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Harmonic analysis, cohomology, and the large-scale geometry of amenable groups

by

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To my father Abraham, and to our baby daughter Noga

Contents

1. Introduction

Geometric group theory has made remarkable progress in the last decade, producing a wealth of striking and deep results which incorporate a wide range of mathematical tools. Following the success of the theory in dealing with "large" groups, such as the quasi-isometry classification of lattices in semisimple Lie groups (cf. the survey [21]; see also [26] and [13]), more attention has been paid recently to "smaller" (e.g. solvable) groups, which do not generally enjoy similar richness of geometric structure. Here the theory has had some outstanding achievements (cf. [22], [23], [24], [46]), although yet with a limited scope, and leaving open many fundamental questions (cf. the survey [25]). The purpose of the present paper is to introduce new ideas, techniques and results, to the geometric group theory of amenable groups, involving representation theory and cohomology. Some of the results and notions developed from spectral theory should be of independent interest, an aspect elaborated upon after stating the main results.

Statement and discussion of the main results. The following result, a fundamental rigidity theorem in itself, serves as a good motivation for our approach:

THEOREM 1.1. (Quasi-isometric rigidity of $\mathbf{Z}^d$.) *If Γ is a group quasi-isometric to $\mathbf{Z}^d$, then Γ has a finite index subgroup isomorphic to $\mathbf{Z}^d$.*

See below, or Definition 2.1.1, for the definition of quasi-isometry (abbreviated q.i.). Theorem 1.1 is known to hold using Gromov's polynomial growth theorem [30] (which reduces it to the case where Γ is nilpotent). Yet a natural question raised by several authors (e.g. [17], [28], [30], [33]) is whether one could find an elementary argument, particularly, avoiding the heavy ingredient related to Montgomery–Zippin's work on Hilbert's fifth problem [42], which is involved in Gromov's proof. We shall indeed present such an argument, which illustrates the main ideas of our approach with very little effort in terms of the spectral theory involved. Considerably more effort in this direction is needed for the other, new results obtained below.

A central notion in the sequel will be that of Betti numbers. Recall that for a discrete group Γ and $n \geqslant 0$, the nth Betti and virtual Betti numbers are defined by

$$b_n(\Gamma) = \dim_{\mathbf{R}} H^n(\Gamma, \mathbf{R}) \quad \text{and} \quad vb_n(\Gamma) = \sup\{b_n(\Gamma_0) \mid \Gamma_0 < \Gamma \text{ and } [\Gamma : \Gamma_0] < \infty\},$$

respectively.

Note that $b_1(\Gamma)$ is also the rank of the abelianization of Γ (after tensoring with $\mathbf{Q}$). To put some of our next results in a better perspective, it is good to keep in mind that the virtual Betti numbers are *not* q.i. invariants, neither in general, nor in the class of amenable groups. One example is that of the wreath product $\mathbf{Z} \wr \mathbf{Z}$, which has infinite vb_1, and is q.i. to the group $\mathbf{Z}\wr D$ satisfying $vb_1=2$, where D is the infinite dihedral group [19]. Another one, in the much tamer class of polycyclic groups (which all have finite vb_n's), is shown in §5.5 below.

THEOREM 1.2. (Quasi-isometry invariance of Betti numbers for nilpotent groups.) *If Γ and Λ are (finitely generated) quasi-isometric nilpotent groups, then for all n one has $b_n(\Gamma)=b_n(\Lambda)$.*

All the previously known q.i. invariants for nilpotent groups follow from Pansu's well-known theorem [44], which states that the *graded* (real) Lie algebra of the (Mal′tsev completion of the) nilpotent group is a q.i. invariant. Following the proof of Theorem 1.2 in §4.1 below, we present an example due to Yves Benoist, showing that this is not the case here. Namely, there exist nilpotent groups (in every dimension $\geqslant 7$) which have the same asymptotic cone, and hence cannot be q.i. distinguished by Pansu's theorem, but are nevertheless not q.i. by Theorem 1.2.

We continue one step further, to the class of polycyclic groups, whose large-scale geometry understanding remains a major challenge. Prior to the result stated next, there seems to be no polycyclic group (which is not virtually nilpotent) for which some "non-trivial" property (unlike amenability, finite presentability, etc.) is known to hold for all groups q.i. to it.

THEOREM 1.3. (Quasi-isometric to polycyclic implies $vb_1>0$.) *If Γ is quasi-isometric to a polycyclic group, then Γ has a finite index subgroup with infinite abelianization.*

As shown in §5.5, no better bound on vb_1 in the theorem can be given, other than its positivity. We next consider amenable groups which are more "complicated" in at least one of two ways: Firstly, non-(virtually) solvable groups, and secondly, groups which are not finitely presentable. In the first category we will deal with the same groups recently shown to exhibit somewhat surprising *non-rigid* behavior [19], namely, lamplighter-type groups. Concerning the second category, we note that the same remark mentioned before Theorem 1.3 concerning polycyclic groups applies to non-finitely presentable ones equally well. Unlike with polycyclicity, however, here there seems to be some inherent difficulty, as all the geometric approaches require some "nice model space" for the group, a space which automatically implies finiteness properties like finite presentability. For example, in the recent q.i. rigidity theorem for nilpotent-by-cyclic groups [46] (generalizing [24]), finite presentability—along with not being polycyclic—is specifically assumed, so here, again, our results seem to complement the literature. We next describe one class of non-finitely presentable, abelian-by-cyclic groups, to which our methods apply.

For any pair $n, m \in \mathbf{Z}$ of non-zero co-prime integers, let $\mathbf{Z}[1/nm]$ be the ring of rational numbers generated (as a ring) by $1/nm$, and let

$$\Gamma(n,m) = \mathbf{Z} \ltimes \mathbf{Z}\left[\frac{1}{nm}\right],$$

where $\mathbf{Z}$ acts through multiplication by (powers of) m/n. It is easy to see that $\Gamma(n,m)$ is finitely generated. In the case where $|m|=1$ or $|n|=1$, we merely recover the solvable Baumslag–Solitar groups, but otherwise $\Gamma(n,m)$ is only a quotient of the (non-solvable) Baumslag–Solitar group $\mathrm{BS}(n,m)$. The case of a Baumslag–Solitar group is the only one where $\Gamma(n,m)$ is finitely presented, as follows from [10].

We can now state our next result:

THEOREM 1.4. (Non-finitely presentable amenable groups.) *Let Γ be either a group of the form $\Gamma(n,m)$ as above, or a lamplighter group $L(F)$ associated with some* finite *group F, i.e. a wreath product*

$$L(F) = \mathbf{Z} \ltimes \bigoplus_{i \in \mathbf{Z}} F_i,$$

where each F_i denotes a copy of F and the semi-direct product is with respect to the shift $\mathbf{Z}$-action. If Λ is a group quasi-isometric to Γ then $vb_1(\Lambda)=1$.

Furthermore, there exists a family of $2^{\aleph_0}$ 3-step solvable groups Γ to which the same conclusion applies.

In fact, the same result holds also for the family of lattices Γ in SOLV. This, together with further results proved below, provides some evidence for the generally believed conjecture that the *family* of all these lattices is q.i. rigid (cf. [25]). Note also the marked difference between the case of a lamplighter group $L(F)$ with F finite (Theorem 1.4), and F infinite (the remark preceding Theorem 1.2 above).

The "algebraic" setting. It turns out that our approach admits an algebraic counterpart, which gives rise to a different set of results. Here we will actually be able to deal with a notion more general than a quasi-isometry, namely, a uniform embedding. We now define it in a way which is free of a choice of a generating set, thereby making it (and the notion of quasi-isometry with it) meaningful also for non-finitely generated groups.

Definition. Let Λ and Γ be discrete countable groups.

(i) A map $\varphi: \Lambda \to \Gamma$ is called a *uniform embedding* (abbreviated *UE*) if for every sequence of pairs $(\alpha_i, \beta_i) \in \Lambda \times \Lambda$ one has

$$\alpha_i^{-1}\beta_i \to \infty \text{ in } \Lambda \quad \Longleftrightarrow \quad \varphi(\alpha_i)^{-1}\varphi(\beta_i) \to \infty \text{ in } \Gamma$$

(where $\to\infty$ means eventually leaving every finite subset).

(ii) Λ and Γ are said to be *quasi-isometric* if there exists a uniform embedding $\varphi: \Lambda \to \Gamma$ and a *finite* subset $C \subseteq \Gamma$ such that as sets, $\varphi(\Lambda) \cdot C = \Gamma$.

The equivalence of (ii) above with the usual notion of quasi-isometry in the case of finitely generated groups (see Definition 2.1.1) is a part of Theorem 2.1.2 below. Even though the above definition makes use of the group structure, the notion of uniform embedding extends naturally to all metric spaces. It was introduced by Gromov and studied in relation to the Baum–Connes conjecture, in the context of a Hilbert space target (see e.g. [48] and the references therein). Rather little has been said about it in the framework of geometric group theory, partially because it is a much more flexible notion than a quasi-isometry (cf. [43] where it is called "uniformly proper embedding" or [11] where it is termed "packing", and the references therein). Among prime motivating examples to keep in mind, note that any subgroup inclusion is a UE, and any discrete subgroup Λ of a locally compact group G uniformly embeds in a co-compact discrete subgroup $\Gamma < G$ (see §6.2 below for more on the "converse" situation). Somewhat more counter-intuitively, any nilpotent group uniformly embeds in $\mathbf{Z}^d$ for d large enough, some hyperbolic groups do receive a UE of high-rank abelian groups, and non-abelian free groups uniformly embed in all (non-virtually nilpotent) solvable groups (see §§ 6.1 and 6.3 below). In general, the restriction of a quasi-isometry to a subgroup is only a UE; hence a study of this notion is valuable even if one is interested in quasi-isometries only.

We next recall the following fundamental notion:

Definition. For a group Γ and a ring R, define the *cohomological dimension* of Γ over R, $\operatorname{cd}_R \Gamma$, by

$$\operatorname{cd}_R \Gamma = \sup\{n \mid \text{there exists an } R\Gamma\text{-module } V \text{ with } H^n(\Gamma, V) \neq 0\}.$$

Note that the cohomological dimension may be infinite. For the next theorem, recall that a commutative ring R with unit is *divisible* if every $0 \neq n \in \mathbf{Z}$ is invertible in R.

THEOREM 1.5. (Uniform embeddings "respect" cohomological dimension.) *Let R be a commutative divisible ring with unit, and let Λ and Γ be any countable groups with Λ amenable. If Λ uniformly embeds in Γ then $\operatorname{cd}_R \Lambda \leqslant \operatorname{cd}_R \Gamma$.*

In particular, if Λ and Γ are quasi-isometric then $\operatorname{cd}_R \Lambda = \operatorname{cd}_R \Gamma$.

In all the applications of Theorem 1.5, it will be enough to consider the ring $\mathbf{Q}$ of rational numbers. Although cohomological dimension is often regarded over the integers $\mathbf{Z}$ in the literature, not only that divisibility of R is necessary for the theorem, but it actually seems the natural assumption in the framework of geometric group theory. Indeed, when R is not divisible, any group is commensurable to a group with infinite cd_R, and as remarked by Gersten [27], there are such examples where passing to the *virtual* cohomological dimension does not amend the problem. Another advantage of working over $\mathbf{Q}$ rather than $\mathbf{Z}$ (for example) is illustrated with the lamplighter group Γ (defined above). One has $\mathrm{cd}_{\mathbf{Z}}\,\Gamma=\infty$ but $\mathrm{cd}_{\mathbf{Q}}\,\Gamma=2$; hence by Theorem 1.5 any amenable group admitting a UE into Γ satisfies the strong restriction $\mathrm{cd}_{\mathbf{Q}}\,\Lambda\leqslant 2$. Gromov asked in [31, 1.H] whether the cohomological dimension is a q.i. invariant. Under the $\mathfrak{F}_\infty$-finiteness assumption on the groups (see §3.3 below), it was shown by Gersten [27] that q.i. groups have the same cohomological dimension, but provided it is a priori assumed to be finite for *both* of them. This condition makes the result more difficult to implement when trying to prove q.i. rigidity-type results without making some finiteness assumption on the "mystery group". On the other hand, [27] gives information also over the ring $R=\mathbf{Z}$, and although limited to the case of quasi-isometries, it applies to *all* (not only amenable) groups. Obviously, it would be desirable to remove the amenability assumption on Λ in Theorem 1.5; interestingly, this is possible at least for some arithmetic groups, such as $\mathrm{SL}_n(\mathbf{Z})$ (see §6.1).

To state some applications of Theorem 1.5 for solvable groups, recall first the basic invariant of such a group Γ—its *Hirsch number*, $h\Gamma$—which is defined by

$$h\Gamma=\sum_{i\geqslant 0}\dim_{\mathbf{Q}}((\Gamma^{(i)}/\Gamma^{(i+1)})\otimes_{\mathbf{Z}}\mathbf{Q}),$$

where $\Gamma^{(i)}$ denotes the ith term in the derived series of Γ.

THEOREM 1.6. (Uniform embeddings of solvable groups.) *Let Λ and Γ be solvable groups. If Λ uniformly embed in Γ then*:

(1) *$h\Lambda\leqslant h\Gamma+1$, and $h\Lambda\leqslant h\Gamma$ if Γ is of type* (FP) (*e.g. if Γ is polycyclic, see below*);

(2) *in particular, if Λ and Γ are quasi-isometric then $|h\Lambda-h\Gamma|\leqslant 1$;*

(3) *if Γ is of bounded rank and Λ is torsion free, then Λ is also of bounded rank.*

Recall that a group Γ is of bounded (Prüfer) rank, if there is some $d=d(\Gamma)$ such that every *finitely* generated subgroup of Γ can be generated by at most d elements.

To sharpen our results we need another result of cohomological type. Recall that a group Γ is said to be of type (FP) over a ring R, if R, as a trivial $R\Gamma$-module, has a finite-length projective resolution over $R\Gamma$, with every module being finitely generated.

THEOREM 1.7. (Type (FP) is a quasi-isometry invariant.) *Let R be as in Theorem* 1.5. *In the class of amenable groups, being of type* (FP) *over R is a quasi-isometry invariant.*

This implies the following sharpening of Theorem 1.6 (2) for "tame" groups, which we actually expect to hold in complete generality (see §6.4):

THEOREM 1.8. (Quasi-isometry invariance of Hirsch number.) *Let Γ be a solvable group of type* (FP) *over* $\mathbf{Q}$. *If Λ is any solvable group quasi-isometric to Γ then $h\Lambda = h\Gamma$. The same conclusion holds if we drop the type* (FP) *assumption on Γ, but assume instead that both Γ and Λ are torsion free.*

The first result of this type was established by Bridson and Gersten [14] when both Γ and Λ are polycyclic, a property which implies being of type (FP) over $\mathbf{Q}$. More generally, any group of type (FP) over $\mathbf{Z}$ is of type (FP) over $\mathbf{Q}$, and the torsion-free solvable groups with the former property admit a concrete algebraic characterization (of being "constructible")—see §3.4 below.

As a concrete application of our results for solvable groups, we return to the groups $\Gamma(n,m)$ defined and discussed earlier.

THEOREM 1.9. (Solvable groups quasi-isometric to $\Gamma(n,m)$.) *Let m and n be co-prime integers. If Λ is a torsion-free solvable group which is quasi-isometric to $\Gamma(n,m)$, then there exists some co-prime integers m' and n' so that Λ has a finite index subgroup isomorphic to $\Gamma(n',m')$.*

We remark that in the case $|n|, |m| \neq 1$, it does not seem possible to prove even that Λ has finite Hirsch number without appealing to Theorem 1.5 above.

The approach. The following three notions will be central to all that follows:

Definition. Let Γ be a discrete group. Say that Γ has property H_{FD}, H_F or H_T, if for every unitary Γ-representation π with $\overline{H}^1(\Gamma,\pi)\neq 0$, there is a Γ-subrepresentation $\sigma\subseteq\pi$ which is *finite-dimensional, finite* or *trivial*, respectively.

By a finite representation we mean one factoring through a finite quotient of Γ. Here $\overline{H}^1$ denotes the first *reduced* cohomology group, i.e. the quotient $Z^1/\overline{B}^1$, where $\overline{B}^1$ is the closure of the space of 1-coboundaries in the topology of pointwise convergence on Γ. Before explaining the relevance of these notions to geometric group theory, a few words may be in place to put them in some perspective.

In [50] it was shown that any compactly generated group Γ without Kazhdan's property (T) admits some (continuous) *irreducible* unitary representation π with $\overline{H}^1(\Gamma,\pi)\neq 0$ (recall that if Γ does have (T) then $\overline{H}^1(\Gamma,\pi)=H^1(\Gamma,\pi)=0$ for all π). This result, which is completely non-constructive, is shown in [50] to have various applications. Of course, it applies to all finitely generated infinite amenable groups, and since the latter family lies in the extreme opposite to the Kazhdan property, one might expect that for amenable Γ the set of such π's is "wild". Non-abelian free groups, for example, satisfy $\overline{H}^1\neq 0$ for "most" unitary representations π (and $H^1\neq 0$ for all of them), whereas in the well-behaved class of simple Lie groups, the classification of these "cohomological" π's needs considerable work. Combined with the well-known fact that there is no available description of the irreducible unitary dual of (non-virtually abelian) discrete groups, one may be led to expect that no classification of such cohomological π should be possible in the class of amenable groups. However, groups with properties H_T, H_F or H_{FD} have an extremely restricted set of such representations π, e.g. the *trivial representation alone* for the sharpest of them—property H_T. Keeping this in mind, the highlights of our (spectral) approach are:

(i) From a purely spectral-theoretic point of view, it is actually a rather general, interesting phenomenon, that many amenable groups do satisfy one of these properties;

(ii) This phenomenon has concrete applications to geometric group theory, such as those described in the previous subsection.

Let us elaborate now on these two issues, starting with the central result connecting the harmonic analysis to geometric group theory (proved in Theorem 4.3.3 below):

THEOREM 1.10. (Quasi-isometry invariance of H_{FD}.) *In the class of amenable groups, having property H_{FD} is a quasi-isometry invariant.*

Although no longer geometric, the other properties help to sharpen the applications:

THEOREM 1.11. (Consequences of H_{FD} and H_F.) (1) *Any finitely generated amenable group Λ with property H_{FD} contains a finite index subgroup with infinite abelianization. In particular, if Λ is quasi-isometric to a group with property H_{FD}, then the same conclusion holds.*

(2) *If Λ is quasi-isometric to an amenable Γ with property H_F, then $vb_1(\Lambda) \leqslant vb_1(\Gamma)$.*

We now illustrate the use of these notions by describing the proof of q.i. rigidity of $\mathbf{Z}^d$. For this purpose, we present the first examples of groups with property H_T:

THEOREM 1.12. (Abelian groups.) *Every abelian group has property H_T.*

Unlike the case for other groups with property H_T, this turns out to be elementary. Another easy result we shall prove is a simple sufficient and necessary condition for a group with property H_{FD} to have the stronger property H_F (see Proposition 4.2.3), of which a special case is the following result:

LEMMA. *Let Γ be a finitely generated group with property H_{FD}. If Γ has subexponential growth then Γ has property H_F.*

Proof of Theorem 1.1. Let Λ be q.i. to $\mathbf{Z}^d$. By Theorem 1.12, $\mathbf{Z}^d$ has in particular property H_{FD}; hence by Theorem 1.10, Λ has it as well. By the lemma above Λ satisfies property H_F, so both groups have this property, and from Theorem 1.11 (2) it follows that $vb_1(\Lambda) = vb_1(\mathbf{Z}^d) = d$. This shows that a finite index subgroup of Λ surjects onto $\mathbf{Z}^d$, and because both groups have polynomial growth of the same degree d, the kernel of the surjection must be finite, as required. □

See Theorem 4.3.6 below for the complete details. The proof of the q.i. invariance of Betti numbers for nilpotent groups uses a result analogous to Theorem 1.11 (2), once they are shown to possess a generalized property H_T for *any* cohomology degree (this, however, requires considerably more efforts than in the abelian case above). In particular, nilpotent groups also have property H_T, but this is no longer the case in general (even virtually) for polycyclic groups.

THEOREM 1.13. (Polycyclic groups.) *Every polycyclic group has property* H_{FD}.

The starting point of the proof of this result is the (non-trivial) fact that every polycyclic group virtually embeds as a co-compact lattice in a connected solvable Lie group. This enables one to appeal to a rather involved work of Delorme [16], concerning the cohomology of irreducible unitary representations of such groups. It would be intriguing (and we believe also rewarding—see §6.6 below) to find a proof of Theorem 1.13 even for some polycyclic (non-nilpotent) groups, without embedding them co-compactly in a connected Lie group. While Delorme's work relies heavily on Lie algebra cohomology, we develop an alternative, more geometric approach, which enables us to treat also the aforementioned, more "exotic" groups:

THEOREM 1.14. (Non-finitely presentable groups.) *Let* Γ *be a group in one of the following classes*:

(1) *A group* $\Gamma(n,m)$ *as defined before Theorem* 1.4 *above*;

(2) *A lamplighter group* $L(F)$ *associated with some* finite *group* F.

Then Γ *has property* H_T. *Furthermore, there exist* $2^{\aleph_0}$ *non-isomorphic finitely generated* 3*-step solvable groups* Γ *with property* H_T.

Since all the groups appearing in the theorem satisfy $b_1=1$, by Theorem 1.11 any group q.i. to them satisfies $vb_1=1$, and hence Theorem 1.4 follows. An interesting aspect of the proof of Theorem 1.14 is that in both (1) and (2) we construct a locally compact (non-discrete) group G containing Γ discretely and co-compactly, and prove first that G has property H_T (using "Mackey's machinery"). This is then "transferred" to Γ. Another interesting feature of the proof is that it provides one instance where it is actually geometric group theory which is used for spectral theory, and not the other way around. This concerns part (2), which we prove directly when F is *abelian*. We then use geometric group theory (through [19]), together with the q.i. invariance of property H_{FD}, to deduce the theorem for every finite F. A second application of geometric group theory to spectral theory comes in the proof of the following fact, which balances our foregoing results:

THEOREM 1.15. (Solvable groups without H_{FD}.) *The wreath products* $\mathbf{Z}\wr\mathbf{Z}$ *and* $\mathbf{Z}\wr D$, *where* D *is the infinite dihedral group,* do not *have property* H_{FD}.

Roughly, once geometric group theory tells us that the two groups are quasi-isometric, the fact that one has finite and the other infinite vb_1, together with Theorem 1.11 (2), accounts (modulo some details) for this result—see §5.4 below.

Finally, to mention a few words about our basic idea—relating the representation theory and cohomology of quasi-isometric amenable groups—we recall first the following observation of Gromov [31], which is our departure point: The groups Γ and Λ are quasi-isometric if and only if there exists a locally compact space X on which both groups act properly discontinuously, co-compactly, and in a commuting way (see §2.1 below). Consequently, one gets a bundle-type structure $X \to X/\Lambda$, on which Γ acts, allowing for an induction map from Λ-modules to Γ-modules. When unitary representations are involved, the amenability is used to get a (σ-additive, σ-finite) $\Gamma\times\Lambda$-invariant measure on X, giving rise to an L^2-unitary induction à la Mackey. The boundedness of the fundamental domains enables one to define an induction map on the corresponding cohomology groups, and show its injectivity. In the algebraic approach, an analogous "smooth" induction functor on the category of R-modules is defined, along with a map between the cohomology groups. Here, amenability implies the injectivity of this map via the existence of a finitely additive R-valued invariant measure, entering the definition of an appropriate "transfer operator". Curiously, our proof here makes use of additive homomorphisms from $\mathbf{R}$ to $\mathbf{Q}$.

Suggestion to the reader. Readers interested in the first (resp. second) set of results, i.e. Theorems 1.1–1.4 (resp. Theorems 1.5–1.9), are recommended to start reading §4.1 (resp. §3.3). Those interested primarily in the spectral theory aspects, may prefer to look first at §5.

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2. Preliminaries

2.1. Topological and measurable couplings

We begin by recalling the standard definition of quasi-isometric groups.

Definition 2.1.1. Let Γ and Λ be groups generated by the finite (symmetric) sets S_Γ and S_Λ, respectively, and let d_Γ and d_Λ be the corresponding word metrics on Γ and Λ. Then Γ and Λ are called *quasi-isometric* if there exists a map $\varphi:\Lambda\to\Gamma$ and constants $\alpha\geqslant 1$ and $K\geqslant 0$ such that for all $\lambda_1,\lambda_2\in\Lambda$,

$$\alpha^{-1}d_\Lambda(\lambda_1,\lambda_2)-K\leqslant d_\Gamma(\varphi(\lambda_1),\varphi(\lambda_2))\leqslant \alpha d_\Lambda(\lambda_1,\lambda_2)+K, \tag{1}$$

and any element $\gamma\in\Gamma$ lies within distance $\leqslant K$ from $\varphi(\Lambda)$.

