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Quasiconformal 4-manifolds

by

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§1. Introduction

For any pseudo-group of homeomorphism of Euclidean space one can define the corresponding category of manifolds. The most familiar examples in Topology are the full pseudo-group of homeomorphisms, giving rise to the theory of topological manifolds, and the subgroup of smooth differomorphisms giving rise to the theory of C^∞ manifolds. In this paper, we discuss an intermediate category—*quasiconformal* homeomorphisms and manifolds.

Recall that a homeomorphism $\varphi: D \to \mathbf{R}^n$ from a domain D in $\mathbf{R}^n$ to its image $\varphi(D)$ is K quasiconformal if for all x in D

$$H_\varphi(x) = \limsup_{r\to 0} \frac{\max\{|\varphi(y)-\varphi(x)| \mid |y-x|=r\}}{\min\{|\varphi(y)-\varphi(x)| \mid |y-x|=r\}} \leqslant K.$$

φ is quasiconformal (QC) if it is K quasiconformal for some $K \geqslant 1$. Roughly, a quasiconformal map distorts the relative distances of nearby points by a bounded factor. Contrast this with the *Lipschitz* condition: a homeomorphism φ is bi-*Lipschitz* if for some $C \geqslant 1$ and all x,y in D:

$$C^{-1}|x-y| \leqslant |\varphi(x)-\varphi(y)| \leqslant C|x-y|.$$

Both these conditions define pseudo-groups of homeomorphism and hence *quasiconformal* and *Lipschitz n-manifolds*; Hausdorff spaces made from domains in $\mathbf{R}^n$ pieced together by, respectively, quasiconformal and Lipschitz homeomorphisms. We also have the obvius notions of *equivalence* in the two categories.

For $n \neq 4$ it is known that these two categories are both essentially equivalent to that of topological manifolds. We have:

THEOREM (Sullivan [25]). *If $n \neq 4$ any topological n manifold admits a quasiconformal structure. Also, any two quasiconformal structures are equivalent by a homeomorphism isotopic to the identity.*

With a similar statement, also proved in [25], for the Lipschitz case. In this paper we show that neither part of the above theorem can extend to dimension 4. We will prove:

THEOREM 1. *There are topological 4-manifolds which do not admit any quasiconformal structure.*

THEOREM 2. *There are quasiconformal (indeed smooth) 4-manifolds which are homeomorphic but not quasiconformally equivalent.*

(The corresponding Lipschitz statements are trivial consequences.)

These theorems illustrate the special nature of manifold theory in four dimensions. It is now well known that there is a radical divergence between the theories of smooth and topological 4-manifolds. This has been discovered by a combination of the classification theory of Freedman [14] on the topological side and, on smooth manifolds, the use of new information coming from Yang-Mills fields. In this paper also we take our topological input straight from the results of Freedman and our theorems will follow, transferring arguments developed in the smoth theory, if we can lay down the foundations of Yang-Mills theory over quasiconformal 4-manifolds. This task takes up the bulk of the paper. In §7 we return to give the proofs of Theorems 1 and 2. The whole programme is similar in spirit to Taubes work on end periodic manifolds [27]. As there, one could hope that once the basic theory is in place one could extend all the results for smooth manifolds proved using Yang-Mills theory to the quasiconformal case. We will make some detailed remarks on this in §7.

It is instructive to isolate more precisely the point at which the general theory for $n \neq 4$ breaks down in four dimensions. Let Γ be a pseudogroup contained in the pseudogroup of quasiconformal homeomorphism of n-space. It is a general fact that two properties of Γ suffice to provide unique Γ structures on topological n-manifolds.

(i) *n-deformation.* C^0 close Γ homeomorphisms can be deformed to one another through Γ homeomorphisms (together with a suitable relative version of the statement).

(ii) *n-approximation.* Any homeomorphism of a ball B^n into $\mathbf{R}^n$ can be C^0-approximated by a Γ homeomorphism.

Now in [25] the n-deformation property is proved for quasiconformal and Lipschitz homeomorphisms in all dimensions n. So our Theorems 1, 2 show that n-approximation fails for these peudogroups when $n=4$. Thus we have:

COROLLARY. *There is a homeomorphism of the* 4-*ball into* $\mathbf{R}^4$ *which cannot be approximated by a quasiconformal homeomorphism.*

We now turn back to the central topics of this paper—the global analysis of Yang-Mills theory on quasiconformal manifolds—and review the standard theory in the smooth case (see [13] for example). There we start with a smooth, compact, oriented 4-manifold Y with Riemannian metric g. Let $P\to Y$ be a principal bundle with compact structure group G. One forms the space $\mathcal{A}$ of L^p_k connections on Y and the "gauge group" $\mathcal{G}$ of L^p_{k+1} bundle automorphisms. There is a great choice in the possible Sobolev spaces L^p_k to use—the key constraint is that $\mathcal{G}$ should consist of continuous automorphisms, i.e. that a Sobolev embedding

$$L^p_{k+1}\hookrightarrow C^0$$

should hold. This requires $(k+1)-n/p>0$. In this case $\mathcal{G}$ is a Banach Lie group acting smoothly on $\mathcal{A}$. One then constructs slices for the action (away from "reducible connections") using the Coulomb gauge condition. For A in $\mathcal{A}$ there is a coupled opererator d^*_A acting on bundle valued 1-forms and

$$T_{A,\varepsilon}=\{A+a|\ d^*_A a=0, |a|<\varepsilon\}$$

gives a local transversal for the $\mathcal{G}$-orbits. These make the quotient space $\mathcal{B}=\mathcal{A}/\mathcal{G}$ into a Banach manifold (except for singularities at reducible connections). Next, for suitably chosen Sobolev spaces L^p_k (e.g. L^2_k, $k>1$) the *curvature* F_A lies in L^p_{k-1} and defines a smooth $\mathcal{G}$-equivariant map on $\mathcal{A}$—or section of a Banach bundle over $\mathcal{B}$. Using the Riemannian metric g we split the curvature into self-dual and anti-self-dual parts:

$$F_A=F^+_A+F^-_A.$$

The anti-self-dual (ASD) moduli space M is the subset of $\mathcal{B}$ cut out by the zeros of F^+_A. Elliptic regularity gives that an anti-self-dual connection (i.e. one with $F^+_A=0$) is $\mathcal{G}$ equivalent to a smooth connection, so the precise Sobolev spaces used are not too important. The equation $F^+_A=0$ is, on $\mathcal{B}$, a Fredholm equation and the moduli space M has a virtual dimension given by the Fredholm index of the linearisation:

$$d=\operatorname{index}(d^*_A+d^+_A).$$

Under suitable restrictions one achieves a manifold M of this dimension either by an abstract perturbation of the set-up ([5], [7], [15]) or by varying the metric g slightly. Similarly one arranges that under smooth change of parameters M changes by a cobordism ([8], [9]). These moduli manifolds are then the input for various simple topological arguments by which one deduces conclusions about the original 4-manifold Y.

Turning now to a quasiconformal base 4-manifold X, the first point is that quasiconformal maps are differentiable almost everywhere. This allows one to set up some differential geometric structures, in particular, we can choose a *measurable conformal structure* on X. The Yang-Mills equations are conformally invariant so this conformal structure defines ASD connections. Two main changes are needed to take the standard theory over to the quasiconformal case. The first concerns the slice condition and the d_A^* operator. This enters already in the linear set up of the Hodge theory (signature operator) coupled to an auxiliary connection. For Lipschitz manifolds a theory of signature operators has been developed by Teleman [28], [29]. He shows that one can define d_A^* and it has sufficient good properties to mimic the usual linear elliptic analysis. However despite some efforts we have not been able to use this operator to construct slices in the non-linear problem, even in the Lipschitz case. The basic difficulty is that

$$d_A^* = * d_A *$$

involves differentiating the $*$ operator and for our manifolds $*$ is at best bounded, measurable; with no control of its regularity. Thus we use a different approach based on the constructing of a (right) *parametrix* for the d_A^+ operator. The latter is better behaved since

$$d_A^+ = \frac{1}{2}(1+*)\, d_A$$

and the measurable $*$ occurs outside the differentiation. The basic analytical lemma for handling this measurable-coefficient operator we learnt from the book of Ahlfors ([1] Chapter V) who deals with the analogous 2-dimensional problem. This lemma is discussed in §2 and, as the reader will see, underpins the whole theory. (In Appendix 2 we should show how this approach can be used to reproduce some of Teleman's results).

The second main change has to do with choosing a suitable functional-analytic framework. The L^4 norm on 1-forms is conformally invariant in 4 dimensions and it is in many ways most natural to try to work with connection matrices which are locally in L^4

(with curvature in L^2). However the natural class of gauge transformations would then be those in L_1^4, and these are not continuous—the exponent being the borderline one where the Sobolev embedding fails. In the Lipschitz situation one can work (thanks to the lemma of §2) in $L^{4+\varepsilon}$ for some fixed small ε and restore the Sobolev embedding theorem but quasi-conformal maps do not preserve the space of $L^{4+\varepsilon}$ 1-forms for $\varepsilon>0$.

We overcome this difficulty by the introducing new function spaces; smaller than L^4 but larger than any $L^{4+\varepsilon}$, which are on the one hand preserved by quasi-conformal maps and on the other hand yield continuous gauge transformations. As in the standard theory, these function spaces depend on real parameters and the precise choice we make is not in the end too important. Similarly, we have "elliptic regularity", that any solution is locally gauge equivalent to an $L^{4+\varepsilon}$ one (analogous to smooth for us). The key here is a theorem of Gehring that a quasiconformal map in n dimensions has derivative in $L_{\mathrm{loc}}^{n+\varepsilon}$ for some $\varepsilon>0$ [16]. We give a new proof of this fact (for $n=4$) using our basic lemma.

Indeed, in Ahlfors' book this result is proved (following [4]) for $n=2$ and our proof is the natural generalisation of that one from 2 to 4 dimensions (in Appendix 2 we discuss the general even dimensional situation.)

§2. Local theory

(i) Conformal classes

Let E be a 4-dimensional oriented real vector space. A conformal structure on E is an equivalence class $[g]$ of Euclidean metrics g on E:

$$[g]=[\lambda^2 g].$$

There is, however, a more concrete description, special to 4 dimensions, using the $*$ operators $*_g$ on 2-forms $\Lambda^2(E^*)$. The operator $*_g$ depends only on the conformal class of g and gives the familiar splitting:

$$\Lambda^2(E^*)=\Lambda^+\oplus\Lambda^-$$

into self-dual and anti-self-dual parts. The eigenspaces Λ^+, Λ^- are respectively, maximal positive and negative subspaces for the wedge product form:

$$\omega\to\omega\wedge\omega$$

on $\Lambda^2(E^*)$. (Of course we need to fix a volume element to define this as an **R**-valued quadratic form). Λ^+ is the annihilator of Λ^- under the wedge product so $*_g$ on Λ^2 is completely determined by Λ^-. If we have some fixed reference metric g_0 with positive and negative subspaces Λ_0^+, Λ_0^- we can represent the negative subspace $\Lambda^-(g)$ for any other metric g as the graph Γ_μ of a linear map:

$$\mu: \Lambda_0^- \to \Lambda_0^+. \tag{2.1}$$

The condition that Λ be negative on Γ_μ goes over the condition:

$$|\mu(\omega)| < |\omega| \tag{2.2}$$

for all non-zero forms ω in Λ_0^-.

LEMMA 2.3. *The map $[g] \to \Lambda^-(g)$ yields a bijection between the conformal structures on E and the space of negative 3-planes in $\Lambda^2(E^*)$*

Proof. Fix a reference metric g_0 and standard g_0-orthonormal basis e_1, e_2, e_3, e_4, for E. If $\{\varepsilon_i\}$ is the dual basis then:

$$\Lambda_0^+ = \langle \varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4,\ \varepsilon_1\varepsilon_3+\varepsilon_4\varepsilon_2,\ \varepsilon_1\varepsilon_4+\varepsilon_2\varepsilon_3 \rangle$$

$$\Lambda_0^- = \langle \varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4,\ \varepsilon_1\varepsilon_3-\varepsilon_4\varepsilon_2,\ \varepsilon_1\varepsilon_4-\varepsilon_2\varepsilon_3 \rangle.$$

Let g_1 be a new metric, diagonal relative to this basis.

$$g_1(e_i, e_j) = 0, \quad i \neq j$$

$$g_1(e_i, e_i) = \lambda_i^2.$$

We normalize so that $\lambda_1\lambda_2\lambda_3\lambda_4 = 1$. Then $\Lambda^-(g_1)$ is represented by a map μ, as above, with:

$$\mu(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) = \frac{(\lambda_1^2\lambda_2^2-1)}{(\lambda_1^2\lambda_2^2+1)}(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4)$$

and symmetrically for the other basis elements. Now the function

$$f(x) = \frac{(x-1)}{(x+1)}$$

gives a bijection from $(0, \infty)$ to $(-1, 1)$ and for any prescribed values of $\lambda_1^2\lambda_2^2$, $\lambda_1^2\lambda_3^2$, $\lambda_1^2\lambda_4^2$ we can solve uniqely for λ_i with $\lambda_1\lambda_2\lambda_3\lambda_4 = 1$. So we have a bijection between the e_i-diagonal conformal classes and maps $\mu: \Lambda^- \to \Lambda_0^+$ with operator norm $|\mu| < 1$, diagonal relative to the given orthonormal bases. But we know that the Λ^2 representation gives a double covering:

$$SO(4) \to SO(3) \times SO(3)$$

So, up to a sign, pairs of orthonormal bases in Λ_0^+, Λ_0^- correspond precisely to orthonormal bases in E. Since any metric is diagonalisable the assertion follows.

There is a natural metric on the set of conformal structures:

$$d([g_0],[g_1]) = \max_{|\zeta|_0 = |\eta|_0 = 1} \log\left(\frac{|\zeta|_1}{|\eta|_1}\right) \tag{2.4}$$

In the notation above

$$d([g_0],[g_1]) = \log(\lambda_1/\lambda_4)$$

if $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. On the other hand the operator norm $|\mu|$ of the associated linear map is:

$$|\mu| = \frac{\lambda_1^2\lambda_2^2 - 1}{\lambda_1^2\lambda_2^2 - 1} \tag{2.5}$$

this is in fact symmetric in g_0, g_1. We have:

$$\tfrac{1}{2}\log(\left(\frac{1+|\mu|}{1-|\mu|}\right) \leq d([g_0],[g_1]) \leq \log\left(\frac{1+|\mu|}{1-|\mu|}\right).$$

So the metric and the operator norm define equivalent "distance functions" on the space of conformal classes.

