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The deformation theorem for flat chains

by

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The deformation theorem of Federer and Fleming [FF] is a fundamental tool in geometric measure theory. The theorem gives a way of approximating (in the so-called flat norm) a very general k-dimensional surface (flat chain) A in $\mathbf{R}^N$ by a polyhedral surface P consisting of k-cubes from a cubical lattice in $\mathbf{R}^N$. Unfortunately, the theorem requires the original surface to have finite mass and finite boundary mass. In this paper, we remove these finiteness restrictions. That is, we show (in Theorem 1.1, Corollaries 1.2 and 1.3) that the Federer–Fleming deformation procedure gives good approximations to an arbitrary flat chain A. Also, the approximating polyhedral surface depends only on the way in which typical translates of A intersect the $(N-k)$-skeleton of the dual lattice. This lets us answer several open questions about flat chains:

(1) For an arbitrary coefficient group, a nonzero flat k-chain cannot be supported in a set of k-dimensional measure 0.

(2) For an arbitrary coefficient group, a flat chain of finite mass and finite size must be rectifiable. In particular, for any discrete group, finite mass implies rectifiability.

(3) Let G be a normed group with $\sup\{|g|:g\in G\}=\lambda<\infty$. Then for any flat chain with coefficients in G, $\mathrm{M}(A)\leqslant\lambda\mathcal{H}^k(\operatorname{spt} A)$.

(Special cases of (1) and (2) are mentioned as open questions in [Fl], and the special case $G=\mathbf{Z}_p$ of (3) is mentioned as an open question in [Fe1]. Federer and Fleming [FF] proved (1) for real flat chains (and therefore also for integral flat chains). Almgren [A] introduced the notion of size and proved (2) for real flat chains; Federer [Fe2] then gave a much shorter proof.)

Furthermore, in another paper [W3] we use the deformation theorem proved here to give a simple necessary and sufficient condition on a coefficient group G in order for every finite-mass flat chain to be rectifiable.

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The proof is in the context of flat chains over an arbitrary abelian group G together with a norm that makes G into a complete metric space. These were introduced by Fleming in [Fl]. The reader is referred to the first half (through §6) of Fleming's paper for definitions and the properties of flat chains needed here. The main distinguishing feature of Fleming's approach (versus the approach via differential forms in [FF], [Fe1], [S], and [Mo]) is that Fleming defines the space of flat chains to be the metric space completion (with respect to the flat metric) of the space of polyhedral chains, and he defines mass (for flat chains) to be the largest lower semicontinuous functional that extends the obvious notion of mass for polyhedral chains. Thus, by definition, every flat chain can be approximated by a polyhedral chain of nearly the same mass. This allows one to prove many theorems (such as those in this paper) for flat chains by proving them for polyhedral chains, and then concluding by continuity that they hold for arbitrary flat chains. For that reason, the proof of the strengthened deformation theorem here is simpler than the proof of the deformation theorem in [FF], [Fe1], and [S]. In case the group is $\mathbf{Z}$ or $\mathbf{R}$ (with the standard norm), Fleming proves that his flat chains are equivalent to the integral flat chains or real flat chains, respectively, as usually defined via differential forms.

Until recently, most research on flat chains dealt with the coefficient groups $\mathbf{Z}$ and $\mathbf{R}$ with the standard norm. However, flat chains over other normed groups are also quite interesting and arise naturally in various contexts. For example, using the integers mod 2 allows one to find least-area surfaces among all surfaces, possibly non-orientable, with a given boundary. (Integral currents are all oriented.) Other coefficient groups are ideally suited to modelling least-energy immiscible fluid configurations, as well as soap films and soap-bubble clusters [W1]. They are also useful in proving that certain surfaces (such as the cone over the 1-skeleton of a regular tetrahedron) minimize area [ML]. And flat chains mod p have been used as a tool to prove theorems about integral flat chains [W4].

The terminology and notation here differ slightly from Fleming's. In particular, we use $\mathcal{F}$ instead of $\mathcal{W}$ for the flat norm, and $A \llcorner S$ instead of $A \cap S$ for the portion of A in the set S. Also, the term "flat chain" here refers to what would be called "flat chain with compact support" in Fleming's paper.

1. The deformation theorem

For $\varepsilon > 0$, let $X(\varepsilon)$ be the standard partition of $\mathbf{R}^N$ into cubes of size ε. In particular, the vertices of $X(\varepsilon)$ are the points in $\mathbf{R}^N$ each of whose coordinates is an integer multiple of ε. A k-dimensional cube of side length ε whose vertices are vertices of $X(\varepsilon)$ will be called a "k-cube of $X(\varepsilon)$". Let $X^k(\varepsilon)$ denote the k-skeleton of $X(\varepsilon)$, that is, the union of all the k-cubes of $X(\varepsilon)$. Let $Y(\varepsilon)$ be the dual partition of $\mathbf{R}^N$, formed by translating each cube of $X(\varepsilon)$ by $\left(\frac{1}{2}\varepsilon, \ldots, \frac{1}{2}\varepsilon\right)$. Thus the vertices of $Y(\varepsilon)$ are the centers of the N-cubes of $X(\varepsilon)$, and vice versa.

Let $\mathcal{P}_k$ and $\mathcal{F}_k$ denote the spaces of k-dimensional polyhedral chains and flat chains, respectively, in $\mathbf{R}^N$.