We next establish the framework in which we shall work throughout the paper.

THEOREM 2.1.2. *For countable groups Λ and Γ, consider the following statements*:

(i) *There exists a uniform embedding $\varphi:\Lambda\to\Gamma$ (as defined in the introduction)*;

(ii) *There exists a locally compact space X on which both Λ and Γ act continuously and properly, such that the two actions commute, and the Γ-action is co-compact, i.e. there exists a bounded subset $X_\Gamma\subseteq X$ with $\Gamma\cdot X_\Gamma=X$*;

(iii) *There exists φ as in* (i) *and a finite subset $C\subseteq\Gamma$ such that $\varphi(\Lambda)\cdot C=\Gamma$*;

(iv) *There exists X as in* (ii), *but with both actions being co-compact.*

Then (i) *is equivalent to* (ii), *and* (iii) *is equivalent to* (iv). *Furthermore, if any of* (i)–(iv) *holds, then after replacing Γ with a direct product $\Gamma\times M$ for some finite group M, a space X can be found with the following three* additional *properties*:

(1) *Both actions on X are free*;

(2) *There exist fundamental domains X_Λ and X_Γ which are both open and closed* (*with X_Γ compact in case* (i)$\Leftrightarrow$(ii), *and both X_Λ and X_Γ compact in case* (iii)$\Leftrightarrow$(iv));

(3) $X_\Gamma\subseteq X_\Lambda$.

Finally, let d_Λ and d_Γ be left Λ- and Γ-invariant metrics on Λ and Γ, respectively, which are proper (*all balls are finite*). *Then for any φ as in* (i) *there exist non-decreasing unbounded real functions $F_1,F_2:\mathbf{R}\to\mathbf{R}$ such that for all $\lambda_1,\lambda_2\in\Lambda$,*

$$F_1(d_\Lambda(\lambda_1,\lambda_2))\leqslant d_\Gamma(\varphi(\lambda_1),\varphi(\lambda_2))\leqslant F_2(d_\Lambda(\lambda_1,\lambda_2)), \tag{2}$$

and if moreover d_Λ and d_Γ are word metrics on the finitely generated groups Λ and Γ, respectively, then F_2 can be taken as a linear function. Consequently, any φ as in (iii) *is a quasi-isometry of Λ with Γ, and* (iii) *or* (iv) *are equivalent to Λ and Γ being quasi-isometric in the sense of Definition* 2.1.1.

Everything in the theorem is elementary, and many claims are trivial. We shall only elaborate here, for completeness, on those statements which we shall actually make use of, leaving to the reader to complete the missing details (possibly using the reference below).

The point of view of this theorem is the one adopted by Gromov [31, 0.2.C_2'] in the topological characterization of quasi-isometric groups, described at the last paragraph of the introduction: The finitely generated groups Λ and Γ are q.i. if and only if statement (iv) in Theorem 2.1.2 holds. A guided exercise proving this equivalence can be found in [33, p. 98], to which we refer for more details. Since we need a space X with the *additional* properties (1)–(3) above, we shall need to go into some detail concerning this construction.

Proof. Assume that d_Λ and d_Γ are left invariant proper metrics on Λ and Γ, respectively. It is easy to see that any countable group, not necessarily finitely generated, admits such a metric. Let $\varphi: \Lambda \to \Gamma$ be a uniform embedding. Defining

$$F_1(t) = \inf\{d_\Gamma(\varphi(\lambda_1), \varphi(\lambda_2)) \mid d_\Lambda(\lambda_1, \lambda_2) \geqslant t\},$$
$$F_2(t) = \sup\{d_\Gamma(\varphi(\lambda_1), \varphi(\lambda_2)) \mid d_\Lambda(\lambda_1, \lambda_2) \leqslant t\},$$

it is clear by the assumption on φ that both functions are finite, non-decreasing, unbounded, and (2) is satisfied. We now proceed to show how the existence of such a φ implies that a space X satisfying the additional properties (1)–(3) in the theorem exists, thereby establishing also the implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv).

By assumption on φ, it is clear that there is a finite subset $Q \subseteq \Lambda$ such that if $\varphi(\lambda_1) = \varphi(\lambda_2)$ then $\lambda_2^{-1}\lambda_1 \in Q$. Hence, by taking any finite group M of order greater than $|Q|$, and replacing Γ with $\Gamma \times M$, we may take φ to be injective. Let us still denote by F_1 and F_2 modified functions such that (2) is satisfied for φ. Consider the space X of all *injective* maps $\Lambda \hookrightarrow \Gamma$, satisfying the same uniform estimates (2) as for φ, equipped with the pointwise convergence topology. It is easy to verify that X is locally compact (this follows from the right-hand side of (2), and also from the fact that there is a compact-open fundamental domain for the action of the discrete group Γ—see below). The groups Λ and Γ admit natural commuting actions on X, by pre- and postcomposing maps with their self-left actions (being isometric, X is stable under these two actions). The Γ-action on X is obviously free, proper and co-compact. Indeed, the set $X_\Gamma = \{\psi \in X \mid \psi(e) = e\}$ is a compact-open fundamental domain for its action. As for the Λ-action on X, the left-hand side of (2) implies that it is proper, and because X is a space of injective maps it is also free, although not co-compact in general. However, if there is a finite subset $C \subseteq \Gamma$ such that $\varphi(\Lambda) \cdot C = \Gamma$, and we add the ($\Gamma \times \Lambda$-invariant and "closed") condition that all maps ψ in X satisfy the same property, then the compact subset $K = \{\psi \in X \mid \psi(e) \in C^{-1}\}$ satisfies $\Lambda K = X$, and hence Λ acts co-compactly (this latter argument establishes the implication (iii) $\Rightarrow$ (iv) of the theorem).

We now wish to show that one can find a closed and open (abbreviated *clopen*) fundamental domain X_Λ for the Λ-action, with $X_\Gamma \subseteq X_\Lambda$ (recall that X_Γ above is clopen). For an element $\gamma \in \Gamma$ define the subsets

$$E_\gamma = \{\psi \in X \mid \psi(e) = \gamma\} \quad \text{and} \quad K_\gamma = \Lambda \cdot E_\gamma = \{\psi \in X \mid \psi \text{ takes the value } \gamma\}.$$

CLAIM 2.1.3. *For any $\gamma_1, \gamma_2 \in \Gamma$, the set $K_{\gamma_2}^C \cap E_{\gamma_1}$ is open, where $K_\gamma^C = X - K_\gamma$.*

Indeed,

$$K_{\gamma_2}^C \cap E_{\gamma_1} = \bigcap_{\lambda \in \Lambda} (\lambda E_{\gamma_2}^C \cap E_{\gamma_1}) = \bigcap_{\lambda \in \Lambda} \{\psi \mid \psi(\lambda) \neq \gamma_2 \text{ and } \psi(e) = \gamma_1\}.$$

If $\psi(e) = \gamma_1$, then by the existence of F_1, for λ far enough we have $\psi(\lambda) \neq \gamma_2$, and hence $K_{\gamma_2}^C \cap E_{\gamma_1}$ is a finite intersection of open subsets.

Now, enumerate the elements of Γ as $\gamma_0 = e, \gamma_1, \gamma_2, \ldots$ and define

$$X_\Lambda = E_e \cup \bigcup_{i=1}^{\infty} E_{\gamma_i} \cap K_{\gamma_{i-1}}^C \cap \ldots \cap K_e^C.$$

In other words, for $\psi \in X$, if n is the minimal integer so that ψ takes the value γ_n, then $\psi \in X_\Lambda$ if and only if $\psi(e) = \gamma_n$ (note that if $\psi(e) \neq \gamma_n$ then by injectivity there is obviously a *unique* element $\lambda \in \Lambda$ which translates ψ back to X_Λ). The claim above implies that every subset in the union is open, hence X_Λ is open. Since X_Λ is obviously a fundamental domain for the Λ-action containing $E_e = X_\Gamma$, and X_Γ is obviously clopen, we are only left with verifying that X_Λ is closed as well. Indeed, assume that $\psi_i \in X_\Lambda$ and $\psi_i \to \psi$. By passing to a subsequence we may assume that for all i, $\psi_i(e) = \psi(e) = \gamma_n$ for some fixed $\gamma_n \in \Gamma$. If $n=0$, i.e. $\gamma_n = e$, then the claim is clear. Otherwise, we need to show that ψ *does not take* any of the values $\gamma_0, \ldots, \gamma_{n-1}$; but if it did, say $\psi(\lambda_0) = \gamma_i$, then for all i large enough, ψ_i would satisfy this as well, contradicting the assumption that $\psi_i \in X_\Lambda$ for all i. Notice that when condition (iii) is satisfied, then as soon as $C^{-1} \subseteq \{\gamma_0, \gamma_1, \ldots, \gamma_{i-1}\}$, the ith set in the union defining X_Λ is empty, and hence X_Λ is indeed compact.

We have thus shown that (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv). We shall not need here the reverse implications, but since they are easy, we indicate the main idea. Let $X_\Gamma \subseteq X$ be a compact subset such that $X = \Gamma X_\Gamma$. Pick $x_0 \in X$, and for each $\lambda \in \Lambda$ let $\gamma \in \Gamma$ be some element such that $\lambda x_0 \in \gamma X_\Gamma$. Then the map $\varphi(\lambda) = \gamma$ is a uniform embedding of Λ in Γ, and if $\Lambda \backslash X$ is compact, there exists a finite subset $C \subseteq \Gamma$ such that $\varphi(\Lambda) \cdot C = \Gamma$ (hint: if X_Γ was a fundamental domain, one could take C^{-1} as a finite subset such that $C \cdot X_\Gamma$ contains X_Λ—see [33, p. 98] for more details).

Finally, we are left with the case where d_Λ and d_Γ are word metrics on the finitely generated groups Λ and Γ, respectively. First of all, if φ as in (iii) exists, and one of the groups is finitely generated, then by (iv) and [33, p. 98, Item 34] the other is as well. In the word metric cases, if $\alpha<\infty$ is such that $d_\Lambda(\lambda_1,\lambda_2)=1 \Rightarrow d_\Gamma(\varphi(\lambda_1),\varphi(\lambda_2))\leqslant\alpha$, it follows from a triangle inequality that φ satisfies $d_\Gamma(\varphi(\lambda_1),\varphi(\lambda_2))\leqslant\alpha d_\Lambda(\lambda_1,\lambda_2)$. If φ satisfies also the property in (iii), then this can be reversed, with an additive constant coming from the diameter of the finite set C. Hence φ is a quasi-isometry. □

Remark. Notice that if the groups Γ and Λ are bi-Lipschitz equivalent, i.e. if there is a φ as in the theorem which is also a bijection, then the fundamental domain X_Γ defined in the proof is a mutual fundamental domain for both actions. In fact, the existence of such a space X with a mutual (bounded) fundamental domain is equivalent to the bi-Lipschitz equivalence of the groups.

Definition 2.1.4. Given two groups Λ and Γ, call a locally compact space X satisfying the conditions in Theorem 2.1.2 a *topological coupling* of Λ and Γ.

In the definition we do not distinguish between the situations where both, or only one of the actions is co-compact (the quasi-isometry vs. the uniform embedding case). This should always be clear from the context, and is also encoded in the following standing notation, which is justified by the commutativity of the actions and will be used throughout the paper:

Γ acts on X via a left action, and Λ acts (from the right) via a right action.

Usually, we will use a coupling X satisfying the three additional properties in Theorem 2.1.2, in particular, $X_\Gamma\subseteq X_\Lambda$, which is an asymmetric condition as well. Thus, it is convenient to think of Γ as being the "large" group, whose fundamental domain is included in that of the "smaller" group Λ (this is consistent with the uniform embedding setup, as well as with our later approach to induce representations from the "smaller" to the "larger" group).

In all of our results which apply methods from harmonic analysis, the topological structure will be almost (though not entirely) immaterial. It is the analogous measurable setting which will become central.

Definition 2.1.5. (Gromov [31]) The discrete countable groups Γ and Λ are called *measure equivalent* (*ME*) if there exists a σ-finite measure space (X,μ) on which both Γ and Λ act (essentially) freely, preserving μ, in a commuting way, such that there exist finite-measure fundamental domains $X_\Gamma\subseteq X\supseteq X_\Lambda$ for the two actions. The groups are called *uniformly measure equivalent* (*UME*) if the following additional property holds: For all $\gamma\in\Gamma$ there is a *finite* subset $S_\gamma\subseteq\Lambda$ such that

$$\gamma X_\Lambda\subseteq X_\Lambda S_\gamma$$

and, similarly, for all $\lambda \in \Lambda$ there is a *finite* subset $S_\lambda \subseteq \Gamma$ such that

$$S_\lambda X_\Gamma \supseteq X_\Gamma \lambda.$$

Notation 2.1.6. If Γ, Λ and (X,μ) are as above, we call X a *ME* or *UME coupling* of Γ and Λ. By X_Γ and X_Λ we shall always mean fundamental domains in X satisfying the conditions in the definition. If the $\Gamma \times \Lambda$-action on (X,μ) (equivalently, the Γ-action on X/Λ, or the Λ-action on $\Gamma \backslash X$) is ergodic, we call X an *ergodic* coupling.

The use of amenability in the harmonic analysis approach comes via the following theorem:

THEOREM 2.1.7. *If amenable groups Λ and Γ are quasi-isometric, then for some finite group M there is an ergodic UME coupling X of Λ and $\Gamma \times M$, with $X_{\Gamma \times M} \subseteq X_\Lambda$.*

Proof. This is clear by Theorem 2.1.2 and the invariant measure property for actions of amenable groups on compact spaces, recalling that by Krein–Milman's theorem, if there is a finite invariant measure, there is also an ergodic one. Taking an ergodic Λ-invariant measure μ on $\Gamma \backslash X$, identifying the latter with $X_\Gamma \subseteq X$, and tessellating μ under the Γ-action, defines a σ-finite $\Lambda \times \Gamma$-invariant ergodic measure on X. □

2.2. The actions and cocycles associated with a coupling

Let Γ and Λ be discrete groups, and suppose that X is a topological coupling of them (Definition 2.1.4), with both actions on X being free. Then associated with this coupling we have a natural Γ-action (from the left) on X/Λ and a Λ-action (from the right) on $\Gamma \backslash X$. Let X_Γ and X_Λ be the fundamental domains associated with the coupling in Definition 2.1.4. Then we define the cocycle $\alpha: \Gamma \times X_\Lambda \to \Lambda$ by the rule

$$\alpha(\gamma, x) = \lambda \iff (\gamma^{-1} x)\lambda \in X_\Lambda. \tag{3}$$

Note that because X_Λ is a fundamental domain for Λ, $\alpha(\gamma, x)$ is uniquely defined. Similarly, we define the cocycle $\beta: X_\Gamma \times \Lambda \to \Gamma$ by

$$\beta(x, \lambda) = \gamma \iff \gamma^{-1}(x\lambda) \in X_\Gamma. \tag{4}$$

Of course, by commutativity of the actions, the parentheses in the right-hand sides of (3) and (4) are redundant; we have put them here only for clarity.

We also have the following fact, which will turn out very useful:

$$X_\Gamma \subseteq X_\Lambda \implies \alpha(\beta(x,\lambda), x) = \lambda \text{ for all } \lambda \in \Lambda \text{ and } x \in X_\Gamma. \tag{5}$$

This is obvious from the definitions (3) and (4).

Once we fix the fundamental domains X_Λ and X_Γ, we may identify them with X/Λ and $\Gamma\backslash X$, respectively. With these identifications, the Γ- and Λ-natural actions on the latter spaces take the forms

$$\gamma\cdot x = \gamma x\alpha(\gamma^{-1}, x), \quad x\in X_\Lambda \quad \text{(compare with (3))}, \tag{6}$$

$$x\cdot\lambda = \beta(x,\lambda)^{-1}x\lambda, \quad x\in X_\Gamma \quad \text{(compare with (4))}. \tag{7}$$

Hereafter we shall keep the dot notation for the actions defined above on the fundamental domains, to distinguish them from the usual actions on X (which we continue to denote as before). It is easy to check that $(\gamma,x)\to\gamma\cdot x$ and $(x,\lambda)\to x\cdot\lambda$ define left and right actions of Γ and Λ on X_Λ and X_Γ, respectively.

We can now make our previous statement that α and β are cocycles precise. They are in fact cocycles over the Γ- and Λ-actions on X_Λ and X_Γ, respectively:

$$\alpha(\gamma_1\gamma_2, x) = \alpha(\gamma_1, x)\alpha(\gamma_2, \gamma_1^{-1}\cdot x) \quad \text{for all } \gamma_1,\gamma_2\in\Gamma \text{ and } x\in X_\Lambda, \tag{8}$$

$$\beta(x,\lambda_1\lambda_2) = \beta(x,\lambda_1)\beta(x\cdot\lambda_1,\lambda_2) \quad \text{for all } \lambda_1,\lambda_2\in\Lambda \text{ and } x\in X_\Gamma. \tag{9}$$

Since the cocycle identities are both crucial in what follows and easy to get confused with, let us verify them quickly. To prove (8) notice first that by (6) one has the identity $\gamma_1^{-1}x=(\gamma_1^{-1}\cdot x)\alpha(\gamma_1,x)^{-1}$, and hence

$$\begin{aligned}(\gamma_1\gamma_2)^{-1}x &= \gamma_2^{-1}(\gamma_1^{-1}x) = \gamma_2^{-1}((\gamma_1^{-1}\cdot x)\alpha(\gamma_1,x)^{-1}) = (\gamma_2^{-1}(\gamma_1^{-1}\cdot x))\alpha(\gamma_1,x)^{-1}\\ &= (\gamma_2^{-1}\cdot(\gamma_1^{-1}\cdot x)\alpha(\gamma_2,\gamma_1^{-1}\cdot x)^{-1})\alpha(\gamma_1,x)^{-1}\\ &= \gamma_2^{-1}\cdot(\gamma_1^{-1}\cdot x)(\alpha(\gamma_2,\gamma_1^{-1}\cdot x)^{-1}\alpha(\gamma_1,x)^{-1}).\end{aligned}$$

Because $\gamma_2^{-1}\cdot(\gamma_1^{-1}\cdot x)\in X_\Lambda$, by (3) it follows that (8) is satisfied. To prove (9) we first notice that by (7) one has $x\lambda=\beta(x,\lambda)(x\cdot\lambda)$, and hence

$$\begin{aligned}x(\lambda_1\lambda_2) &= (x\lambda_1)\lambda_2 = (\beta(x,\lambda_1)(x\cdot\lambda_1))\lambda_2 = \beta(x,\lambda_1)((x\cdot\lambda_1)\lambda_2)\\ &= \beta(x,\lambda_1)(\beta(x\cdot\lambda_1,\lambda_2)((x\cdot\lambda_1)\cdot\lambda_2)) = \beta(x,\lambda_1)\beta(x\cdot\lambda_1,\lambda_2)((x\cdot\lambda_1)\cdot\lambda_2).\end{aligned}$$

Because $(x\cdot\lambda_1)\cdot\lambda_2\in X_\Gamma$, by (4) it follows that (9) holds.

2.3. Algebraic, continuous and reduced cohomology

We briefly review here the three group cohomology notions, while introducing some notation which will be needed later on. The first is the classical one. A comprehensive treatment of the last two in the general setting we shall be needing can be found in [32].

Let R be a commutative ring with unit, G be a locally compact second countable group, and let RG be the group ring of finitely supported R-valued functions on G. For an RG-module V we denote the G-action by $(g,v)\mapsto\pi(g)v$ and define

$$C^n(G,V)=\{\omega: G^n\to V \mid \omega(gg_0,\dots,gg_{n-1})=\pi(g)\omega(g_0,\dots,g_{n-1})\}. \tag{10}$$

Let $d_n: C^n(G,V)\to C^{n+1}(G,V)$ be the standard (homogeneous) differential:

$$[d_n\omega](g_0,\dots,g_n)=\sum_{i=0}^{n}(-1)^i\omega(g_0,\dots,\hat{g}_i,\dots,g_n). \tag{11}$$

Denote then the spaces of n-cocycles, n-coboundaries and n-cohomology by

$$\begin{aligned}Z^n(G,V)&=\operatorname{Ker} d_{n+1}\subseteq C^{n+1}(G,V),\\ B^n(G,V)&=\operatorname{Im} d_n\subseteq C^{n+1}(G,V),\\ H^n(G,V)&=Z^n(G,V)/B^n(G,V),\end{aligned}$$

respectively.

We shall often use the following basic property of group cohomology:

THEOREM 2.3.1. *Let G be a discrete group and $N\lhd G$ be a* finite *normal subgroup. Then for every G-module V which is a vector space over the field $\mathbf{Q}$, and every $n\geqslant 1$, one has $H^n(G,V)\cong H^n(G/N,V^N)$, where V^N denotes the space of N-fixed vectors, and the isomorphism is induced by the inclusion of V^N in V.*

Proof. Use Proposition 3.2 on p. 18, and Proposition 8.1 on p. 47, in [32]. □

When V is a topological vector space on which G acts continuously, the continuous cohomology $H^*_{\mathrm{ct}}(G,V)$ is defined in a similar manner to algebraic cohomology, only that here one insists that all maps be continuous with respect to the corresponding topologies. In all the cases of continuous cohomology we shall be concerned with, V will be a complex Hilbert space and the G-action π unitary. In this case, $Z^n(G,V)$ supports the natural topology of uniform convergence on compact subsets, in which it is easily seen to be a Fréchet space. However, $B^n(G,V)$ need not be a closed subspace. Denoting by $\overline{B}^n(G,V)$ its closure, we define the nth *reduced cohomology* group with coefficients in V (or π) by

$$\overline{H}^n_{\mathrm{ct}}(G,V)=Z^n(G,V)/\overline{B}^n(G,V).$$

Let us assume now that V is an (always separable) Hilbert space on which the locally compact second countable group G acts (continuously and) unitarily via the representation π. A crucial advantage of the reduced cohomology is its disintegration property (which does not hold in general for the ordinary continuous cohomology):

THEOREM 2.3.2. [32, p. 190, Proposition 2.6] *If* $\pi=\int^{\oplus}\pi_x\,d\mu(x)$ *is a direct integral decomposition of the unitary* G*-representation* π*, and for* μ*-a.e.* x *one has* $\overline{H}^n_{\mathrm{ct}}(G,\pi_x)=0$*, then* $\overline{H}^n_{\mathrm{ct}}(G,\pi)=0$.

2.4. A closer look at the first cohomology

Assume for this subsection that G is a locally compact group and (π, V) is a unitary G-representation. In the previous subsection we have defined cohomology using homogeneous cocycles, but for the first cohomology the inhomogeneous formulation turns out to be convenient. We define

$$\begin{aligned}
Z^1(G,V) &= \{b\colon G\to V \mid b \text{ is continuous and } b(gh)=\pi(g)b(h)+b(g)\},\\
B^1(G,V) &= \{b\in Z^1(G,V)\mid b(g)=\pi(g)v-v,\ v\in V\},\\
H^1_{\mathrm{ct}}(G,V) &= Z^1(G,V)/B^1(G,V),\\
\overline{H}^1_{\mathrm{ct}}(G,V) &= Z^1(G,V)/\overline{B}^1(G,V),
\end{aligned}$$

where $\overline{B}^1(G,V)$ denotes the closure of $B^1(G,V)$ as before.

In degree one a simple criterion enables us to identify reduced cohomology:

PROPOSITION 2.4.1. *Assume that* π *does not weakly contain the trivial representation (i.e. in the terminology of Definition* 2.4.3 *below, its linear action on the unit* sphere *of* V_π *is uniform). Then* $B^1(G,V_\pi)$ *is closed, and hence* $\overline{H}^1_{\mathrm{ct}}(G,\pi)=H^1_{\mathrm{ct}}(G,\pi)$.

The result is easy, see for example [50, Proposition 1.6]. In fact it admits a converse (due to Guichardet), which is less trivial—cf. [34, p. 48].