We now take these ideas over to manifolds. Let Y be a smooth oriented 4-manifold, and fix a smooth Riemannian metric g_0 on Y. We can define a *bounded, measurable conformal structure* on Y to be an equivalence class of measurable sections g of $S^2(T^*Y)$ with

$$\sup_{y \in Y} d([g],[g_0])_{TY_y} < \infty .$$

If Y is compact this notation is plainly independent of our reference metric g_0. Equivalently we can define the structure by a measurable bundle map.

$$\mu \colon \Lambda_Y^- \to \Lambda_Y^+ \tag{2.6}$$

with

$$\|\mu\| = \sup_y |\mu_y| < 1.$$

There is then a measurable field of subspaces $\Gamma_\mu = \Lambda^-(\mu)$ in Λ^2_Y. In the smooth case we have first order operators.

$$d^+ : \Omega^1_Y \to \Omega^+_Y = \Gamma(\Lambda^+_Y)$$

$$d^- : \Omega^1_Y \to \Omega^-_Y = \Gamma(\Lambda^+_Y)$$

with $d = d^+ + d^-$. Similarly if we compose with the projection from $\Lambda^+(\mu)$ to Λ^+ the d^+ operator relative to the new conformal structure is represented as:

$$d^+_\mu = d^+ + \mu d^- : \Omega^1_Y \to \Omega^+_Y. \tag{2.7}$$

(Strictly we should replace Ω^+_Y here by a space of bounded sections of Λ^+_Y.) Thus d^+_μ is a first order operator with bounded measureable coefficients.

(ii) Elliptic theory and measureable coefficients

In this sub-section we prove the basic analytical lemma for the d^+_μ operators. The proof is elementary and is a direct translation of an idea due to Boyarskii [4] for the 2-dimensional problem discussed by Ahlfors [1]. See also [17] Chapter V.

Let Y be a smoth compact oriented Riemannian 4-manifold and introduce standard differential operators:

$$\begin{tikzcd} & & \Omega^+_Y \\ \Omega^0_Y \arrow[r, shift left, "d"] & \Omega^1_Y \arrow[l, shift left, "d^*"] \arrow[ur, "d^+"] \arrow[dr, "d^-"'] & \\ & & \Omega^-_Y \end{tikzcd}$$

Then $d^+ d = d^- d = 0$ and we have a pair of elliptic complexes, whose cohomology can be readily identified by Hodge theory. First for α in Ω^1_Y the relation between the norm and wedge on Λ^2 gives:

$$\int_Y |d^+\alpha|^2 - |d^-\alpha|^2 \, d\mu = \int d\alpha \wedge d\alpha = 0. \tag{2.9}$$

So $\|d^+\alpha\|_{L^2} = \|d^-\alpha\|_{L^2}$. (Notice that the L^2 norm on 2-forms appearing here is conformally invariant.) In particular:

$$\frac{\operatorname{Ker} d^+}{\operatorname{Im} d} = \frac{\operatorname{Ker} d^-}{\operatorname{Im} d} \cong H^1(Y; \mathbf{R}). \tag{2.10}$$

On the other hand:

$$\text{(2.11)} \qquad \operatorname{coker}(d^+) = H_Y^+, \quad \operatorname{coker}(d^-) = H_Y^-;$$

the $\pm$ self dual harmonic forms representing maximal positive and negative subspaces for the cup product form on $H^2(Y; \mathbf{R})$. In particular, if Y is a homology 4-sphere the only cohomology appearing is the constants $\operatorname{Ker} d \subset \Omega_Y^0$. Under this hypothesis standard elliptic theory gives us a Hodge decomposition:

$$\text{(2.12)} \qquad \Omega_Y^1 = d(\tilde{\Omega}_Y^0) \oplus \operatorname{Ker} d^*$$

where $\tilde{\Omega}_Y^0$ represents the functions of integral zero. There is an inverse

$$\text{(2.13)} \qquad Q\colon \Omega_Y^+ \to \operatorname{Ker} d^* \subset \Omega_Y^1 \quad \text{with} \quad d^+Q(\omega) = \omega.$$

All of this is compatible with the usual Sobolev norms, so Q is a bounded operator $L_{k-1}^p \to L_k^p$ and $S = d^- \circ Q\colon \Omega_Y^+ \to \Omega_Y^-$ is bounded on L_{k-1}^p. In fact, S is a singular integral operator (essentially the signature operator) of order zero, of the kind considered in the Calderon-Zygmund theory [24]. The identity (2.9) shows that S gives an isometry on L^2-spaces. It then follows from an interpolation argument (see [1], pp. 113–115 or [24] p. 22) that we have:

$$\text{(2.14)} \qquad \|S(\omega)\|_{L^p} \leq C_p \|\omega\|_{L^p} \quad \text{with } C_p \to 1 \text{ as } p \to 2.$$

Now let μ be a bounded conformal structure as above with

$$c = \|\mu\| = \sup |\mu| < 1.$$

For p in $(0, \infty)$ we can consider d_μ^+ as a bounded operator on the Sobolev spaces:

$$\text{(2.15)} \qquad d_\mu^+ = d^+ + \mu d^-\colon L_1^p(\Omega_Y^1) \to L^p(\Omega_Y^+).$$

LEMMA 2.16. *There is an $\eta > 0$ (depending only on c) such that for $|p-2| < \eta$ there is a bounded inverse:*

$$Q_\mu\colon L^p(\Omega_Y^+) \to L_1^p(\Omega_Y^1) \quad \textit{for } d_\mu^+,$$

mapping to $\operatorname{Ker} d^* \subset \Omega_Y^1$.

Proof. Consider:

$$(1 + \mu \circ S) = (d^+ + \mu d^-) \circ Q\colon \Omega_Y^+ \to \Omega_Y^-.$$

This is bounded on L^p and the L^p-operator norm of $\mu \circ S$ is at most $c \cdot C_p$. Since $C_p \to 1$ as $p \to 2$ and $c<1$ we can chose η such that for p in the given range $\|\mu \circ S\|<1$. Then $1+\mu S$ is invertible with inverse

$$(1+\mu S)^{-1} = 1-\mu S+(\mu S)^2-\dots$$

Now put

$$Q_\mu = Q \circ (1+\mu S)^{-1}.$$

The stated properties of Q_μ follow from those of Q. (Notice that the $L^p \to L_1^p$ operator norm of Q_μ *is* bounded on any closed subinterval in $(2-\eta, 2+\eta)$ and the bound depends only on c.)

We have then a version of the usual elliptic theory for d_μ^+ in the given range of function spaces. Notice that the operator d^* we have used to define our inverse Q_μ is essentially an auxiliary tool—we do *not* use a metric in the given bounded conformal class to define it. While we have carried out this argument on a compact manifold Y our main application will be in local setting, for a d_μ^+ operator over a bounded domain $D \subset \mathbf{R}^4$. We take $Y=S^4=\mathbf{R}^4 \cup \{\infty\}$ and transfer our forms to S^4 using a cut off function β, supported in D and equal to 1 on some subdomain $D' \subset\subset D$. We then extend μ to S^4 and deduce from the above result:

COROLLARY 2.17. *There are constants $\eta(c,D,D')$, $A(c,D,D')$ such that if μ is a bounded conformal structure over D with $\|\mu\|<c$ then for $|p-2| \leq \eta$:*

(i) *For any form $\omega \in L^p(\Omega_D^+)$ there is an $\alpha = Q_\mu(\omega)$ in $L_1^p(\Omega_D^+)$ with*

$$d_\mu^+ \alpha = \omega \quad \text{on } D'.$$

$$\|\alpha\|_{L_1^p,D} \leq A\|\omega\|_{L^p,D}.$$

(ii) *For any α in $L_1^p(\Omega_D^1)$ there is a u in $L_2^p(\Omega_D^0)$ such that*

$$\|\alpha - du\|_{L_1^p,D'} \leq A\left(\|d_\mu^+\alpha\|_{L^p,D} + \|\alpha\|_{L^p,D}\right).$$

(Notice that in (ii) we can also choose u to have control of $\|u\|_{L_2^p}$.)

(iii) Quasi-conformal maps

In the introduction we gave the most geometric definition

$$H_\varphi(x) \leq K$$

of the K-quasiconformality of a homeomorphism $\varphi : D \to \mathbf{R}^4$. There are many other difinitions which turn out to be equivalent, see [32]. Let us note first that a point x where φ is differentiable $H_\varphi(x)^2$ is the ratio of the maximum and minimum eigenvalues of the matrix $(\nabla\varphi)_x^*(\nabla\varphi)_x$. In the notation above: $H_\varphi(x)^2 = \exp(d(\varphi_x^*(g_0), g_0))$ where $\varphi_x^*(g_0)$ is the pull back of the Euclidean metric g_0. Now quasiconformal maps are certainly not everywhere differentiable but they have the following main regularity properties:

PROPOSITION 2.18 ([32]. *If $\varphi : D \to \mathbf{R}^4$ is a quasiconformal map then:*
(i) *φ is differentiable almost everywhere;*
(ii) *φ preserves Lebesgue null sets:*

$$\mu(A) = 0 \quad \Rightarrow \quad \varphi(\mu(A)) = 0.$$

(iii) *The derivative $\nabla\varphi$ is locally in L^4.*
(iv) *$\nabla\varphi$ is a weak derivative:*

$$\int_D f \cdot \frac{\partial \varphi}{\partial x_i} = -\frac{\partial f}{\partial x_i} \cdot \varphi$$

for smooth compactly supported test functions f on D.

We could take these properties (i)–(iv) as the defining properties for quasi conformality together with the key condition that

$$d([g_0], [\varphi^*(g_0)]) \leqslant e^K.$$

Properties (ii) and (iii) are related. The Radon-Nikodym theorem, together with (i) and (ii), implies that the usual integration-by-substitution formula is valid:

$$\int_{\varphi(D)} g(y)\, d\mu_y = \int_D g(\varphi(x))^* \, |(J_\varphi)_x| \, d\mu_x \tag{2.19}$$

where $J_\varphi = \det(\nabla\varphi)$, defined almost everywhere). The meaning here is that if g is in $L^1(\varphi(D))$ then $(g \circ \varphi) J_\varphi$ is in $L^1(D)$ and the two integrals agree. In particular if we restrict the domain so that $\varphi(D)$ has finite measure and take $g = 1$ we have:

$$\int_D |J_\varphi| \, d\mu_x = \mu(\varphi(D)) < \infty.$$

Now if $\lambda_1^2 \geqslant \lambda_2^2 \geqslant \lambda_3^2 \geqslant \lambda_4^2$ are the eigenvalues of $(\nabla\varphi)_x^*(\nabla\varphi)_x$ at a point of differentiability x:

$$|J_\varphi| = \lambda_1 \lambda_2 \lambda_3 \lambda_4$$

while $|\nabla\varphi|^2 = \Sigma\, \lambda_i^2$. If $\lambda_1/\lambda_4 \leq K$ we have

$$|\nabla\varphi|^2 \leq 4K^{3/2}|J_\varphi|^{1/2}$$

so

$$\int_D |\nabla\varphi|^4 \, d\mu_x \leq 16K^3\mu(\varphi(D)) < \infty.$$

As we shall see in §2(v) below, the derivative $\nabla\varphi$ is in fact locally in $L^{4+\eta}$ for some $\eta>0$. In the proof we give we will make use of a rather obvious variant of the argument above. Let σ be any postive function on the cone of positive matrices which is homogenous of degree d. Then there is a constant $C_{K,\sigma}$ such that for any K quasiconformal map φ:

$$|\nabla\varphi| \leq C_{K,\sigma}\, \sigma((\nabla\varphi)^*(\nabla\varphi))^{1/d}.$$

This follows from the fact that the e^K-ball about g_0 in the space of conformal structures is compact.

(iv) Differential forms

For $n=1,2,3,4$, the $L^{4/n}$ norm on n-forms on $\mathbf{R}^4$ is conformally invariant. Let $\varphi: D \to \mathbf{R}^4$ be a K quasiconformal map and write $L^{4/n}(\Omega^n_{\varphi(D)})$ for the Banach space on n-forms on $\varphi(D)$ with $L^{4/n}$ coefficients. We define the integral

$$\int : L^1(\Omega^4_{\varphi(D)}) \to \mathbf{R}$$

in the obvius way and also the pull back forms

$$\varphi^*(\omega)$$

on D using the usual formula and the (almost everywhere defined) derivative of φ. We assume that D is connected. Then we can define the orientation $\sigma_\varphi = \pm 1$ of φ by, say, the action on cohomology with compact support. This agrees with sign J_φ at points of differentiability. The verification of the following is straightforward.

PROPOSITION 2.20. (i) *For ω in $L^{4/n}(\Omega^n_{\varphi(D)})$, $\varphi^*(\omega)$ lies in $L^{4/n}(\Omega^n_D)$ and φ^* gives a bounded map between these Banach spaces.*

(ii) *The wedge product*

$$L^{4/n}(\Omega^n_{\varphi(D)}) \otimes L^{4/n'}(\Omega^{n'}_{\varphi(D)}) \to L^{4/n+n'}(\Omega^{n+n'}_{\varphi(D)})$$

is defined and $\varphi^*(\alpha \wedge \beta) = \varphi^*(\alpha) \wedge \varphi^*(\beta)$.

(iii) *For* ω *in* $L^1(\Omega^4_{\varphi(D)})$

$$\int_D \varphi^*(\omega) = \sigma_\varphi \int_{\varphi(D)} \omega.$$

We now define an exterior derivative d. If $\omega \in L^{4/n}(\Omega^n_D)$ and $\theta \in L^{4/n+1}(\Omega^{n+1}_D)$ $(n>0)$ we say $d\omega=\theta$ if

$$\int_D \theta \wedge \alpha = (-1)^{n+1} \int_D \omega \wedge d\alpha \tag{2.21}$$

for all smooth compactly supported test forms α. Another definition is to say $d\omega=\theta$ if there are smooth ω_i converging to ω in $L^{4/n}$ with $d\omega_i$ converging to θ in L^{4+n}—the equivalence of the two approaches follows from a regularisation ([18] Theorem 7.4). Similarly when $n=0$ we define a derivative d on the functions on D and if D has, say, a smooth boundary the Sobolev embedding theorems imply that a function f with df in L^4 lies in $L^N(\Omega^0_D)$ for any $N>0$ ([18], Chapter 7).

LEMMA 2.22. *If* $\varphi: D \to \varphi(D) \subset \mathbf{R}^4$ *is a quasi conformal homeomorphism,* $n \geq 1$ *and* $\omega \in L^{4/n}(\Omega^n_D)$ *with* $d\omega = \theta$ *then*

$$d(\varphi^*(\omega)) = \varphi^*(\theta).$$

Similarly when $n=0$, *if* $df \in L^4$ *then* $d(f\circ\varphi) = \varphi^*(df)$ *is also in* L^4.

