mu} e^{-nu}.$$

The last term is bounded by $2c\|f\| e^{-n\xi} e^{-mv_1}$ (and therefore by $2c\|f\| e^{-n\xi} e^{-mu} e^{-nu/6}$ for all $u < \frac{1}{2}v_1$) by Proposition 4.3. For the second term, note that on $\{\bar{\gamma}^n \in \Gamma\} \setminus \mathcal{V}$ there is a $j \in \{0, 1, 2, ..., 23\}$ such that $\bar{\gamma}^n$ has a downcrossing from $e^{-jn/24}$ to $e^{-(jn+m)/24}$. By estimating the contribution of each of these 24 possible values of j separately, one easily gets, using the strong Markov property, (4.6) and (3.2), that

$$\begin{aligned}\mathbf{E}[1_{\neg\mathcal{V}} e^{-\lambda\psi_n}] &\leqslant \sum_{j=0}^{23} \mathbf{E}[e^{-\lambda\psi_{jn/24}} 2c_0' e^{-v_0' m/24} c_1 e^{-(24-j)n\xi/24}] \\ &\leqslant c e^{-n\xi} e^{-v_0' m/24} \\ &\leqslant c e^{-n\xi} e^{-um} e^{-un/6}\end{aligned}$$

for all $u < \frac{1}{48}v_0'$. Combining these three estimates shows that for all sufficiently small u, there exists $c = c(u, \lambda)$ such that for all $n \geqslant 1$,

$$N_u(T^n(\cdot) - h(\cdot)T^n 1) \leqslant c e^{-n\xi} e^{-nu/6}. \tag{5.4}$$

Step 3. Note that $T^{n+1}1(\gamma)/T^n 1(\gamma) \to h(T1)$ and $T^{n+j}1(\gamma)/T^n 1(\gamma) \to h(T^j 1)$ as $n \to \infty$. Since $T^n 1(\gamma) \approx e^{-n\xi}$ for $\gamma \in \Gamma^+$, we get $h(T1) = e^{-\xi}$ and $h(T^j 1) = e^{-j\xi}$. Recall also that $\|T^j 1\|_u \leqslant c e^{-j\xi}$ by Proposition 4.3. Hence, (5.4) for $f = T^j 1$ shows that

$$\|T^{n+j} 1 - e^{-j\xi} T^n 1\|_u \leqslant c e^{-n\xi} e^{-j\xi} e^{-nu/6}.$$

Hence, $R_n = e^{n\xi} T^n 1$ converges in $\mathcal{A}_u$ to some limit R, and

$$\|R_n - R\|_u = \|e^{n\xi} T^n 1 - R\|_u \leqslant c e^{-nu/6}. \tag{5.5}$$

Since T is continuous we have $TR = \lim_{n\to\infty} TR_n = \lim_{n\to\infty} e^{-\xi} R_{n+1} = e^{-\xi} R$. Since h is continuous, $h(R) = \lim_{n\to\infty} h(R_n) = 1$. In particular, this shows that $h(\cdot)R$ is a continuous projection on the vector space spanned by R.

If $h(f) = 0$, then $T^n f / T^n 1 \to 0$ on Γ and therefore $T^{n+1} f / T^n 1 \to 0$, which implies $h(Tf) = 0$. That is, $T \ker(h) \subset \ker(h)$. In particular, the nth iterate of the operator $T(\cdot) - e^{-\xi} h(\cdot) R$ is equal to $T^n(\cdot) - e^{-n\xi} h(\cdot) R$. (5.4) and (5.5) show that

$$N_u(e^{n\xi} T^n(\cdot) - h(\cdot)R) \leqslant c' e^{-nu/6},$$

so that the operator norm of $T(\cdot) - e^{-\xi} h(\cdot) R$ is bounded by $e^{-\xi - u/6}$. This implies Proposition 2.1 (ii). □

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