An advantage of the first cohomology is that it admits a useful geometric interpretation. Indeed, given $b\in Z^1(G,\pi)$ one can deform the linear action π to a continuous affine isometric G-action ϱ on V, by letting $\varrho(g)v=\pi(g)v+b(g)$. The following result is straightforward from the definition:

LEMMA 2.4.2. *With the above notation,* $b\in B^1(G,\pi)$ *if and only if the* G*-action* ϱ *admits a fixed point.*

In the same vein, we would like to identify cocycles $b\in\overline{B}^1(G,\pi)$. For this, we recall first the following notion, which will show up frequently in the sequel:

Definition 2.4.3. A continuous action of a locally compact group G on a metric space (X,d) is called *uniform*, if there exists a compact subset $Q\subseteq G$ and $\varepsilon>0$ such that for every $x\in X$ there is $g\in Q$ with $d(gx,x)>\varepsilon$. Otherwise, we say that the action admits *almost fixed points.*

We now leave it to the reader to verify the following lemma as an easy exercise in the definitions:

LEMMA 2.4.4. *With the above notation, $b \in \overline{B}^1(G,\pi)$ if and only if the G-action ϱ has almost fixed points (or in other words, it is not uniform).*

2.4.5. *A remark on cohomology of finite-dimensional representations.* When V is finite-dimensional (of dimension n), a unitary G-representation (π, V) corresponds to a continuous homomorphism $\pi: G \to U(n)$ (well defined up to conjugation), and 1-cocycles $b \in Z^1(G,\pi)$ correspond to homomorphisms ϱ to the group of rigid motions

$$\varrho: G \to U(n) \ltimes \mathbf{C}^n$$

whose linear part is π. Because of Proposition 2.4.1 and the compactness of the unit sphere, the reduced and ordinary cohomology are identical for finite-dimensional unitary representations, so every such affine action without a global fixed point is in fact uniform.

3. Induction through couplings and some first applications

3.1. Induction of representations—the unitary setting

Let Γ and Λ be discrete groups and assume that (X,μ) is a ME coupling of them with fundamental domains X_Γ and X_Λ, respectively (Notation 2.1.6). Given this structure, we introduce the following:

Definition 3.1.1. Let (π, V_π) be a unitary Λ-representation on the Hilbert space V_π. We define the unitary Γ-representation $\mathrm{Ind}_\Gamma^\Lambda$ induced from Λ to Γ, by first considering the representation space

$$L^2(X_\Lambda, V_\pi) = \left\{ f: X_\Lambda \to V_\pi \,\middle|\, \int_{X_\Lambda} \|f(x)\|_{V_\pi}^2 \, d\mu(x) < \infty \right\}$$

with the obvious inner product. To define the Γ-action, recall from §2.2 the Γ-action and cocycle defined on X_Λ (see (3) and (6)), and define the Γ-operation on $L^2(X_\Lambda, V_\pi)$ by

$$(\gamma f)(x) = \pi(\alpha(\gamma, x)) f(\gamma^{-1} \cdot x).$$

This is easily verified to define a unitary Γ-representation. In fact, the construction extends naturally the well-known unitary induction for locally compact groups à la Mackey. Indeed, in the special case where $X = \Gamma$ and $\Lambda < \Gamma$, we obtain the usual unitary induction from Λ to Γ, and when Γ and Λ are subgroups of the locally compact group G, and $(X,\mu)=(G, \text{Haar measure})$ with left and right actions of Γ and Λ, the above construction amounts to unitarily inducing from Λ to G, and then restricting to Γ. At the expense of abusing notation (but gaining simplicity of it), we do not explicitly indicate the information on the coupling space X in our notation $\mathrm{Ind}_\Lambda^\Gamma$ for induction, as this should always be clear from the context.

As in Mackey's usual induction, we have an equivalent "equivariant" model of induction, which has the advantage of being free of a choice of fundamental domain. Here we let the representation space of the induced representation be

$$L^2(X,V_\pi)^\Lambda := \{f: X\to V_\pi \mid f(x\lambda)=\pi(\lambda^{-1})f(x)\}$$

with the same L^2-condition as above relative to some (hence every) Λ-fundamental domain, and Γ acts by $(\gamma f)(x)=f(\gamma^{-1}x)$. For later use we also remark here that in a dual way one may induce a representation π from Γ to Λ, where the induction space consists of L^2-functions satisfying $f(\gamma x)=\pi(\gamma)f(x)$, and Λ operates by $(\lambda f)(x)=f(x\lambda)$ (this is a *left* Λ-action; recall that the Λ-action on X is a *right* one).

The following result will turn out crucial for our harmonic analysis approach to geometric group theory:

THEOREM 3.1.2. *Let X be a ME coupling of Γ and Λ (Notation 2.1.6), and π be a unitary Λ-representation. If $\mathrm{Ind}_\Lambda^\Gamma \pi$ contains a finite-dimensional Γ-subrepresentation ($\neq\{0\}$), then π contains a finite-dimensional Λ-subrepresentation ($\neq\{0\}$).*

Hereafter, denote by 1_Γ the trivial (1-dimensional) representation of a group Γ. The following Frobenius reciprocity-type result, and its proof, highlight the duality in ME couplings:

PROPOSITION 3.1.3. *Let X, Γ and Λ be as above, let σ be a unitary Γ-representation, and π be a unitary Λ-representation. Then $1_\Gamma\subseteq \mathrm{Ind}_\Lambda^\Gamma \pi\otimes\sigma$ if and only if $1_\Lambda\subseteq \mathrm{Ind}_\Gamma^\Lambda \sigma\otimes\pi$.*

Proof of the proposition. We use throughout the proof the second "equivariant" model of induction described above. Assume first that X is an *ergodic* ME coupling.

We may identify the representation space of $\mathrm{Ind}_\Lambda^\Gamma \pi\otimes\sigma$, $L^2(X/\Lambda, V_\pi)\otimes V_\sigma$, with $L^2(X, V_\pi\otimes V_\sigma)^\Lambda$, where the Λ-equivariance condition on $f: X\to V_\pi\otimes V_\sigma$ reads $f(x\lambda)=(\pi(\lambda^{-1})\otimes \mathrm{id})f(x)$ (this isomorphism is the linear extension of the map $f\otimes v\mapsto [f\otimes v](x)=f(x)\otimes v$ on pure tensors). With this identification, the group Γ acts by $(\gamma f)(x)=(\mathrm{id}\otimes\sigma(\gamma))f(\gamma^{-1}x)$, so the Γ-invariance condition reads $f(\gamma x)=(\mathrm{id}\otimes\sigma(\gamma))f(x)$. Notice that by ergodicity of the $\Gamma\times\Lambda$-action on X, the norm of any measurable f satisfying the equivariance and invariance conditions is essentially constant, hence such an f always defines back a Γ-invariant vector in $\mathrm{Ind}_\Lambda^\Gamma \pi\otimes\sigma$.

Now, reversing the situation and considering a Λ-invariant function in $\mathrm{Ind}_\Gamma^\Lambda \sigma\otimes\pi$, we arrive at exactly the same conditions on $f\colon X\to V_\pi\otimes V_\sigma$ as above, where the original Λ-equivariance condition here becomes Λ-invariance of f, and vice versa for Γ. (The apparent lack of symmetry comes from the fact that the Γ-action on X is a left action, while the Λ-action is a right one—see the paragraph preceding the statement of Theorem 3.1.2.) This proves the proposition in the ergodic case.

For the general, non-ergodic case, we use the fact that any ME coupling (X,μ) can be disintegrated into ergodic couplings $\mu=\int\mu_t\,dt$, realized on the space X. This is achieved by a usual ergodic decomposition of the Γ-action on the finite measure space X/Λ (or vice versa), and "lifting" the measures back to X. We may now perform induction of representations from one group to the other with respect to each of the μ_t's, although this creates a little notational difficulty, as our notation $\mathrm{Ind}_\Lambda^\Gamma$ left behind the information on the (coupling space and) measure. To simplify notation, let us drop for the rest of the argument the group notation (which will be self-explanatory), and write Ind and Ind_t for inductions in the ME couplings (X,μ) and (X,μ_t), respectively.

Suppose that $1_\Gamma\subseteq(\mathrm{Ind}\,\pi)\otimes\sigma$. By the definition of disintegration it is easy to see that we have $\mathrm{Ind}\,\pi\cong\int^\oplus\mathrm{Ind}_t\,\pi\,dt$, hence $1_\Gamma\subseteq(\int^\oplus\mathrm{Ind}_t\,\pi\,dt)\otimes\sigma\cong\int^\oplus(\mathrm{Ind}_t\,\pi\otimes\sigma)\,dt$. This implies that for a positive measure set of t's, we have $1_\Gamma\subseteq\mathrm{Ind}_t\,\pi\otimes\sigma$, and for them by the ergodic case, $1_\Lambda\subseteq\mathrm{Ind}_t\,\sigma\otimes\pi$. Thus $1_\Lambda\subseteq\int^\oplus(\mathrm{Ind}_t\,\sigma\otimes\pi)\,dt\cong(\int^\oplus\mathrm{Ind}_t\,\sigma\,dt)\otimes\pi\cong(\mathrm{Ind}\,\sigma)\otimes\pi$, as required. By symmetry this completes the proof of the proposition. $\square$

For later use we record the following consequence of the proof (for which no originality is claimed):

COROLLARY 3.1.4. *Let Λ be a discrete co-compact subgroup of the locally compact group G. Let π be a unitary Λ-representation and σ be a unitary G-representation. If $1_G\subseteq\sigma\otimes\mathrm{Ind}_\Lambda^G\pi$ then $1_\Lambda\subseteq\sigma|_\Lambda\otimes\pi$.*

Proof. One can use a similar proof, replacing Γ in the proof of the proposition by G, and the coupling space (X,μ) by $(G,\text{Haar measure})$. The transitivity of the G-action on X $(=G)$ implies here immediately that a function f as in the first paragraph is a.e. equal to a continuous map whose value at the identity is the required Λ-invariant vector. We leave the details to the reader. $\square$

Before proving Theorem 3.1.2 we briefly recall some basic facts in representation theory which will be crucial for the proof. These are brought up for the benefit of non-specialists, in order to make the proof of the theorem, which is fundamental to all that follows, essentially self-contained.

Let G be a locally compact group and (π,V_π) be a unitary G-representation. Recall that the contragredient dual $\bar\pi$ is the unitary G-representation defined as follows: The space of representations for $\bar{\pi}$, denoted $V_{\bar{\pi}}$, is a (setwise) "copy" of V_π through the identification $v \mapsto \bar{v}$. The inner product on $V_{\bar{\pi}}$ is defined by $\langle \bar{v}, \bar{u} \rangle_{V_{\bar{\pi}}} = \overline{\langle v, u \rangle_{V_\pi}}$, where in the right-hand side we have complex conjugation, the module operations are defined by $\bar{v} + \bar{u} = \overline{v+u}$, and $\lambda \bar{v} = \overline{\bar{\lambda} v}$ ($\bar{\lambda}$ = complex conjugation of λ). The unitary G-action on $V_{\bar{\pi}}$ is then defined by $\bar{\pi}(\gamma)\bar{v} = \overline{\pi(\gamma)v}$.

Next, let (π, V_π) and (σ, V_σ) be unitary G-representations. The representation space $V_{\bar{\pi}} \otimes V_\sigma$ may be identified with the space of Hilbert–Schmidt operators $T: V_\pi \to V_\sigma$ ($\mathrm{tr}(T^*T) < \infty$), by extending linearly the natural map $\bar{v} \otimes u \mapsto T_{\bar{v} \otimes u}(w) = \langle w, v \rangle u$, defined on pure tensors. The G-action on the latter space then takes the form

$$(g, T) \longmapsto \sigma(g) \circ T \circ \pi(g)^{-1},$$

and thus invariant vectors in $\bar{\pi} \otimes \sigma$ correspond to G-equivariant Hilbert–Schmidt operators $T: V_\pi \to V_\sigma$. Therefore, if π is irreducible then $1_G \subseteq \bar{\pi} \otimes \sigma$ is possible only if $\pi \subseteq \sigma$, and if σ is irreducible as well, by Schur's lemma the two must be isomorphic and T be a scalar (after identification). However, a (non-zero) scalar operator is Hilbert–Schmidt (if and) only if π is *finite-dimensional.* By decomposing a general representation σ into a direct integral of irreducibles, one can easily conclude that for an irreducible π and *any* σ, $1 \subseteq \bar{\pi} \otimes \sigma$ if and only if π is finite-dimensional, and is a subrepresentation of σ. Finally, by fixing σ and now taking any (not necessarily irreducible) π, decomposing the latter into a direct integral of irreducibles and using the above, one can make the following conclusion (cf. [6]):

Let G be a locally compact second countable group, and π be a unitary G-representation. There exists a unitary G-representation σ with $1_G \subseteq \pi \otimes \sigma$ if and only if π contains a finite-dimensional subrepresentation (i.e. by the usual terminology, π is not *weakly mixing*).

Proof of Theorem 3.1.2. Retain the assumptions and notations of the theorem. By the previously quoted result and the assumption, there exists a unitary Γ-representation σ such that $1_\Gamma \subseteq \mathrm{Ind}_\Lambda^\Gamma \pi \otimes \sigma$, and hence by Proposition 3.1.3, $1_\Lambda \subseteq \mathrm{Ind}_\Gamma^\Lambda \sigma \otimes \pi$. Applying once again the preceding result completes the proof. □

We end by observing the following additional consequences:

COROLLARY 3.1.5. (1) *Let $\Lambda < G$ be a discrete co-compact subgroup, and let π and τ be unitary representations of Λ and G, respectively, with π irreducible and τ* finite-dimensional. *If $\tau \subseteq \mathrm{Ind}_\Lambda^G \pi$ then $\pi \subseteq \tau|_\Lambda$.*

(2) *Let (X, μ) be a ME coupling of the groups Λ and Γ. Let π and τ be unitary representations of Λ and Γ, respectively, with π irreducible and τ* finite-dimensional. *If $\tau \subseteq \mathrm{Ind}_\Lambda^\Gamma \pi$ then $\pi \subseteq \mathrm{Ind}_\Gamma^\Lambda \sigma$.*

Proof. Use Corollary 3.1.4 for (1), and Proposition 3.1.3 for (2), with $\sigma=\bar{\tau}$, applying the result preceding the proof of Theorem 3.1.2 and the fact that taking contragredient dual commutes with induction. □

3.2. Induction of cohomology—the unitary setting

Let (X,μ) be a UME coupling of the discrete groups Γ and Λ, with fundamental domains X_Γ and X_Λ, respectively. In this subsection we shall construct for any unitary Γ-representation π a map $I\colon H^n(\Lambda,\pi)\to H^n(\Gamma,\mathrm{Ind}_\Lambda^\Gamma\pi)$, and our main concern will be its injectivity (on the reduced cohomology as well). We begin by defining the map I at the level of cocycles. We retain here all the notation of §2.2.

For an element $\omega\in C^{n+1}(\Lambda,V_\pi)$ (see (10) above) define

$$\begin{gathered} I\omega\colon\Gamma^{n+1}\to L^2(X_\Lambda,V_\pi),\\ I\omega(\gamma_0,\dots,\gamma_n)(x)=\omega(\alpha(\gamma_0,x),\alpha(\gamma_1,x),\dots,\alpha(\gamma_n,x)). \end{gathered}\tag{12}$$

The fact that $I\omega$ indeed ranges in L^2 follows from the property that for any given γ, $\alpha(\gamma,x)$ takes essentially finitely many values. Thus the right-hand side of (12) is no more than a finite sum over values of ω, with weights defined by the γ's, the cocycle α and the measure μ. It is exactly (and only) here that we make use of the extra finiteness property of the *UME* coupling, compared to an ordinary ME coupling.

We now claim that $I\omega\in C^{n+1}(\Gamma,\mathrm{Ind}_\Lambda^\Gamma\pi)$, i.e. $I\omega$ is Γ-equivariant for the appropriate actions. Indeed, using (8) for the second equality and Λ-equivariance for the third, we have

$$\begin{aligned} I\omega(\gamma\gamma_0,\dots,\gamma\gamma_n)(x) &= \omega(\alpha(\gamma\gamma_0,x),\dots,\alpha(\gamma\gamma_n,x))\\ &= \omega(\alpha(\gamma,x)\alpha(\gamma_0,\gamma^{-1}\cdot x),\dots,\alpha(\gamma,x)\alpha(\gamma_n,\gamma^{-1}\cdot x))\\ &= \pi(\alpha(\gamma,x))[\omega(\alpha(\gamma_0,\gamma^{-1}\cdot x),\dots,\alpha(\gamma_n,\gamma^{-1}\cdot x))]\\ &= \pi(\alpha(\gamma,x))[I\omega(\gamma_0,\dots,\gamma_n)(\gamma^{-1}\cdot x)]. \end{aligned}$$

The map I is clearly linear. It is also easy to verify that, denoting the coboundary operator $d=d_n$ in (11), we have

$$I\circ d=d\circ I.\tag{13}$$

Therefore $d\omega=0\Rightarrow Id\omega=0\Rightarrow dI\omega=0$ and $\omega=d\sigma\Rightarrow I\omega=Id\sigma=dI\sigma$, so I takes cocycles to cocycles and coboundaries to coboundaries, i.e. I induces maps (still denoted by I) $H^n(\Lambda,\pi)\to H^n(\Gamma,\mathrm{Ind}_\Lambda^\Gamma\pi)$. We observe also that I induces a map on the reduced cohomology groups, which follows from the obvious continuity of I on Z^n. We now arrive at our main goal:

THEOREM 3.2.1. *Assume that the UME coupling X of Γ and Λ satisfies $X_\Gamma \subseteq X_\Lambda$. Then the map I above is injective on both the ordinary and reduced cohomology groups. In particular,*

$$H^n(\Lambda,\pi)\neq 0 \;\Rightarrow\; H^n(\Gamma,\mathrm{Ind}_\Lambda^\Gamma\pi)\neq 0 \quad \textit{and} \quad \overline{H}^n(\Lambda,\pi)\neq 0 \;\Rightarrow\; \overline{H}^n(\Gamma,\mathrm{Ind}_\Lambda^\Gamma\pi)\neq 0.$$

Proof. We shall construct a (so-called) transfer map

$$T\colon C^{n+1}(\Gamma,\mathrm{Ind}_\Lambda^\Gamma\pi)\to C^{n+1}(\Lambda,V_\pi)$$

satisfying the properties:

(i) T is linear and continuous;

(ii) $d\circ T=T\circ d$;

(iii) $TI\omega=\omega$ for all $\omega\in C^{n+1}(\Lambda,V_\pi)$.

This suffices since (i) and (ii) imply that T defines a linear map between the corresponding cohomology (and reduced cohomology) groups, and (iii) shows that $I\omega=0$ in H^n (resp. $\overline{H}^n$) implies that $\omega=TI\omega=0$ in H^n (resp. $\overline{H}^n$).

To define the map T we retain the notation of §2.2, particularly that of the cocycle $\beta\colon X_\Gamma\times\Lambda\to\Gamma$. We define for $\sigma\in C^{n+1}(\Gamma,\mathrm{Ind}_\Lambda^\Gamma\pi)$,

$$T\sigma(\lambda_0,\dots,\lambda_n)=\int_{X_\Gamma}\sigma(\beta(x,\lambda_0),\dots,\beta(x,\lambda_n))(x)\,d\mu(x). \tag{14}$$

We first remark that although $\sigma(\cdot,\dots,\cdot)$ is a map defined on X_Λ, we integrate it over X_Γ. This makes sense as X_Γ is a subset of positive measure, which we actually assume hereafter to be 1. We further observe that since for every fixed $\lambda\in\Lambda$, $\beta(\cdot,\lambda)$ takes μ-essentially finitely many values, the set X_Γ can be broken into a disjoint union of finitely many subsets A_i, on each of which $\sigma(\beta(x,\lambda_0),\dots,\beta(x,\lambda_n))=\sigma(\gamma_0,\dots,\gamma_n)$ for some $(n+1)$-tuple of γ's not depending on x. For each such fixed $(n+1)$-tuple, σ is integrable as a function on X_Λ, hence it is so over any given subset A_i, and the right-hand side in (14) is well defined.

We first check that $T\sigma\in C^{n+1}(\Lambda,V_\pi)$, i.e. that $T\sigma$ is Λ-equivariant. Indeed, using (9) we have

$$\begin{aligned}T\sigma(\lambda\lambda_0,\dots,\lambda\lambda_n)&=\int_{X_\Gamma}\sigma(\beta(x,\lambda\lambda_0),\dots,\beta(x,\lambda\lambda_n))(x)\,d\mu(x)\\&=\int_{X_\Gamma}\sigma(\beta(x,\lambda)\beta(x\cdot\lambda,\lambda_0),\dots,\beta(x,\lambda)\beta(x\cdot\lambda,\lambda_n))(x)\,d\mu(x).\end{aligned}$$

The Γ-equivariance of σ reads $\sigma(\gamma\gamma_0,...,\gamma\gamma_n)(x)=\pi(\alpha(\gamma,x))[\sigma(\gamma_0,...,\gamma_n)(\gamma^{-1}\cdot x)]$, and hence we get

$$\begin{aligned}T\sigma(\lambda\lambda_0,...,\lambda\lambda_n) &= \int_{X_\Gamma} \pi(\alpha(\beta(x,\lambda),x))[\sigma(\beta(x\cdot\lambda,\lambda_0),...,\beta(x\cdot\lambda,\lambda_n))(\beta(x,\lambda)^{-1}\cdot x)]\,d\mu(x)\\ &= \pi(\lambda)\int_{X_\Gamma} \sigma(\beta(x\cdot\lambda,\lambda_0),...,\beta(x\cdot\lambda,\lambda_n))(\beta(x,\lambda)^{-1}\cdot x)\,d\mu(x).\end{aligned}$$

For the last equality we have used (5), i.e. the fact that $X_\Gamma\subseteq X_\Lambda$. Making the (measure-preserving) change of variable $x\mapsto x\cdot\lambda^{-1}$ on X_Γ, we get further

$$T\sigma(\lambda\lambda_0,...,\lambda\lambda_n)=\pi(\lambda)\int_{X_\Gamma} \sigma(\beta(x,\lambda_0),...,\beta(x,\lambda_n))(\beta(x\cdot\lambda^{-1},\lambda)^{-1}\cdot(x\cdot\lambda^{-1}))\,d\mu(x).$$

Therefore, to finish the computation it is enough to show that for μ-a.e. $x\in X_\Gamma$ one has $\beta(x\cdot\lambda^{-1},\lambda)^{-1}\cdot(x\cdot\lambda^{-1})=x$, which, replacing x by $x\cdot\lambda$, is equivalent to $\beta(x,\lambda)^{-1}\cdot x=x\cdot\lambda$. Indeed, by (7), then (6), and finally (5), we have

$$\begin{aligned}x\cdot\lambda &= (\beta(x,\lambda)^{-1}x)\lambda=((\beta(x,\lambda)^{-1}\cdot x)\alpha(\beta(x,\lambda),x)^{-1})\lambda\\ &= (\beta(x,\lambda)^{-1}\cdot x)(\alpha(\beta(x,\lambda),x)^{-1}\lambda)=(\beta(x,\lambda)^{-1}\cdot x)\lambda^{-1}\lambda=\beta(x,\lambda)^{-1}\cdot x,\end{aligned}$$

as required. Thus T maps equivariant co-chains to equivariant co-chains, and since its continuity, linearity and commutativity with the co-boundary operator(s) d are easy to verify, we are left with one last computation, which is verifying property (iii).

Let $\omega:\Lambda^{n+1}\to V_\pi$. Applying (14), then (12), and then (5), yields (recalling the normalization $\mu(X_\Gamma)=1$):

$$\begin{aligned}[T(I\omega)](\lambda_0,...,\lambda_n) &= \int_{X_\Gamma} I\omega(\beta(x,\lambda_0),...,\beta(x,\lambda_n))(x)\,d\mu(x)\\ &= \int_{X_\Gamma} \omega(\alpha(\beta(x,\lambda_0),x),...,\alpha(\beta(x,\lambda_n),x))\,d\mu(x)\\ &= \int_{X_\Gamma} \omega(\lambda_0,...,\lambda_n)\,d\mu(x)\\ &= \omega(\lambda_0,...,\lambda_n).\end{aligned}$$

This completes the proof of Theorem 3.2.1. □

We conclude this subsection by remarking on a special case of the framework treated here, namely, that of a discrete co-compact subgroup Λ in a locally compact group G. Strictly speaking, this situation is not covered by our previous analysis, but if we regard $\Gamma=G=X$ then all our formulae may be used with little modification. First, the map I

defined in (12) remains exactly the same. For the transfer map T we obviously cannot integrate over $X_\Gamma = X_G$, which is a "point", so we shall simply integrate (14) over the fundamental domain X_Λ. Because G acts transitively, the map then takes the form

$$T\sigma(\lambda_0, \dots, \lambda_n) = \int_{X_\Lambda} \sigma(g\lambda_0 g^{-1}, g\lambda_1 g^{-1}, \dots, g\lambda_n g^{-1})(g)\, d\mu(g).$$

Thus, the proof of Theorem 3.2.1 shows that the map

$$I: \overline{H}^n(\Lambda, \pi) \to \overline{H}^n_{\mathrm{ct}}(G, \mathrm{Ind}_\Lambda^G \pi)$$

is injective. The surjectivity of this map is well known; in fact, it is known to induce an isomorphism at the level of *ordinary* cohomology—see [32, p. 208, Proposition 4.5] (for more on the literature around this subject see [32, p. 213]). Thus, for later use we may record the following result:

THEOREM 3.2.2. *For any locally compact second countable group G, a co-compact discrete subgroup $\Lambda < G$, and a unitary Λ-representation π, one has an isomorphism*

$$\overline{H}^n(\Lambda, \pi) \cong \overline{H}^n_{\mathrm{ct}}(G, \mathrm{Ind}_\Lambda^G \pi).$$

3.3. Induction of representations and cohomology—the algebraic setting

We begin by discussing the condition on the ring R over which we shall work.