DEFORMATION THEOREM 1.1. *For every $\varepsilon > 0$, there are operators*

$$P = P^\varepsilon : \mathcal{F}_k \to \mathcal{P}_k,$$
$$H = H^\varepsilon : \mathcal{F}_k \to \mathcal{F}_{k+1}$$

such that for all flat k-chains A and B, the following properties hold for almost every $y \in \mathbf{R}^N$:

(1)

$$H(\tau_y(A+B)) = H(\tau_y A) + H(\tau_y B),$$
$$P(\tau_y(A+B)) = P(\tau_y A) + P(\tau_y B),$$
$$\partial P(\tau_y A) = P(\partial \tau_y A).$$

(2) *$P(\tau_y A)$ is a sum (with G-coefficients) of k-cubes of $X(\varepsilon)$.*

(3) *$\tau_y A = P(\tau_y A) + \partial H(\tau_y A) + H(\partial \tau_y A)$.*

(4) *$P(\tau_y A)$ and $H(\tau_y A)$ are supported in a $\sqrt{N}\varepsilon$-neighborhood of* $\operatorname{spt} \tau_y A$.

(5) *If A is a k-chain disjoint from the $(N-k)$-skeleton of the dual partition $Y(\varepsilon)$, then $P^\varepsilon(A) = 0$.*

(6)

$$\int_{y \in [0,1]^N} \mathrm{M}(H^\varepsilon(\tau_{\varepsilon y} A))\, dy \leqslant c\varepsilon \mathrm{M}(A),$$
$$\int_{y \in [0,1]^N} \mathrm{M}(P^\varepsilon(\tau_{\varepsilon y} A))\, dy \leqslant c \mathrm{M}(A),$$
$$\int_{y \in [0,1]^N} \mathcal{H}^k(\operatorname{spt} P^\varepsilon(\tau_{\varepsilon y} A))\, dy \leqslant c \mathcal{H}^k(\operatorname{spt} A).$$

(7) *If A is polyhedral, then $H(\tau_y A)$ is polyhedral.*

Here and throughout the paper we write X, Y, P, and H instead of $X(\varepsilon)$, $Y(\varepsilon)$, P^ε, and H^ε, except when we want to emphasize the dependence on ε. The expression $\tau_y A$ denotes the chain obtained by translating A by y. Constants (such as c above) may depend on N, but not on other quantities.

Conclusions (1) and (3) of Theorem 1.1 can also be stated in the language of homological algebra as follows. Let $\mathcal{C}$ be the chain complex generated by any finite or countable collection of flat chains in $\mathbf{R}^N$. Then for almost every $y\in\mathbf{R}^N$, the map

$$P\circ\tau_y\colon \mathcal{C}_*\to\mathcal{F}_*$$

is a chain homomorphism, and $H\circ\tau_y\colon\mathcal{C}_*\to\mathcal{F}_{*+1}$ is a chain homotopy between $\tau_y\colon\mathcal{C}_*\to\mathcal{F}_*$ and $P\circ\tau_y\colon\mathcal{C}_*\to\mathcal{F}_*$.

In the following corollary, we write $P(y,A)$ and $H(y,A)$ for $P(\tau_yA)$ and $H(\tau_yA)$, respectively, and we let $\mathrm{Av}=\mathrm{Av}^\varepsilon$ denote average over $[0,\varepsilon]^N$:

$$\mathrm{Av}\, f=\varepsilon^{-N}\int_{y\in[0,\varepsilon]^N} f(y)\,dy=\int_{y\in[0,1]^N} f(\varepsilon y)\,dy.$$

COROLLARY 1.2. *For all k-chains A and B, we have*

(1) $\mathrm{Av}\,\mathcal{F}(P(y,A))\leqslant c\mathcal{F}(A)$,

(2) $\mathrm{Av}\,\mathcal{F}(A-P(y,A))\leqslant c\varepsilon N(A)$,

(3) $\mathrm{Av}\,\mathcal{F}(A-P(y,A))\leqslant c\mathcal{F}(A-B)+c\varepsilon N(B)$,

(4) $\lim_{\varepsilon\to0}\mathrm{Av}\,\mathcal{F}(A-P(y,A))=0$,

where $N(A)=\mathrm{M}(A)+\mathrm{M}(\partial A)$.

Proof. Let $A=B+\partial C$. Then $P(y,A)=P(y,B)+\partial P(y,C)$, so

$$\mathcal{F}(P(y,A))\leqslant \mathrm{M}(P(y,B))+\mathrm{M}(P(y,C))$$

and

$$\begin{aligned}\mathrm{Av}\,\mathcal{F}(P(y,A))&\leqslant \mathrm{Av}\,\mathrm{M}(P(y,B))+\mathrm{Av}\,\mathrm{M}(P(y,C))\\&\leqslant c(\mathrm{M}(B)+\mathrm{M}(C))\end{aligned}$$

by (6) of the deformation theorem. Taking the infimum over all such B and C gives (1).

To prove (2), note that $\tau_yA-P(y,A)=\partial H(y,A)+H(y,\partial A)$ (by (3) of the deformation theorem), so

$$\mathcal{F}(\tau_yA-P(y,A))\leqslant \mathrm{M}(H(y,A))+\mathrm{M}(H(y,\partial A)).$$

Averaging over y and using (6) of the deformation theorem gives

$$\mathrm{Av}\,\mathcal{F}(\tau_yA-P(y,A))\leqslant c\varepsilon\,\mathrm{M}(A)+c\varepsilon\,\mathrm{M}(\partial A)=c\varepsilon N(A). \tag{$*$}$$

If we translate A to τ_yA, it sweeps out a $(k+1)$-dimensional polyhedral chain S whose boundary is $A-\tau_yA$ together with the k-chain T swept out by ∂A. Thus if $|y|<\varepsilon$,

$$\mathcal{F}(A-\tau_yA)\leqslant \mathrm{M}(S)+\mathrm{M}(T)\leqslant \varepsilon\,\mathrm{M}(A)+\varepsilon\,\mathrm{M}(\partial A)=\varepsilon N(A).$$

Combining this with $(*)$ gives (2).