Proof. Consider first the case $n=0$; then the assertion is just the chain rule for distributional derivatives:

$$\frac{\partial(f\circ\varphi)}{\partial x_\alpha} = \sum_\beta \left(\frac{\partial f}{\partial x_\beta}\right)_{\varphi(x)} \left(\frac{\partial \varphi_\beta}{\partial x_\alpha}\right).$$

To establish this we use the properties that φ is both in $L^4_{1,\mathrm{loc}}$ and continuous. We suppose first that $f: \varphi(D) \to \mathbf{R}$ is smooth and with compact support and choose an approximating sequence $\varphi^{(i)} \to \varphi$ in $L^4_{1,\mathrm{loc}} \cap C^0$. Then

$$\left(\frac{\partial f}{\partial x_\beta}\right)_{\varphi^{(i)}(x)} \frac{\partial \varphi^{(i)}_\beta}{\partial x_\alpha} \to \left(\frac{\partial f}{\partial x_\beta}\right)_{\varphi(x)} \left(\frac{\partial \varphi_\beta}{\partial x_\alpha}\right)$$

in L^4, and $f\circ\varphi^{(i)}\to f\circ\varphi$ in C^0 so $d(f\circ\varphi)=\varphi^*(df)$ by our second definition. To extend to general f we note that the equation $d(f\circ\varphi)=\varphi^*(df)$ is local and we can approximate a function in the neighbourhood of a point in $\varphi(D)$ by smooth functions of compact support.

For larger values of n the naturality of d, when φ is smooth, expresses the symmetry of partial derivatives:

$$\frac{\partial^2\varphi_\alpha}{\partial x_\beta\partial x_\gamma}=\frac{\partial^2\varphi_\alpha}{\partial x_\gamma\partial x_\beta}.$$

For our situation we formulate a weak version of this:

$$\int_D\left(\frac{\partial\sigma}{\partial x_\beta}\frac{\partial\varphi_\alpha}{\partial x_\gamma}-\frac{\partial\sigma}{\partial x_\gamma}\frac{\partial\varphi_\alpha}{\partial x_\beta}\right)d\mu=0, \tag{2.23}$$

for smooth test functions σ over D. This holds for our φ by the definition of the weak derivative. In turn it holds for any σ in $L^{4/3}_{1,0}$: the closure of $C_c^\infty(D)$ in the $L_1^{4/3}$ norm.

For simplicity of notation we treat the case $n=1$. To begin with suppose $\omega=p(y)dy_\lambda$ with p smooth. Then

$$\theta=d\omega\sum\frac{\partial p}{\partial y_\mu}dy_\mu\,dy_\lambda.$$

Let τ be a test form on D of the shape:

$$\tau=t(x)\,dx_1\,dx_2.$$

We have:

$$\int_D\tau\wedge\varphi^*(d\omega)=\int_D t(x)\left(\frac{\partial p}{\partial y_\mu}\left(\frac{\partial\varphi_\mu}{\partial x_3}\frac{\partial y_\lambda}{\partial x_4}-\frac{\partial\varphi_\mu}{\partial x_4}\frac{\partial y_\lambda}{\partial x_3}\right)\right).$$

Now apply our chain rule to $p\circ\varphi=\bar{p}$ to write this as:

$$\int_D t(x)\left(\frac{\partial\bar{p}}{\partial x_3}\frac{\partial\varphi_\mu}{\partial x_4}-\frac{\partial\bar{p}}{\partial x_4}\frac{\partial\varphi_\mu}{\partial x_3}\right).$$

Now put $\sigma(x)=t(x)\bar{p}(x)$ so σ is in $L^4_{1,0}\subset L^{4/3}_{1,0}$ and by the Leibnitz rule for weak derivatives this integral is:

$$\int_D\left(\frac{\partial\sigma}{\partial x_3}\frac{\partial\varphi_\lambda}{\partial x_4}-\frac{\partial\sigma}{\partial x_4}\frac{\partial\varphi_\lambda}{\partial x_3}\right)-\int_D\bar{p}\left(\frac{\partial t}{\partial x_3}\frac{\partial\varphi_\lambda}{\partial x_4}-\frac{\partial t}{\partial x_4}\frac{\partial\varphi_\lambda}{\partial x_3}\right).$$

The first integral vanishes by (2.23) and the second term yields

$$\int_D d\tau \wedge \varphi^*(\omega)$$

as required. Now the formula extends to any τ by linearity and finally to any ω by an approximation argument.

The distribution definition (2.23) gives immediately that $d^2=0$ so if we put:

$$B_D^n = \begin{cases} \{\omega \in \Omega_D^n \mid \omega \in L_{\mathrm{loc}}^{4/n},\ d\omega \in L_{\mathrm{loc}}^{4n+1}\}, & n \geq 1 \\ L_{1,\mathrm{loc}}^4 & , \quad n=0 \end{cases} \tag{2.24}$$

we have a quasiconformally invariant chain complex:

$$B_D^0 \xrightarrow{d} B_D^1 \xrightarrow{d} B_D^2 \xrightarrow{d} B_D^3 \xrightarrow{d} B_D^4. \tag{2.25}$$

One verifies readily enough that if $\alpha \in B_D^n, \beta \in \beta_D^m$ for $n, m \geq 1$ then $\alpha \wedge \beta \in B_D^{n+m}$ and

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^n \alpha \wedge d\beta. \tag{2.26}$$

However it is *not* true that multiplication takes $B_D^0 \times B_D^m$ to B_D^m This is one of the difficulties associated with the failure of the Sobolev embedding theorem at the critical, conformally invariant, exponent:

$$L_{1,\mathrm{loc}}^4 \not\hookrightarrow C^0.$$

For example the function

$$f(x) = \log|\log|x||$$

is in $L_{1,\mathrm{loc}}^4$ but is not bounded around $x=0$. We could get around this by replacing B_D^0 by $B_D^0 \cap C_D^0$ but that would do damage to the following Poincaré lemma:

LEMMA 2.27. *Suppose $n \leq 2$ and $\beta \in B_D^{n+1}$ with $d\beta=0$. Then each point in D has a neighbourhood $\tilde{D}$ on which we can find $\alpha \in B_{\tilde{D}}^n$ such that $d\alpha=\beta$.*

Proof. We transfer the problem to the compact manifold S^4, as in Corollary 2.17, using cut-offs. Then choose the Hodge solution to $d\alpha=\beta$ with $d^*\alpha=0$. Elliptic theory gives, for $4/n+1>1$:

$$\|\alpha\|_{L_1^{4/n+1}} \lesssim \|\beta\|_{L^{4/n+1}}$$

and

$$\|\alpha\|_{L^{4/n}} \lesssim \|\alpha\|_{L_1^{4/n+1}}$$

by the Sobolev embedding theorem. (Here, and throughout the paper, we use the notation $\lesssim$ to denote "bounded by a constant multiple of" when the dependence on parameters is clear.)

The Poincaré lemma for the B-forms fails in the top dimension. There are forms α in $L^1_{\mathrm{loc}}(\Omega^4)$ which are not d of an $L^{4/3}_{\mathrm{loc}}$ 3-form. This can easily be deduced, using the open mapping theorem, from the failure of the Sobolev embedding theorem mentioned above. This breakdown in the conformally invariant theory, which becomes more acute when one goes to the non-linear problems involved in gauge theory, motivates the search for alternative quasiconformally invariant spaces of forms.

(v) Gehring's theorem

Let $\varphi: d \to \varphi(D) \subset \mathbf{R}^4$ be a K quasiconformal map between *bounded* domains D, $\varphi(D)$. We know already that $\nabla\varphi$ is in L^4_D. Gehring showed that one could do a little more.

THEOREM 2.28 [16]. *There is a $\delta=\delta(K)>0$ such that if x_0 is a point in D there is a neighbourhood $D' \subset D$ of x_0 with*

$$\int_{D'} |\nabla\varphi|^{4+\delta}\, d\mu < \infty.$$

(*Note*. Gehring's theorem is valid in any dimension n—a quasiconformal map has derivatives in $L^{n+\delta}_{\mathrm{loc}}$.)

We can deduce this result from our fundamental Corollary 2.17. Let α be a 1-form on $\mathbf{R}^4$ with $d^+\alpha=0$ but $d\alpha=\omega$ nowhere zero—for example:

$$\alpha = y_1dy_2 - y_2dy_1 - y_3dy_4 + y_4dy_3.$$

Pull back the standard flat conformal structure on $\varphi(D)$ by φ to obtain a bounded structure representing by μ on D, with $|\mu| \leq c < 1$. Then

$$\alpha' = \varphi^*(\alpha)$$

satisfies

$$d^+_\mu \alpha' = 0$$

on D.

So on an interior domain $D' \subset\subset D$ we have by Corollary 2.17 (ii) that there exists U with

$$\|\alpha' - du\|_{L^p_1, D'} \leqslant A \|\alpha'\|_{L^p} < \infty$$

for $p \leqslant 2+\eta$, some $\eta > 0$. So $d\alpha' = d(\alpha' - du)$ is in $L^{2+\eta}$ on D' and we have:

$$\int_{D'} |\varphi^*(\omega)|^{2+\eta} < \infty.$$

We now apply the observation in (iii) above. At each point x the function $(\nabla\varphi) \rightarrow |\varphi^*(\omega)_x|$ is homogenous of degree 2, so for K quasiconformal maps:

$$|\nabla\varphi|^{2p} \leqslant C_K^p |\varphi^*(\omega)|^p.$$

Hence

$$\int_{D'} |\nabla\varphi|^{4+2\eta} \leqslant C_K^p \int_{D'} |\varphi^*(\omega)|^{2+\eta} < \infty$$

as required.

Going in the opposite direction to the proof above we can now deduce that quasiconformal maps act on spaces of forms a little beyond the conformally invariant exponents.

LEMMA 2.29. *Let $\varphi: D \rightarrow \varphi(D) \subset \mathbf{R}^4$ be a quasiconformal map between bounded domains with $\int |\nabla\varphi|^{4+\delta} < \infty$. If $\alpha \in L^{(4/n)+\varepsilon}(\Omega^n_{\varphi(D)})$ $(n=1,2,3,4)$ then $\varphi^*(\alpha) \in L^{(4/n)+\varepsilon'}(\Omega^n_D)$ where*

$$\varepsilon' = \frac{\delta\varepsilon}{\dfrac{4+\delta}{n} + \varepsilon}$$

and $\|\varphi^*(\alpha)\|_{L^{(4/n)+\varepsilon'}} \leqslant C_\varphi \|\alpha\|_{L^{(4/n)+\varepsilon'}}$. The constant C_φ can be taken independent of ε in a range $\varepsilon \in [0, E]$.

Proof. Our hypotheses are:

$$I = \int_D |\nabla\varphi|^{4+\delta} < \infty$$

$$J = \int_D |\nabla\varphi|^4 \, |\omega|^{(4/n)+\varepsilon}_{\varphi(x)} < \infty$$

and we wish to bound:

$$\int_D |\varphi^*(\omega)|^{(4/n)+\varepsilon'} \leq \int_D |\nabla\varphi|^{4+n\varepsilon'} |\omega|_{\varphi(x)}^{(4/n)+\varepsilon'}.$$

Let $f(x)=|\nabla\varphi|_x$, $g(x)=|\omega|_{\varphi(x)}$; then Hölder's inequality gives:

$$\int f^{4+n\varepsilon'} g^{4/n+\varepsilon'} \leq \left(\int f^{ap}\right)^{1/p} \left(\int (f^b g^{4/n+\varepsilon'})^q\right)^{1/q}$$

with $a+b=4+n\varepsilon'$ and $1/p+1/q=1$.
We want to choose indices so that:

$$ap = 4+\delta$$

$$bq = 4$$

$$\left(\frac{4}{n}+\varepsilon'\right) q = \frac{4}{n}+\varepsilon,$$

then the expression on the right is $I^{1/p} J^{1/q} < \infty$. The five linear equations in p^{-1}, q^{-1}, a, b, ε' have a unique solution, and the required ε' is

$$\frac{\delta\varepsilon}{\dfrac{4+\delta}{n}+\varepsilon}.$$

We can now define quasiconformally invariant spaces of forms:

(2.30) $$B_D^{+,n} = \left\{\omega \in \Omega_D^n \mid \omega \in L_{\text{loc}}^{(4/n)+\varepsilon},\ d\omega \in L_{\text{loc}}^{(4/n+1)+\varepsilon} \text{ for some } \varepsilon > 0\right\}.$$

The definition of the exterior derivative goes over to this setting to yield a graded differential algebra $(B^{+,*}, d)$. $B^{+,0}$ consists of continuous functions so we avoid the difficulties associated to the failure of the Sobolev embedding at the critical exponent (for example a full Poincaré lemma is valid). However these B^+ forms are not very convenient for analysis since $\bigcup_{p>4/n} L^p$ is not a Banach space. So in the next section we introduce new function spaces lying between the B^n and the $B^{+,n}$ and which enjoy the good properties of both.

§3. Modified Banach spaces

(i) General definitions

Let (S, μ) be a measure space with $\mu(S)<\infty$: then we can regard the function spaces $L^p(S)$ $(1 \geqslant p \geqslant \infty)$ as being ordered by inclusion:

$$L^p(S)\subset L^q(S) \quad \text{if} \quad p\geq q.$$

Fix an exponent p and additional real parameters ε, ϱ with $0<\varepsilon, \varrho<1$. Eventually we shall require $\varrho<\varepsilon$. Define a space of functions $\hat{L}^p(S)$ (depending on ε, ϱ) as follows. A function f is in $\hat{L}^p$ if it can be written as an L^p-convergent sum:

$$f=\sum_1^\infty f_i \quad \text{with} \quad f_i\in L^{p+\varepsilon^i} \quad \text{and} \quad \sum_1^\infty \varrho^{-i}\|f_i\|^2_{L^{p+\varepsilon^i}}<\infty. \tag{3.1}$$

We define a norm on $\hat{L}^p$ by:

$$\|f\|_{\hat{L}^p}=\inf\left(\sum_1^\infty \varrho^{-i}\|f_i\|^2_{L^{p+\varepsilon^i}}\right)^{1/2} \tag{3.2}$$

where the infimum is taken over all possible such decompositions $f=\Sigma f_i$.