LEMMA 3.3.1. *For a commutative unital ring R, the following are equivalent:*

(1) *The field of rationals $\mathbf{Q}$ embeds as a subring of R;*

(2) *$H^n(G, V) = 0$ for all $n \geqslant 1$, all finite groups G and all RG-modules V;*

(3) *R is divisible.*

Proof. (1) $\Rightarrow$ (2). This is a special case of Theorem 2.3.1 above.

(2) $\Rightarrow$ (3). As is well known, if $\mathrm{cd}_R\, G$ is finite then G has no R-torsion, i.e. the order of any element of G must be invertible in R (cf. [9, Proposition 4.11]), namely, R is divisible.

(3) $\Rightarrow$ (1). For every $m \in N$, an $x_m \in R$ satisfying $m \cdot x_m = 1$ is unique, because if $m \cdot x_m = m \cdot y_m = 1$ then multiplying by x_m and using commutativity yields $1 \cdot x_m = 1 \cdot y_m$. Thus $n/m \mapsto x_m \cdot n$ is clearly a ring embedding of $\mathbf{Q}$ in R. □

We now arrive at the main purpose of this subsection:

Proof of Theorem 1.5. Let (π, V) be an $R\Lambda$-module with $H^n(\Lambda, \pi) \neq 0$ for some $n > 0$. We shall be done by constructing an induced $R\Gamma$-module, $\mathrm{Ind}\, \pi$, satisfying

$H^n(\Gamma, \operatorname{Ind}\pi)\neq 0$. This module is the algebraic analogue of the induced module constructed previously in the unitary setting, with a similar strategy of proof, yet with some significant changes as well.

By Theorem 2.3.1 we have for any finite group M, $\operatorname{cd}_R\Gamma=\operatorname{cd}_R(\Gamma\times M)$. Hence using Theorem 2.1.2 we may assume without loss of generality that X is a topological coupling of Λ and Γ satisfying the additional properties (1), (2) and (3) stated in that theorem. As in Definition 2.1.4 (and thereafter), we continue to denote the Γ-action on X from the left, and that of Λ from the right. Thus, we have clopen (closed and open) fundamental domains X_Λ and X_Γ for the two actions, with X_Γ compact and $X_\Gamma\subseteq X_\Lambda$. Recall that to these fundamental domains we have associated cocycles $\alpha:\Gamma\times X_\Lambda\to\Lambda$ and $\beta: X_\Lambda\times\Gamma\to\Lambda$ which are defined in (3) and (4). We denote by $\mathfrak{C}$ the algebra of clopen subsets of X. Call a function defined on a clopen subset of X, and ranging in some discrete set, *$\mathfrak{C}$-measurable*, if the inverse image of any point is a clopen subset. We now define the induced module

$$W=\operatorname{Ind}_\Lambda^\Gamma V=\{f: X_\Lambda\to V\mid f \text{ is } \mathfrak{C}\text{-measurable}\},$$

where Γ operates exactly as in Definition 3.1.1:

$$(\gamma f)(x)=\pi(\alpha(\gamma,x))f(\gamma^{-1}\cdot x)$$

($\gamma\cdot x$ is the action of γ on $X/\Lambda\cong X_\Lambda$ as defined in (6)). Because the Γ- and Λ-actions on X, being continuous, preserve $\mathfrak{C}$, and both fundamental domains are clopen subsets, it follows that for any fixed $\gamma\in\Gamma$ and $\lambda\in\Lambda$, the sets $\{x\mid\alpha(\gamma,x)=\lambda\}$ and $\{x\mid\beta(x,\lambda)=\gamma\}$ are clopen. Thus all the functions and operations which we make hereafter are readily seen to be measurable with respect to $\mathfrak{C}$ (where a target space is always viewed with a discrete structure).

Next, given an element $\omega:\Lambda^{n+1}\to V$, we define $I\omega:\Gamma^{n+1}\to W$ by the very same formula (12), and repeat verbatim the discussion there to show that I defines a map in cohomology (still denoted I): $H^n(\Lambda,V)\to H^n(\Gamma,W)$. We wish to show now that I is injective in cohomology, and for that purpose we adopt the same strategy as in the proof of Theorem 3.2.1, i.e. we construct a transfer map $T: C^{n+1}(\Gamma,W)\to C^{n+1}(\Lambda,V)$ with the properties (i), (ii) and (iii) as in the proof of that theorem. To define T we use the same formal formula appearing in (14), namely, for $\sigma\in C^{n+1}(\Gamma,W)$ set

$$T\sigma(\lambda_0,\dots,\lambda_n)=\int_{X_\Gamma}\sigma(\beta(x,\lambda_0),\dots,\beta(x,\lambda_n))(x)\,d\mu(x). \tag{15}$$

Of course, one should explain now how to interpret this formula, and particularly, what is the measure μ involved in it.

Suppose that μ is some R-valued, finitely additive measure, defined on the algebra of clopen subsets of X_Γ, which is also normalized. That is, μ assigns to each clopen subset $A\subseteq X_\Gamma$ an element $\mu(A)\in R$ so that

(i) $\mu(X_\Gamma)=1$ (where 1 is the unit element of R);

(ii) $A\cap B=\varnothing \Rightarrow \mu(A\cup B)=\mu(A)+\mu(B)$.

Since every $\mathfrak{C}$-measurable function on X is locally constant, and by compactness it takes only finitely many values on X_Γ, one can define naturally for any V-valued function $\varphi: X_\Gamma\to V$ its integration $\int_{X_\Gamma}\varphi(x)\,d\mu(x)\in V$, which is simply a finite linear combination of elements in V with coefficients in R (themselves determined by the measures of subsets on which φ takes a given value). If in addition the measure μ is Λ-invariant, namely,

(iii) $\mu(A\cdot\lambda)=\mu(A)$ for all $\lambda\in\Lambda$ and a clopen subset $A\subseteq X_\Gamma$,

then we have $\int_{X_\Gamma}\varphi(x\cdot\lambda)\,d\mu(x)=\int_{X_\Gamma}\varphi(x)\,d\mu(x)$ for all $\mathfrak{C}$-measurable φ, and $\lambda\in\Lambda$.

Returning to the formula (14) for the map T, notice that given an R-valued measure μ as above, T is well defined because it is easily verified that the expression in the integrand is a $\mathfrak{C}$-measurable V-valued function on X_Γ. Thus, to show that T is a transfer map, one can appeal to the same exact formal computation shown in the proof of Theorem 3.2.1, once the R-valued measure μ is also Λ-invariant.

We are therefore reduced to the question of finding such an invariant measure, and it is here that we use the amenability of Λ. First, because $\mathbf{Q}\subseteq R$ (Proposition 3.3.1), it is enough to find such a $\mathbf{Q}$-valued measure. Now, by amenability of Λ and the compactness of X_Γ, there exists a (positive) real-valued Λ-invariant probability measure m, defined on all the Borel (in particular clopen) subsets of X_Γ. Let $\psi:\mathbf{R}\to\mathbf{Q}$ be a homomorphism of abelian groups with $\psi(1)=1$. Then $\mu(A)=\psi(m(A))$ defines a normalized finitely additive $\mathbf{Q}$-valued measure, as required. This completes the proof of Theorem 1.5. $\square$

Proof of Theorem 1.7. The result follows from two basic facts: one is that being of type (FP) is characterized by having both finite cd_R and the property that H^k commutes with direct sums, and the other is that the induction operation defined in the proof of the previous theorem commutes with direct sums when Λ and Γ are quasi-isometric.

We begin by proving the second fact: if $V=\bigoplus_{i\in I}V_i$ then any $f: X_\Lambda\to V$ which is $\mathfrak{C}$-measurable takes finitely many values by compactness of X_Λ (recall that Γ and Λ are assumed now to be q.i.—see Theorem 2.1.2), and hence f ranges as a function (and not just pointwise) in the sum of finitely many V_i's. This shows that the natural embedding of $\bigoplus_{i\in I}\operatorname{Ind}V_i$ in $\operatorname{Ind}V$ is onto in this case, and hence an isomorphism, i.e. induction commutes with direct sums.

The first fact mentioned follows, e.g., from [9, p. 134, Theorem 8.20] (in fact, it is enough by Corollary 8.21 there to test the case where each V_i is the group ring itself). Now, to prove the theorem assume that Γ is of type (FP) over R and that Λ is quasi-isometric to Γ. Because $\mathrm{cd}_R \Gamma < \infty$, it follows from Theorem 1.5 that $\mathrm{cd}_R \Lambda < \infty$. We are left to show that the natural map $v: \bigoplus_{i \in I} H^k(\Lambda, V_i) \to H^k(\Lambda, \bigoplus_{i \in I} V_i)$ is an isomorphism for all k and all $R\Lambda$-modules V_i, where the issue of course is surjectivity. Since Γ is of type (FP) over R, so is $\Gamma \times M$ for any finite group M (by divisibility of R). Thus, applying Theorem 2.1.2 we may retain the framework and notation used in the proof of Theorem 1.5. Let $V = \bigoplus_i V_i$ and let $w \in H^k(\Lambda, V)$. Then by the previous discussion we have a natural isomorphism $Iw \in H^k(\Gamma, \mathrm{Ind}\, V) \cong H^k(\Gamma, \bigoplus_i \mathrm{Ind}\, V_i)$. Since Γ is of type (FP) over R, we have the natural isomorphism $H^k(\Gamma, \bigoplus_i \mathrm{Ind}\, V_i) \cong \bigoplus_i H^k(\Gamma, \mathrm{Ind}\, V_i)$, i.e. we may write $Iw = \sigma_1 \oplus \ldots \oplus \sigma_n$ (as cohomology classes), where $\sigma_i \in H^k(\Gamma, \mathrm{Ind}\, V_i)$. Applying the transfer operator T (as defined in the proof of Theorem 1.5) to both sides yields $w = TIw = T\sigma_1 \oplus \ldots \oplus T\sigma_n = w_1 \oplus \ldots \oplus w_n$ with $w_i \in H^k(\Lambda, V_i)$, as required. This completes the proof of Theorem 1.7. □

We now present a result of a similar type, this time making some regularity assumption on the group. Recall that a finitely generated group G is said to be of type $\mathfrak{F}_n$ if it has a $K(G,1)$-complex with finite n-skeleton, and it is of type $\mathfrak{F}_\infty$ if it is of type $\mathfrak{F}_n$ for all n. Unlike making many finiteness conditions, the advantage of making a type $\mathfrak{F}_\infty$ assumption is that it is itself (like each $\mathfrak{F}_n$ separately) a q.i. invariant, as was observed by Gromov [31, 1.C_2']. Recall also that G is said to be a *duality* group (of dimension n) over a ring R, if there is a (right) RG-module D (the so-called "dualizing module") such that one has natural isomorphisms

$$H^k(G, V) \cong H_{n-k}(G, D \otimes_R V)$$

for all k and all RG-modules V (where G acts diagonally on the tensor product). It can be shown that a dualizing module D as above must be isomorphic to $H^n(G, RG)$, and in case the latter is isomorphic to R, we say that G is moreover a *Poincaré duality* group—see [9] for more details.

THEOREM 3.3.2. *Let Γ be a finitely generated amenable group of type $\mathfrak{F}_\infty$. If Γ is a duality (resp. Poincaré duality) group over a divisible ring R, and a group Λ is quasi-isometric to Γ, then Λ is also a duality (resp. Poincaré duality) group over R.*

Proof. We shall use the following equivalent characterization (cf. [9, Theorem 9.2]): G is a duality group (of dimension n) over R if and only if

(i) G is of type (FP) over R;

(ii) $H^k(G, RG) = 0$ for $k \neq n$;

(iii) $H^n(G, RG)$ is flat as an R-module (in the case of Poincaré duality, this is replaced by $H^n(G, RG) = R$).

Thus, it is enough to show that properties (i), (ii) and (iii) are q.i. invariant. The first follows from Theorem 1.7, whereas (ii) and (iii) follow from a general result of Gersten [27]: For groups of type $\mathfrak{F}_\infty$ and all rings R, the cohomology groups $H^n(G, RG)$ are q.i. invariants. □

3.4. Applications of the algebraic approach

We preface the proofs of our next results with a brief discussion on cohomological aspects of solvable groups. The following theorem is a combination of works of several authors: Bieri, Gildenhuys–Strebel and Kropholler. We refer to [29, 1.4] and to the review of [37] for a comprehensive exposition.

THEOREM 3.4.1. *Let G be a torsion-free countable solvable group, hG be its Hirsch number, and let $\mathrm{cd}_R G$ (resp. $\mathrm{hd}_R G$) be its cohomological (resp. homological) dimension over R. Then the following are equivalent:*

(1) *G is constructible (cf. [29, 1.3]);*
(2) *G is a duality group (over $\mathbf{Z}$);*
(3) *G is of type (FP) (over $\mathbf{Z}$);*
(4) *G satisfies $hG = \mathrm{cd}_{\mathbf{Z}} G < \infty$;*
(5) *$hG = \mathrm{cd}_{\mathbf{Q}} G < \infty$.*

Recall that by a result of Stammbach [51], for every solvable G (not necessarily torsion free) $hG = \mathrm{hd}_{\mathbf{Q}} G$, and that for every countable group G and a ring R one has $\mathrm{hd}_R G \leqslant \mathrm{cd}_R G \leqslant \mathrm{hd}_R G + 1$ (cf. [9, Theorem 4.6]). Thus, Theorem 3.4.1 may be reformulated by saying that for every torsion-free countable solvable group G one has $hG = \mathrm{hd}_{\mathbf{Q}} G = \mathrm{hd}_{\mathbf{Z}} G$ and $\mathrm{cd}_{\mathbf{Q}} G = \mathrm{cd}_{\mathbf{Z}} G$, and these two quantities are equal (or else differ by 1) if and only if G satisfies any one of conditions (1), (2) and (3) above.

Proof of Theorem 1.6. Part (1) (and hence also (2)) follows now readily from Theorem 1.5 and the above discussion, since if Λ uniformly embeds in Γ, one has $h\Lambda = \mathrm{hd}_{\mathbf{Q}} \Lambda \leqslant \mathrm{cd}_{\mathbf{Q}} \Lambda \leqslant \mathrm{cd}_{\mathbf{Q}} \Gamma \leqslant \mathrm{hd}_{\mathbf{Q}} \Gamma + 1 = h\Gamma + 1$. In the case where Γ is of type (FP) over $\mathbf{Q}$, one has $\mathrm{hd}_{\mathbf{Q}} \Gamma = \mathrm{cd}_{\mathbf{Q}} \Gamma$ (see [9, Theorem 4.6]), thereby implying the sharper result $h\Lambda \leqslant h\Gamma$. For the proof of the last statement recall that by a result of Merzlyakov [39], if a torsion-free solvable group G has a bound on the rank of all its finitely generated abelian subgroups, then it has bounded rank. Because by "monotonicity" of cohomological dimension all such ranks are bounded by the $\mathbf{Q}$-cohomological dimension, it is enough to show that the latter is finite, which in turn would follow from the finiteness of $\mathrm{cd}_{\mathbf{Q}} \Gamma$ by Theorem 1.5. The finiteness of $\mathrm{cd}_{\mathbf{Q}} \Gamma$ is equivalent to the finiteness of $\mathrm{hd}_{\mathbf{Q}} \Gamma = h\Gamma$, and the latter is indeed finite by our assumption on Γ (see [39]). □

Proof of Theorem 1.8. The second statement, where Γ is assumed to be of type (FP) over $\mathbf{Q}$, follows readily from Theorem 1.7 and part (1) in Theorem 1.6. The first now follows immediately from this if any of the groups is of type (FP) over $\mathbf{Q}$. Otherwise, both are not of type (FP) over $\mathbf{Q}$, hence also not over $\mathbf{Z}$, and from Theorem 3.4.1 (see the discussion preceding it) we deduce that $\mathrm{hd}_{\mathbf{Q}}\,\Lambda=\mathrm{cd}_{\mathbf{Q}}\,\Lambda-1$ and $\mathrm{hd}_{\mathbf{Q}}\,\Gamma=\mathrm{cd}_{\mathbf{Q}}\,\Gamma-1$. By Theorem 1.5 it follows that $h\Lambda=\mathrm{hd}_{\mathbf{Q}}\,\Lambda=\mathrm{cd}_{\mathbf{Q}}\,\Lambda-1=\mathrm{cd}_{\mathbf{Q}}\,\Gamma-1=\mathrm{hd}_{\mathbf{Q}}\,\Gamma=h\Gamma$, which completes the proof. $\square$

Proof of Theorem 1.9. The result follows immediately from the q.i. invariance of the Hirsch number for torsion-free solvable groups, once we observe the lemma below. $\square$

LEMMA 3.4.2. *Every finitely generated torsion-free solvable group Γ with $h\Gamma=2$ has a finite index subgroup isomorphic to some $\Gamma(n,m)$ as in the theorem (here $|n|=|m|=1$, i.e. $\Gamma\cong\mathbf{Z}^2$, is also allowed).*

Proof. By finite generation and solvability, there is some finite index subgroup of Γ with infinite abelianization. We keep the notation Γ for this subgroup and show that it is isomorphic to some $\Gamma(n,m)$. Indeed, let N be the kernel of an epimorphism of Γ onto $\mathbf{Z}$. Then necessarily $hN=1$, and since N is (solvable and) torsion free, it may be identified with a subgroup of the additive group of $\mathbf{Q}$. Because $\mathbf{Z}$ is free, we have a splitting $\Gamma=\mathbf{Z}\ltimes N$; let $z_0\in\mathbf{Z}$ denote a generator. A standard argument shows that for all $r\in N$ we have $z_0(r)=z_0(1)\cdot r$, where $z_0(r)=z_0rz_0{}^{-1}$, so that for some co-prime integers n and m we have $z_0(r)=m/n\cdot r$ for all $r\in N$. Denote by S_∞ the set of primes which have unbounded powers appearing in the denominators of elements in N, by S' the set of primes which divide *some* denominator of some element in N, and by S'' the set of primes which divide *all* numerators of the elements of N. The set S'' is of course finite, and an examination of the situation shows that by finite generation of Γ the sets S' and S_∞ must be finite as well, with the latter being *equal* to the set of primes dividing $m\cdot n$. These three sets determine completely the group N: For an appropriate rational number r (whose numerator and denominator prime divisors come from the sets S' and S'', respectively), $rN=\mathbf{Z}[S_\infty]$, and the map $(z,n)\mapsto(z,rn)$ is an isomorphism of Γ with some $\Gamma(n,m)$. $\square$

4. Properties H_T, H_F, H_{FD} and their applications

4.1. Quasi-isometry invariance of Betti numbers for nilpotent groups

In this subsection we prove Theorem 1.2 in the introduction. We will actually not yet make an explicit use of properties H_T, H_F and H_{FD}; we argue, however, in a way which will enable us to use them in the sequel. The following result is fundamental to our approach:

THEOREM 4.1.1. *Let n be a natural number and let Γ be a finitely generated amenable group with the following property: Any unitary Γ-representation π with $\overline{H}^n(\Gamma,\pi)\neq 0$ contains the trivial representation 1_Γ. Let Λ be a finitely generated group which is quasi-isometric to Γ, and assume that $b_n(\Lambda)$ is finite (which is always the case if $n=1$). Then $b_n(\Lambda)\leqslant b_n(\Gamma)$.*

Proof. By definition,

$$b_n(\Lambda)=\dim_{\mathbf{R}} H^n(\Lambda,\mathbf{R})=\dim_{\mathbf{C}} H^n(\Lambda,\mathbf{C})=\dim_{\mathbf{C}} \overline{H}^n(\Lambda,1),$$

where 1 denotes the trivial Λ-representation. The last equality uses the assumption that $H^n(\Lambda,1)$ is finite-dimensional, hence it is automatically reduced. Now, replacing Γ by $\Gamma\times M$ for a finite group M does not affect the property assumed in the theorem (e.g. by Theorem 2.3.1). Hence by Theorem 2.1.7 we may assume that there exists an *ergodic* UME coupling (X,μ) of Γ and Λ, with $X_\Gamma\subseteq X_\Lambda$. An application of Theorem 3.2.1 then yields an injection

$$I:\overline{H}^n(\Lambda,1)\hookrightarrow \overline{H}^n(\Gamma,\operatorname{Ind}_\Lambda^\Gamma 1)\cong \overline{H}^n(\Gamma,L^2(X/\Lambda)).$$

Because X is an ergodic coupling, i.e. Γ is ergodic on X/Λ, $L^2(X/\Lambda)=\mathbf{C}\oplus L_0^2(X/\Lambda)$, where $\mathbf{C}$ is the subspace of constant functions and the second summand—the subspace of zero-mean functions—does not contain 1_Γ. By our assumption on Γ we have $\overline{H}^n(\Gamma,L_0^2(X/\Lambda))=0$, and hence $\overline{H}^n(\Gamma,L^2(X/\Lambda))\cong\overline{H}^n(\Gamma,1)$. We conclude that

$$b_n(\Lambda)=\dim \overline{H}^n(\Lambda,1)\leqslant \dim \overline{H}^n(\Gamma,L^2(X/\Lambda))=\dim \overline{H}^n(\Gamma,1)\leqslant \dim H^n(\Gamma,1)=b_n(\Gamma),$$

as required. □

To verify the property assumed in Theorem 4.1.1, the following result will be useful:

THEOREM 4.1.2. *Let G be locally compact, and $\Gamma<G$ be discrete and co-compact.*

(1) *If G has the property that $\overline{H}^n_{\mathrm{ct}}(G,\pi)\neq 0$ implies $1_G\subseteq\pi$ for any (continuous) unitary G-representation π (where $\overline{H}^n_{\mathrm{ct}}$ is the continuous cohomology—see §2.3), then Γ has this property as well. In particular (the case $n=1$), if G has property H_T (in its obvious modification to locally compact groups), then so does Γ.*

(2) *The group G has the property assumed in (1) if (and only if) $\overline{H}^n_{\mathrm{ct}}(G,\pi)=0$ for every non-trivial* irreducible *unitary G-representation π.*

Proof. (1) The argument is similar to the one showing that a lattice in a Kazhdan group is itself Kazhdan: Apply the injectivity of $\overline{H}^n(\Gamma,\pi)$ in $\overline{H}^n_{\mathrm{ct}}(G,\operatorname{Ind}_\Gamma^G\pi)$ (see Theorem 3.2.2), and the easy fact that $1_G\subseteq\operatorname{Ind}_\Gamma^G\pi$ implies $1_\Gamma\subseteq\pi$ (cf. the proof of [34, p. 33, Theorem 4]).

(2) Apply a direct integral decomposition of π into irreducibles and Theorem 2.3.2. □

Remark. For the case $n=1$ we give an example in §5.5 below of a co-compact lattice $\Gamma<G$ which itself satisfies the conclusion of part (1), and such that G does not satisfy it (even virtually).

Proof of Theorem 1.2. The proof clearly follows from Theorem 4.1.1 and the following key fact:

THEOREM 4.1.3. *For every n, any finitely generated nilpotent group Γ satisfies the property assumed in Theorem* 4.1.1, *namely, every unitary Γ-representation π with $\overline{H}^n(\Gamma,\pi)\neq0$ contains the trivial representation 1_Γ.*

Proof. We may assume that Γ is torsion free. Indeed, the elements of finite order in Γ form a finite normal subgroup N such that Γ/N is torsion free (cf. [38, Corollary 9.18]), and by Theorem 2.3.1 it is enough to prove the statement for Γ/N. We now apply Mal′tsev's well-known theorem (cf. [45, Theorem 2.18]) to find a connected nilpotent Lie group G in which Γ is embedded discretely and co-compactly. Our result now follows from the two parts of the previous theorem, once we call on the following result of Blanc for connected nilpotent Lie groups G (see [32, p. 243, Proposition 8.2]): For any irreducible non-trivial unitary G-representation π, and any n, one has $H^n_{\mathrm{ct}}(G,\pi)=0$ (and in particular $\overline{H}^n_{\mathrm{ct}}(G,\pi)=0$). □

Comparison with Pansu's theorem. We end this subsection by showing that Theorem 1.2 does not follow from Pansu's theorem [44], or more precisely: *There exist two finitely generated nilpotent groups which have isomorphic graded Lie algebras, but different b_2's.* This example was brought to our attention by Yves Benoist, who kindly verified the precise calculations [4].