To prove (3), observe that

$$\begin{aligned}\mathcal{F}(A-P(y,A)) &\leqslant \mathcal{F}(A-B)+\mathcal{F}(B-P(y,B))+\mathcal{F}(P(y,B)-P(y,A))\\ &=\mathcal{F}(A-B)+\mathcal{F}(B-P(y,B))+\mathcal{F}(P(y,B-A)).\end{aligned}$$

Now averaging and using (1) and (2), we get

$$\operatorname{Av}\mathcal{F}(A-P(y,A))\leqslant \mathcal{F}(A-B)+c\varepsilon N(B)+c\mathcal{F}(A-B),$$

which is (3).

To prove (4), note that by (3) we have

$$\limsup_{\varepsilon\to 0}\operatorname{Av}\mathcal{F}(A-P^{\varepsilon}(y,A))\leqslant (1+c)\mathcal{F}(A-B)$$

for every k-chain B with $N(B)<\infty$. In particular, this holds for every polyhedral chain B. But by definition, the polyhedral chains are dense in the flat chains, so we can choose polyhedral chains B making $\mathcal{F}(A-B)$ arbitrarily small. □

COROLLARY 1.3. *Let A be a flat chain, and $\varepsilon(i)$ a sequence of positive numbers converging to* 0. *Then there is a subsequence $\varepsilon'(i)$ such that*

$$P^{\varepsilon'(i)}(\tau_{\varepsilon'(i)y}A)\to A$$

for almost every $y\in[0,1]^N$.

Proof. By Corollary 1.2 (4),

$$\lim_i \int_{y\in[0,1]^N}\mathcal{F}(A-P^{\varepsilon(i)}(\tau_{\varepsilon(i)y}A))\,dy=0.$$

Thus we can find a subsequence $\varepsilon'(i)$ such that

$$\sum_i\int_{y\in[0,1]^N}\mathcal{F}(A-P^{\varepsilon'(i)}(\tau_{\varepsilon'(i)y}A))\,dy<\infty.$$

Hence

$$\int_{y\in[0,1]^N}\sum_i\mathcal{F}(A-P^{\varepsilon'(i)}(\tau_{\varepsilon'(i)y}A))\,dy<\infty.$$

Thus for almost every y, the integrand is finite, which implies that

$$\mathcal{F}(A-P^{\varepsilon'(i)}(\tau_{\varepsilon'(i)y}A))\to 0.$$

□

2. Proof of the deformation theorem

We first define retraction maps

$$\psi_k: \mathbf{R}^N \setminus Y^{N-k-1} \to X^k.$$

These retractions will have the property that

$$\operatorname{dist}(\psi_k x, Y^j) \geqslant \operatorname{dist}(x, Y^j) \tag{1}$$

for all k and j. In particular, if $x \notin Y^j$, then $\psi_k x \notin Y^j$.

We define the retractions inductively as follows. First,

$$\psi_N: \mathbf{R}^N \to \mathbf{R}^N$$

is the identity map. Now given ψ_k, we define ψ_{k-1} as follows. Let $x \in \mathbf{R}^N \setminus Y^{N-k}$. Then by (1), $\psi_k x \notin Y^{N-k}$. That is,

$$\psi_k x \in X^k \setminus Y^{N-k}.$$

Thus $\psi_k x$ will lie in, but not at the center of, some k-cube Q of the grid X. Let $\psi_{k-1} x$ be the point in ∂Q such that $\psi_k x$ is in the line segment joining the center of Q to $\psi_{k-1} x$.

Now define a map $\Phi(t, x)$ on a subset (to be specified below) of $[0, N] \times \mathbf{R}^N$ as follows. We let

$$\Phi(j, x) = \psi_j x \quad (j = 0, 1, \ldots, N)$$

and we let $\Phi(t, x)$ be linear in t on each interval $j \leqslant t \leqslant j+1$ (for $j = 0, 1, \ldots, N-1$). Note that for each $j = 1, 2, \ldots, N$ the map Φ is well-defined and continuous on

$$[j, N] \times (\mathbf{R}^N \setminus Y^{N-j-1})$$

and is in fact a deformation retraction from $\mathbf{R}^N \setminus Y^{N-j-1}$ onto X^j. Note also that if $x \notin Y^j$, then $\Phi(t, x) \notin Y^j$ (for all t such that $\Phi(t, x)$ is defined).

We say that a polyhedral k-chain A is in *good position* (with respect to Y) provided $\operatorname{spt} A$ is disjoint from Y^{N-k-1} and $\operatorname{spt} \partial A$ is disjoint from Y^{N-k}. We say that A is in *very good position* if it can be expressed as $\sum g_i[\sigma_i]$ where each $g_i[\sigma_i]$ is in good position. Let A be any polyhedral chain. Note that for almost every $y \in \mathbf{R}^N$, the translate $\tau_y A$ will be in very good position (and therefore also in good position).