Proposition 3.3. *$\hat{L}^p$ is a reflexive Banach under $\|\ \|_{\hat{L}^p}$. Both $\hat{L}^p$ and its dual space are separable. There are bounded inclusions $L^r\subset\hat{L}^p\subset L^p$ if $r>p$.*

Proof. Consider the space

$$Z=l^2(\oplus L^{p+\varepsilon^i})$$

consisting of infinite sequences (f_i) with the weighted norm

$$\|(f_i)\|_Z^2=\sum \varrho^{-i}\|(f_i)\|^2_{L^{p+\varepsilon^i}}.$$

It is a standard fact that Z is a Banach space. If $(f_i)\in Z$

$$\sum\|f_i\|_{L^p}<\sum\|f_i\|_{L^{p+\varepsilon^i}}\leq\left(\sum\|f_i\|^2_{L^{p+\varepsilon^i}}\varrho^{-i}\right)^{1/2}\left(\sum\varrho^i\right)^{1/2}.$$

Now $(\Sigma\varrho^i)<\infty$ so

$$\sum\|f_i\|_{L^p}\lesssim\|(f_i)\|_Z$$

and there is a bounded sum map:

$$\sigma: Z\to L^p,$$
$$\sigma(f_i)=\sum f_i.$$

$\hat{L}^p$ is, by definition, the image of σ and the norm is the usual quotient norm on $Z/\mathrm{Ker}\,\sigma$. Thus $\hat{L}^p$ is a Banach space.

The dual space Z^* is well known to be:

$$\left\{(\alpha_i): \alpha_i \in L^{q_i}, \sum \|\alpha_i\|^2_{L^{q_i}} \varrho^i \leq \infty\right\}$$

where q_i is defined by:

$$\frac{1}{q_i} + \frac{1}{p+\varepsilon^i} = 1.$$

Moreover, Z is reflexive. By the Hahn–Banach theorem $\hat{L}^p$ is also reflexive and $(\hat{L}^p)^* \subset Z^*$ is the space of functions α such that $\alpha \in L^q$ for all $q<p$ and $\Sigma\, \|\alpha\|^2_{L^{q_i}} \varrho^i < \infty$. It follows that the bounded functions, and a fortiori L^{p+1}, are dense in $\hat{L}^p$ and the separability of $\hat{L}^p$ follows from that of L^{p+1}. Similarly for $(\hat{L}^p)^*$.

We will use a simple lemma many times in our argument below. Let $c>0$ and define a space $L^{p,c}$ to consist of functions f which can be written $f = \Sigma f_i$ with

$$\sum \|f_i\|^2_{L^{p+c\varepsilon^i}} \varrho^i < \infty.$$

Let $\| \;\|_{p,c}$ be the obvious norm on this space.

LEMMA 3.4. *$L^{p,c} = \hat{L}^p$ and the two norms $\| \;\|_{p,c}$ and $\| \;\|_{\hat{L}^p}$ are equivalent.*

Proof. Suppose $c>1$. Then $L^{p+c\varepsilon^i} \subset L^{p+\varepsilon^i}$ and we obviously have $\|(f)\|_{\hat{L}^p} < \|f\|_{p,c}$. Conversely, choose k such that $\varepsilon^k c<1$. Then if $f = \Sigma f_i$, $f_i \in L^{p+\varepsilon^i}$ put:

$$\tilde{f}_i = \begin{cases} f_{i-k}, & i \geq k+1 \\ 0, & i \leq k. \end{cases}$$

So $f = \Sigma \tilde{f}_i$ and $\tilde{f}_i \in L^{p+c\varepsilon^i}$. Also

$$\sum \|\tilde{f}_i\|^2_{L^{p+c\varepsilon^i}} \varrho^{-i} \leq \left(\sum \|\tilde{f}_i\|^2_{p+\varepsilon^{i-k}} \varrho^{-i}\right) = \varrho^{-k} \sum \|f_i\|^2_{p+\varepsilon^i} \varrho^{-i}$$

and the two norms are equivalent. The proof when $c<1$ is exactly parallel.

LEMMA 3.5. *If $1/p + 1/q = 1/r$, $f \in \hat{L}^p$ and $g \in L^q$ then $fg \in \hat{L}^r$.*

Proof. If $f = \Sigma f_i$, $f_i \in L^{p+\varepsilon^i}$ then $fg = \Sigma f_i g$ and $f_i g \in L^{r+\delta_i}$,

$$\|f_i g\|_{L^{r+\delta_i}} \leq \|f_i\|_{L^{p+\varepsilon^i}} \|g\|_{L^q}$$

where

$$\frac{1}{r+\delta_i} = \frac{1}{p+\varepsilon^i} + \frac{1}{q}.$$

That is:

$$\delta_i = \frac{q^2\varepsilon^i}{(q+p+\varepsilon^i)(q+p)} > c\varepsilon^i \quad \text{where} \quad c = \frac{q^2}{(q+p+1)^2}.$$

Now apply Lemma 3.4. Similarly we have a multiplication theorem for the dual spaces.

Recall that the dual $(\hat{L}^p)^*$ is:

(3.6) $$\left\{ \alpha \,|\, \alpha \in L^n \text{ for all } n < \frac{p}{p-1} \text{ and } \sum \|\alpha\|^2_{L^{q_i}} \varrho^i < \infty \right\},$$

where

$$\frac{1}{q_i} + \frac{1}{p+\varepsilon^i} = 1.$$

LEMMA 3.7. *Let $1/q+1/s=1/r$ and $1/q+1/p=1$. Then if $f\in(\hat{L}^p)^*, g\in\hat{L}^s$ we have $fg\in L^r$.*

Proof. Let $f\in(\hat{L}^p)^*$, so $\Sigma\|f\|^2_{L^{q_i}}\varrho^i<\infty$ $(1/q_i=1/(p+\varepsilon^i))$. For any $c>0$ we can, by Lemma 3.4 write $g\in\hat{L}^s$ as Σg_i with

$$\sum \|g_i\|^2_{L^{s+c\varepsilon^i}} \varrho^{-i} < \infty.$$

Then if $fg_i = h_i$:

$$\|h_i\|_{L^{r+\delta_i}} \leq \|f\|_{L^{q_i}} \|g^i\|_{L^{s+c\varepsilon^i}}$$

where

$$\frac{1}{r+\delta_i} = \frac{1}{r} - \frac{c\varepsilon^i}{s(s+c\varepsilon^i)} + \frac{\varepsilon^i}{p(p+\varepsilon^i)}.$$

We can choose c large enough so that $\delta_i>0$. Then $fg=\Sigma h_i$ whith

$$\sum \|h_i\|_{L^r} \lesssim \sum \|h_i\|_{L^{r+\delta_i}}$$

$$\lesssim \sum \|g_i\|_{L^{s+\varepsilon\varepsilon^i}} \|f\|_{L^{q_i}}$$

$$\lesssim \left(\sum \|g_i\|^2_{L^{s+\varepsilon\varepsilon^i}} \varrho^i \right)^{1/2} \left(\sum \|f_i\|^2_{L^{q_i}} \varrho^{-i} \right)^{1/2}$$

by Cauchy-Schwarz.

Now suppose that μ is Lebesgue measure on a convex bounded domain $D \subset \mathbf{R}^4$. We extend the $\hat{L}^p$-function spaces to differential forms in the obvious way.

Let x_0 be a point in D.

LEMMA 3.8 (Sobolev embedding theorem). *Suppose $\varrho < \varepsilon^{3/4}$, then there is a constant $A = A(D, x_0)$ such that*

$$|f(x_0)| \leq A \|df\|_{\hat{L}^4}$$

for all smooth compactly supported functions f on D.

Proof. We can take $x_0 = 0$ and work in "polar" co-ordinates r, θ with $\theta \in S^3$. Then:

$$\begin{aligned} f(0) &= \int_0^\infty \frac{\partial f}{\partial r} dr \\ &= \frac{1}{\operatorname{Vol}(S^3)} \int \int_0^\infty \frac{\partial f}{\partial r} dr d\theta \\ &= \frac{1}{\operatorname{Vol}(S^3)} \int \int_0^\infty \frac{\partial f}{\partial r} \frac{1}{r^3} r^3 dr d\theta \\ &= \frac{1}{\operatorname{Vol}(S^3)} \int \int_0^\infty \frac{\partial f}{\partial r} \frac{1}{r^3} d\mu. \end{aligned}$$

So

$$|f(0)| \leq \frac{1}{\operatorname{Vol}(S^3)} \int_D |df| \frac{1}{r^3} d\mu.$$

Now suppose $df = \Sigma \alpha_i$ with

$$\sum \|\alpha_i\|^2_{L^{4+\varepsilon^i}} \varrho^{-i} = C^2 < \infty.$$

Then

$$\int_D |\alpha_i| \frac{1}{r^3} d\mu \leq \|1/r^3\|_{L^{q_i}} \|\alpha_i\|_{L^{4+\varepsilon^i}} \quad \text{where} \quad q_i = \frac{4+\varepsilon^i}{2+\varepsilon^i}.$$

But, since D is bounded,

$$\|1/r^3\|_{q_i} \leq \left(\int_0^1 r^{3(1-q_i)} dr\right)^{1/q_i} = \left(\frac{3+\varepsilon^i}{\varepsilon^i}\right)^{\frac{3+\varepsilon^i}{4+\varepsilon^i}}.$$

So

$$|f(0)| < \sum \|\alpha_i\|_{L^{4+\varepsilon^i}} \varepsilon^{-3/4^i}$$

and if $\varrho < \varepsilon^{3/4}$ this sum is bounded by a multiple of C.

In fact the same integral formula used in Lemma 3.8 here shows that ([18] Lemma 7.16):

(3.9) $\quad |f(x)-f(y)| \leq C \cdot g(|x-y|) \quad$ where $\quad C = \|df\|_{\hat{L}^4} \quad$ and $\quad g(S) = \sum S^{\varepsilon^i/4} \varrho^i \varepsilon^{-3i/4}$.

g is a monotone function and $g(S) \to 0$ as $S \to 0$. So we have by the Ascoli-Arzela theorem:

COROLLARY 3.10. *The space of functions f in $C_0^\infty(D)$ with $\|df\|_{\hat{L}^4} < C$ is precompact in $C^0(D)$.*

We can now define the analogous of the usual Sobolev spaces

$$\hat{L}^4_{1,\mathrm{c}} = \text{closure of } C_c^\infty \text{ in norm } \|df\|_{\hat{L}^4}$$

$$\hat{L}^4_{1,\mathrm{loc}} = \text{functions } f \text{ on } D \text{ with } df \text{ locally in } \hat{L}^4.$$

Both of these consist of continuous functions and Corollary 3.10 gives that there is a *compact* embedding.

$$\hat{L}^4_{1,c} \to C_c^0(D) \tag{3.11}$$

Moreover we have a composition rule:

PROPOSITION 3.12. *If $F: \mathbf{R} \text{—} \mathbf{R}$ is a smooth function then for all f in $\hat{L}^4_{1,\mathrm{loc}}$ the composite $F \circ f$ also lies in $L^4_{1,\mathrm{loc}}$*

The proof is straightforward. We define $\hat{B}^n_{D,\mathrm{loc}} \subset B^n_D$ to be those forms $\omega \in \hat{L}^{4/n}_{\mathrm{loc}}$ with $d\omega$ in $\hat{L}^{4/n+1}_{\mathrm{loc}}$ $(n \geq 1)$ and to be $\hat{L}^4_{1,\mathrm{loc}}$ when $n=0$. Then:

$$\hat{B}^0_{D,\mathrm{loc}} \xrightarrow{d} B^1_{D,\mathrm{loc}} \xrightarrow{d} \hat{B}^2_{D,\mathrm{loc}} \xrightarrow{d} \hat{B}^3_{D,\mathrm{loc}} \xrightarrow{d} \hat{B}^4_{D,\mathrm{loc}} \tag{3.13}$$

is a graded differential algebra.

LEMMA 3.14. *Let $\varphi: D \to \varphi(D) \subset \mathbf{R}^4$ be a quasiconformal map between bounded domains as in Lemma* 2.29. If $\omega \in \hat{L}^{4/n}(\Omega^n_{\varphi(D)})$ *then* $\varphi^*(\omega) \in \hat{L}^{4/n}(\Omega^n_D)$ *and*

$$\|\varphi^*(\omega)\|_{\hat{L}^{4/n}} \leq \hat{C}_\varphi \|\omega\|_{\hat{L}^{4/n}} \quad \textit{for some constant } \hat{C}_\varphi.$$

Proof. Let $\omega = \Sigma\, \omega_i$ where $\omega_i \in L^{4/n+\varepsilon^i}$, then $\varphi^*(\omega) = \Sigma \varphi^*(\omega_i)$ and by Lemma 2.29 we can suppose:

$$\|\varphi^*(\omega_i)\|_{L^{4/n+\zeta_i}} \leq C_\varphi \|\omega_i\|_{L^{4/n+\varepsilon^i}}$$

where

$$\zeta_i = \frac{\delta \varepsilon^i}{\dfrac{4+\delta}{n} + \varepsilon^i}.$$

So $\zeta_i \leq c\varepsilon^i$ for some c and

$$\sum \|\varphi^*(\omega_i)\|^2_{L^{4/n+c\varepsilon^i}} \varrho^i \leq \sum \|\omega_i\|^2_{L^{4/n-\varepsilon^i}} \varrho^{-i}.$$

Now the result follows from (3.4).

To sum up we have for any ε, ϱ with $0<\varepsilon, \varrho<1$, $\varrho<\varepsilon^{3/4}$ a quasi-conformally invariant differential graded algebra $(\hat{B}^*_{\mathrm{loc}}, d)$.

Let $D' \subset\subset D$ be a bounded domain in $\mathbf{R}^4$ and μ represent a bounded conformal structure on D, as in Corollary 2.17. We consider:

$$d^+_\mu : \hat{B}^1_D \to \hat{L}^2(\Omega^+_D). \tag{3.15}$$

PROPOSITION 3.16. (i) *There is a bounded map*

$$Q_\mu : \hat{L}^2(\Omega^+_D) \to \hat{B}^1_D$$

with

$$d^+_\mu Q_\mu \omega = \omega \quad \textit{on } D'.$$

(ii) *For any α in $\hat{B}^1_D$ we can find u in $\hat{B}^0_{D,c}$ such that*

$$\|\alpha - du\|_{\hat{B}^1(D')} \leq A \|d^+_\mu \alpha\|_{\hat{L}^2(D)} + \|\alpha\|_{\hat{L}^2(D)}.$$

Proof. (i) Let $\omega = \Sigma\, \omega_i \in \hat{L}^2(\Omega^+_D)$ with $\omega^i \in L^{2+\varepsilon^i}$. By Corollary 2.17

$$\omega_i = d_\mu^+ Q_\mu(\omega_i)$$

on D', where:

$$\|Q_\mu(\omega_i)\|_{L_1^{2+\varepsilon^i}} < \|\omega_i\|_{L^{2+\varepsilon^i}}.$$

So by the Sobolev embedding theorem, $\|Q_\mu(\omega_i)\|_{L^{r_i}} \lesssim \|\omega_i\|_{L^{2+\varepsilon^i}}$ where $1-4/(2+\varepsilon^i)=-4/r_i$ i.e. $r_i=4(2+\varepsilon^i)/(2-\varepsilon^i)>4+4\varepsilon^i$. So

$$\|Q_\mu(\omega)\|_{\hat{L}^4} \lesssim \|\omega\|_{\hat{L}^2}$$

as required. Part (ii) is similar.