Recall again that by Mal′tsev's theorem every finitely generated torsion-free nilpotent group Γ is a lattice in a unique simply-connected nilpotent Lie group G, and thus has associated with it a well-defined (real) Lie algebra $\mathfrak{g}$. As is well known, one has for all Betti numbers $b_n(\Gamma)=b_n(G)$ (this can be easily deduced also from Theorems 3.2.2 and Blanc's result mentioned at the end of the proof of Theorem 4.1.3 above). The latter is equal to $b_n(\mathfrak{g})$ (see [32, Chapter (ii)] for more on Lie algebra cohomology). Recall also that if a nilpotent real Lie algebra admits a basis for which the structural constants are rational, then (and only then) the corresponding simply-connected nilpotent Lie group admits a (co-compact) lattice. To provide the required example, we shall construct a nilpotent Lie algebra $\mathfrak{g}$ defined over $\mathbf{Q}$ such that for the graded Lie algebra $\mathfrak{g}'$ one has $b_2(\mathfrak{g})<b_2(\mathfrak{g}')$. We remark in passing that it can be shown that for any $\mathfrak{g}$, the Betti numbers of the graded $\mathfrak{g}$ are always greater than or equal to the corresponding Betti numbers of $\mathfrak{g}$. The point here is to find an example where some Betti number (the second in our example) strictly increases.

The construction comes from a finite-dimensional quotient of the (positive part of the) so-called infinite-dimensional Virasoro Lie algebra. More precisely, fix a positive integer n, and consider the n-dimensional Lie algebra $\mathfrak{g}$ (so-called "filiform" or "thread-like") generated by $e_1, ..., e_n$ with

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j}, & \text{if } i+j \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

To compute the corresponding graded Lie algebra $\mathfrak{g}'$ denote first $C^k(\mathfrak{g})=[\mathfrak{g}, C^{k-1}(\mathfrak{g})]$ ($C^1(\mathfrak{g})=\mathfrak{g}$), $\mathfrak{g}'_k=C^k(\mathfrak{g})/C^{k+1}(\mathfrak{g})$, and finally $\mathfrak{g}'=\bigoplus_{k=1}^{n-1}\mathfrak{g}'_k$ with the associated "graded" Lie brackets. Then $\mathfrak{g}'$ is n-dimensional and spanned by $e'_1, ..., e'_n$, where $\mathfrak{g}'_1$ is spanned by e'_1, e'_2, and every other $\mathfrak{g}'_i$ $(2\leqslant i\leqslant n-1)$ is 1-dimensional and generated by e'_{i+1}. Thus, as for all $i, j\geqslant 2$ one has that $e_i\in C^{i-1}(\mathfrak{g})$, $e_j\in C^{j-1}(\mathfrak{g})$, but $[e_i, e_j]\in C^{(i-1)+(j-1)+1}(\mathfrak{g})$, the latter bracket vanishes in $\mathfrak{g}'_{(i-1)+(j-1)}$. Subsequently, $\mathfrak{g}'$ is spanned by $e'_1, ..., e'_n$ with the relations $[e'_i, e'_j]=0$ for $i, j\geqslant 2$, $[e'_1, e'_i]=(i-1)e'_{i+1}$ $(i<n)$ and $[e'_1, e'_n]=0$; i.e. all the relations in $\mathfrak{g}$ not involving the first vector got annihilated in $\mathfrak{g}'$. A computation [4] shows that for $n\geqslant 7$ one has $b_2(\mathfrak{g})=3<\left[\frac{1}{2}(n+1)\right]=b_2(\mathfrak{g}')$, where $[\cdot]$ denotes the integer part. Because both $\mathfrak{g}$ and $\mathfrak{g}'$ are defined over $\mathbf{Q}$ and have isomorphic graded Lie algebras, for $n\geqslant 7$ the associated finitely generated nilpotent groups cannot be q.i. distinguished using Pansu's theorem, although they are indeed not quasi-isometric by Theorem 1.2.

4.2. Properties H_T, H_F and H_{FD}

The three properties were defined in the introduction. In this subsection we establish some basic properties, and examine the connections between them. Throughout this subsection Γ denotes a finitely generated group. We first characterize the three properties in terms of irreducible representations.

THEOREM 4.2.1. (1) *Γ has property H_T if and only if the only irreducible unitary Γ-representation π with $\overline{H}^1(\Gamma, \pi)\neq 0$ is the trivial representation 1_Γ.*

(2) *Γ has property H_F if and only if every representation π as above is finite.*

(3) *Γ has property H_{FD} if and only if there are at most countably many representations π as above, and all of them are finite-dimensional.*

Proof. (1) One direction is obvious. The other is Theorem 4.1.2 (2).

(2) We prove only the non-trivial direction. Since Γ has countably many different finite index subgroups, and by the σ-additivity of the measure μ in a direct integral decomposition $\pi=\int^{\oplus}\pi_x\,d\mu(x)$, Theorem 2.3.2 shows that one of these subgroups must have a non-zero invariant vector in any representation π with $\overline{H}^1(\Gamma,\pi)\neq 0$.

(3) This is the least easy claim of the three. However, since we will not make any use of it in the sequel, we will only indicate the idea. In one direction, if there are only countably many irreducible representations with non-zero $\overline{H}^1$, then by Theorem 2.3.2 one of them must appear discretely in any π with $\overline{H}^1\neq 0$, and hence Γ has property H_{FD} if they are all finite-dimensional. In the other direction, the point is of course to show that there cannot be uncountably many finite-dimensional irreducible representations with $\overline{H}^1\neq 0$. If this were not the case, then we would find such a collection of representations in one fixed dimension, say n. Then one can form a direct integral without atoms over these representations which also satisfies $\overline{H}^1\neq 0$ (by suitably integrating the 1-cocycles), and this representation contradicts the assumed property H_{FD} of Γ (see the end of §5.4 for an explicit construction of this type for the group $\mathbf{Z}\wr\mathbf{Z}$). $\square$

LEMMA 4.2.2. *If* $[\Gamma:\Gamma_0]<\infty$ *then* Γ *has property* H_F *if and only if* Γ_0 *has it.*

Proof. Suppose that Γ_0 has property H_F, and let π be a unitary Γ-representation with $\overline{H}^1(\Gamma,\pi)\neq 0$. Then the same holds for the restriction to Γ_0, and by property H_F of the latter, there is a finite index subgroup $\Gamma_0'<\Gamma_0$ fixing a non-zero vector. Since $\Gamma_0'<\Gamma$ has finite index, this shows that Γ has property H_F.

Assume now that Γ has property H_F, and let π be an *irreducible* unitary Γ_0-representation with $\overline{H}^1(\Gamma_0,\pi)\neq 0$. Using Theorem 3.2.2, and Corollary 3.1.5 applied to the finite Γ-representation τ guaranteed by property H_F of Γ, we deduce that π must be finite. By Theorem 4.2.1 (2) this shows that Γ_0 has property H_F. $\square$

We now discuss the relations between the three properties, starting with properties H_{FD} and H_F.

PROPOSITION 4.2.3. *Let* Γ *be a finitely generated amenable group which has property* H_{FD}. *Then* Γ *has property* H_F *if and only if no finite index subgroup* $\Gamma_0<\Gamma$ *admits a homomorphism* $\varrho:\Gamma_0\to\mathrm{SO}(2)\ltimes\mathbf{R}^2\cong S^1\ltimes\mathbf{C}$ *with dense image. In particular, if* Γ *has property* H_{FD} *and it is of subexponential growth, then* Γ *has property* H_F.

Proof. The condition is necessary, because by the previous lemma Γ_0 should also have property H_F. But denoting by $\pi:\Gamma_0\to S^1$ the linear part of ϱ (which is infinite by density), we get a non-zero element in $\overline{H}^1(\Gamma_0,\pi)$ $(=H^1(\Gamma_0,\pi))$ defined by ϱ (see §2.4.5), while no finite index subgroup of Γ_0 has an invariant vector in π.

To show that the condition is sufficient, let π be a unitary Γ-representation with $\overline{H}^1(\Gamma,\pi)\neq 0$. Because Γ has property H_{FD}, π has a finite-dimensional subrepresentation with $\overline{H}^1\neq 0$ (indeed, if all such subrepresentations had vanishing cohomology, then the representation on the complement to their direct sum would contradict property H_{FD} by Theorem 2.3.2). Thus, we may assume that π is finite-dimensional, and hence, by complete reducibility, that it is also irreducible. The image $\pi(\Gamma)$, being an amenable linear group, has a finite index solvable subgroup (see [49] for a short proof of this consequence of Tits' alternative). The connected component of its closure is a compact connected solvable Lie group, hence abelian, so by intersecting it with $\pi(\Gamma)$ and pulling back to Γ, we may assume that there exists a finite index normal subgroup $\Gamma_0<\Gamma$ with $\pi(\Gamma_0)$ abelian. Of course, the restriction $\pi|_{\Gamma_0}$ still has non-vanishing $\overline{H}^1$. On the other hand, $\pi|_{\Gamma_0}$ decomposes now as a sum of 1-dimensional subrepresentations π_i, which, by normality of Γ_0 and irreducibility of π, are all finite or all infinite. We will be done by showing that the second possibility cannot occur. Indeed, since at least one of the π_i's has non-vanishing H^1, a non-vanishing 1-cocycle would give a homomorphism to $S^1\ltimes\mathbf{C}$ (§2.4.5) whose linear part, π_i, is infinite, and thereby defining a homomorphism $\varrho:\Gamma_0\to S^1\ltimes\mathbf{C}$ with dense image, a contradiction. (The density of $\varrho(\Gamma_0)$ follows by looking at the connected component of its closure, using the fact that any connected proper subgroup of $S^1\ltimes\mathbf{C}$ is abelian, and hence it is either contained in $\mathbf{C}$, or conjugate to S^1. We leave the easy verification to the reader.)

Finally, if Γ has subexponential growth, then so does every (finite index) subgroup of it, and hence also $\varrho(\Gamma_0)<\mathrm{SO}(2)\ltimes\mathbf{R}^2$. But again being amenable, $\varrho(\Gamma_0)$ is virtually solvable, and then by a result of Milnor and Wolf ([40], [54]) it is virtually nilpotent. Hence it cannot be dense in $\mathrm{SO}(2)\ltimes\mathbf{R}^2$, and the condition in the first part of the proposition is satisfied. □

We next analyze the precise relation between properties H_F and H_T.

PROPOSITION 4.2.4. *Suppose that Γ has property H_F.*

(1) *Γ has property H_T if and only if $vb_1(\Gamma)=b_1(\Gamma)$.*

(2) *A finite index subgroup of Γ has property H_T if and only if $vb_1(\Gamma)<\infty$.*

Proof. By the (simplest version of) the Shapiro lemma (cf. Theorem 3.2.2 above) applied to the trivial representation of Γ_0, we have for any finite index subgroup $\Gamma_0<\Gamma$ that $H^1(\Gamma,l^2(\Gamma/\Gamma_0))\cong H^1(\Gamma_0,1)$. By decomposing the left-hand side representation into a sum of the constants and the zero-sum functions, we deduce that

$$b_1(\Gamma)+\dim H^1(\Gamma,l_0^2(\Gamma/\Gamma_0))=b_1(\Gamma_0). \tag{16}$$

Now, to prove (1) notice that if Γ has property H_T then by definition and the fact that $l_0^2(\Gamma/\Gamma_0)$ has no Γ-invariant vectors, we have by (16) that $b_1(\Gamma)=b_1(\Gamma_0)$ for every finite index subgroup Γ_0, and hence $vb_1(\Gamma)=b_1(\Gamma)$. In the other direction, assume that the latter equality holds. Then by (16) we have $H^1(\Gamma, l_0^2(\Gamma/\Gamma_0))=0$ for all finite index subgroups. Let π be an irreducible Γ-representation with $\overline{H}^1(\Gamma,\pi)\neq 0$. By Theorem 4.1.2 (2) it is enough to show that π is trivial. Indeed, because Γ is assumed to have property H_F, and by irreducibility of π, there exists a finite index normal subgroup $\Gamma_0<\Gamma$ such that π factors through the quotient Γ/Γ_0, i.e. it occurs as a subrepresentation of $l^2(\Gamma/\Gamma_0)$. If π is not trivial then it must be a subrepresentation of $l_0^2(\Gamma/\Gamma_0)$, which we now know to have vanishing first cohomology. This completes the proof of (1).

Part (2) now follows immediately from (1) by considering a finite index subgroup $\Gamma_0<\Gamma$ with either $b_1(\Gamma_0)=vb_1(\Gamma)$ for one direction, or with property H_T for the other. □

We conclude this subsection with a stability result:

THEOREM 4.2.5. *All three families of groups with properties H_{FD}, H_F or H_T are closed under taking direct products and central extensions.*

Proof. The assertion follows readily from the following two results proved in [50, Theorem 3.1 and Corollary 3.7]:

(i) For any locally compact, second countable groups G_1 and G_2, and any unitary representation π of $G=G_1\times G_2$, one has

$$\overline{H}^1(G,\pi)\cong \overline{H}^1(G_1,\pi^{G_2})\oplus \overline{H}^1(G_2,\pi^{G_1}).$$

(ii) For any locally compact, second countable group G, and a closed central subgroup $C<Z(G)$, and for any unitary G-representation π with $1_G\not\subseteq\pi$, one has

$$\overline{H}^1(G,\pi)\cong \overline{H}^1(G/C,\pi^C).$$

(In both statements π^N denotes the N-invariants of π. Actually, part (i) implies (ii).) For an alternative simple proof of the central extension part, which will be used in the sequel, see Corollary 5.1.3 below. □

Remark. As shown in §5.4 below, *none* of these properties is closed under taking semi-direct products.

4.3. The relation to b_1 and to the geometry of amenable groups

We begin by establishing the first claim in Theorem 1.11 (1):

THEOREM 4.3.1. *Let Γ be a finitely generated infinite amenable group with property H_{FD}. Then there exists a finite index subgroup $\Gamma_0<\Gamma$ with infinite abelianization.*

Proof. Because Γ is infinite and amenable, it does not have Kazhdan's property (T). Because it is also finitely generated, Theorem 6.1 in [50] implies that it admits an irreducible unitary representation π with $\overline{H}^1(\Gamma,\pi)\neq 0$ (cf. the proof of Theorem 0.2 in [50, p. 30]). By property H_{FD} it follows that π is finite-dimensional, hence it defines a homomorphism $\varrho:\Gamma\to U(n)\ltimes\mathbf{C}^n$ ($n=\dim\pi$) with infinite image (see §2.4.5 above). Thus, $\varrho(\Gamma)$, being infinite, finitely generated, amenable and linear, has a finite index solvable subgroup ("Tits' alternative"). Taking the inverse image under ϱ of this subgroup completes the proof of the theorem. □

Turning to Theorem 1.11 (2) of the introduction, we now show how the stronger properties H_F and H_T give additional information.

THEOREM 4.3.2. *Assume that the finitely generated amenable group Γ has property H_F. If the group Λ is quasi-isometric to Γ, then $vb_1(\Lambda)\leqslant vb_1(\Gamma)$. If moreover Γ has property H_T, then $vb_1(\Gamma)$ may be replaced by $b_1(\Gamma)$.*

Proof. Because we only assume that Λ is q.i. to Γ, it is of course enough to prove the result with $vb_1(\Lambda)$ replaced by $b_1(\Lambda)$. The second statement is a special case of Theorem 4.1.1 above with $n=1$. As for the first, if $vb_1(\Gamma)$ is infinite then there is nothing to prove. If it is finite, then by Proposition 4.2.4 there is a finite index subgroup $\Gamma_0<\Gamma$ with property H_T, so the claim now follows from the above. □

Remark. In §5.5 below we describe an example of a polycyclic group Γ with property H_T which is q.i. to a polycyclic group Λ for which a strict inequality holds:

$$1=vb_1(\Lambda)<vb_1(\Gamma)\ (=b_1(\Gamma)).$$

We now turn to Theorem 1.10, which forms a bridge between spectral and geometric group theory:

THEOREM 4.3.3. *In the class of amenable groups, property H_{FD} is a q.i. invariant.*

Proof. Assume that Γ has property H_{FD} and that Λ is q.i. to Γ. Let π be a unitary Λ-representation with $\overline{H}^1(\Lambda,\pi)\neq 0$. By Theorem 2.1.7, after replacing Γ with $\Gamma\times M$ for some finite group M, there exists a UME coupling (X,μ) of Γ and Λ with the property $X_\Gamma\subseteq X_\Lambda$ (retaining the notation in Definition 2.1.5). Note that $\Gamma\times M$ still has property

H_{FD} (by Theorem 4.2.5 or Theorem 2.3.1). Induce π from Λ to Γ as described in §3.1. From Theorem 3.2.1 it follows that $\overline{H}^1(\Gamma, \mathrm{Ind}_\Lambda^\Gamma \pi) \neq 0$. Because Γ has property H_{FD} it then follows that $\mathrm{Ind}_\Lambda^\Gamma \pi$ contains a finite-dimensional Γ-subrepresentation. By Theorem 3.1.2 this implies that π contains a finite-dimensional Λ-subrepresentation, as required. □

Remark. In §5.5 below we show an example of a group with property H_T which is q.i. to a group without property H_F. Thus the latter is not a q.i. invariant.

Theorem 1.11 (1) now follows immediately:

COROLLARY 4.3.4. *Let Γ be a finitely generated amenable group with property H_{FD}. If Λ is quasi-isometric to Γ, then a finite index subgroup of Λ has infinite abelianization.*

Proof. This follows readily from Theorems 4.3.1 and 4.3.3. □

In the following situation, of which we shall see many examples in §5, our results yield sharper information:

COROLLARY 4.3.5. *Let Γ be an amenable group with property H_T which satisfies $b_1(\Gamma)=1$. If Λ is a group which is quasi-isometric to Γ, then $vb_1(\Lambda)=1$.*

Proof. This follows readily from the previous corollary and Theorem 4.3.2. □

We now return to the proof of q.i. rigidity of abelian groups, already sketched in the introduction, completing some missing details.

THEOREM 4.3.6. *If Λ is quasi-isometric to $\mathbf{Z}^d$, then a finite index subgroup of Λ is isomorphic to $\mathbf{Z}^d$.*

Proof. The first step is showing that $\mathbf{Z}^d$ has property H_T. This is, for example, a special case of Theorem 4.2.5 above (going back to [50]), but in the next section we shall bring an alternative self-contained simple proof of this fact. We remark that while the proof presented here (see Theorem 5.1.1 below) applies one non-elementary ingredient—the use of a direct integral decomposition—the proof in [50, Corollary 3.6] is completely geometric and elementary (though less transparent), thereby making elementary the whole proof of Theorem 4.3.6.

Next, because $\mathbf{Z}^d$ in particular has property H_{FD}, it follows from Theorem 4.3.3 that Λ has it as well. However, being q.i. to a group of polynomial growth, Λ itself has such growth, so Proposition 4.2.3 implies that Λ actually has property H_F.

Now, applying Theorem 4.3.2, we get inequalities in both sides and therefore an equality $vb_1(\Lambda)=vb_1(\mathbf{Z}^d)=d$, so after passing to a finite index subgroup we may assume that $b_1(\Lambda)=d$. Observe that this implies that there exists a surjective homomorphism

$\Psi: \Lambda \twoheadrightarrow \mathbf{Z}^d$. Indeed, if $\psi_1, ..., \psi_d$ are linearly independent elements (over $\mathbf{R}$) in $\mathrm{Hom}(\Lambda, \mathbf{R})$, consider the homomorphism $\Psi: \Lambda \to \mathbf{R}^d$ which is defined by

$$\Psi(\lambda) = (\psi_1(\lambda), \psi_2(\lambda), ..., \psi_d(\lambda)).$$

Then $\Psi(\Lambda)$ is a finitely generated abelian subgroup of $\mathbf{R}^d$, hence isomorphic to $\mathbf{Z}^m$ for some m. If $m<d$ then $\Psi(\Lambda)$ spans over $\mathbf{R}$ a proper subspace, and hence any non-zero vector $(a_1, ..., a_d)$ orthogonal to that subspace gives an identity $\sum_{i=1}^d a_i \psi_i(\lambda)=0$ for all $\lambda \in \Lambda$, contradicting the linear independence of the ψ_i. Thus $m \geqslant d$, and we conclude that $\Lambda/N \cong \mathbf{Z}^d$ for some normal subgroup N.

We shall be done by showing that N is finite. Fix a finite generating set $S=\{\lambda_1, ..., \lambda_k\}$ for Λ, and let $b(n)$ denote the number of elements in Λ of S-length $\leqslant n$. Because Λ is q.i. to $\mathbf{Z}^d$, there is a constant α such that $b(n) \leqslant \alpha n^d$. Let $a(n)$ be the number of elements in N of S-length $\leqslant n$. Let $c(n)$ be the number of elements in $\mathbf{Z}^d$ of length $\leqslant n$ in the image of the generators $\bar{\lambda}_1, ..., \bar{\lambda}_k$. Then there is some constant $\beta>0$ such that $\beta n^d \leqslant c(n)$. Obviously, we must have $a(n)c(n) \leqslant b(2n)$, and hence $a(n)\beta n^d \leqslant \alpha(2n)^d = \alpha 2^d n^d$. Therefore $a(n) \leqslant \alpha 2^d/\beta$, and so N is finite, thereby completing the proof of the theorem. $\square$

5. Amenable groups and their reduced cohomology

5.1. Abelian, nilpotent and polycyclic groups

The following result in the case $G=\mathbf{Z}^d$ is central to the proof of its q.i. rigidity (see Theorem 4.3.6 above).

THEOREM 5.1.1. *A locally compact second countable abelian group has property H_T.*

Proof. Because of Theorem 4.1.2 (2), it is enough to prove that for every *irreducible* non-trivial unitary G-representation π, one has $\overline{H}^1(G, \pi)=0$. In fact, for any such π one has $H^1(G, \pi)=0$. Indeed, because G is abelian, π is 1-dimensional, and hence any element $b \in Z^1(G, \pi)$ defines a homomorphism $\varrho: G \to S^1 \ltimes \mathbf{C}$ whose linear part is π (§2.4.5). Being abelian, the image $\varrho(G)$ is either contained in $\mathbf{C}$, which means that π is trivial, or is conjugate to S^1, which means precisely that the affine action ϱ has a fixed point, so $[b]=0$ in $H^1(G, \pi)$ (see Lemma 2.4.2). $\square$

In fact, the previous theorem is a special case of a more general result concerning central extensions, which was already cited in Theorem 4.2.5 above, going back to [50]. However, since the central extension theorem will be used in the sequel, we present here an alternative short proof:

PROPOSITION 5.1.2. *Let G be a locally compact group, and $C<Z(G)$ be a closed and central subgroup. If π is an* irreducible non-trivial *unitary G-representation with $H^1(G,\pi)\neq 0$, then C acts trivially in π, and $H^1(G,\pi)=H^1(G/C,\pi)$.*

Proof. Let $b\in Z^1(G,\pi)$ and let $\varrho(g)v=\pi(g)v+b(g)$ be the corresponding affine G-action on V_π. By Schur's lemma, C acts via some 1-dimensional character χ. If χ is non-trivial, then, as shown in Theorem 5.1.1, the restriction of b to C must be a coboundary, so that C has a fixed point v_0 for its action through ϱ (Lemma 2.4.2). Now, for any $g\in G$, $\varrho(g)v_0$ is also fixed by C, so if $\varrho(g)v_0\neq v_0$ for some $g\in G$ then the line determined by v_0 and $\varrho(g)v_0$ is fixed by C, contradicting the assumption that χ is not trivial. Therefore v_0 must be fixed by all of G, so ϱ has a global fixed point, and $[b]$ vanishes in H^1. Thus C acts trivially in π. We now show that as a 1-cocycle b must vanish identically on C, which proves the proposition. Indeed, assume that $b(c)\neq 0$ for some $c\in C$. Then for all $g\in G$ we have $b(gc)=b(cg)$, which by the 1-cocycle identity gives $\pi(g)b(c)+b(g)=\pi(c)b(g)+b(c)$. Because C acts trivially we deduce that $\pi(g)b(c)=b(c)$, so $b(c)$ is a non-zero $\pi(G)$-invariant vector, and by irreducibility π must be trivial, in contradiction to the assumption. □

COROLLARY 5.1.3. *For* any *unitary G-representation π not containing 1_G, one has $\overline{H}^1(G,\pi)\cong\overline{H}^1(G/C,\pi^C)$, where π^C denotes the C-invariants in π. In particular, all the three properties H_T, H_F and H_{FD} are stable under passing to central extensions, and every nilpotent group has property H_T.*

Proof. Decompose $\pi=\sigma\oplus\tau$, where $\sigma=\pi^C$ is the representation on the subspace of C-invariant vectors, and τ is its orthogonal complement (by centrality of C both are stable under G). Decomposing τ into a direct integral of irreducibles and using Theorem 2.3.2 with Proposition 5.1.2, it follows that $\overline{H}^1(G,\tau)=0$. Hence one only needs to use the same computation as in the last part of the proof of Proposition 5.1.2, to conclude that any $b\in Z^1(G,\sigma)$ vanishes on C, so $\overline{H}^1(G,\pi)\cong\overline{H}^1(G/C,\sigma)$. □

In fact, as seen in Theorem 4.1.3, nilpotent groups share a similar property for all degrees. We remark in passing that Proposition 5.1.2, and the equivalence of property (T) with the non-existence of an irreducible unitary representation with $\overline{H}^1\neq 0$ for compactly generated groups [50], immediately give an alternative proof of the following result of Serre [34, p. 28, Theorem 12]: If G is compactly generated, $G/\overline{[G,G]}$ is compact, and G/C has property (T) for some closed central subgroup $C<G$, then G has property (T) as well (noting that the assumption on $G/\overline{[G,G]}$ implies cohomology vanishing with trivial coefficients).