Now let A be a polyhedral k-chain in good position. We let $H(A)$ be the polyhedral $(k+1)$-chain swept out by A under the deformation retraction from $\mathbf{R}^N \setminus Y^{N-k-1}$ to X^k:

$$H(A) = \Phi_{\#}([k, N] \times A).$$

We also define

$$P(A) = A - \partial H(A) - H(\partial A).$$

LEMMA 2.1. *Let A and B be polyhedral k-chains in good position. Then:*

(1)

$$H(A+B)=H(A)+H(B),$$
$$P(A+B)=P(A)+P(B),$$
$$\partial P(A)=P(\partial A).$$

(2) *$P(A)$ is a combination of k-cubes of the grid X.*

(3) *$A=P(A)+\partial H(A)+H(\partial A)$.*

(4) *$P(A)$ and $H(A)$ are supported in a $\sqrt{N}\varepsilon$-neighborhood of* $\operatorname{spt} A$.

(5) *If* $\operatorname{spt} A$ *is disjoint from Y^{N-k}, then $P(A)=0$.*

(6)

$$\mathbf{M}(H^{\varepsilon}(A))\leqslant c\varepsilon^{k+1}\int(\operatorname{dist}(x,Y^{N-k-1}))^{-k}\,d\mu_A x.$$

Proof. Assertion (1) follows immediately from the definitions of H and P. Note that by the homotopy formula [Fl, 6.3] we have

$$\begin{aligned}\partial H(A)&=(\psi_N)_{\#}A-(\psi_k)_{\#}A-\Phi_{\#}([k,N]\times\partial A)\\&=A-(\psi_k)_{\#}A-\Phi_{\#}([k,N]\times\partial A).\end{aligned}$$

Hence

$$\begin{aligned}P(A)&=A-\partial H(A)-H(\partial A)\\&=(\psi_k)_{\#}A+\Phi_{\#}([k,N]\times\partial A)-\Phi_{\#}([k-1,N]\times\partial A)\qquad(*)\\&=(\psi_k)_{\#}A-\Phi_{\#}([k-1,k]\times\partial A).\end{aligned}$$

Of course ψ_k takes values in X^k, as does $\Phi(t,\cdot)$ for $t\leqslant k$. Thus from $(*)$ we see that $P(A)$ is supported in X^k. Likewise $\partial P(A)=P(\partial A)$ is supported in X^{k-1}. It follows that A is a combination of k-cubes of the grid X, which is assertion (2).

Assertion (3) is just the definition of $P(A)$. Assertion (4) is obvious.

Now suppose that the support of A is disjoint from Y^{N-k}. Recall that if $x\notin Y^{N-k}$, then $\psi_k x\notin Y^{N-k}$ and $\Phi(t,x)\notin Y^{N-k}$ for $k-1\leqslant t\leqslant N$. Consequently, according to $(*)$, $\operatorname{spt} P(A)$ is disjoint from Y^{N-k}. This means (by assertion (2)) that $P(A)$ is a combination of k-cubes of the grid X, but does not touch the centers of those cubes. Of course that means $P(A)=0$, which is assertion (5).

Finally, assertion (6) is a straightforward calculation. □

The following proposition gives another characterization of $P(A)$. For each k-cube Q of the grid X, note that there is a unique $(N-k)$-cube Q^* of Y such that Q and Q^* intersect at their common center. Give Q an orientation, and then orient Q^* so that $Q\oplus Q^*$ induces the standard orientation on $\mathbf{R}^N$.

Let σ and τ be oriented polyhedra of dimensions k and $N-k$, respectively. We define an intersection number $I(\sigma,\tau)$ as follows.

(1) If σ and τ are disjoint, then $I(\sigma,\tau)=0$.

(2) If the intersection of σ and τ is a single point that is not in the boundary of either σ or τ, then $I(\sigma,\tau)$ is 1 or -1, according to whether the orientation on $\mathbf{R}^N$ induced by $\sigma\oplus\tau$ is the standard one or not.

(3) Otherwise, $I(\sigma,\tau)$ is undefined.

PROPOSITION 2.2. *Let $A=\sum g_i[\sigma_i]$ be a polyhedral k-chain in very good position. Then*

$$P(A)=\sum_Q\sum_i g_i I(\sigma_i,Q^*)[Q],$$

where the first summation is over all k-cubes in the grid X.

Proof. We will show in fact that for each i,

$$P(g_i[\sigma_i])=\sum_Q g_i I(\sigma,Q^*)[Q]. \tag{$*$}$$

By subdividing, we may assume that each σ_i gets mapped by ψ_k into a single k-cube of X. If σ_i is disjoint from Y^{N-k}, then both sides of $(*)$ are 0 (by (5) of the lemma). If σ_i intersects some Q, we leave it to the reader to check $(*)$. □

Proof of the deformation theorem. Assume first that A and B are polyhedral k-chains. Then $\tau_y A$ and $\tau_y B$ are in very good position for almost every $y\in\mathbf{R}^N$, and thus we can apply the results of this section to them. In particular, from Lemma 2.1, we see that assertions (1)–(5) of the deformation theorem hold when A and B are polyhedral.

Furthermore, by (6) of the lemma, we have

$$\begin{aligned}\int_{y\in[0,1]^N}\mathrm{M}(H(\tau_{\varepsilon y}A))\,dy &\leqslant \int_{y\in[0,1]^N}\varepsilon^{k+1}\int_x c\,(\mathrm{dist}(x,Y^{N-k-1}))^{-k}\,d\mu_{\tau_{\varepsilon y}A}x\,dy\\ &=\int_{y\in[0,1]^N}\varepsilon^{k+1}\int_x c\,(\mathrm{dist}(x+\varepsilon y,Y^{N-k-1}))^{-k}\,d\mu_A x\,dy\\ &=\varepsilon^{k+1}\int_x\int_{y\in[0,1]^N}c\,(\mathrm{dist}(x+\varepsilon y,Y^{N-k-1}))^{-k}\,dy\,d\mu_A x\\ &\leqslant \varepsilon\int_x c'\,d\mu_A x\\ &\leqslant c'\varepsilon\,\mathrm{M}(A),\end{aligned}$$

which is the first inequality in assertion (6) of the deformation theorem.