Now suppose γ is in $\hat{B}_D^1$. Wedge product followed by projection to Ω_D^+ gives a map

$$M_\gamma\colon \hat{L}^4(\Omega_D^1) \to \hat{L}^2(\Omega_D^+).$$

LEMMA 3.17. *For μ, Q_μ as above the maps*
(i) $M_\gamma \circ Q_\mu\colon \hat{L}^2(\Omega_D^+) \to \hat{L}^2(\Omega_D^+)$
(ii) $Q_\mu \circ M_\mu\colon \hat{L}^4(\Omega_D^1) \to \hat{L}^4(\Omega_D^1)$
are compact.

Proof. (i) Let γ_i be bounded with $\gamma_i \to \gamma$ in $\hat{L}^4$. Then $M_\gamma \circ Q_\mu \to M_\gamma \circ Q_\mu$ in operator norm, so it suffices to consider the case when γ is bounded. If $\varphi^{(j)}$ is a sequence bounded in $\hat{L}^2(\Omega_D^+)$, $Q_\mu(\varphi^{(j)})$ is bounded in $\hat{L}_1^2$ so taking a subsequence we can suppose $Q_\mu(\varphi^{(j)})$ is convergent in L^q for any $q<4$. In particular $Q_\mu(\varphi^{(j)})$ converges in $\hat{L}^2$ and so also does $M_\gamma Q_\mu(\varphi^{(j)})$.

(ii) is similar.

So our elliptic theory for the d_μ^+ operators behaves very well on the $\hat{B}$ spaces. In the same way we have a full Poincaré lemma.

LEMMA 3.18. *Suppose $\varrho<\varepsilon$. If $\beta \in \hat{B}^{n+1}_{\mathrm{loc},D}$ and $d\beta=0$ then each point in D has a neighbourhood $\tilde{D}$ on which we can find $\alpha \in \hat{B}^n_{\mathrm{loc},D}$ such that $d\alpha=\beta$.*

Proof. Once again we use cut-offs to transfer to a compact manifold and apply the usual elliptic estimates and Sobolev embedding theorem. The interesting case is when $n=3$. The operator norm of the map

$$S\colon L^p(\Omega^4) \to L^p(\Omega^3)$$

solving $dS\beta=\beta$, $d^*S\beta=0$, blows up as $p\to 1$. In fact by ([24] p. 22)

$$\|S(\beta)\|_{L_1^p} \leq \left(\frac{A}{p-1}\right)\|\beta\|_{L^p}$$

as $p \to 1$. So if $\beta = \Sigma \beta_i$ with $\beta_i \in L^{1+\varepsilon^i}$

$$S(\beta_i) \in L_1^{1+\varepsilon} \to L^{4(1+\varepsilon^i)/(3-\varepsilon^i)}$$

and

$$\|S(\beta_i)\|_{L^{4(1+\varepsilon^i)/(3-\varepsilon^i)}} \leq A\varepsilon^{-1}\|\beta_i\|_{L^{1+\varepsilon^i}}.$$

It follows that S is a bounded map $\hat{L}^1(\Omega^4) \to \hat{L}^{4/3}(\Omega^3)$ if $\varrho < \varepsilon$.

§ 4 Global theory

(i) Quasiconformal manifolds and conformal structures

Definition 4.1. (i) A quasiconformal 4-manifold X is a topological 4-manifold equipped with a maximal atlas of charts

$$\psi_a : U_a \to X$$

such that the overlap maps $\psi_a^{-1}\psi_\beta$ are quasiconformal mappings on their domains of definition in $\mathbf{R}^4$.

(ii) The quasiconformal 4-manifolds $(X, \{\psi_a\}), (X', \{\psi'_\lambda\})$ are quasiconformally equivalent if there is a homeomorphism $f: X \to X'$ such that the $(\psi'_\lambda)^{-1} f \psi_a$ are quasiconformal maps on their domains of definition.

Thus a smooth manifold, for example, has a quasiconformal structure. We will assume our manifolds are oriented, Hausdorff and paracompact—most often we will be concerned with compact manifolds. Using the results in §§ 2, 3 we can now develop some global analysis on quasiconformal 4-manifolds quite parallel to the standard theory in the smooth case.

First, the quasiconformal invariance of differential forms allows us to define the following spaces of n-forms on a quasiconformal manifold X:

$$L^{4/n}_{\mathrm{loc}}(\Omega^n_X), \quad L^{4/n+}_{\mathrm{loc}}(\Omega^n_X), \quad \hat{L}^{4/n}_{\mathrm{loc}}(\Omega^n_X)$$
$$B^n_{\mathrm{loc},X}, \quad B^{n,+}_{\mathrm{loc},X}, \quad \hat{B}^n_{\mathrm{loc},X}.$$

The definitions are local so we have the obvious associated sheaves. If X is compact we write:

$$L^{4/n}_{\text{loc}}(\Omega^n_X) = L^{4/n}(\Omega^n_n)$$

$$B^n_{\text{loc},X} = B^n_X \quad \text{etc.}$$

and the $L^{4/n}(\Omega^n_X), \hat{L}^{4/n}(\Omega^n_X), B^n_X, \hat{B}^n_X$ are Banach spaces. The norms can be defined using a $\hat{B}^0_X$ partition of unity and are unique up to equivalence. For $n=0$ the space $\hat{B}^0_X$ of functions with derivatives in $\hat{L}^4$ consists of continuous functions, and the inclusion $\hat{B}^0_X \hookrightarrow C^0(X)$ is compact.

The $B^{+,n}_{\text{loc}}, \hat{B}^n_{\text{loc}}$ spaces yield chain complexes with the differential d and the Poincaré lemmas of § 3 combined with the usual sheaf theory argument give:

PROPOSITION 4.2. *The de Rham cohomology groups of* $(\hat{B}^*_{\text{loc}}, d), (\hat{B}^{+,*}_{\text{loc}}, d)$ *are naturally isomorphic to the singular coholomogy groups of* X.

Moreover if X is compact the fundamental class in H_4 is represented by the integration of forms over X.

Definition 4.3. A bounded conformal structure on a quasiconformal 4-manifold X is given by bounded structures $[g_\alpha]$ (or μ_α) on each chart $U_\alpha \subset \mathbf{R}^4$ (in the sense of § 2(i)) compatible under the (a.e. defined) derivatives of $\psi_\alpha^{-1}\psi_\beta$.

Just as in § 2(i) we have a distance function $d([g_1],[g_2]) \in L^\infty_{\text{loc},X}$ defined for every pair of such conformal structures. If X is compact we put a metric on the set of structures:

$$\operatorname*{ess\,sup}_X d([g_1],[g_2]).$$

PROPOSITION 4.4. *If X is a compact quasiconformal* 4*-manifold there exist bounded conformal structures on X and any two can be joined by a continuous path (in the above* sup *norm topology).*

Proof. For any two Euclidean metrics g_1, g_2 on $\mathbf{R}^4$:

$$d([g_1+g_2]),[g_1]) \leq d[g_2],[g_1]).$$

So if γ_α is a partition of unity subordinate to a locally finite cover of X by coordinate charts ψ_α, and we let $[g_\alpha]$ on U_α be:

$$[g_\alpha] = \left[\sum_\beta (\gamma_\beta \circ \psi_\alpha)(\psi_\beta^{-1}\psi_\alpha)^*(g_0)\right]$$

(where g_0 is the standard Euclidean metric) the $[g_\alpha]$ are bounded and compatible under the overlap maps. Similarly if $[g_\alpha], [g'_\alpha]$ represent bounded structures in a system of charts, and we normalise so that $\det g_\alpha = \det g'_\alpha$ then $[tg_\alpha + (1-t)g'_\alpha]$ gives a path between the two structures.

Now if $(X, [g])$ is a quasiconformal 4-manifold with bounded conformal structure we have a $*$ operator on

$$L^2(\Omega^2_X),$$

and so self-dual and anti-self-dual forms. As usual the L^2-metric is

$$\|\alpha\|_{L^2} = \pm \int \alpha \wedge \alpha$$

for $\alpha \in \Omega^\pm_X$. Similarly we have $\hat{L}^2(\Omega^+_X), L^2(\Omega^+_X)$ etc., and an operator:

$$d^+ : \hat{B}^1_X \to \hat{L}^2(\Omega^+_X) \quad \text{(or from } B^{+,1}_X \text{ to } L^{2+}(\Omega^+_X)).$$

Locally, in coordinate charts these are of course represented by

$$d^+ + \mu_\alpha d^-$$

where we identify g_α-self-dual forms with the Euclidean ones by the graph construction of § 2(i).

(ii) Bundles and connections

Here the usual definitions in the smooth category go over wholesale in both the $\hat{B}$ and B^+ frameworks on a quasiconformal 4-manifold. Abstractly, if $\mathscr{C}$ is a subsheaf of the sheaf of continuous functions over a topological manifold X, closed under the usual algebraic operations (including inversion of non-vanishing functions); we can define a category of $\mathscr{C}$ vector bundles. If E is a $\mathscr{C}$ vector bundle its local sections form a sheaf of $\mathscr{C}$-modules. Suppose $(\mathscr{C}^*, d)$ is a graded differential algebra with $\mathscr{C}^0 = \mathscr{C}$; we then form

$$\mathscr{C}^p(E) = \mathscr{C}^p \otimes_{\mathscr{C}} \Gamma(E)$$

("E valued forms"). A $\mathscr{C}$ connection on E can then be defined to be a linear map

$$d_A : \mathscr{C}^0(E) \to \mathscr{C}^1(E)$$

such that $d_A(fs) = f d_A s + df s$. In the familiar way we have a curvature F_A in $\mathscr{C}^2(\operatorname{End} E)$

such that $d_A^2 = F_A$. The gauge transformations or bundle automorphisms (modelled on $\mathscr{C}^0$) act on the affine space $\mathscr{A}$ of connections.

In our case we take the sheaves $\hat{B}, B^+$ with the corresponding complexes of forms, to get two classes of bundles, connections, curvatures and gauge transformations over our quasiconformal 4-manifold X. For brevity we will stick to the $\hat{B}$ set-up.

We can work with bundles having different structure groups G and the Chern-Weil theory applies so that for an appropriate constant $c(G)$:

$$c(G)\int \operatorname{Tr}(F_A \wedge F_A) \in \mathbf{Z} \tag{4.5}$$

represents a topological characteristic number. (To see this one can either develop $\hat{B}$ classifying maps $X \to BG$ or reduce to the case where E is trivial on $X \setminus$point cf. § 5.)

The space $\mathscr{A}$ of $\hat{B}$ connections is an affine space modelled on the Banach space $\hat{B}^1(\mathfrak{g}_E)$. (Here $\mathfrak{g}_E \subset \operatorname{End} E$ represents the Lie algebra of G.) If $\mathscr{G}$ is the $\hat{B}^0$ gauge group acting on $\mathscr{A}$ we have:

PROPOSITION 4.6. *$\mathscr{G}$ is a Banach Lie group modelled on $\hat{B}^0(\mathfrak{g}_E)$ and the action $\mathscr{G} \times \mathscr{A} \to \mathscr{A}$ is a smooth.*

This is a simple consequence of the composition property in Proposition 3.12 applied to the exponential map—the proof is exactly as in [13] Appendix A.

Finally, suppose X has a bounded conformal structure $[g]$ and define

$$d_A^+ : \hat{B}^1(\mathfrak{g}_E) \to \hat{L}^2(\Omega^+(\mathfrak{g}_E))$$

by pointwise projection to the self dual forms. Similarly for d_A^-. We decompose the curvature into

$$F_A = F_A^+ + F_A^-, \quad F_A^\pm \in L^2(\Omega^\pm(\mathfrak{g}_E))$$

and

$$F_{A+a}^+ = F_A^+ + d_A^+ a + \{a \wedge a\}^+ .$$

PROPOSITION 4.7. *The map sending A to F_A is a smooth $\mathscr{G}$ invariant map from $\mathscr{A}$ to $\hat{L}^2(\Omega^+(\mathfrak{g}_E))$ with derivative d_A^+.*

Proof (see [13] Appendix A). $A \mapsto F_A$ is smooth from $\mathscr{A}$ to $\hat{B}^2$ and the projection $\hat{B}^2 \to \hat{L}^2(\Omega^+)$ is bounded linear.

(iii) Linear analysis; the parametrix

Let A be a $\hat{B}$ connection on a bundle E over a compact quasiconformal 4-manifold X with a conformal structure $[g]$. From now on we shall suppose that the structure group G is compact and E has an A-invariant metric.

Define $H_A^2 \subset \hat{B}^2(\mathfrak{g}_E) \cap L^2(\Omega^+(\mathfrak{g}_E))$ to be the space of coupled self-dual harmonic forms; self-dual 2-forms ω with $d_A\omega=0$. In this subsection we abbreviate H_A^2 to H. Let $H^\perp \subset \hat{L}^2(\Omega^+(\mathfrak{g}_E))$ be the annihilator of H under the standard L^2 inner product

$$\langle \alpha, \beta \rangle = -\int_X \operatorname{Tr}(\alpha \wedge \beta)$$

on $\Omega^+(\mathfrak{g}_E)$ (using the metric on E).

THEOREM 4.8. (i) *H is finite dimensional, so $\hat{L}^2(\Omega^+(\mathfrak{g}_E)) = H \oplus H^\perp$.*

(ii) *The image of $d_A^+: \hat{B}^1(\mathfrak{g}_E) \to \hat{L}^2(\Omega^+(\mathfrak{g}_E))$ is $H^\perp$.*

(iii) *There is a bounded right inverse*

$$Q_A: H^\perp \to \hat{B}^1(\mathfrak{g}_E)$$

with

$$d_A^+ Q_A = 1_{H^\perp}.$$

Proof. We begin by constructing a right parametrix

$$P: \hat{L}^2(\Omega^+(\mathfrak{g}_E)) \to \hat{B}^1(\mathfrak{g}_E)$$

for d_A^+ by the familiar patching procedure. Choose a finite set of co-ordinate charts $\psi_\alpha: U_\alpha \to X$ such that $\psi_\alpha(U'_\alpha)$ cover X for $U'_\alpha \subset\subset U_\alpha$.