Remark. One can also show that nilpotent groups have property H_T using the following general result of Vershik and Karpushev [53]: Any irreducible representation with non-vanishing first (ordinary) cohomology is not Hausdorff separated from the trivial representation. Indeed, using Schur's lemma applied inductively to the center, it is easy to see that for nilpotent groups the only irreducible representation not separated from (1) is (1) itself. This strategy of proof, which shows (as does our previous argument) that already the non-reduced H^1 vanishes for the trivial representation only (among *irreducible* representations), cannot work in the rest of the examples we shall analyze, and we do not know any finitely generated non-virtually nilpotent group for which it does.

Polycyclic groups. Recall that a solvable group Γ is called *polycyclic* if it admits a filtration

$$\{e\}=\Gamma_k \triangleleft \ldots \triangleleft \Gamma_1 \triangleleft \Gamma_0=\Gamma,$$

where each successive quotient is cyclic. As we show in §5.5, unlike nilpotent groups, these groups need not have property H_F, let alone H_T. However, we have the following result, generalizing Theorem 1.13 of the introduction:

THEOREM 5.1.4. *Every polycyclic group Γ has property H_{FD}. Moreover, a polycyclic group Γ has only finitely many irreducible unitary representations with $\overline{H}^1\neq 0$, which are all finite-dimensional. In fact, for a finite index subgroup of Γ, all such representations are 1-dimensional (and finite in number).*

Proof. The proof relies on the following quite involved result of Delorme [16, Theorem V.6 and Corollary V.2]. We thank Alain Valette for pointing it out to us.

THEOREM 5.1.5. *Let G be a connected solvable Lie group. Then every irreducible unitary G-representation π with $\overline{H}^1(G,\pi)\neq 0$ is 1-dimensional. Furthermore, there exist only finitely many such π's (which can all be classified explicitly).*

The first statement is the content of Theorem V.6 (p. 323) in [16]. As for the second, in Corollary V.2 (p. 333) there, it is observed that a non-trivial 1-dimensional π satisfies $H^1(G,\pi)\neq 0$ if and only if π is a quotient of the adjoint G-representation on $[\mathfrak{g},\mathfrak{g}]_{\mathbf{C}}$. Thus, the number of such π's is bounded by the dimension $\dim_{\mathbf{C}}[\mathfrak{g},\mathfrak{g}]_{\mathbf{C}}$. We remark that this gives an (in principle) "explicit" way to determine whether, in particular, every cohomological π is trivial, a property which indeed holds for "many" groups G.

Returning to the proof of Theorem 5.1.4, observe first that it is enough to prove the very last statement of the theorem. Indeed, if $\Gamma_0<\Gamma$ is a finite index subgroup satisfying that claim, and $\pi_1, \ldots, \pi_n$ are its irreducible representations with non-vanishing $\overline{H}^1$, then using the injectivity of the restriction of $\overline{H}^1$ to a finite index subgroup one can easily deduce that any irreducible Γ-representation σ with non-vanishing $\overline{H}^1$ must be contained in one of the (finitely many) $\operatorname{Ind}_{\Gamma_0}^{\Gamma}\pi_i$, which are all finite-dimensional.

Now, to find this subgroup Γ_0, recall that by a well-known theorem of Auslander, because Γ is polycyclic it has a finite index subgroup Γ_0 which is a co-compact lattice in a connected solvable Lie group G (cf. [45, Theorem 4.28]). Let π be any irreducible unitary Γ_0-representation with $\overline{H}^1(\Gamma_0, \pi)\neq 0$, and consider the induced G-representation $\sigma=\mathrm{Ind}_{\Gamma_0}^G \pi$. By Theorem 3.2.2 we have $\overline{H}^1(G, \sigma)\neq 0$. Hence by decomposing σ into a direct integral of irreducibles, applying Theorem 2.3.2, and using Delorme's Theorem 5.1.5, we deduce that for at least one of the finitely many characters, say χ, appearing in Theorem 5.1.5, one has $\chi\subseteq\sigma$. Corollary 3.1.5 (1) now shows that $\pi=\chi|_{\Gamma_0}$, as required. □

Remark. For any polycyclic group which is not virtually nilpotent, the "moreover" statement in Theorem 5.1.4 fails when reduced is replaced by non-reduced cohomology: it can be shown that such groups always admit an infinite-dimensional irreducible unitary representation with non-vanishing first cohomology.

An example of a connected solvable Lie group G which has property H_T is the 3-dimensional simply-connected $G=\mathrm{SOLV}=\mathbf{R}^*\ltimes\mathbf{R}^2$, where $\mathbf{R}^*$ acts linearly on $\mathbf{R}^2$ through a 1-parameter volume-preserving group of hyperbolic transformations. Here it is easy to verify directly that the 1-dimensional quotients of the G-adjoint representation on $[\mathfrak{g},\mathfrak{g}]_{\mathbf{C}}=\mathbf{C}^2$ are not unitary but have real eigenvalues (coming from the hyperbolic action). Therefore Delorme's theorem implies that G (and hence, by Theorem 4.1.2 (1), its co-compact lattices as well) actually has property H_T. In the next subsection we present another proof of this fact, within a different general approach which can be used to deal also with the (considerably more involved) full result of Delorme. Our approach avoids Lie algebra cohomology, which is essential in Delorme's proof.

5.2. The lamplighter and some related groups

The lamplighter group $L(F)$ associated with a (finite) group F was defined in Theorem 1.4. Obviously, it is an amenable group, but it is solvable only when F is. It is easy to check that $L(F)$ is finitely generated, and it is known not to be finitely presented unless F is trivial [2].

Our aim here is to prove Theorem 1.14 (2) of the introduction:

THEOREM 5.2.1. *For any finite group F, the group $L(F)$ has property H_T.*

By Corollary 4.3.5 this implies the following result (already mentioned in the introduction):

COROLLARY 5.2.2. *If Λ is q.i. to $L(F)$ for some finite F, then $vb_1(\Lambda)=1$.*

We shall prove Theorem 5.2.1 directly only when F is abelian. The general case follows from this, using the fact that when $|F|=|F'|$, the groups $L(F)$ and $L(F')$ are quasi-isometric (and even isometric with respect to the natural choices of generating sets, as is easy to verify directly—see [19] for the origin and first use of this observation). Hence, for every finite F the group $L(F)$ has property H_{FD}. To promote it to property H_T one can, for example, observe first that by Proposition 4.2.3 it has property H_F, since every homomorphism of it into $\mathrm{GL}_n(\mathbf{C})$ has in its kernel a finite index subgroup of the normal torsion subgroup (this follows immediately from the fact that a finite index subgroup of the image, being a finitely generated linear group of characteristic 0, is torsion free). It then follows that $L(F)$ actually has property H_T by Proposition 4.2.4 (1). For this reason, we may actually assume that $F=\mathbf{Z}/n\mathbf{Z}$ is a cyclic group, and we do so hereafter for convenience.

An essential step in the proof of Theorem 5.2.1 is the embedding of $L(F)$ as a discrete co-compact subgroup of a non-discrete, locally compact group $G=G(F)$. This construction may be of interest in its own right, and we first describe it.

Embedding $L(F)$ co-compactly in a locally compact group $G(F)$. Recall that we assume F to be the cyclic group $\mathbf{Z}/n\mathbf{Z}$, which has also a natural ring structure. We first reinterpret the abelian group $\bigoplus_{j\in\mathbf{Z}} F_j$ as follows: Let t be a variable, and denote by $F[t,t^{-1}]$ the ring of polynomials over F in t and t^{-1}. Then $(a_j)_{j\in\mathbf{Z}}\mapsto\sum_j a_j t^j$ is a natural group isomorphism of $\bigoplus_{j\in\mathbf{Z}} F_j$ with $F[t,t^{-1}]$. Next, we want to embed $F[t,t^{-1}]$ in a product of two locally compact rings. In the case where F is a finite field these will be no more than two completions of $F[t,t^{-1}]$ with respect to the valuations at zero and infinity, resulting in local fields of positive characteristic. However, a similar construction can be made in our situation as follows. Consider the completion $F((t))$ of $F[t,t^{-1}]$ with respect to the additively invariant metric for which $t^n\to 0$ when $n\to\infty$. That is, $F((t))=\{\sum_{i=i_0}^{\infty} a_i t^i \mid a_i\in F \text{ and } i_0\in\mathbf{Z}\}$ is the ring of Laurent series, and the metric is induced by the valuation $v^+\colon \sum_{i=i_0}^{\infty} a_i t^i\mapsto i_0$, where i_0 is the minimal index with $a_i\neq 0$ (so "positive high powers" go to zero). Similarly, define $F((t^{-1}))$ with the valuation v^-, where here "negative high powers" go to zero. Notice that the diagonal embedding of $F[t,t^{-1}]$ in $F((t))\oplus F((t^{-1}))$ is discrete and co-compact, with a fundamental domain being the compact subring

$$K=\{(x,y)\in F((t))\oplus F((t^{-1})) \mid v^+(x)\geqslant 0 \text{ and } v^-(y)\geqslant 1\}.$$

(Indeed, for $(x,y)\in F((t))\oplus F((t^{-1}))$ define $p\in F[t,t^{-1}]$ as the polynomial whose part of negative powers is identical to that of x, and whose part of non-negative powers is identical to that of y. Then $(x-p,y-p)\in K$, and it is clear that p satisfying this property is unique.)

Next, notice that the $\mathbf{Z}$-shift action on $\bigoplus_{j\in\mathbf{Z}} F_j$ translates to a multiplication by the corresponding power of t. This action extends naturally to a continuous action on the rings $F((t))$ and $F((t^{-1}))$, and we may define

$$G(F)=\mathbf{Z}\ltimes(F((t))\oplus F((t^{-1}))),$$

where $\mathbf{Z}\cong\langle t^m\rangle_{m\in\mathbf{Z}}$ acts as above, via diagonal multiplication by t^m. It is now clear that $G(F)$ is a locally compact group, containing $\mathbf{Z}\ltimes F[t,t^{-1}]\cong L(F)$ as a discrete co-compact subgroup.

The unitary dual of $F((t))$. For the proof of Theorem 5.2.1 we shall need a convenient description of the unitary duals of the rings $F((t))$ and $F((t^{-1}))$, in which the dual $\mathbf{Z}$-action is transparent. For brevity, we shall sometimes denote these rings by $K=F((t))$ and $\widetilde{K}=F((t^{-1}))$.

Let $\widehat{F}$ be the dual of the abelian group F. Written additively, we have that $\widehat{F}=\{\chi: F\to\mathbf{R}/\mathbf{Z}\,|\,\chi(g+h)=\chi(g)+\chi(h)\}$. Then $\widehat{F}\cong F\cong\mathbf{Z}/n\mathbf{Z}$ is the cyclic group generated by the character χ_1 sending the generator $1\in F$ to $1/n\in\mathbf{R}/\mathbf{Z}$.

Next, we investigate the dual $\widehat{K}=\widehat{F((t))}$. View χ_1 above as a character: $\chi_1: K\to\mathbf{R}/\mathbf{Z}$ by first projecting to the "zero-coordinate" and then applying χ_1. Notice that for every $k\in K$, $\chi_k(x)=\chi_1(kx)$ is an element of $\widehat{K}$.

CLAIM 5.2.3. *The assignment* $\varphi: K\to\widehat{K}$ *defined by* $k\mapsto\chi_k(x)$ *is a topological group isomorphism of the locally compact abelian groups* K *and* $\widehat{K}$.

Proof. Recall that $\widehat{K}$ is a topological group, as usual, with respect to the topology of uniform convergence on compact subsets of K. It is easy to check that φ is a continuous homomorphism. It is injective, because if for all $x\in K$, $\chi_1(kx)=0$, writing $k=\sum_j a_j t^j$ and choosing $x=t^{-j}$ shows that $a_j=0$. This being so for all j, it follows that $k=0$ (note that here the fact that $F=\mathbf{Z}/n\mathbf{Z}$, and not just an abelian group, is used).

Next, $\varphi(K)$ is closed in $\widehat{K}$. Indeed, since K is locally compact it is enough to check that $k_i\to\infty$ in K implies that $\varphi(k_i)\to\infty$ in $\widehat{K}$ (i.e. $\varphi(k_i)$ has no converging subsequence in $\widehat{K}$), which is easy to verify directly. Thus, to complete the proof of the claim, i.e. to show that φ is onto, it is enough by Pontryagin duality to verify that the image $\varphi(K)$ separates the points of K, namely, that for every $x\in K$ there is some $k\in K$ such that $\chi_1(kx)\neq 0$. Here one can use a similar argument as for the injectivity. This completes the proof of the claim. □

Remark 5.2.4. More generally, if F is any finite abelian group and $\widehat{F}$ is its dual, then $\widehat{F((t))}\cong\widehat{F}((t^{-1}))$, where the action is defined by

$$\left\langle\sum_i^{\infty}\chi_i t^i,\sum_i^{-\infty}a_i t^i\right\rangle=\sum_i\langle\chi_i,a_i\rangle,\quad \chi_i\in\widehat{F},\ a_i\in F$$

(the sum in the right-hand side is finite).

We can therefore identify K and $\widehat{K}$ through φ. A useful feature of this identification is its compatibility with the automorphism action of $\mathbf{Z}=\langle t^m\rangle$ on K. Recall that an endomorphism A of the abelian group K defines a dual endomorphism of $\widehat{K}$ by $(A\chi)(x)=\chi(Ax)$. If A is a multiplication by t^m and $\chi=\chi_k$ for some $k\in K$, we have $(A\chi)(x)=\chi_k(Ax)=\chi_k(t^m x)=\chi(k\cdot t^m x)=\chi_{t^m k}(x)$, and hence the action on $\widehat{K}\cong K$ is just the usual multiplication in the ring K. Obviously, the whole discussion applies to the "twin" ring $\widetilde{K}$ as well, with the obvious modifications.

COROLLARY 5.2.5. *Consider the multiplication diagonal action of* $\mathbf{Z}\cong\langle t^m\rangle$ *on* $K\oplus\widetilde{K}$, *and the induced action on the dual*

$$\widehat{K\oplus\widetilde{K}}\cong\widehat{K}\oplus\widehat{\widetilde{K}}\cong K\oplus\widetilde{K}.$$

Then the orbits of the action fall into three types:

(1) *an orbit of an element* (χ_1,χ_2), *with* $\chi_1,\chi_2\neq 0$, *which is closed* (*call such an element, or its orbit, "regular"*);

(2) *an orbit of an element* $(\chi_1,0)$ *or of an element* $(0,\chi_2)$, *with* $\chi_1,\chi_2\neq 0$, *which is of the form* $(k\chi_1,0)$ *or* $(0,k\chi_2)$, *and has a unique limit point* $(0,0)$;

(3) *the zero-character* $(0,0)$.

Proof. If $m\to\infty$ then $t^m k\to 0$ for all $k\in K$ and $t^m\tilde{k}\to\infty$ for all $0\neq\tilde{k}\in\widetilde{K}$. The converse holds when $m\to-\infty$. Using the above identifications of K and $\widetilde{K}$ with their duals, this accounts for the three possible options. □

We can now prove Theorem 1.14 (2) of the introduction:

Proof of Theorem 5.2.1. As before, we may assume that $F\cong\mathbf{Z}/n\mathbf{Z}$, and we continue to do so henceforth. Because $L(F)<G(F)$ is discrete and co-compact (see the discussion above), it is enough by Theorem 4.1.2 (1) to prove that $G(F)$ has property H_T. By part (2) of that theorem it is enough to show that for every *irreducible* non-trivial (continuous) unitary representation π of $G(F)$, one has $\overline{H}^1(G(F),\pi)=0$. For that purpose we shall need the following lemma:

LEMMA 5.2.6. (1) *Assume that $G(F)$ acts continuously and isometrically on a metric space (X,d). Then any $\mathbf{Z}$-fixed point is fixed by all of $G(F)$.*

(2) *Let $J=\mathbf{Z}\ltimes F((t))$, where $\mathbf{Z}=\langle t^m\rangle$ acts as above on $F((t))$ via multiplication. If a continuous isometric J-action on (X,d) has almost fixed points for $\mathbf{Z}$, then it admits almost fixed points for all of J (see Definition* 2.4.3 *for this notion).*

Proof. (1) Let $x\in X$ satisfy $t^m x=x$ for all $m\in\mathbf{Z}$. Let $\bar{\alpha}\in F((t))\oplus F((t^{-1}))$ be of the form $(\alpha,0)$ or $(0,\alpha)$, and take a sequence of $m\,(\to\pm\infty)$ with $t^{-m}\bar{\alpha}t^m\to 0$. Then

$d(\bar{\alpha}x, x) = d(\bar{\alpha}t^m x, t^m x) = d(t^{-m}\bar{\alpha}t^m x, x) \to 0$. Hence x is fixed by $F((t))$, $F((t^{-1}))$ and $\mathbf{Z}$ separately, and thus by all of $G(F)$.

(2) Given $\varepsilon > 0$ and a compact subset $Q \subseteq F((t))$, we need to find an $x \in X$ which is ε-invariant for both t and Q. By assumption there exists some $y \in X$ with $d(ty, y) < \varepsilon$. Let $U \subseteq F((t))$ be a neighborhood of 0 small enough so that $d(uy, y) < \varepsilon$ for $u \in U$. Take m such that $t^{-m}qt^m \in U$ for all $q \in Q$. Then the point $x = t^m y$ satisfies both

$$d(tx, x) = d(t \cdot t^m y, t^m y) = d(t^m ty, t^m y) = d(ty, y) < \varepsilon$$

and

$$d(qx, x) = d(qt^m y, t^m y) = d(t^{-m}qt^m y, y) < \varepsilon \quad \text{for all } q \in Q. \qquad \square$$

An immediate consequence is the following corollary:

COROLLARY 5.2.7. *Let $J = \mathbf{Z} \ltimes F((t))$, where $\mathbf{Z} = \langle t^m \rangle$ acts on $F((t))$ as above. Then J has property H_T.*

Proof. Let π be a unitary J-representation and $b \in Z^1(J, \pi)$.

Case (i): $1_{\mathbf{Z}} \not\subseteq \pi|_{\mathbf{Z}}$. Because $\mathbf{Z}$ is abelian, it has property H_T (Theorem 5.1.1), so the isometric action defined by b has almost $\mathbf{Z}$-fixed points (see Definition 2.4.3 and Lemma 2.4.4). By Lemma 5.2.6 (2) there are almost fixed points for J, i.e. $b \in \overline{B}^1(J, \pi)$ and $[b] = 0$ in $\overline{H}^1_{\mathrm{ct}}(J, \pi)$.

Case (ii): $1_{\mathbf{Z}} \subseteq \pi|_{\mathbf{Z}}$. Then by Lemma 5.2.6 (1) we get $1_J \subseteq \pi|_J$, as required. $\square$

Continuing the proof of Theorem 5.2.1, let π be an irreducible unitary representation of $G(F)$. Corollary 5.2.5 shows that the orbits of the $\mathbf{Z}$-action on the dual of $F((t)) \oplus F((t^{-1}))$ are locally closed. Hence by Mackey's machinery for representations of semi-direct products $A \ltimes N$ (cf. [55, 7.3.1], where it is assumed that $N \cong \mathbf{R}^n$ only for convenience), we may continue by analyzing the spectral measure describing the restriction of π to the abelian normal subgroup $N = F((t)) \oplus F((t^{-1}))$, according to the three types of orbits in Corollary 5.2.5:

(1) *A regular orbit.* Here the restriction of π to N has spectral measure supported on a regular orbit. In particular, since the orbit does not accumulate at 0, $\pi|_N$ does not have almost invariant vectors. Now, let $b \in Z^1(G(F), \pi)$ and let $\varrho(g)v = \pi(g)v + b(g)$ be the associated isometric action on V_π. If N has a fixed point for this action, say v_0, then by normality $\varrho(t^m)v_0$ is also fixed by N for all $m \in \mathbf{Z}$. If $\varrho(t^m)v_0 \neq v_0$ for some m_0, then the line between $\varrho(t^{m_0})v_0$ and v_0 is also preserved by N, i.e. $1_N \subseteq \pi|_N$, which is impossible (since we are dealing with a regular orbit). Therefore $\varrho(t^m)v_0 = v_0$ for all $m \in \mathbf{Z}$, so v_0 is a global fixed point for ϱ and $b \in B^1(G(F), \pi)$ (Lemma 2.4.2). Suppose

on the other hand that N does not have a fixed point for the isometric action, i.e. $b|_N \notin B^1(N, \pi|_N)$. The fact that the orbit is regular, and hence does not accumulate at 0, shows that the restriction of π to N does not have *almost* invariant vectors, and hence by Proposition 2.4.1, $\overline{B}^1(N, \pi|_N) = B^1(N, \pi|_N)$. It follows that $b|_N \notin \overline{B}^1(N, \pi|_N)$, and because N is abelian, Theorem 5.1.1 implies that $1_N \subseteq \pi|_N$, which is again impossible by our assumption that the orbit is regular. We conclude that necessarily $H^1(G(F), \pi) = 0$, and in particular $\overline{H}^1(G(F), \pi) = 0$, for π coming from a regular orbit.

(2) *An orbit of an element* $(\chi, 0)$ *or* $(0, \chi)$ *with* $\chi \neq 0$. By symmetry we may assume that we are in the first case. Here $F((t^{-1}))$ acts trivially in π, i.e. the representation π factors through a representation of the group J as defined in Corollary 5.2.7. Consider now $b \in Z^1(G(F), \pi)$ and examine the following two possibilities: either $b|_{F((t^{-1}))} = 0$ (pointwise, not only as a cohomology class), in which case both π and b factor through the homomorphism to $J = \mathbf{Z} \ltimes F((t))$, so by Corollary 5.2.7 it follows that $\overline{H}^1(G(F), \pi) = \overline{H}^1(J, \pi) = 0$ (since π is not the trivial representation), or $b|_{F((t^{-1}))} \neq 0$. But the latter case is impossible because $b|_{F((t^{-1}))}$ is then automatically reduced (since π is trivial on $F((t^{-1})))$, hence b is reduced on the whole abelian group $F((t) \oplus F((t^{-1}))$, and by Theorem 5.1.1 the latter should have an invariant vector in π, which is not the case here.

(3) *The* 0*-orbit.* This means that π, being irreducible, factors through a character χ of $\mathbf{Z}$. Let $b \in Z^1(G(F), \chi)$ and ϱ be the associated isometric action (on $\mathbf{C}$). If $\chi \neq 1$ then by Theorem 5.1.1 and Proposition 2.4.1, $b|_{\mathbf{Z}} \in B^1(\mathbf{Z}, \chi)$, and hence $\varrho|_{\mathbf{Z}}$ has a fixed point. By Lemma 5.2.6 (1) this point is fixed by all of $G(F)$, so Lemma 2.4.2 shows that $b \in B^1(G(F), \pi)$. It follows that if $[b] \neq 0$ then $\chi = 1$, thereby completing the proof of Theorem 5.2.1. $\square$

We remark that here (and in the sequel), one could also make use of a standard spectral sequence for cohomology of group extensions, in the cases where the computation of reduced cohomology is reduced to that of ordinary cohomology.

The various arguments used in the proof of Theorem 5.2.1 for the ambient group $G(F)$ can in fact be used, together with some structure theory, to treat the case of a general connected solvable Lie group, as in Delorme's Theorem 5.1.5. The main ingredients of this approach are:

(1) Reduction through direct products and centers (see the proof of Theorem 4.2.5 above);

(2) The use of Mackey's machinery through a distinction between orbits of characters which respectively contain, or do not contain, the trivial representation (1) in their closure;

(3) In the case when (1) is *not* in the closure, Proposition 2.4.1 and then Theo-

rem 5.1.1 are applicable.