The other two inequalities of (6) follow immediately from Proposition 2.2 by elementary integral geometry.

Hence we have proved all of the deformation theorem in the case of polyhedral chains. It follows that Corollary 1.2 also holds for polyhedral chains.

Now let A be an arbitrary flat k-chain. Choose a sequence of polyhedral chains A_i that converge rapidly to A, i.e., such that

$$\sum \mathcal{F}(A_i - A) < \infty.$$

Of course it follows that

$$\sum \mathcal{F}(A_i - A_{i+1}) < \infty.$$

Now by Corollary 1.2 (1),

$$\begin{aligned} \operatorname{Av} \sum_i \mathcal{F}(P^\varepsilon(\tau_y A_i) - P^\varepsilon(\tau_y A_{i+1})) &= \sum_i \operatorname{Av} \mathcal{F}(P(\tau_y A_i - \tau_y A_{i+1})) \\ &\leqslant \sum_i c\mathcal{F}(A_i - A_{i+1}) \\ &< \infty. \end{aligned}$$

Consequently, for almost every y, the chains $P^\varepsilon(\tau_y A_i)$ converge to a limit. We define $P^\varepsilon(\tau_y A)$ to be this limit. Note that if A'_i is any other sequence converging rapidly to A, then the sequence $A_1, A'_1, A_2, A'_2, \ldots$ also converges rapidly to A, so (for almost every y) the sequence

$$P^\varepsilon(\tau_y A_1),\ P^\varepsilon(\tau_y A'_1),\ P^\varepsilon(\tau_y A_2),\ P^\varepsilon(\tau_y A'_2),\ \ldots$$

converges. Hence (for almost every y) the chains $P^\varepsilon(\tau_y A'_i)$ also converge to the same limit $P^\varepsilon(\tau_y A)$.

An exactly analogous argument shows that for almost every y the chains $H^\varepsilon(\tau_y A_i)$ converge to a limit, which we define to be $H^\varepsilon(\tau_y A)$. Likewise (for almost every y) the chains $H^\varepsilon(\tau_y A'_i)$ converge to the same limit $H^\varepsilon(\tau_y A)$.

Now assertions (1)–(3) of the deformation theorem (for arbitrary flat chains A and B) follow immediately from the same assertions for the polyhedral chains A_i and B_i (where the B_i converge rapidly to B), which we already know to hold.

By definition of support, we can choose polyhedral chains A'_i converging rapidly to A with $\operatorname{spt} A'_i \to \operatorname{spt} A$. Assertions (4) and (5) now follow immediately from the same assertions for the A'_i.

Similarly, we can choose polyhedral chains A''_i converging rapidly to A with $\mathrm{M}(A''_i) \to \mathrm{M}(A)$. The first two inequalities of assertion (6) now follow immediately from the corresponding inequalities for the A''_i.

Assertion (7) is trivially true.

This completes the proof of all of the deformation theorem except for the last inequality in (6), which will not be used except in §5. We postpone proof of the inequality until that section. □

3. Concentration

It is important to know that flat chains cannot be concentrated in small sets. This is crucial, for example, in establishing rectifiability of integral currents. Most accounts of flat chains ([FF], [Fe1], [Mo], [S]) prove this when the group is $\mathbf{Z}$ or $\mathbf{R}$ using differential forms. That proof does not work for other coefficient groups since integration of differential forms over such groups does not make sense. Fleming [Fl] gave a different proof that works precisely for finite groups. Here we give a proof that works for any group. Rectifiability theorems (§4 below) then follow immediately from Federer's structure theorem, exactly as in [FF].

THEOREM 3.1. *Let A be a flat k-chain in $\mathbf{R}^N$ with $\mathcal{H}^k(\operatorname{spt} A)=0$. Then $A=0$. More generally, if the projection of* $\operatorname{spt} A$ *on each coordinate k-plane has k-dimensional measure* 0, *then $A=0$.*

Proof. By Corollary 1.2 (4),

$$\lim_{\varepsilon\to 0} \operatorname{Av} \mathcal{F}(A-P^{\varepsilon}(y,A))=0. \qquad (*)$$

The hypothesis about projections of $\operatorname{spt} A$ is equivalent to saying that $\tau_y \operatorname{spt} A=\operatorname{spt} \tau_y A$ is disjoint from the $(N-k)$-skeleton of $Y(\varepsilon)$ for almost every y. Hence by (5) of the deformation theorem, $P(y,A)=0$ for almost every y. Thus $(*)$ becomes

$$\lim_{\varepsilon\to 0} \operatorname{Av} \mathcal{F}(A)=0$$

or $\mathcal{F}(A)=0$. Thus $A=0$. □

4. Rectifiability

A finite-mass flat k-chain A in $\mathbf{R}^N$ is said to be *rectifiable* if there is a countable collection $M_1, M_2, \ldots$ of k-dimensional C^1-submanifolds of $\mathbf{R}^N$ such that

$$\mu_A(\mathbf{R}^N \setminus \textstyle\bigcup_i M_i)=0$$

(or, equivalently, such that $A=A \llcorner \bigcup_i M_i$). The *Hausdorff size* of a finite-mass flat k-chain A is the infimum of $\mathcal{H}^k(S)$ among Borel sets S such that $\mu_A(\mathbf{R}^N\setminus S)=0$ (or, equivalently, such that $A=A\llcorner S$). Note that there is an S that attains the infimum.