Fix trivialisations of $\psi_\alpha^*(E)$ over the U_α, then the d_A^+ operator is represented in the U_α by:

$$d_\mu^+ + A^+ = d^+ + \mu d^- + A^+$$

where $\mu = \mu_\alpha$, $A^+ \in \hat{L}^4$ and d_μ^+ is as in § 2(ii). Let Q_α be the inversion operator for d_μ^+ given by Corollary 2.17:

$$d_\mu^+ Q_\alpha(\theta) = \theta \quad \text{on } U'_\alpha.$$

Choose a $\hat{B}^0$ partition of unity $\{\gamma_\alpha\}$ subordinate to the cover and set

$$P(\Phi) = \sum_\alpha \gamma_\alpha Q_\alpha(\Phi|_{U_\alpha}).$$

(Here we have made an obvious simplification in notation.) Then:

$$\begin{aligned} d_A^+ P(\Phi) &= \sum_\alpha \gamma_\alpha d_A^+ Q_\alpha(\Phi|_{U'_\alpha}) \\ &\quad + \sum_\alpha (\nabla\gamma_\alpha) Q_\alpha(\Phi|_{U'_\alpha}) \\ &= \sum_\alpha \gamma_\alpha \Phi|_{U'_\alpha} + \sum_\alpha K_\alpha(\Phi) \\ &= \Phi + \sum K_\alpha(\Phi) \end{aligned}$$

where $K_\alpha(\Phi) = (\gamma_\alpha A^+ + \nabla\gamma_\alpha) Q_\alpha(\Phi|_{U_\alpha})$.
The coefficient $\gamma_\alpha A^+ + \nabla\gamma_\alpha$ is in $\hat{L}^4$ so, by Lemma 3.17, each K_α is compact and

$$d_A^+ P = 1 + K$$

with K compact. Thus

$$\operatorname{Im} d_A^+ \supset \operatorname{Im}(1+K)$$

is of finite codimension (hence closed) and we have proved part (i).

To prove (ii) begin by observing that $\operatorname{Im} d_A^+ = \operatorname{Ann}(H')$ where $H' \subset (\hat{L}^2(\Omega^+(\mathfrak{g}_E)))^*$ is the set of functionals vanishing on $\operatorname{Im} d_A^+$. Now, proceeding from the local situation of § 3, we can identify this dual space with self-dual 2-forms having $(\hat{L}^2)^*$ coefficients in coordinate charts. We claim that if such a form ω annihilates $\operatorname{Im} d_A^+$ then ω in fact lies in H (hence in $\hat{L}^2 \subset (\hat{L}^2)^*$). This is a mild version of elliptic regularity. The condition that ω vanishes on $\operatorname{Im} d_A^+$ is equivalent locally in a U_α to

$$d_A \omega = (d+A)\omega = 0$$

in the distributional sense. But $A \in \hat{L}^4$ so our multiplication property (3.7) gives $d\omega \in L^{4/3}$. Then by a Sobolev version of the Hodge theory, as in § 2(ii) we can find χ in Ω^1 such that over $U'_\alpha \subset\subset U_\alpha$:

$$\omega + d\chi \in L_1^{4/3} \subset L^2.$$

But now $d_\mu^- \chi \in L^2$ (since ω is self dual) and by Corollary 2.17 we can suppose $d\chi$, and hence ω is in L^2. Now repeat the argument using Lemma 3.5:

$$\omega \in L^2, A \in \hat{L}^4 \quad \Rightarrow \quad A\omega \in \hat{L}^2$$

to deduce that $\omega \in \hat{L}^2$. Then since $d_A \omega = 0$ it follows that $d\omega \in \hat{L}^{4/3}$ and $\omega \in \hat{B}^2 \cap \hat{L}^2(\Omega^+)$. So $H' = H$ and $\operatorname{Im} d_A^+ = H^\perp$, as asserted.

For (iii) we consider the operator $\pi \circ d_A^+ \circ P \colon H^\perp \to H^\perp$ where π is projection to $H^\perp$. This differs from the identity by a compact operator so its kernel and cokernel have the same dimension d.

Let $\omega_1, \ldots, \omega_d$; $d_A^+ \alpha_1, \ldots, d_A^+ \alpha_d$ be a basis for the kernel and cokernel and set $T = (\Pi d_A^+ P) + \Sigma(\omega_i, \ )(d_A^+ \alpha_i)$. Then $T \colon H^\perp \to H^\perp$ is an isomorphism, and

$$Q_A = \left(P|_{H^\perp} + \sum (\omega_i, \quad) \alpha_i \right) T^{-1}$$

gives a map from $H^\perp$ to $\hat{B}^1(\mathfrak{g}_E)$ with $d_A^+ Q_A = 1$.

(iv) Index theory

Let A be a $\hat{B}$ connection on a bundle over a quasiconformal 4-manifold X with bounded conformal structure. In the smooth situation one introduces an elliptic operator:

(4.9) $$d_A^* + d_A^+ \colon \Omega^1(E) \to \Omega^0(E) \oplus \Omega^+(E).$$

This is defined for any smooth connection; if A is anti-self-dual the index of this operator is minus the Euler characteristic of the elliptic complex:

(4.10) $$\Omega^0(E) \xrightarrow{d_A} \Omega^1(E) \xrightarrow{d_A^+} \Omega^+(E).$$

We will now develop a different approach for the quasiconformal situation—avoiding the d^* operator.

Extending the operator Q_A of Theorem 4.8 to $\hat{L}^2(\Omega^+(E))$ by projection we have:

(4.11) $$\begin{tikzcd} \hat{B}^0(E) \arrow[r, "d_A"] & \hat{B}^1(E) \arrow[r, "d_A^+"] & \hat{L}^2(\Omega^+(E)) \arrow[l, bend left, "Q_A"] \end{tikzcd}$$

with $d_A^+ Q_A$ the projection to $H^\perp$. Let:

$$\delta_A = d_A - Q_A F_A^+ : \hat{B}^0(E) \to \hat{B}^1(E), \tag{4.12}$$

so $d_A^+ \delta_A = 0$. Then we have a complex:

$$\hat{B}^0(E) \xrightarrow{\delta_A} \hat{B}^1(E) \xrightarrow{d_A^+} \hat{L}^2(\Omega^+(E)). \tag{4.13}$$

(This depends of course on the choice of Q_A.) Let $H^i_{A,Q}$ be the cohomology groups of this complex.

PROPOSITION 4.11. *The image of δ_A is closed in $\hat{B}^1(E)$ and the cohomology groups $H^i_{A,Q}$ are finite dimensional.*

Proof. First notice that we have proved $H = H^2_A = H^2_{A,Q}$ is finite dimensional in Theorem 4.8. To see that $H^0_{A,Q} = \operatorname{Ker} \delta_A$ has finite dimension, consider a sequence u_i in $H^0_{A,Q}$ with $\|u_i\|_{\hat{B}^0} = 1$. Then:

$$d_A u_i = Q_A F_A^+(u_i)$$

is bounded in $\hat{L}^4$ and, by the compact embedding $\hat{L}^4 \hookrightarrow C^0$, we can suppose u_i convergent in C^0 to u_∞. Then $d_A u_i \to Q_A F_A^+ u_\infty$ in $\hat{L}^4$ and it follows that u_∞ lies in $\hat{B}^0$ and Ker δ_A. Moreover u_i converges to u_∞ in $\hat{B}^0$. Thus $H^0_{A,Q}$ is finite dimensional.

Next, choose a closed complementary subspace T in $\hat{B}^0$ to $\operatorname{Ker} \delta_A$, so

$$\delta_A|_T : T \to \hat{B}^1(E)$$

is an injection. We claim that $\|\delta_A t\| \geq c\|t\|$ for some $c > 0$ and all t in T. For, if not, there would be a sequence t_i in T with $\|\delta_A t_i\|_{\hat{B}^1} \to 0$, $\|t_i\|_{\hat{B}^0} = 1$ and, arguing just as above, we would find a non-zero limit t_∞ in T with $\delta_A t_\infty = 0$ giving a contradiction. Thus $\delta_A T = \operatorname{Im} \delta_A$ is complete and so also closed in $\hat{B}^1$.

Finally, we prove that $H^1_{A,Q}$ is finite dimensional. Suppose φ_i is a sequence in $\operatorname{Ker} d_A^+ \subset \hat{B}^1(E)$ and $\|\varphi_i\|_{\hat{B}^1} = 1$. Choose a cover of X by coordinate patches (U_α, ψ_α) and functions β_α supported in $\psi_\alpha U_\alpha$ with $\Sigma \beta_\alpha^2 = 1$. Then if $\varphi_{i,\alpha} = \beta_\alpha \cdot \psi_i$ we have, in local coordinates

$$d_\mu^+ \varphi_{i,\alpha} = (-\beta_\alpha A^+ + \nabla \beta_\alpha) \varphi_i.$$

According to Proposition 3.16 we can find $u_{i,\alpha}$ such that if $\varphi'_{i,\alpha} = \varphi_{i,\alpha} + du_{i,\alpha}$

$$\|\varphi_{i,\alpha}\|_{\hat{B}^1} \lesssim \|d_\mu^+ \varphi_{i,\alpha}\|_{\hat{L}^2}$$

and by our compactness results (3.17) we can suppose the $\varphi'_{i,\alpha}$ converge in $\hat{L}^4$. Similarly we way suppose that the $u_{i,\alpha}$ converge in C^0.

Now put

$$\varphi'_i = \varphi_i + \delta_A\left(\sum_\alpha \beta_\alpha u_{i,\alpha}\right).$$

Then

$$\varphi'_i = \varphi_i + \sum_\alpha \beta_\alpha du_{i,\alpha} + \sum \beta_\alpha A_\alpha u_{i,\alpha} + \sum (\nabla\beta_\alpha) u_{i,\alpha} + Q_A F_A^+\left(\sum_\alpha \beta_\alpha u_{i,\alpha}\right)$$

and the last three terms converge in $\hat{L}^4$. But

$$du_{i,\alpha} = \varphi'_{i,\alpha} - \beta_\alpha \varphi_i \quad \text{and} \quad \sum \beta_\alpha^2 = 1,$$

so

$$\varphi'_i - \sum_\alpha \beta_\alpha \varphi'_{i,\alpha}$$

converges in $\hat{L}^4$, hence also the φ'_i. Finally $d_A^+ \varphi'_i = d_A^+ \varphi_i = 0$ so, locally, $d_\mu^+ \varphi'_i = -A^+ \varphi_i$ converges in $\hat{L}^2$ and φ'_i converge in $\hat{B}^1$. This proves that $H^1_{A,Q}$ is finite dimensional.

Let us now consider an abstract set up—a family of chain complexes:

$$V_0 \xrightarrow{D_t} V_1 \xrightarrow{D_t} V_2 \tag{4.12}$$

parametrised by a connected space T, with the following properties:

(i) $D_t^2 = 0$ for all t in T.

(ii) $D_t(V_0)$ is closed in V_1 and the cohomology groups H_t^i are finite dimensional.

(iii) The D_t vary continuously (in the operator norm topology) with t.

(iv) For each t there is a bounded right inverse $Q_t : D_t V_i \to V_1$.

In this context we can define an index of the family of complexes in $KO(T)$—formally equal to the "Euler characteristic"

$$H_t^0 - H_t^1 + H_t^2.$$

(This is a special instance of a theory developed by Segal [20].) This index can be defined as follows: for any compact subset $T' \subset T$ we "stabilise" the complex by choosing:

$$\tilde{V}_1 = V_1 \oplus \mathbf{R}^n$$
$$\tilde{V}_0 = V_0 \oplus \mathbf{R}^m$$
$$\tilde{D}_t = D_t \oplus \psi_t \colon V_1 \to V_2$$

and

$$\tilde{D}_t = D_t \oplus \chi_t \colon \tilde{V}_0 \to \tilde{V}_1$$

in such a way that the $\tilde{D}_t$ complex has no cohomology in dimension 1 or 2, for t in T'. Then $\operatorname{Ker} \tilde{D}_t \subset \tilde{V}_0$ yield a vector bundle over T' and the index is defined to be:

$$\operatorname{ind}(D_t) = \{\operatorname{Ker} \tilde{D}_t\} - \mathbf{R}^m + \mathbf{R}^n.$$

This is independent (as an element of $KO(T')$) of the choices made in the stabilisation. Simple linear algebra gives:

$$\dim H_t^0 - \dim H_t^1 + \dim H_t^2 = \dim(\operatorname{ind} D_t)$$

so in particular this Euler characteristic is independent of the point t in T.

In our application we define the integer:

$$i(E) = \dim H^1_{A,Q} - \dim H^0_{A,Q} - \dim H^2_{A,Q}. \tag{4.13}$$

This depends, a priori, on the conformal structure of X, the connection A and E and the choice of right inverse Q_A; but we can prove that it is in fact independent of these choices by appealing to the theory above. First, if Q_0 and Q_1 are two choices for the inverse the linear family

$$Q_s = sQ_1 + (1-s)Q_0, \quad s \in [0,1]$$

interpolates between them and the family $d_A - Q_s F_A^+$ is continuous in operator norms. To handle variations in A we introduce a stabilisation in the manner above. If A_0, A_1 are two connections, joined by a path A_s we choose

$$\psi_s \colon \mathbf{R}^{\mathscr{C}} \to \hat{L}^2(\Omega^+(E))$$

such that

$$d^+_{A_s} \oplus \psi_s \colon \hat{B}^1 \oplus \mathbf{R}^{\mathscr{C}} \to \hat{L}^2$$

is surjective for every s. We can then find a continuous family of right inverses Q_s with:

$$(d^+_{A_s} \oplus \psi_s) \circ \tilde{Q}_s = 1.$$

So the Euler characteristics of the complexes $d_{A_s} - \tilde{Q}_s F^+_{A_s}, d^+_{A_s} \oplus \psi_s$ are constant in s. On the other hand when $s=0,1$ one easily sees that these agree with the definitions of $-i(E)$ made using A_0, A_1 respectively. Similarly, to see that $i(E)$ is independent of the bounded conformal structure $[g]$ on X we first observe that the d^+_A operators vary continuously in operator norm with $[g]$ (identifying the $\hat{L}^2(\Omega^+)$ spaces in the familiar way). Then, after stabilisation, we can find a continuous family of inverses Q and fit into the framework above.