(4) In the case when (1) *is* in the closure, one has contracting automorphisms, and an argument as in Lemma 5.2.6 can be applied.

We shall see further illustration of this approach in the next subsection, but meanwhile remark that the proof of Theorem 5.2.1 can be applied virtually verbatim in the case where K is a local field and $G=K^*\ltimes K^2$, where K^* acts on K^2 through a homomorphism

$$t\longmapsto\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\in \mathrm{SL}_2(K).$$

Here K^* plays the role of $\mathbf{Z}$, and K^2 replaces $K\oplus\widetilde{K}$ in Theorem 5.2.1. Thus, G as above has property H_T, and by Theorem 4.1.2 (1) so do its co-compact lattices. Specializing to the case $K=\mathbf{R}$ yields the group SOLV, which was already discussed earlier. Alternatively, taking K to be of positive characteristic yields other interesting examples when considering co-compact lattices (and using Theorem 4.1.2 (2)), such as in the following result:

COROLLARY 5.2.8. *Let F be a finite field. Then the group $\Gamma=\mathbf{Z}\ltimes_A(F[t])^2$, where $F[t]$ is the ring of polynomials over F, and $\mathbf{Z}$ acts through multiplication by powers of the matrix*

$$A=\begin{pmatrix} t & t-1 \\ 1 & 1\end{pmatrix},$$

is finitely generated and has property H_T. Consequently, by Corollary 4.3.5, if Λ is any group quasi-isometric to Γ then $vb_1(\Gamma)=1$.

We remark that it can be shown that a group Γ as above is *not* finitely presented.

Finally, as an application of our result for the lamplighter group, we have the following theorem, establishing the relevant parts of Theorems 1.4 and 1.14:

THEOREM 5.2.9. *There exists a family of $2^{\aleph_0}$ non-isomorphic finitely generated 3-step solvable groups with property H_T, and having $b_1=1$. Consequently, by Corollary 4.3.5, any group Λ quasi-isometric to one of these groups satisfies $vb_1(\Lambda)=1$.*

Proof. The groups which we construct are all (torsion-)central extensions of one given lamplighter group $L(F)$, where F is a, say, cyclic group on p elements. Recall that by Theorem 5.2.1 and Corollary 5.1.3 any such group has property H_T, so we only need to construct a continuum of those.

Consider the group Γ generated by an infinite set $\{x_n, z_m, t\}$, where $n\in\mathbf{Z}$ and $1\leqslant m\in\mathbf{Z}$, with the following relations: $[x_i,x_j]=z_{i-j}$ for all $i>j$; $tx_it^{-1}=x_{i+1}$ for all i; ${x_i}^p={z_i}^p=1$ for all i; and all the z_m's are central. The group G is generated by t and x_0

(say), so it is actually finitely generated. Also, it is easy to see that G is a "Heisenberg-like" central extension of the lamplighter group $L(F)$, with center C generated by the z's, i.e. isomorphic to the infinite direct sum of cyclic p-groups. Therefore C admits a continuum of different subgroups, C_α, all central. An easy classical argument due to Hall (cf. [33, p. 69, Item III.42]), making use only of the finite generation of Γ, then shows that any given group can be isomorphic to G/C_α for only countably many α's. Hence the family of quotients of the form G/C_α must contain a continuum of non-isomorphic groups (all central extensions of $L(F)$). This completes the proof of the theorem. $\square$

5.3. Other abelian-by-cyclic groups

Recall that the groups $\Gamma(n,m)$ were defined in the introduction (we assume throughout that $|n\cdot m|>1$). As we shall see, the $\Gamma(n,m)$'s are the "($\mathbf{S}$-arithmetic) characteristic-0" analogues of the "positive characteristic" lamplighter groups (discussed in the previous section). Their treatment will rely on a very similar approach, with some additional technicalities.

The following result completes the proof of Theorems 1.4 and 1.14 in the introduction:

THEOREM 5.3.1. *The groups $\Gamma(n,m)$ have property H_T. Consequently, by Corollary* 4.3.5, *if Λ is a group which is quasi-isometric to $\Gamma(n,m)$, then $vb_1(\Lambda)=1$.*

As in the case of the lamplighter groups, we preface the proof by constructing a locally compact group $G(n,m)$ in which $\Gamma(n,m)$ embeds discretely and co-compactly. For any prime p, we denote by $\mathbf{Q}_p$ the field of p-adic numbers, and follow the notation $\mathbf{Q}_\infty=\mathbf{R}$. We fix hereafter n and m with $(n,m)=1$, denote by $\mathbf{S}$ the set of primes dividing $n\cdot m$, and let $\bar{\mathbf{S}}=\mathbf{S}\cup\{\infty\}$. It is easy to verify that the diagonal embedding of $\mathbf{Z}[1/nm]=\mathbf{Z}(\mathbf{S})$ in $\bigoplus_{p\in\bar{\mathbf{S}}}\mathbf{Q}_p$ is discrete and co-compact. We set $G(n,m)=\mathbf{Z}\ltimes(\bigoplus_{p\in\bar{\mathbf{S}}}\mathbf{Q}_p)$, where $\mathbf{Z}$ acts through multiplication by (powers of) m/n diagonally on each one of the summands. We then have a natural embedding of $\Gamma(n,m)$ in $G(n,m)$, which is easily seen to be discrete and co-compact.

Proof (*of Theorem* 5.3.1). By Theorem 4.1.2 (1) it is enough to prove that $G(n,m)$ has property H_T, and by (2) of that theorem it suffices to see that if π is any non-trivial irreducible unitary representation of $G(n,m)$, then one has $\overline{H}^1(G(n,m),\pi)=0$. Let $b\in Z^1(G(n,m),\pi)$ be a 1-cocycle, and let $\mathbf{S}'\subseteq\bar{\mathbf{S}}$ be the *minimal* subset for which both b and π factor through the natural quotient map from $G(n,m)$ to $\mathbf{Z}\ltimes(\bigoplus_{p\in\mathbf{S}'}\mathbf{Q}_p)$, i.e. $\bar{\mathbf{S}}-\mathbf{S}'$ is the set of *all* primes p for which both $b=0$ and $\pi=1$ on $\mathbf{Q}_p$. We shall see that if $[b]\neq 0$ in $\overline{H}^1(G(n,m),\pi)$ then $\mathbf{S}'$ is *empty*, so π and b factor through a representation of the acting $\mathbf{Z}$, thereby implying that π is trivial (as $\mathbf{Z}$ has property H_T).

Assume by contradiction that $\mathbf{S}'$ is not empty and set $G=\mathbf{Z}\ltimes(\bigoplus_{p\in\mathbf{S}'}\mathbf{Q}_p)$.

Let $|\cdot|_p$ denote the usual absolute value in the field $\mathbf{Q}_p$ (including the case $p=\infty$). Notice that $|m/n|_p\neq 1$ for all $p\in\bar{\mathbf{S}}$. We distinguish between two cases:

(i) Either $|m/n|_p>1$ for all $p\in\mathbf{S}'$, or $|m/n|_p<1$ for all $p\in\mathbf{S}'$;

(ii) There are $p,q\in\mathbf{S}'$ with $|m/n|_p>1$ and $|m/n|_q<1$.

Suppose that (i) holds. This means that high powers (of appropriate sign) of m/n contract all of $\bigoplus_{p\in\mathbf{S}'}\mathbf{Q}_p$ to the identity. Then a proof identical to that of Lemma 5.2.6 and Corollary 5.2.7 shows that G has property H_T, hence π (being irreducible) must be trivial, and b factors through the acting $\mathbf{Z}$ $(=G/[G,G])$, so $\mathbf{S}'$ is empty.

Suppose that (ii) holds. For every p the dual $\widehat{\mathbf{Q}}_p$ is isomorphic to $\mathbf{Q}_p$, and

$$\widehat{\bigoplus_{p\in\mathbf{S}'}\mathbf{Q}_p}\cong\bigoplus_{p\in\mathbf{S}'}\mathbf{Q}_p,$$

an isomorphism respecting the action of $\mathbf{Z}=\langle m/n\rangle$ (this follows, e.g., as in the analogous discussion in the proof of Theorem 5.2.1). We make this identification henceforth. It is easy to check that every orbit of the latter $\mathbf{Z}$-action is locally closed (in fact, we classify them below), so we may continue by Mackey's machinery, as in the lamplighter group case, to examine the possibilities for the $\mathbf{Z}$-orbit corresponding to the spectral measure of the restriction of π to $\bigoplus_{p\in\mathbf{S}'}\mathbf{Q}_p$. Let $\bar{\chi}=(\chi_{p_1},...,\chi_{p_k})\in\bigoplus_{p\in\mathbf{S}'}\mathbf{Q}_p$ $(k=|\mathbf{S}'|)$ be a point in that orbit, and consider the following possibilities:

(1) $\bar{\chi}$ is "regular", i.e. $\chi_{p_i}\neq 0$ for all i. In this case the orbit $\mathbf{Z}\bar{\chi}$ is closed, in particular, does not accumulate at 0 (this is exactly where the assumption in (ii) is used). Then a discussion completely analogous to the one in the corresponding part of the proof of Theorem 5.2.1 shows that $H^1(G,\pi)=0$ in this case, contradicting the assumption on the 1-cocycle b.

(2) $\bar{\chi}\neq 0$ but $\chi_{p_i}=0$ for some i. Then by normality of $\mathbf{Q}_{p_i}$, π is trivial on it. By minimality of $\mathbf{S}'$, we must then have $b|_{\mathbf{Q}_{p_i}}\neq 0$. Hence $b|_{\mathbf{Q}_{p_i}}$ is not in $\overline{B}^1(\mathbf{Q}_{p_i},\pi|_{\mathbf{Q}_{p_i}})$, and the same must hold for b when restricted to all of $\bigoplus_{p\in\mathbf{S}'}\mathbf{Q}_p$. Being abelian, Theorem 5.1.1 then implies that the latter group admits an invariant vector in π. Hence by normality and irreducibility it acts trivially, which is impossible in our case $\bar{\chi}\neq 0$.

(3) $\bar{\chi}=0$, i.e. π factors through a 1-dimensional character χ of $\mathbf{Z}$. Then, if χ is not trivial, it follows from Theorem 5.1.1 and Proposition 2.4.1 that $b|_{\mathbf{Z}}$ is a 1-coboundary. Hence the isometric G-action it defines has a $\mathbf{Z}$-fixed point (Lemma 2.4.2). By an obvious modification of Lemma 5.2.6 (1) to our case, it follows that this $\mathbf{Z}$-fixed point is actually fixed by all of G. Hence $b\in B^1(G,\pi)$ (Lemma 2.4.2), in contradiction to our assumption. We are thus left with the remaining case $\pi=\chi=1_G$, as claimed. □

5.4. Solvable groups without property H_{FD}

There are many known infinite, finitely generated amenable groups which do not admit a finite index subgroup with infinite abelianization, such as torsion groups (e.g. the Grigorchuk groups). By Theorem 4.3.1 it follows that they cannot have property H_{FD}. Of course, this line of argument cannot work for solvable groups. This, together with the various examples shown in the previous sections, may give a wrong impression, as we now make the following observation:

THEOREM 5.4.1. *Let G be a discrete group. Then* every *unitary representation π of the wreath product $\Gamma=G\wr\mathbf{Z}$ factoring through a representation of G satisfies $\overline{H}^1(\Gamma,\pi)\neq 0$. In particular, if G is infinite then Γ does not have property H_{FD}.*

Thus, taking G in the theorem to be solvable yields a solvable group without property H_{FD}. In fact, whenever G is not virtually abelian, Theorem 5.4.1 gives a continuum of *infinite-dimensional irreducible* unitary representations with $\overline{H}^1\neq 0$. The particularly interesting case of $\mathbf{Z}\wr\mathbf{Z}$ is discussed further below.

Notation. Hereafter we use the notation $\mathbf{Z}^G$ for the direct sum of G copies of $\mathbf{Z}$, namely, the *finitely supported* functions from G to $\mathbf{Z}$. The set of functions supported on one fixed element $g_0\in G$ is denoted by $\mathbf{Z}^{g_0}$.

Proof. Pick any non-zero vector v in the representation space V_π, and define a homomorphism $f:\mathbf{Z}^G\to V_\pi$ as follows: On the "copy" $\mathbf{Z}^e$ of $\mathbf{Z}$, define $f(m)=mv$ (scalar multiplication of v by $m\in\mathbf{Z}$). For an arbitrary $g\in G$ define f on the "g-copy" $\mathbf{Z}^g$ of $\mathbf{Z}$ by $f(m)=\pi(g)(mv)=m\pi(g)v$. Then f extends uniquely to a homomorphism $\tilde{f}:\mathbf{Z}^G\to V$. Denoting elements of $\mathbf{Z}^G$ by $\overline{m}$, it is easy to see that $(\overline{m},g)v=\pi(g)v+\tilde{f}(\overline{m})$ defines an isometric uniform action of Γ on V_π whose linear part is π. This gives a non-zero element $b\in\overline{H}^1(\Gamma,\pi)$, namely, $b(\overline{m},g)=\tilde{f}(\overline{m})$. □

Notice that if $\mathbf{Z}$ is replaced by any group H which has $\mathbf{Z}$ as a quotient, then the group $\Gamma=G\wr H$ satisfies the same conclusion of the theorem, by repeating the same construction, this time letting the homomorphism $\tilde{f}$ factor through the quotient. However, when H does not have infinite abelianization, the strategy of the proof of Theorem 5.4.1 breaks down. The simplest case to examine in this regard is that of $\Gamma=\mathbf{Z}\wr D$, where D is the infinite dihedral group. Note that this group is not virtually torsion free. Let us see how it can be approached using geometric group theory. In contrast to the previous construction, this establishes the existence of cohomological (irreducible) representations of wreath products which *do not* factor through a representation of the acting subgroup. Note that the proof below is completely non-constructive, and relies almost entirely on the relation developed earlier with geometric group theory.

THEOREM 5.4.2. *The group* $\Gamma=\mathbf{Z}\wr D$ *admits an irreducible* infinite-dimensional *unitary representation* π *with* $\overline{H}^1(\Gamma,\pi)\neq 0$.

Proof. We first claim that for any non-trivial *finite-dimensional* irreducible Γ-representation π, one has $H^1(\Gamma,\pi)=0$. (Recall that for finite-dimensional representations the first reduced and ordinary cohomology coincide—see §2.4.5 above.) Indeed, suppose by contradiction that $0\neq[b]\in H^1(\Gamma,\pi)$, and consider the restriction of b to $D^{\mathbf{Z}}$. We examine two possibilities:

(1) $b|_{D^{\mathbf{Z}}}=0$ in $H^1(D^{\mathbf{Z}},\pi|_{D^{\mathbf{Z}}})$;

(2) $b|_{D^{\mathbf{Z}}}\neq 0$ in $H^1(D^{\mathbf{Z}},\pi|_{D^{\mathbf{Z}}})$.

If case (1) holds, the affine isometric Γ-action defined by b has a fixed point for $D^{\mathbf{Z}}$ (Lemma 2.4.2). By normality, the set of its fixed points is Γ-invariant (affine subspace), and hence by irreducibility of π it is either one point, or the whole space. The first is impossible because then this point would be fixed by all of Γ, contradicting Lemma 2.4.2 and our assumption on the 1-cocycle. The second implies that the $D^{\mathbf{Z}}$-isometric action is trivial, so both the representation and the cocycle factor through $\mathbf{Z}$, and π must be trivial by property H_T of $\mathbf{Z}$.

Suppose that (2) holds, and let $\varrho: D^{\mathbf{Z}}\to U(n)$ ($n=\dim\pi$) be the homomorphism corresponding to $b|_{D^{\mathbf{Z}}}$ (see §2.4.5 above). Since ϱ is non-trivial, there is at least one copy of D in $D^{\mathbf{Z}}$, denoted D_1, on which the restriction of ϱ remains non-trivial. Denoting the "complement" of D_1 in $D^{\mathbf{Z}}$ by D_2, we then deduce from Theorem 4.2.5 (passing to the reduced cohomology using §2.4.5) that there is a non-zero vector v on which the $D^{\mathbf{Z}}$-linear action factors through an action of D_1. By irreducibility and finite-dimensionality of π, finitely many G-translations of v span the whole space, and hence all but finitely many of the $\mathbf{Z}$-copies of D in $D^{\mathbf{Z}}$ act trivially in π. But then all the $\mathbf{Z}$-conjugates of one of these copies act trivially as well, and they generate all of $D^{\mathbf{Z}}$. Thus $D^{\mathbf{Z}}$ acts trivially in π, and $b|_{D^{\mathbf{Z}}}$ is just a homomorphism into $\mathbf{C}^n$, which must vanish as D has finite abelianization. This contradicts the assumption on b, and completes the proof of the claim.

Returning to the proof of Theorem 5.4.2, suppose by contradiction that Γ does not admit an infinite-dimensional irreducible representation π with $\overline{H}^1(\Gamma,\pi)\neq 0$. Then this, the above claim and Theorem 4.1.2 (2) imply together that Γ must have property H_T. However, recall now that since D is bi-Lipschitz equivalent to $\mathbf{Z}$, by [19] Γ is q.i. to $\mathbf{Z}\wr\mathbf{Z}$, which has *infinite* vb_1. This contradicts Theorem 4.3.2 and completes the proof of the theorem. $\square$

In fact, examining the proof more carefully shows that it works equally well for wreath products $G\wr H$ if H is bi-Lipschitz equivalent to a group with infinite abelinization, and G is residually finite. These assumptions can be weakened further, but at any rate we expect a much more general phenomenon (see §6.6 below).

Notice that using Theorem 4.3.3, Theorem 5.4.2 also shows that the group $\mathbf{Z}\wr\mathbf{Z}$ does not have property H_{FD}, without making any computations. Thus Theorem 5.4.2 proves Theorem 1.15 of the introduction. Unfortunately, it does not shed light on the intriguing question of whether or not this group admits an *infinite-dimensional* irreducible representation with $\overline{H}^1\neq 0$. We conclude this subsection by discussing this group a little further, making some explicit computations.

The proof of Theorem 5.4.1 shows that for *every* 1-dimensional unitary character χ of $\Gamma=\mathbf{Z}\wr\mathbf{Z}$, factoring through a character of the acting $\mathbf{Z}$, one has $\dim H^1(\Gamma,\chi)=1$. Explicitly, given any such χ, to define a homomorphism $\varrho:\Gamma\to S^1\ltimes\mathbf{C}$ with linear part χ, one chooses the ($\mathbf{C}$-)value of the generator of the $\mathbf{Z}$-copy in the, say, 0-copy, and this extends *uniquely* to a homomorphism as in the proof of Theorem 5.4.1 above. Now, integrating this representation (and 1-cocycles) with respect to any non-atomic measure μ on S^1 gives rise to a weakly mixing unitary representation with $\overline{H}^1\neq 0$. More concretely, take any such measure μ on S^1 whose points χ are identified with the characters of $\mathbf{Z}$, and consider the Hilbert space $V_\mu=L^2(S^1,\mu)$ with the usual scalar product. Writing the wreath product elements as (h,z), where $h:\mathbf{Z}\to\mathbf{Z}$ is finitely supported and $z\in\mathbf{Z}$ acts as usual by translations, we have a unitary Γ-action π and a (reduced) 1-cocycle b ranging in V_μ, defined by

$$(\pi(h,z)(F))(\chi)=\chi(z)F(\chi),\quad b((h,z))(\chi)=\sum_{n\in\mathbf{Z}}h(n)\chi(n).$$

We do not know if this construction accounts for all the first reduced cohomology of the unitary representations of Γ. The possibility that the only *irreducible* unitary Γ-representations with non-vanishing $\overline{H}^1$ are 1-dimensional is equivalent to the fact that a representation as above occurs discretely in any π with $\overline{H}^1(\Gamma,\pi)\neq 0$. An equivalent characterization of this phenomenon would be a "relative property H_T"-type situation: For every unitary π with $\overline{H}^1(\Gamma,\pi)\neq 0$, there exists a non-zero vector which is invariant under the normal subgroup $\mathbf{Z}^{\mathbf{Z}}$.

5.5. An example of quasi-isometric polycyclic groups

The purpose of this section is to prove the following result:

THEOREM 5.5.1. *There exist polycyclic groups Γ and Λ which are co-compact lattices in the same Lie group, such that Γ satisfies $b_1(\Gamma)=vb_1(\Gamma)=3$ and has property H_T, while Λ satisfies $b_1(\Lambda)=vb_1(\Lambda)=1$ and does not have property H_T or H_F (but only H_{FD}). Moreover, such examples exist with Γ's having arbitrarily large b_1.*

As Γ and Λ are in particular quasi-isometric, this highlights the following phenomena, complementing various results proved earlier for the class of amenable groups:

(1) Unlike property H_{FD}, property H_F is *not* a q.i. invariant;

(2) Even among groups with finite vb_1, this quantity is not a q.i. invariant (compare with the remark preceding Theorem 1.2 in the case of infinite vb_1);

(3) Even if the group Γ has (the strongest) property H_T and large b_1 (as one wishes), no better lower estimate on vb_1 of groups q.i. to it may be obtained in general, other than its positivity;

(4) Unlike nilpotent groups, there exist polycyclic groups without property H_F.

We begin the discussion by describing a general method to construct polycyclic groups without property H_F, which will then be used in our concrete examples.

Consider a ring of integers $\mathcal{O}$ of a number field K, and a field embedding $\tau: K\to\mathbf{C}$. Assume that there exists an element $\mu\in\mathcal{O}$ which satisfies $|\tau(\mu)|=1$, and is not a root of 1. We have $\mathcal{O}\cong\mathbf{Z}^d$ as an abelian additive group, and define $\Lambda=\mathbf{Z}\ltimes\mathcal{O}$, where a generator of $\mathbf{Z}$ acts on $\mathcal{O}$ via multiplication by μ. Then Λ is a polycyclic group, and τ defines a natural homomorphism of it into $S^1\ltimes\mathbf{C}$ with an infinite character χ as its linear part. This gives a non-zero class in $H^1(\Lambda,\chi)$ (§2.4.5), so Λ does not have property H_F.

We therefore proceed to find such $\mathcal{O}$ and μ. It is easy to see that in extension degrees 2 and 3, any algebraic integer μ of absolute value 1 has to be a root of 1. It is in degree 4 that we will find (and in fact classify) our examples. Suppose that $f(x)=x^4-ax^3+bx^2-cx+d$ is the minimal polynomial of μ over $\mathbf{Q}$, having integral coefficients (recall that μ is an algebraic integer, not a root of 1, with $|\mu|=1$). Because $\bar{\mu}=\mu^{-1}$ is also a root of f, μ is a root of $x^4f(x^{-1})$, which (using minimality of f) must then be equal to f. Hence we can write $f(x)=x^4-ax^3+bx^2-ax+1$. As the product of all roots is 1, if the other two roots of f are not real then they are complex conjugates whose product is 1, so all roots would have modulus 1. However, as is well known, an algebraic unit all of whose Galois conjugates have modulus 1 must be a root of 1. Therefore the other two roots of f must be real. Conversely, if we find an irreducible polynomial f of the above form, then its roots come as two reciprocal pairs. If furthermore it has exactly two real and two complex roots, then the reciprocal of one complex root is its complex conjugate, hence it is of modulus 1, and the fact that the other two are real ensures that it is not a root of 1, as required.

We are thus reduced to finding an integral polynomial of the form $f(x)=x^4-ax^3+bx^2-ax+1$ which is irreducible and has two real and two complex roots. For this purpose, observe that if x is a root of f as above, then $y=x+x^{-1}$ is a root of $h(y)=y^2-ay+b-2$. It is easy to see that the fact that h has one real root with $|y_1|>2$, and another with $|y_2|<2$, is *equivalent* to the requirement that f above has two real and two complex roots (if all four roots were real (resp. non-real) then we would have $|y_j|>2$ (resp. $|y_j|<2$) for both y_j). Furthermore, once we have f of the above form with two real and two complex roots, it fails to be irreducible only if it has a quadratic factor whose roots are the two complex roots of f, and therefore they are roots of 1. In that case a root y of h must be an integer, which together with the condition $|y|<2$ leaves only the possibilities $y=0,\pm 1$. Consequently, adding the condition on h of *non-vanishing* at these three values to the previous one gives a set of conditions on the coefficients a and b in f which forms a *complete* description of all the degree-4 polynomials satisfying the properties required for our construction.