The following holds for any group:

THEOREM 4.1. *Let A be a flat k-chain of finite mass and finite Hausdorff size. Then A is rectifiable.*

COROLLARY. *Suppose that G is a discrete group (i.e., the norms of nonzero elements of G are bounded away from 0). Then every finite-mass flat chain is rectifiable.*

Proof of the corollary. By [Fl, 8.3], the hypothesis on G implies that finite-mass chains also have finite size. □

Proof of Theorem 4.1. Suppose that A were a counterexample to the theorem, and let S be a Borel set with $A=A\llcorner S$ and $\mathcal{H}^k(S)<\infty$. Then the purely unrectifiable part of S would contain a compact set K of positive μ_A-measure, and $A'=A\llcorner K$ would be a nonzero flat k-chain supported in a purely unrectifiable compact set K of finite $\mathcal{H}^k$-measure.

Thus the theorem follows from

THEOREM 4.2. *Let $K\subset\mathbf{R}^N$ be a purely unrectifiable compact set of finite $\mathcal{H}^k$-measure. Let A be a flat k-chain supported in K. Then $A=0$.*

Proof. By Federer's structure theorem (see [Fe1, 3.3.15], [Ma, §18], or the simplified proof in [W3]), there is a rotation $\theta\colon\mathbf{R}^N\to\mathbf{R}^N$ such that the projection of

$$\theta(K)$$

onto each of the coordinate k-planes has measure 0. By Theorem 3.1, $\theta_\# A=0$. Hence $A=0$. □

5. Groups with bounded norms

THEOREM 5.1. *Suppose* $\sup_{g\in G}|g|=\lambda<\infty$. *Let A be a flat k-chain in $\mathbf{R}^N$ with coefficients in G. Then*

$$\mathrm{M}(A)\leqslant c\lambda\mathcal{H}^k(\operatorname{spt}A). \qquad (*)$$

Remark. Suppose $\mathcal{H}^k(\operatorname{spt}A)<\infty$. Once we know that $\mathrm{M}(A)$ is finite, it follows from the rectifiability theorem, Theorem 4.1, that $(*)$ is actually true with $c=1$. (See §6.)

Proof. By Corollary 1.3 to the deformation theorem, there is a sequence $\varepsilon(i)\to 0$ such that

$$P^{\varepsilon(i)}(\tau_{\varepsilon(i)y}A)\to A$$

for almost every $y\in[0,1]^N$. Therefore

$$\begin{aligned}\mathrm{M}(A)&\leqslant\liminf\,\mathrm{M}(P^{\varepsilon(i)}(\tau_{\varepsilon(i)y}A))\\&\leqslant\liminf\,\lambda\mathcal{H}^k(\operatorname{spt}P^{\varepsilon(i)}(\tau_{\varepsilon(i)y}A))\end{aligned}$$

since $(*)$ is clearly true (with $c=1$) for polyhedral chains. If we integrate this inequality over $y\in[0,1]^N$ and use Fatou's lemma and Theorem 1.1 (6), we get

$$\begin{aligned} \mathbf{M}(A) &\leqslant \liminf \int_{y\in[0,1]^N} \lambda\mathcal{H}^k(\operatorname{spt} P(y\varepsilon(i),A))\,dy \\ &\leqslant \liminf \lambda c\mathcal{H}^k(\operatorname{spt} A) \\ &= c\lambda\mathcal{H}^k(\operatorname{spt} A). \end{aligned}$$ □

In this proof we used the last inequality in the deformation theorem. That inequality was not proved with the rest of the deformation theorem, so we give the proof now:

PROPOSITION 5.2. *Let A be a flat k-chain and $\varepsilon>0$. Then*

$$\int_{y\in[0,1]^N} \mathcal{H}^k(\operatorname{spt} P^\varepsilon(\tau_{\varepsilon y}A))\,dy \leqslant c\mathcal{H}^k(\operatorname{spt} A).$$

Proof. We assume $\mathcal{H}^k(\operatorname{spt} A)<\infty$, as otherwise the proposition is trivially true. Note that for any Borel set S, we have

$$\varepsilon^j \int_{y\in[0,1]^N} \mathcal{H}^0(Y^{N-j}(\varepsilon)\cap\tau_{\varepsilon y}S)\,dy = \sum_{\Pi} \int_{x\in\mathbf{R}^j} \mathcal{H}^0(\Pi^{-1}x\cap S)\,dx$$

(where the summation is over orthogonal projections Π from $\mathbf{R}^N$ to the various coordinate j-planes)

$$\begin{aligned} &\leqslant \sum_{\Pi} \mathcal{H}^j(S) \qquad (1)\\ &= \binom{N}{j}\mathcal{H}^j(S). \end{aligned}$$

(This follows, for example, from [Ma, 7.7] or [Fe1, 2.10.25].) In particular, letting $S=\operatorname{spt} A$ and $j=k+1$, we see that

$$Y^{N-k-1}(\varepsilon)\cap\operatorname{spt}(\tau_{\varepsilon y}A)=\varnothing \quad \text{for almost all } y. \qquad (2)$$