(v) Moduli spaces

Let $\mathscr{A}$ be the space of $\hat{B}$ connections on a bundle $E \to X$ and $\mathscr{G} = \operatorname{Aut} E$ the $\hat{B}$ gauge group, as above. We let $\mathscr{B} = \mathscr{A}/\mathscr{G}$ be the quotient space, with the quotient topology.

LEMMA 4.13. *$\mathscr{B}$ is a Hausdorff space.*

Proof. The topology on $\mathscr{B}$ is induced from a $\mathscr{G}$-invariant metric on $\mathscr{A}$ so it suffices to show that the $\mathscr{G}$ orbits are closed. If $g_i(A) \to B$ in $\mathscr{A}$ then $d_A g_i$ is a bounded sequence in $\hat{L}^4$ and we can suppose the g_i converge in C^0 to a limit g_∞. Then

$$d_A g_i \to A g_\infty - B g_\infty$$

so $d_A g_\infty$ exists (distributionally), lies in $\hat{L}^4$ and $g_\infty(A) = B$.

We can describe the local structure of $\mathscr{B}$, using the implicit function theorem in Banach spaces, beginning with the linear theory of §4(iii),(iv). For this we should replace the bundle E by the bundle of Lie algebras $\mathfrak{g}_E$ associated to the structure group. First, just as in the smooth situation, the stabiliser subgroups $\Gamma_A \subset \mathscr{G}$ of connections A in $\mathscr{A}$ are compact Lie groups. The Lie algebra of Γ_A is $\operatorname{Ker} d_A \subset \hat{B}^0(\mathfrak{g}_E)$ and the argument of Proposition 4.11 shows that this is finite dimensional. Likewise $\operatorname{Im} d_A \subset \hat{B}^1(\mathfrak{g}_E)$ is closed. Now in the smooth action

$$\mathscr{G} \times \mathscr{A} \to \mathscr{A}$$

the derivatives at a point $(1_{\mathscr{G}}, A)$ is given by

$$(a, u) \to a - d_A u$$

for $u \in \hat{B}^0(\mathfrak{g}_E)$—the Lie algebra of the gauge group (cf. Proposition 4.6). It follows then, given only the abstract Banach manifold set-up, that the orbit of A is a submanifold of $\mathscr{A}$ with tangent space $\operatorname{Im} d_A$ at A.

LEMMA 4.14. *For each connection A in $\mathscr{A}$ there exists a subspace $T_A \subset T\mathscr{A}_A \cong \hat{B}^1(\mathfrak{g}_E)$ transverse to* $\operatorname{Im} d_A$ *(that is with $T\mathscr{A} \cong T_A \oplus \operatorname{Im} d_A$ as topological vector spaces).*

Proof. Let Q_A be a right inverse to d_A^+, as in § 4 (iii). Then

$$\operatorname{Im} Q_A = \operatorname{Ker}(\varphi \mapsto \varphi - Q_A\, d_A^+\, \Phi)$$

is closed in $\hat{B}^1(\mathfrak{g}_E)$. We claim that the intersection of the closed subspaces $\operatorname{Im} d_A$, $\operatorname{Im} Q_A$ is finite dimensional. For

$$\alpha_i = d_A\, u_i = Q_a\, \psi_i \in \operatorname{Im} d_A \cap \operatorname{Im} Q_A$$

with $\|\alpha_i\|_{B^1} = 1$, the compact embedding $\hat{B}^0 \hookrightarrow C^0$ allows us to suppose that the u_i converge in C^0. Then $d_A^+\, \alpha_i = [F^+, u_i] = \psi_i$ is $\hat{L}^2$ convergent, so $\alpha_i = Q_A(\psi_i)$ is $\hat{B}$ convergent. Similarly, if

$$p : \hat{B}^1(\mathfrak{g}_E) \to \hat{B}^1(\mathfrak{g}_E)/\operatorname{Im} Q_A$$

is the projection map

$$p(\operatorname{Im}(d_A + Q_A\, F^+)) = \operatorname{Im}(p(d_A + Q_A\, F^+)$$

has finite codimensions by Proposition 4.11. But QF^+ is a compact operator so

$$p(\operatorname{Im} d_A)$$

is also of finite codimension. Hence $\operatorname{Im} d_A + \operatorname{Im} Q_A$ has finite codimension in $\hat{B}^1(\mathfrak{g}_E)$ and we can modify $\operatorname{Im} Q_A$ by finite rank changes to achieve the desired transversal.

Given these transversals it is straightforward Banach space differential topology to construct local models for $\mathscr{B}$, just as in the smooth situation (when we can take the standard transversal $T_A = \operatorname{Ker} d_A^*$). We call a connection A *irreducible* if Γ_A is equal to the centre of G; then we have:

PROPOSITION 4.15. *If A is an irreducible connection the restriction of the projection map:*

$$A + T_A \to \mathscr{B}$$

gives a homeomorphism from a neighbourhood of the origin in T_A to a neighbourhood of $[A]$ in $\mathscr{B}$, and these give charts making the space $\mathscr{B}^*$ of irreducible connections into a Banach manifold.

At reducible connections we have to modify the description to take account of the Γ_A action. We choose a Γ_A invariant transversal T_A; then a neighbourhood of $[A]$ is modelled on T_A/Γ_A.

Finally, to complete the abstract Banach manifold picture, we consider the *moduli space* $M \subset \mathscr{B}$. By definition this is the set of equivalence classes of anti-self-dual connections.

PROPOSITION 4.16. *If A is an anti-self-dual connection a neighbourhood of $[A]$ in $M \subset \mathscr{B}$ is represented in the local model by $\psi^{-1}(0)/\Gamma_A$ where ψ is a smooth, Γ_A-equivariant, Fredholm map from a neighbourhood of 0 in T_A to $\hat{L}^2(\Omega^+(\mathfrak{g}_E))$. The Fredholm index of ψ is $i(\mathfrak{g}_E) - \dim \operatorname{Ker} d_A$.*

Proof. By definition M is given locally by the solutions of

$$F^+(A+a) = d_A^+ a + (a \wedge a)^+ = 0, \quad \text{for } a \text{ in } T_A.$$

This expression represents a smooth map (cf. Proposition 4.7) and the index is

$$\operatorname{ind}(d_A^+|_{T_A}) = i(\mathfrak{g}_E) - \dim \operatorname{Ker} d_A.$$

To sum up we have duplicated the essential parts of the usual description of the moduli space in the quasiconformal situation. For example we know that the $\mathscr{G}$-equivalence classes of irreducible anti-self-dual connections A with $H_A^2 = 0$ are parametrised by a smooth manifold of dimension $i(\mathfrak{g}_E)$. More abstractly we can set up the anti-self-dual equations $F_A^+ = 0$ as the zeros of a Fredholm section (with index $i(\mathfrak{g}_E)$) of a Banach space bundle over $\mathscr{B}^*$; with the usual extension to reducible connections.

§ 5. Index calculation

(i) Index formula

In § 4 (iv) we have defined for every $\hat{B}$ vector bundle $E \to X$ over a compact quasiconformal 4-manifold X an integer $i(E)$. In this section we want to prove that

$$i(E) = -(2p_1(E) + \operatorname{rank}(E)(1 - b_1(X) + b_2^+(X)). \tag{5.1}$$

Here $p_1(E) \in H^4(X; \mathbf{Z}) \cong \mathbf{Z}$ is the Pontryagin class, $b_1(X)$ is the first Betti number and $b_2^+(X)$ is the rank of a maximal positive subspace for the cup product form on $H^2(X)$. In the smooth situation this formula follows from the Atiyah-Singer index theorem ([2]), and in the Lipschitz case could be deduced from the work of Teleman [29]. We need to show that the same formula is valid in the quasiconformal setting. The general scheme of our proof—an excision argument to reduce to easily calculable cases—is a very familiar one, so we will pass quickly over some details.

(ii) Hodge theory

The discussion of § 4 applies, a fortiori, in the case when A is the product connection on $E=X\times\mathbf{R}$. Just as in the smooth case we have, for α in $\hat{B}^1$;

$$\int_X |d^+\alpha|^2-|d^-\alpha|^2\,d\mu=\int_X d\alpha\wedge d\alpha=0. \tag{5.2}$$

So $\operatorname{Ker} d^+/\operatorname{Im} d=\operatorname{Ker} d/\operatorname{Im} d=H^1(X;\mathbf{R})$ by the de Rham theorem 4.2, also we clearly have $(\operatorname{Ker} d\subset B^0)\cong H^0(X;\mathbf{R})$. To verify formula (5.1) in this case $(p_1(E)=0,\operatorname{rank}(E)=1)$ it suffices to show that the dimension of the second cohomology group of the complex

$$\hat{B}^0\xrightarrow{d}\hat{B}^1\xrightarrow{d^+}\hat{L}^2(\Omega^+)$$

is $b_2^+(X)$.

LEMMA 5.3. *There is a natural inclusion* $H^+\subset H^2(X;\mathbf{R})$ *of* $H^+\cong\operatorname{Coker} d^+\cong\operatorname{Ker} d\cap\hat{L}^2(\Omega^+)$ *as a maximal positive subspace for the cup product form.*

Proof. We know that forms in H^+ are closed so there is a natural map $i: H^+\to H^2(X;\mathbf{R})$. i is injective since, by (5.2),

$$\omega\in H^+,\ \omega=d\alpha \;\Rightarrow\; d^-\alpha=0 \;\Rightarrow\; d^+\alpha=0 \;\Rightarrow\; \omega=0.$$

Furthermore for ω in H^+

$$\int_X \omega\wedge\omega=\int_X|\omega|^2\,d\mu$$

so H^+ is a positive subspace. Symmetrically we have a negative subspace H^-.

It remains to prove that $H^2(X;\mathbf{R})=H^+\oplus H^-$ i.e. that for any closed form ω in $\hat{B}^2$ we can find α in $\hat{B}^1$ such that

$$\omega+d\alpha=\omega_++\omega_-$$

with $\omega_\pm$ in $H^\pm$. But we know that $\hat{L}^2(\Omega^+)=(\operatorname{Im} d^+)\oplus H^+$, so we can find α such that

$$\omega_+ + d^+ a = \omega$$

with ω_+ in H^+. Then

$$\omega + da - \omega_+$$

is closed and anti-self-dual as required.

(iii) Connected sums

Suppose $E_1 \to X_1, E_2 \to X_2$ are bundle of the same rank over quasiconformal 4-manifolds. Then there is an obvious notion of a "connected sum" bundle E over the manifold $X = X_1 \# X_2$. In this subsection we shall establish the following formula (which serves as our version of the Atiyah-Singer "excision axiom" [3]).

PROPOSITION 5.4. $i(E) = i(E_1) + i(E_2) + \mathrm{rank}(E)$.

The remainder of this section consists of a proof of this formula. We begin by introducing some notation.

We can suppose that the 4-manifolds X_1, X_2 have flat conformal structures in neighbourhoods of points x_1, x_2; and we use Euclidean co-ordinates ξ_1, ξ_2 in these neighbourhoods. We write $B_i(\varrho)$ for the ϱ-ball in X_i about x_i defined via these co-ordinates. Suppose, for simplicity, that the bundles E_1, E_2 have anti-self-dual connections A_1, A_2 (this will be the case in our application of Proposition 5.4 below). The idea of the proof if to "cut and paste" bundles and connections using cut-off functions. These cut-off functions depend on three real parameters r, λ, K; where r and λ will be small and K large. These parameters are introduced now.

The parameter r. For each (small) r we fix a connection $\tilde{A}_i$ on E_i which agrees with A_i outside $B_i(2r)$ but is flat in $B_i(r)$. For example we can take

$$\tilde{A}_i = \chi_r A_i$$

where $\chi_r(z) = \chi(|z|/r)$ is a cut-off and the right hand side refers to the connection matrix of A_i in a local trivialisation of E_i. For a suitable choice of this trivialisation we get:

(5.5) $$\|F^+_{\tilde{A}_i}\|_{L^2} \text{ and } \|\tilde{A}_i - A_i\|_{L^4} \text{ are } o(r) \text{ as } r \to 0.$$

The parameter λ. This parameter defines the connected sum $X = X_1 \# X_2$ as a quasiconformal 4-manifold with a conformal structure. Choose an orientation reversing isometry $\xi \mapsto \bar{\xi}$ of $\mathbf{R}^4$ and let f_λ be the map from a punctured neighbourhood of x_1 in X_1 to the corresponding punctured neighbourhood of x_2, defined in coordinates by:

$$f_\lambda(\xi_1) = \frac{\lambda}{|\xi_1|^2}\bar{\xi}_1. \tag{5.6}$$

The connected sum X is defined by removing small balls, $B_i(\lambda)$ say, from X_i and identifying the remaining manifolds $U_i = X_i \setminus B_i(\lambda)$ by f_λ. We regard U_i as common open subsets of X_i and X and will make a number of obvious abuses of notation: for example we regard a compactly supported function on U_i as being simultaneously a function on X and X_i. Similarly we extend a function on U_i which is constant outside a compact set to X and X_i and we will not distinguish between these functions.

Let $X_i' = X_i \setminus B_i(\frac{1}{2}\lambda^{1/2})$, so $X = X_1' \cup X_2'$ and $X_1' \cap X_2'$ is an annulus. To define the $\hat{L}^p$ spaces on X we can introduce a metric on the "neck" region in the connected sum. We will not need detailed formulas: the important point is that there is a constant c (independent of λ) and a choice of $\hat{L}^{4/n}$-norm on $\hat{B}^n_X$ with the two properties: for a form α supported on X_i'

$$c^{-1}\|\alpha\|_{L^{4/n}(X_i)} \leq \|\alpha\|_{\hat{L}^{4/n}(X)} \leq c\|\alpha\|_{\hat{L}^{4/n}(X_i)}; \tag{5.7a}$$

for any form α supported on the common open set U_i we have

$$\|\alpha\|_{\hat{L}^{4/n}(X_i)} \leq c\|\alpha\|_{\hat{L}^{4/n}(X)}. \tag{5.7b}$$

This just reflects the fact that f_λ increases distances in the set $|\xi| \leq \lambda^{1/2}$ but distorts distances by only a bounded factor on the annulus $\frac{1}{2}\cdot\lambda^{1/2} \leq |\xi_1| \leq 2\lambda^{1/2}$.

The parameter K. For $K > 1$ let $\mathfrak{A}_K$ be the annulus

$$\mathfrak{A}_K = \{z \in \mathbf{R}^4 \mid K^{-1} \leq |z| \leq K\}.$$

The fundamental fact we will exploit is the existence on $\mathfrak{A}_K$ of functions χ_K with:

(i) $\chi_K(z) = 0$ when $|z| = K^{-1}$

(ii) $\chi_K(z) = 1$ when $|z| = K$

(iii) $\|\nabla\chi_K\|_{L^4} \to 0$ as $K \to \infty$.