To write explicit examples, we note that as it will turn out shortly, we shall need the real roots of f to be positive. This is guaranteed by the conditions $a>0$ and $b-2>0$ (which imply that h has positive roots). In this case, the condition on the two roots y_1 and y_2 of h lying at different sides of 2 is equivalent to the condition $h(2)<0$, i.e. (i) $b<2a-2$. For the irreducibility of f it only remains to verify that $h(1)\neq 0$, i.e. (ii) $a\neq b-1$. Subsequently, these two latter conditions, together with the third condition (iii) $a>0$ and $b-2>0$, force the polynomial $f(x)=x^4-ax^3+bx^2-ax+1$ to satisfy all the required properties. For instance, taking any integers $a=b>2$ guarantees all of (i), (ii) and (iii), so $f(x)=x^4-ax^3+ax^2-ax+1$ $(a>2)$ has a root μ as required.

Continuing the proof of Theorem 5.5.1, fix once and for all a polynomial f as above, and denote its complex roots by μ and μ^{-1}, and its real (positive) roots by α and α^{-1}. Denote by $\Lambda=\mathbf{Z}\ltimes\mathcal{O}$ the corresponding group which we know not to possess property H_F, as explained in the first paragraph of the proof. Let $B\in\mathrm{SL}_4(\mathbf{Z})$ be a matrix whose characteristic polynomial is f. Then $\Lambda\cong\mathbf{Z}\ltimes\mathbf{Z}^4$, where $\mathbf{Z}$ acts through multiplication by powers of B. We may embed Λ naturally as a co-compact lattice in the connected polycyclic Lie group $H=A_t\ltimes\mathbf{R}^4$, where A_t is a 1-parameter subgroup such that $A_{t=1}$ is conjugate to B. Indeed, we may take A_t so that $A_{t=1}$ is a block matrix having the diagonal matrix with α and α^{-1} in its upper left 2×2-corner, and a rotation matrix whose eigenvalues are μ and μ^{-1} in the lower right 2×2-corner.

To construct the second group Γ we climb one dimension higher, to a 6-dimensional (unimodular) Lie group G in which H is embedded co-compactly. Thus Λ will also be co-compact in G. The group G is a semi-direct product $(S^1\times\mathbf{R})\ltimes\mathbf{R}^4$ $(S^1\cong\mathbf{R}/\mathbf{Z})$, where $S^1\times\mathbf{R}$ acts through a homomorphism $\psi: S^1\times\mathbf{R}\to\mathrm{SL}_4(\mathbf{R})$ defined by letting $\mathbf{R}$ act as in the group H above through A_t, and S^1 act through the (commuting!) 4×4-block matrices which are the identity in the upper left 2×2-corner, and rotation in the lower right 2×2-corner (in fact, G is isomorphic to $\mathrm{SOLV}\times\mathrm{SO}(2)\ltimes\mathbf{R}^2$). Now, in G we can find another 5-dimensional co-compact subgroup H' isomorphic to $\mathbf{R}\ltimes\mathbf{R}^4$, where this time $\mathbf{R}$ acts through multiplication by the positive diagonal matrices in $\mathrm{SL}_4(\mathbf{R})$ having

1 in the third and fourth diagonal entries. In fact, $H' \cong \mathrm{SOLV} \times \mathbf{R}^2$, and if Γ is the product of some co-compact lattice in SOLV with $\mathbf{Z}^2$, then Γ has property H_T (e.g. by using Corollary 5.1.3 and the fact that the lattices of SOLV have this property—see the discussion preceding Corollary 5.2.8). Thus both Γ and Λ are co-compact lattices in G, and an easy computation of the abelianizations verifies the properties claimed at the first part of the theorem.

To see that one can have similar constructions with Γ having arbitrary large b_1 and Λ having $vb_1=1$, one can use the same technique, but letting $G=(S^1\times\mathbf{R})\ltimes(\mathbf{R}^4)^n$ for any n, where $S^1\times\mathbf{R}$ acts through the homomorphism $\Psi: S^1\times\mathbf{R}\to\mathrm{SL}_{4n}(\mathbf{R})$, which is taken as the previous ψ on each one of the n copies of $\mathrm{SL}_4(\mathbf{R})$ along the main diagonal. Here we get $b_1(\Lambda)=vb_1(\Lambda)=1$ and $b_1(\Gamma)=vb_1(\Gamma)=2n+1$. □

6. Some further results, remarks and related questions

6.1. UEs and lattices in semisimple groups

It is natural to expect that Theorem 1.5 should hold for *all* (not necessarily amenable) finitely generated groups. Some supporting evidence is supplied by the fact (interesting in itself) that it does hold for every arithmetic Chevalley group, such as $\mathrm{SL}_n(\mathbf{Z})$; more precisely, for every group commensurable to $\mathbf{G}(\mathbf{Z})$, where $\mathbf{G}$ is a simple algebraic group defined and split over $\mathbf{Q}$. This follows immediately from Theorem 1.5 and the fact that these groups contain nilpotent (and hence amenable) subgroups of equal cohomological dimension. The latter, in turn, follows from a result of Borel and Serre [12]: For any $\mathbf{G}$ as above (but without any assumption on its $\mathbf{Q}$-rank), $\mathrm{cd}_{\mathbf{Q}}(\mathbf{G}(\mathbf{Z}))$ differs from the dimension of the symmetric space associated with $G=\mathbf{G}(\mathbf{R})$ by the $\mathbf{Q}$-rank of $\mathbf{G}$. Calculating dimensions in the Iwasawa decomposition $G=KAN$, it is now easy to see that in our case $\mathrm{cd}_{\mathbf{Q}}(\mathbf{G}(\mathbf{Z}))=\dim N$, so it only remains to use the fact that one can choose N that intersects $\mathbf{G}(\mathbf{Z})$ with a co-compact lattice of N. In a similar spirit, one can deduce from Theorem 1.5 other related results which are not special to the amenable setting, e.g., if a product of n infinite groups uniformly embeds in a group Γ, then $\mathrm{cd}_{\mathbf{Q}}\Gamma\geqslant n$.

In fact, the framework of non-uniform lattices suggests further intriguing questions: Given a simple Lie group G and two lattices Γ and $\Lambda<G$, when does Γ uniformly embed in Λ? If one of the lattices, say Γ, is uniform, then a complete answer is available: Every discrete subgroup of G uniformly embeds in Γ, and the only discrete subgroups of G in which Γ uniformly embeds are themselves uniform lattices (in which case a UE must be a quasi-isometry—see §6.2 below). We believe that the hierarchy for non-uniform lattices should be determined according to their $\mathbf{Q}$-rank: If Γ uniformly embeds in Λ then $\mathbf{Q}\text{-rank}\,\Gamma\geqslant\mathbf{Q}\text{-rank}\,\Lambda$, with equality only when they are commensurable. It would

be especially interesting to show the converse, namely, that a UE does exist when there is a strict inequality between the $\mathbf{Q}$-ranks (in the right direction). The first, yet challenging, case to understand here is when $G=\mathrm{SL}_2(\mathbf{C})$ (for these purposes, in the rank-1 case regard all non-uniform lattices as having $\mathbf{Q}$-rank 1).

Finally, for a discrete group Γ, call the maximal integer n such that $\mathbf{Z}^n$ uniformly embeds in Γ the *Euclidean degree* of Γ, denoted $d(\Gamma)$. By Theorem 1.5, if Γ has finite $\mathbf{Q}$-cohomological dimension, then $d(\Gamma)<\infty$. (Is the converse true?) Notice that, unlike with quasi-isometric embeddings, this invariant is interesting also for hyperbolic groups. For example, if Γ is the fundamental group of a closed hyperbolic n-manifold, then $d(\Gamma)=n-1$ ($d(\Gamma)\leqslant n-1$ because a uniform embedding of $\mathbf{Z}^n$ into Γ must be a quasi-isometry using §6.2 below, and $d(\Gamma)\geqslant n-1$ using horosphere embeddings in the universal covering). Similarly, the Euclidean degree of non-uniform lattices in $\mathrm{SO}(n,1)$ is $n-1$ (note that here the general upper bound $\mathrm{cd}_{\mathbf{Q}}$ is attained). What about lattices in the other rank-1 Lie groups? What can be said about the Euclidean degree of other symmetric spaces (and, what may turn out to be closely related, of nilpotent groups)?

6.2. UE equivalence and UE rigidity

One can weaken the notion of q.i. of groups Γ and Λ, by calling them *UE equivalent* if each one uniformly embeds in the other. Equivalently, every group which uniformly embeds in one, does so in the other. What properties are UE equivalence invariants? From some point of view UE equivalence seems rather weak; for example, if each of the groups embeds as a subgroup of the other, then the groups are UE equivalent, an information which does not seem too strong geometrically. However, for many classes of groups it is as rigid as the usual notion of quasi-isometry: Whenever any self-UE of Γ is a quasi-isometry, in which case we call Γ *UE rigid*, any group UE equivalent to Γ is q.i. to it. Any group Γ which admits a continuous proper co-compact action on $\mathbf{R}^n$ is UE rigid. More generally, if Λ and Γ admit proper co-compact actions on $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, and Λ uniformly embeds in Γ, then $m\leqslant n$, and in case of equality the uniform embedding must be a q.i. This can easily be seen by extending the uniform embedding to a continuous proper map of the Euclidean spaces, and using the following well-known application of the Borsuk–Ulam theorem: Any continuous proper map of $\mathbf{R}^n$ into itself is onto (cf. [11] and the recent [35] for further related results, using heavier technology). Thus, polycyclic (in particular nilpotent) groups on one hand, and uniform lattices in semisimple Lie groups on the other, are UE rigid, and this also shows that if one of these groups Γ admits a UE φ into some discrete subgroup Λ of the same ambient Lie group where it is embedded co-compactly, then Λ must also be such a lattice and φ a quasi-isometry. Thus, being virtually abelian or virtually nilpotent is a UE equivalence invariant. Are non-uniform lattices (excluding free groups) also UE rigid? (Can the recent [7] and [8] be relevant for such questions?) What about uniform lattices in semisimple algebraic groups over p-adic fields? To avoid hasty conjectures notice that in the positive characteristic case, uniform lattices are *not* UE rigid (even in higher rank), as they admit a self-embedding with infinite index image, induced by the local field embedding $F((t^2)) \to F((t))$. It is not clear, however, if there is a group UE equivalent to such a lattice which is not q.i. to it.

6.3. Uniform embeddings and growth of groups

It is easy to see that if Λ uniformly embeds in Γ then the growth of Γ dominates that of Λ. Thus, every group which uniformly embeds in $\mathbf{Z}^d$ (equivalently, in a finite-dimensional Euclidean space) must have polynomial growth. It is a highly non-trivial fact that the converse is also true, a result which follows from Assouad's "doubling theorem" [1] (we thank Mario Bonk and Bruce Kleiner for the information concerning Assouad's result). Although just knowing that the growth is polynomial is not a priori enough to deduce that a space is doubling (while bounding it between some $c_1 n^d$ and $c_2 n^d$ is enough), a modification of Assouad's theorem, together with passing to a "regular" subsequence of balls in the Cayley graph, shows that one can deduce in this way directly from Theorem 1.5 that groups with polynomial growth have finite $\mathbf{Q}$-cohomological dimension (at any rate, everything follows of course from Gromov's theorem).

Just like admitting a uniform embedding in $\mathbf{Z}^d$ characterizes polynomial growth, we conjecture that in the other extreme, a group has exponential growth if and only if a non-abelian free group uniformly embeds in it (as before, one direction is obvious). That non-amenable groups always receive a uniform embedding of a free group follows easily from the main result of [3], so the issue here is the case of amenable groups. In fact, every group containing a free subsemigroup receives a UE of a non-abelian free group, and hence by [15] the conjecture holds in the class of all elementary amenable groups. Besides being natural in its own right, our interest in this question is motivated by the following result:

THEOREM 6.3.1. *For a discrete group G, consider the following properties:*

(1) *G has subexponential growth (locally, if G is not finitely generated).*

(2) *G has the following translation property: For every non-negative, non-zero, bounded real function $0 \leqslant f: G \to \mathbf{R}$, for all n and elements $g_1, ..., g_n \in G$, if the real linear combination of translations of f, $\sum_{i=1}^{n} a_i (f \circ g_i)$, is a non-negative function on G, then necessarily $\sum_{i=1}^{n} a_i \geqslant 0$.*

(3) *For every continuous G-action on a* locally compact *space X admitting some compact subset $K \subseteq X$ with $X = G \cdot K$, there is a σ-finite G-invariant (Radon) measure on X.*

(4) *There is* no *uniform embedding of a non-abelian free group in G.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

The implication $(1) \Rightarrow (2)$ follows from [47]. We we will not prove here the other two, but only remark that all the four conditions imply amenability of G. Notice that (3) is obviously a strengthening of the invariant measure property for amenable group actions on compact spaces, as one can take $K = X$ when X is compact; the $ax+b$ group (or any dense subgroup of it) acting on the line is an example where it is not satisfied. We do not know how to reverse any of the implications above, but the remaining $(4) \Rightarrow (1)$, conjectured above, would close the circle (1)–(4).

6.4. Some natural extensions

We believe that Theorem 1.5 should hold also for *homological dimension* instead of cohomological dimension. This would improve, for example, Theorem 1.6 to the expected inequality $h\Lambda \leqslant h\Gamma$ in all cases, and remove the assumptions on Γ in Theorem 1.8. It would also give significant information concerning groups q.i. to the lamplighter group, as the latter has $\mathbf{Q}$-homological dimension 1. Since we proved that any group q.i. to the lamplighter group has a finite index subgroup which surjects onto $\mathbf{Z}$, together with a homological dimension argument one might be able to show that the kernel of this surjection must be a locally finite group (these are exactly the groups of $\mathbf{Q}$-homological dimension 0), which we conjecture to be the case. The homological dimension argument predicts that there is no uniform embedding of $\mathbf{Z}^2$ into the lamplighter group, a result which was indeed verified by Benjamini and Schramm (private communication), via a study of the following notion, interesting in itself, which weakens that of a uniform embedding: A map $f: \Lambda \to \Gamma$ between finitely generated groups is a *quasimonomorphism* if it is a Lipschitz map, and it satisfies $\#\{f^{-1}(\gamma)\} < C$ for some global $C < \infty$ and all $\gamma \in \Gamma$. Benjamini and Schramm showed that $\mathbf{Z}^2$ does not admit a quasimonomorphism into the lamplighter group, nor to a tree. Do quasimonomorphisms also "respect" cohomological dimension as in Theorem 1.5?

Finally, since uniform embeddings are defined equally well for non-finitely generated groups, and enable one to define a quasi-isometry in this case as well, it may be of interest to look more closely at some examples. Are two infinite direct sums of finite cyclic p-groups q.i. exactly when they are commensurable? This may be helpful in deciding for which finite groups F and F' the lamplighter groups $L(F)$ and $L(F')$ are quasi-isometric (compare with [33, IV, Item 44]). Note that when $|F|<\infty$ and $|F'|=\infty$ the two groups are not q.i. by Theorem 1.5 (this follows also from recent results of Erschler [20]).

6.5. More on properties H_T, H_F and H_{FD}

We believe that Theorem 4.2.1 (3) should be strengthened to characterizing property H_{FD} by having only *finitely many* irreducible unitary representations with $\overline{H}^1\neq 0$ (all being finite-dimensional). Similarly, we conjecture that the condition on the virtual first Betti number in Proposition 4.2.4 (2) is redundant, namely, that any finitely generated amenable group with property H_F should contain a finite index subgroup with property H_T. Both results would follow from the following plausible dichotomy: A finitely generated group has either finitely many, or uncountably many, irreducible unitary representations with $\overline{H}^1\neq 0$. It is natural also to ask whether property H_{FD} may be characterized by the non-existence of an *irreducible infinite-dimensional* unitary representation with $\overline{H}^1\neq 0$. A possible counterexample would be the group $\mathbf{Z}\wr\mathbf{Z}$ (which does not have H_{FD}—see §5.4). Proving that this group has no such representation is of independent interest, and would seem to require fundamentally new techniques.

Although we mostly concentrated on first cohomology, one may define and study properties analogous to H_T, H_F and H_{FD} for any cohomology degree (as was done in §4.1 for nilpotent groups). A proof similar to that of Theorem 1.10 shows that the latter is again a q.i. invariant in the class of amenable groups, so a further study of these generalized properties would also be applicable to geometric group theory. In fact, one need not insist on reduced cohomology, and define in a similar way "spectral" q.i. invariants based on ordinary cohomology, such as the vanishing or non-vanishing of H^n with coefficients in some weakly mixing, mixing (see below) or the regular representation (in the case of measure equivalence, this approach leads to new results when bounded cohomology is used [41]). Another intriguing question is to find an *amenable* group Γ having a *mixing* representation π (i.e. all matrix coefficients decay to 0 at infinity), with $\overline{H}^n(\Gamma,\pi)\neq 0$. The existence of such a representation can be shown to be a q.i. invariant in the class of amenable groups (for any fixed value of n separately), and in the case $n=1$ it stands in sharp contrast to property H_{FD}. As an application to geometric group theory, one can show that if an amenable group Γ admits such a representation π (for some n), then any group q.i. to it must have finite center.

Finally, in contrast to the amenable case, non-amenable groups satisfying property H_{FD} seem to be quite rare. However, groups with property (T) do have, by definition, property H_{FD}. It is well known by now that property (T) is not a q.i. invariant. Is it possible that one can extend Theorem 1.10 to *all* (not only amenable) groups, so that it is the weaker property H_{FD} which is respected by the geometry of groups? Interestingly, the known non-Kazhdan groups which are q.i. to Kazhdan groups do satisfy property H_{FD}. Even knowing whether this latter fact holds in general would seem to require some new insight. Finding examples of non-amenable groups having property H_{FD}, which are not based on property (T), would also be of interest.

6.6. Property H_{FD} and solvable groups

Various families of solvable groups having, and not having property H_{FD} were shown. The following conjecture would provide a uniform explanation:

CONJECTURE. *A finitely generated solvable group has property H_{FD} if and only if it has finite Hirsch number.*

The more interesting part in terms of applications to geometric group theory (through Theorem 1.11), i.e. the "if" part, seems particularly difficult at this level of generality. Even for polycyclic groups, we know how to establish property H_{FD} using only the (non-trivial) fact that they virtually embed as lattices in connected solvable Lie groups (and from there a long way is still to go). Providing a proof without using this fact would be a challenge worth taking, which could be a first step to an understanding of the general phenomenon. In fact, the above conjecture can be seen as a special case of a significantly more far-reaching speculation: Is it true that a finitely generated *amenable* group has property H_{FD} if and only if it has finite $\mathbf{Q}$-cohomological dimension? As two initial concrete test cases which should be more tractable, one may try to prove that if Γ is locally finite by $\mathbf{Z}$ then Γ *has* property H_T (generalizing the lamplighter group in Theorem 5.2.1), and that for any two infinite finitely generated amenable groups G and H, the wreath product $G \wr H$ *never* has property H_{FD} (compare with Theorem 5.4.1 above). Another concrete related question is to decide whether for a finite group F, the wreath product $\mathbf{Z}^2 \wr F$ has property H_T. Such groups seem considerably more complicated than $\mathbf{Z} \wr F$ in their algebraic structure.

6.7. Property H_{FD} and Gromov's polynomial growth theorem

In this last subsection we suggest a new strategy to proving one of the outstanding results of geometric group theory—Gromov's polynomial growth theorem [30]. This approach has the advantage of avoiding completely the solution to Hilbert's fifth problem [42], and is based on the ideas developed here; more precisely, on the following two results:

THEOREM 6.7.1. *A finitely generated group of polynomial growth has property H_{FD}.*

THEOREM 6.7.2. *The above theorem implies Gromov's polynomial growth theorem.*

Proof of Theorem 6.7.2. The proof goes by induction on the degree d of the polynomial growth (using an idea of Tits in his note [52] appended to Gromov's paper). The point is that in order to prove Gromov's theorem by induction, it is enough to show that every group G of polynomial growth has a finite index subgroup with infinite abelianization. By subexponential growth the kernel of the abelianization can be shown to be finitely generated ([40]), thus of polynomial growth degree $\leqslant d-1$. Hence one can argue by induction to show that every such G is virtually polycyclic. Wolf's classical theorem [54], together with polynomial growth, then finish the proof. The key ingredient of the argument, being that G indeed virtually has infinite abelianization, follows from its amenability and property H_{FD} as in Theorem 1.11. To put matters in perspective, we only remark that the point of property H_{FD} here is that an a priori infinite-dimensional cohomological unitary representation, which exists for any finitely generated group G without Kazhdan's property by [50], must be finite-dimensional in this case, leading to the required abelianization. It may be worth noting also that the proof of the latter existence result in [50] involves a rescaling–limiting construction for isometric actions, which appears also as a crucial ingredient in Gromov's approach. We also remark that very recently we were informed, first by Alain Valette, that for discrete groups a similar and completely independent construction of this type appeared also in [36]. In [50], the fact that the spaces are Hilbert enables one to use negative definite functions to produce a rather elegant construction. $\square$

We are therefore left with the proof of Theorem 6.7.1, observing first that this result follows from Gromov's theorem, since we established it for (virtually) nilpotent groups (see Corollary 5.1.3 and Lemma 4.2.2). Thus, if one could prove Theorem 6.7.1 *without* appealing to Gromov's result, we would be done. Our purpose henceforth is to show that there is reasonable hope and a natural approach to doing so, which reduces to a certain *conjectural* mean ergodic theorem (for groups of polynomial growth). In the case $G=\mathbf{Z}$, this is no more than von Neumann's classical mean ergodic theorem.

Let G be of polynomial growth. To establish property H_{FD} one needs, by Definition 2.4.3 and Lemma 2.4.4, to study affine isometric G-actions without almost fixed points. For amenable G, given a G-action on a Hilbert space V, there is a natural "candidate" for a sequence of almost fixed points, namely, the sequence obtained by averaging over Folner subsets. However, this is too naive in general, as the action is only affine and not necessarily linear. Thus, if we normalize things so that a generating set $S\subseteq G$ translates a base point $v_0\in V$ up to a maximum distance 1, then elements in the sphere of radius n of G (denoted S_n) may shift v_0 up to distance n (this is exactly the case for the natural $\mathbf{Z}^d$-translation action on the Euclidean space $\mathbf{R}^d$, which indeed has no almost fixed points).

Yet, for groups of polynomial growth, (at least some of) the balls, denoted B_n, are much "better" Folner sets:

LEMMA 6.7.3. *For G of polynomial growth there exists a constant C, and an infinite sequence of integers $n_i \to \infty$, so that $|S_{n_i}|/|B_{n_i}| \leqslant C/n_i$.*

Indeed, an elementary argument shows that otherwise we get for every C and all m large enough, $|B_m| \geqslant \prod_{n=1}^{m}(1+C/n) \gg m^{C-1}$, contradicting polynomial growth.

Now, given an isometric G-action on V, fix for convenience the origin $v_0 = o \in V$ as a base point, and define

$$v_n = \frac{1}{|B_n|} \sum_{g \in B_n} g v_0. \tag{17}$$

CONJECTURE. *If the linear part π of the isometric action does not contain a finite-dimensional invariant subspace (namely, it is weakly mixing), then, after passing to a subsequence, v_n is a sequence of almost fixed points: $\|gv_n - v_n\| \to 0$ for all $g \in S$.*

As the conjecture clearly implies Theorem 6.7.1 above, we next concentrate on it. For a general amenable group, it is not even true that $\|gv_n - v_n\|$ should be bounded over n when $g \in S$. A first positive indication in our case comes from the fact that $\|gv_{n_i} - v_{n_i}\| \leqslant C$ for all $g \in S$, which can be deduced directly from Lemma 6.7.3. To better appreciate the nature of this conjecture, it is illuminating to first analyze the case $G = \mathbf{Z}$, with the standard generators $S = \pm 1$. If the generator 1 acts through the affine operator $gu = T_0 u + w$, with T_0 unitary and $w \in V$, then substituting in (17) gives

$$gv_n - v_n = \frac{1}{2n+1}(T_0^n w + \ldots + T_0 w + w + T_0^{-1} w + \ldots + T_0^{-n} w). \tag{18}$$

By von Neumann's mean ergodic theorem, the norm of the expression in (18) goes to zero, unless T_0 has a non-zero invariant vector. Therefore, we have just re-proved that $\mathbf{Z}$ has property H_T! For general G, applying the 1-cocycle identity (§2.4) again transforms the expression $\|gv_n - v_n\|$, similarly to (18), into an average over the balls B_n of the *unitary* part π, and Lemma 6.7.3 shows that the number of summands in this expression is of the "right magnitude" $O(|B_n|)$. One is left to show that under a weak mixing hypothesis on π, this average goes to 0. Note that it is a common phenomenon in ergodic theory that weak mixing implies good averaging properties, typically by guaranteeing that the product (representation or) action be ergodic as well (cf. [5] for one example).

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