Choose polyhedral chains A_i' converging rapidly to A with $\operatorname{spt} A_i'\to\operatorname{spt} A$. If Q is a k-cube of $X^k(\varepsilon)$ such that Q^* is disjoint from $\operatorname{spt}(\tau_y A)$, then Q^* is disjoint from $\operatorname{spt}(\tau_y A_i')$ for all sufficiently large i. Consequently (for large i)

$$\#\{Q: Q^*\cap\operatorname{spt}\tau_y A_i'\neq\varnothing\} \leqslant \#\{Q: Q^*\cap\operatorname{spt}\tau_y A\neq\varnothing\}.$$

Note (by Proposition 2.2) that the left-hand side of this inequality is greater than or equal to

$$\varepsilon^{-k}\mathcal{H}^k(\operatorname{spt} P^\varepsilon(\tau_y A_i'))$$

and (for almost every y) the right-hand side is at most

$$\mathcal{H}^0(Y^{N-k}(\varepsilon)\cap\operatorname{spt}\tau_y A)$$

by (2). Thus

$$\mathcal{H}^k(\operatorname{spt} P^\varepsilon(\tau_y A_i'))\leqslant\varepsilon^k\mathcal{H}^0(Y^{N-k}(\varepsilon)\cap\operatorname{spt}\tau_y A). \tag{3}$$

As $i\to\infty$, $P^\varepsilon(\tau_y A_i')\to P^\varepsilon(\tau_y A)$, so (since these chains are all made up of k-cubes from the same grid)

$$\mathcal{H}^k(\operatorname{spt} P^\varepsilon(\tau_y A))\leqslant\liminf\mathcal{H}^k(\operatorname{spt} P^\varepsilon(\tau_y A_i')).$$

Thus by (3),

$$\mathcal{H}^k(\operatorname{spt} P^\varepsilon(\tau_y A))\leqslant\varepsilon^k\mathcal{H}^0(Y^{N-k}(\varepsilon)\cap\operatorname{spt}\tau_y A).$$

Now integrating (and using (1) with $j=k$ and $S=\operatorname{spt} A$) proves the desired inequality. □

Remark. The proof actually shows that

$$\int_{y\in[0,1]^N}\mathcal{H}^k(\operatorname{spt} P^\varepsilon(\tau_{\varepsilon y}A))\,dy\leqslant\nu(\operatorname{spt} A),$$

where

$$\nu(S)=\sum_{\Pi}\int_{x\in\mathbf{R}^k}\mathcal{H}^0(\Pi^{-1}x\cap S)\,dx,$$

the summation being over orthogonal projections Π from $\mathbf{R}^N$ to coordinate k-planes. Consequently, Theorem 5.1 becomes

$$\mathrm{M}(A)\leqslant c\lambda\nu(\operatorname{spt} A).$$

Now if we apply this to $\theta_\# A$, where θ is a rotation of $\mathbf{R}^N$, and then average over all such rotations, we get

$$\mathrm{M}(A)\leqslant c\lambda\mathcal{I}^k(\operatorname{spt} A),$$

where $\mathcal{I}^k$ is k-dimensional integral-geometric measure.

As in Theorem 5.1, the best constant c is in fact 1. This is because a flat chain of finite mass supported in a set of finite integral-geometric measure must be rectifiable [W2, 6.1], and the inequality (with $c=1$) is easy for rectifiable chains. (See [W2, §8] for related results.)

6. Size and other weighted-area functionals

Mass is a weighted area, with densities given by the group norm of the coefficients. In some situations, it is useful to consider other weighted areas. Let $\phi: G \to [0,\infty]$ be any function such that $\phi(0)=0$ and such that $\phi(-g)\equiv\phi(g)$ (i.e. such that ϕ is even). Then ϕ induces a functional (which we also denote by ϕ) on the space of polyhedral chains:

$$\phi: \sum g_i[\sigma_i] \mapsto \sum \phi(g_i)\,\mathrm{area}(\sigma_i),$$

where the σ_i are non-overlapping. For ϕ to be lower semicontinuous with respect to flat convergence on the space of polyhedral chains, we also need $\phi: G\to[0,\infty]$ to be lower semicontinuous and subadditive: $\phi(g+h)\leqslant\phi(g)+\phi(h)$. In fact, these conditions suffice for $\phi: \mathcal{P}_k(\mathbf{R}^N; G)\to\mathbf{R}$ to be $\mathcal{F}$-lower semicontinuous: the proof is exactly the same as for mass lower semicontinuity [Fl, 2.3]. Then ϕ extends to a lower semicontinuous functional

$$\phi: \mathcal{F}_k(\mathbf{R}^N; G) \to [0,\infty]$$

by setting

$$\phi(A) = \inf\{\liminf \phi(A_i) \colon A_i \to A,\ A_i \in \mathcal{P}_k(\mathbf{R}^N; G)\}.$$

For rectifiable chains A, one can describe $\phi(A)$ more explicitly as follows. Recall that if A is a rectifiable k-chain, then there are k-dimensional C^1-submanifolds M_i such that $A = A \llcorner S$, where $S=\bigcup M_i$. We can choose the M_i to be disjoint and oriented. One can then show, if G is separable, that there is an isometry $f\mapsto S\wedge f$ from

$$\{f\in\mathcal{L}^1(\mathcal{H}^k\llcorner S; G) : f \text{ has compact support}\}$$

to

$$\{A\in\mathcal{F}_k(\mathbf{R}^N; G) : \mathrm{M}(A)<\infty \text{ and } A = A\llcorner S\}.$$

(If G is not separable, simply replace the space $\mathcal{L}^1$ above by the closure in $\mathcal{L}^1$ of the set of functions each of which takes only finitely many values.) In particular, the mass of $S\wedge f$ is the $\mathcal{L}^1$-norm of f:

$$\mathrm{M}(S\wedge f) = \int_S |f|\, d\mathcal{H}^k.$$

Likewise

$$\phi(S\wedge f) = \int_S \phi(f(x))\, d\mathcal{H}^k x. \qquad (*)$$

(To show this, one can first use the radon measure $\mu_{A,\phi}: V\mapsto\phi(A\llcorner V)$ of Theorem 6.1 below to reduce to the case when S is a single manifold M_i. That case is fairly straightforward; see [W1, §4] for the special case when S is a k-plane.)

One such $\phi: G \to [0, \infty]$ of particular interest is

$$\phi_s(g) = \begin{cases} 1 & \text{if } g \neq 0, \\ 0 & \text{if } g = 0. \end{cases}$$

By (*), $\phi_s(A)$ is equal to the Hausdorff size (§4) of A if A is rectifiable. In another paper [W2], we will show that if $\mathrm{M}(A)$ and $\phi_s(A)$ are both finite, then A is rectifiable. This together with the corollary to Theorem 4.1 implies that $\phi_s(A)$ is equal to the Hausdorff size of A for any A of finite mass. If A has infinite mass, the definition of Hausdorff size does not make sense, so we define the *size* of A to be $\phi_s(A)$ for every flat chain A.

THEOREM 6.1. *Let $\phi: G \to [0, \infty]$ be an even lower semicontinuous subadditive function such that $\phi(0)=0$. Let $\phi: \mathcal{F}_k(\mathbf{R}^N; G) \to [0, \infty]$ be the corresponding functional on flat chains. Then:*

(1) $\operatorname{Av} \phi(H^\varepsilon(\tau_y A)) \leqslant c\varepsilon\phi(A)$.

(2) $\operatorname{Av} \phi(P^\varepsilon(\tau_y A)) \leqslant c\phi(A)$.

(3) *If $\phi(A)+\mathrm{M}(A)<\infty$, then $\mu_{A,\phi}: S \mapsto \phi(A \llcorner S)$ defines a radon measure on $\mathbf{R}^N$.*

(4) *Suppose that ϕ is positive on the nonzero elements of G. Then $\phi(A)>0$ for every nonzero flat chain A.*

Proof. Assertions (1), (2), and (3) are proved exactly as for the mass functional (Theorem 1.1 and [Fl, §4]). To prove (4), suppose $\phi(A)=0$. Then by (2), we see that $\phi(P^\varepsilon(\tau_y A))=0$ and hence (by the hypothesis on ϕ) that $P^\varepsilon(\tau_y A)=0$ for almost every y. But then $A=0$ (by Corollary 1.2 (4) or Corollary 1.3). □

This theorem lets us extend Almgren's isoperimetric inequality [A, 2.3] to such functionals:

THEOREM 6.2. *Let ϕ be as in Theorem 6.1 and let $k>0$. If T is a flat k-cycle, then it is the boundary of a flat $(k+1)$-chain A such that*

$$\phi(A) \leqslant c\phi(T)(\operatorname{size} T)^{1/k} \tag{*}$$

and

$$\operatorname{size}(A) \leqslant c(\operatorname{size} T)^{(k+1)/k}.$$

In particular, this is true with $\phi(\cdot)=\mathrm{M}(\cdot)$.

Proof. Let

$$\begin{aligned} S = S(\varepsilon) &= \{y \in [0,1]^N : P^\varepsilon(\tau_{\varepsilon y} T) = 0\} \\ &= \{y \in [0,1]^N : \tau_y T = \partial H^\varepsilon(\tau_{\varepsilon y} T)\}. \end{aligned}$$

Note that if $y \notin S$, then size $P^{\varepsilon}(\tau_y T) \geqslant \varepsilon^k$. Hence by Theorem 6.1 (2),

$$\varepsilon^k \mathcal{L}^N([0,1]^N \setminus S) \leqslant c \operatorname{size} T.$$

Let $\varepsilon^k = 2c \operatorname{size} T$. Then

$$\mathcal{L}^N(S) \geqslant \tfrac{1}{2}.$$

Consequently Theorem 6.1 (1) implies

$$\begin{aligned}\operatorname{Av}_{y\in S} \phi(H^{\varepsilon}(\tau_{\varepsilon y}T)) &\leqslant 2 \operatorname{Av} \phi(H^{\varepsilon}(\tau_{\varepsilon y}T)) \\ &\leqslant 2c\varepsilon\phi(T).\end{aligned}$$

Likewise

$$\operatorname{Av}_{y\in S}(\operatorname{size} H^{\varepsilon}(\tau_{\varepsilon y}T)) \leqslant 2c\varepsilon \operatorname{size} T.$$

Hence we can pick a $y \in S$ with

$$\begin{gathered}\phi H^{\varepsilon}(\tau_{\varepsilon y}T) \leqslant 3c\varepsilon\phi T = 3c(2c \operatorname{size} T)^{1/k}\phi T, \\ \operatorname{size} H^{\varepsilon}(\tau_{\varepsilon y}T) \leqslant 3c\varepsilon \operatorname{size} T = 3c(2c \operatorname{size} T)^{(k+1)/k}.\end{gathered}$$

Then $A = \tau_{-\varepsilon y} H^{\varepsilon}(\tau_{\varepsilon y}T)$ is the desired surface. □

Remark. Given any finite set of ϕ, there is an A that simultaneously satisfies $(*)$ for each ϕ. However, the constant c increases with the number of ϕ.

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