For example we can take:

$$\chi_K(z) = \frac{\log|z| + \log K}{2\log K}.$$

(This leads to the notion of the conformal "modulus" of an annulus [32], and is bound up with the failure of the Sobolev embedding $L_1^p \to C^0$ at the critical exponent $p=4$.)

Now for given K, with

$$K^3\lambda^{1/2} \leq \frac{1}{4} r \tag{5.8}$$

we let θ_1 be the function.

$$\theta_1(\xi) = \chi_K\left(\frac{\xi}{K^2\lambda^{1/2}}\right), \tag{5.9}$$

extended by constants outside the domain of definition. Thus θ_1 is a function on X_1 (or X) which is equal to 1 outside the ball $B_1(K^3\lambda^{1/2})$ and to zero on $B_1(K\lambda^{1/2})$. The derivative of θ_1 is supported in an annulus conformal to $\mathfrak{A}_K$, and, by conformal invariance:

$$\|d\theta_1\|_{L^4} \to 0 \quad \text{as} \quad K \to \infty. \tag{5.10}$$

Similarly define a cut-off θ_2 on X_2 and X.

We will now introduce some more cut-off functions depending on the parameters above. First, regarding θ_2 as a function on X, put

$$\varphi_1 = 1 - \theta_2 : X \to \mathbf{R}. \tag{5.11}$$

By our conventions φ_1 is simultaneously a function on X and X_1, with compact support in U_1. Like θ_1, φ_1 is a cut-off function equal to 1 on "most" of X_1 but the support of φ_1 is larger, in particular:

$$\varphi_1 = 1 \quad \text{on} \quad X_1 \setminus B_1(\lambda^{1/2}). \tag{5.12}$$

Similarly we define φ_2.

Next, let γ_1 be a cut-off function on X_1 equal to 1 on $X_1 \setminus B_1(2\lambda/r)$ and to 0 on $X_1 \setminus B_1(\lambda/r)$. Thus the support of $d\gamma_1$ is contained in a very small annulus in X_1, which is mapped by f_λ to the annulus:

$$\frac{r}{2} \leq |\xi_2| \leq r$$

in X_2. This means that the $\hat{L}^4$ norm of $d\gamma_1$ is estimated by the $\hat{L}^4$ norm measured in X_2 (by (5.7b)) so we can suppose, regarding γ_1 as a function on X, that:

$$\|d\gamma_1\|_{\hat{L}^4(X)} \text{ is independent of } K, \lambda. \tag{5.13}$$

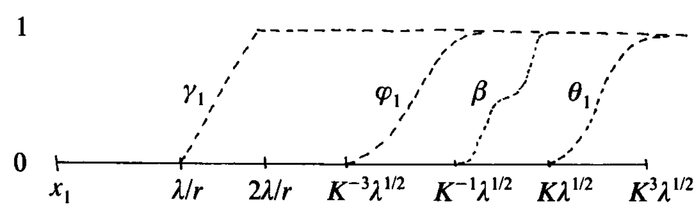

Diagram 5.14

Define γ_2 similarly. Finally, let $\beta: X \to \mathbf{R}$ be a function equal to 0 outside X_2' and to 1 outside X_1', and let $\psi = 1-(\theta_1+\theta_2)$. The various cut-off functions we have defined are summarised in Diagram 5.14.

We will now explain the significance of the term "rank E" in the formula of Proposition 5.4. For any identification between the fibres $(E_i)_{x_i}$ of the two bundles we can form a connection A on a bundle E over X by gluing the flattened connections $\tilde{A}_i$ over the neck. For symmetry, we will regard this identification as an identification of each fibre $(E_i)_{x_i}$ with a fixed space V. So sections of E over the neck region can be viewed as V-valued functions. We define a map:

(5.15) $$j: V \to \operatorname{Ker} d_A^+$$

by letting $j(v) = d_A(\beta v)$ on the neck region and extending by zero over the rest of X. For any right inverse Q we get a corresponding cohomology class

$$[j(v)] \in \operatorname{Ker} d_A^+/\operatorname{Im}(d_A - QF_A^+),$$

and this cohomology class is independent of the particular choice of cut-off function β. So we have a canonical map:

(5.16) $$i: V \to H^1_{A,Q}$$

with $i(v) = [j(v)]$. Proposition 5.4 asserts, roughly speaking, that the cohomology of A is made up of the sum of cohomologies of A_1, A_2 and the image of i. The precise statement is simplest in the case when the A_i are acyclic.

PROPOSITION 5.17. *Suppose* $H^p_{A_1}$ and $H^p_{A_2}$ *are zero for* $p=0, 1, 2$. *Then for a suitable choice of parameters* r, λ, K *we have:*

(i) d_A^+ *is surjective i.e.* $H_A^2 = 0$.

(ii) *There is a right inverse* Q *for* d_A^+ *such that* $H^0_{A,Q} = 0$ *and* $i: V \to H^1_{A,Q}$ *is an isomorphism.*

This immediately implies Proposition 5.4, in the acyclic case. To give the corresponding statement in the general case we proceed as follows. For any harmonic form ω with:

$$\omega \in \hat{L}^2(\Omega_X^+(E_1)), \quad d_A\omega = 0$$

we multiply by the cut-off function θ_1 to get a form $\theta_1\omega$ which we can regard as an element of $\hat{L}^2(\Omega_X^+(E))$; similarly for the A_2 harmonic forms. This gives us a map:

$$k : H^2_{A_1} \oplus H^2_{A_2} \to \hat{L}^2(\Omega_X^+(E)). \tag{5.18}$$

Similarly, pick open sets $G_i \subset X_i$ in the region where the metrics are smooth but not meeting the neck region in the connected sum (for small enough r). Then we can define

$$p : \hat{B}_X^0(E) \to H^0_{A_1} \oplus H^0_{A_2}. \tag{5.19}$$

by L^2 projection over the G_i, i.e.

$$\langle p(s), u \rangle = \left(\int_{G_1} (s, u)\, d\mu, \int_{G_2} (s, u)\, d\mu \right). \tag{5.20}$$

Now for any right inverse Q for d_A^+ we have a diagram

$$\begin{tikzcd}
0 & & 0 \\
H^0_{A_1} \oplus H^0_{A_2} \arrow[u] & & \hat{L}^2(\Omega^+)/\operatorname{Im} k \arrow[u] \\
\hat{B}^0 \arrow[u] \arrow[r, "\delta_A"] & \hat{B}^1 \arrow[r, "d_A^+"] \arrow[ur] & \hat{L}^2(\Omega^+) \arrow[u] \\
\operatorname{Ker} p \arrow[u] \arrow[ur] & & H^2_{A_1} \oplus H^2_{A_2} \arrow[u] \\
0 \arrow[u] & & 0 \arrow[u]
\end{tikzcd} \tag{5.21}$$

Here the columns are exact and the diagonal maps yield another Fredholm complex. It is then easy to prove abstractly that the Euler characteristic of the diagonal complex

$$\operatorname{Ker} p \to \hat{B}^1 \to \hat{L}^2(\Omega^+)/\operatorname{Im} k$$

is

$$-i(E) - \dim(H^0_{A_1} \oplus H^0_{A_2}) - \dim(H^2_{A_1} \oplus H^2_{A_2}).$$

So Proposition 5.4 is a consequence of the following generalisation of Proposition 5.17:

PROPOSITION 5.22. *For a suitable choice of r, λ, K there is a right inverse Q for d_A^+ such that the cohomology of the diagonal complex* $\operatorname{Ker} p \to \hat{B}^1 \to \hat{L}^2(\Omega)/\operatorname{Im} k$ *is zero in dimensions* 0, 2 *and in dimension* 1 *is isomorphic to $H^1_{A_1} \oplus H^1_{A_2} \oplus V$.*

For simplicity we will prove Proposition 5.17; the proof of Proposition 5.22 is the same in all essentials. Note however that there is one case where we can see explicitly how the cohomology groups behave and verify Proposition 5.22: when A_i are the flat connections on the trivial bundles over copies S^4_1, S^4_2 of S^4. This observation will be important in our proof of Proposition 5.17.

Proof of Proposition 5.17.

Step 1. *Construction of Q.* Let Q_1, Q_2 be right inverses for $d^+_{A_1}$, $d^+_{A_2}$ respectively. Given a form f in $\Omega^+_X(E)$ we write $f = f_1 + f_2$ where f_i is the restriction of f to $X_i \setminus B_i(\lambda^{1/2})$. Then define:

$$P : \hat{L}^2(\Omega^+_X(E)) \to \hat{B}^1_X(E)$$

by

$$P(f) = \varphi_1 Q_1(f_1) + \varphi_2 Q_2(f_2).$$

Then

$$d_A^+ P(f) = \sum_i \varphi_i f_i + [d\varphi_i + (\tilde{A}_i - A_i)] Q_i(f_i).$$

Now $\varphi_1 f_1 + \varphi_2 f_2 = f$ so:

$$\|d_A^+ P(f) - f\|_{\hat{L}^2} \leq \sum_i \|d\varphi_i + (\tilde{A}_i - A_i)\|_{L^4} \|Q_i(f_i)\|_{\hat{L}^4}$$

$$\leq \Big(\sum_i \|d\varphi_i\|_{L^4} + \|\tilde{A}_i - A_i\|_{L^4}\Big) \|f\|_{\hat{L}^2}.$$

Here we have used (5.7) to compare the norms on X and X_i. Now by making r (and so $(\tilde{A}_i - A_i)$ small, cf. (5.5)), and K large (hence $\|d\varphi_i\|_{L^4}$ small, cf. (5.10)) we can arrange that $d_A^+ P - 1$ is a contraction on $\hat{L}^2$. (This will require that λ be made small, to maintain

(5.8)). Then d_A^+ is surjective, as asserted in (5.7 a), and we define the right inverse Q to be:

$$Q = P(d_A^+ P)^{-1}.$$

Notice that we get uniform bounds on the $(\hat{L}^2, \hat{L}^4)$ operator norm of Q.

Step 2. *i is injective*. The hypothesis that $H^0_{A_i}=0$ and the $\hat{L}^4_1$ Sobolev embedding give, for sections t_i of E_i over $X_i' \subset X_i$:

$$\|t_i\|_{C^0(X_i')} < \|d_A t_i\|_{\hat{L}^4(X_i')} \tag{5.23}$$

with bounds independent of r and λ (cf. [6] p. 314). Suppose $i(v)=0$ for some v in V, so

$$j(v) = d_A t - QF_A^+(t)$$

for some section t to E. Let t_i be the restriction of t to X_i. So that, on X_1',

$$d_A((1-\beta)v+t_i) = Q(F_A^+(t))|_{X_1'},$$

and on X_2',

$$d_A(\beta v+t_2) = Q(F_A^+(t))|_{X_2'}.$$

Then, by (5.23),

$$\|(1-\beta)v+t_1\|_{C^0(X_1)} \lesssim \|F_A^+\|_{\hat{L}^2} \|t\|_{C^0(X)}$$

and

$$\|\beta v+t_2\|_{C^0(X_2)} \lesssim \|F_A^+\|_{\hat{L}^2} \|t\|_{C^0(X)}.$$

Now choose r and hence $\|F_A^+\|_{\hat{L}^2}$ so small, using (5.5), that these inequalities give:

$$\|(1-\beta)v+t_1\| \leqslant \frac{1}{4}\|t\|,$$

$$\|\beta v+t_2\| \leqslant \frac{1}{4}\|t\|,$$

say. Over the annulus $X_1 \cap X_2$ in X we get:

$$|v| = \|\beta v+(1-\beta)v\| \leqslant \frac{1}{2}\|t\|$$

so, substituting back,

$$\|t_i\| \leq \frac{3}{4}\|t\|$$

and, since t_i is the restriction of t to $X_i \setminus B_i(\sqrt{\lambda})$ and these are two sets cover X, we must have $t=v=0$.

Step 3. $H^0_{A,Q}=0$.

The proof follows that of Step 2 above exactly.

Step 4. *i is surjective*. We will define a map T from $\operatorname{Ker} d_A^+ \subset \hat{B}^1_X(E)$ to itself such that $\|T(\alpha)\|_{\hat{L}^4} \leq \frac{9}{10}\|\alpha\|_{\hat{L}^4}$ and $T(\alpha)$ is equivalent to α modulo $\operatorname{Im}(d_A - QF_A^+) + \operatorname{Im} j$. So if $\alpha^{(n)} = T^n\alpha$, $\alpha^{(n)}$ tends to zero in $\hat{L}^4$ as n tends to infinity, but for all n, $\alpha^{(n)}$ defines the same class in $H^1_{A,Q}/\operatorname{Im} i$. Since $\operatorname{Im}(d_A - QF_A^+) + \operatorname{Im} j$ is *closed* in $\hat{L}^4$ we deduce that i is surjective, as required.

Suppose $d_A^+\alpha=0$; we define $T(\alpha)$ by splitting α into three pieces:

$$\alpha = \theta_1\alpha + \theta_2\alpha + \psi\alpha$$

supported on X_1, X_2 and the neck region respectively.

Consider first $\theta_1\alpha$. We have

$$d^+_{A_1}(\theta_1\alpha) = (d\theta_1)\alpha + (A_1 - \tilde{A}_1)(\theta_1\alpha),$$

so by making r small and K large, as in Step 1, we can make $\|d^+_{A_1}(\theta_1\alpha)\|_{\hat{L}^2(X_1)}$ less than an arbitrarily small multiple of $\|\alpha\|_{\hat{L}^4(X)}$. Now since $H^1_{A_1}=0$ we can write:

$$\theta_1\alpha = d_{A_1}u_1 + Q_1\{d^+_{A_1}(\theta_1)\},$$

and we can suppose, by the above, that for $r \leq r_0$, $K \leq K_0$:

$$\|du_1\|_{\hat{L}^4(X_1)} + \|u_1\|_{C^0(X_1)} \leq C\|\alpha\|_{\hat{L}^4(X)}$$

for a constant C depending only on A_1. Now let $v_1 = u_1(x_1)$ in V: by the equicontinuity of $\hat{L}^4_1$ functions (2.39) we have

$$|u_1(y) - v_1| \leq \varepsilon(\lambda)\|\alpha\|_{\hat{L}^4(X)}$$

for y in the small ball $B_1(2\lambda/r)$ containing the support of $d\gamma_1$, where $\varepsilon(\lambda)\to 0$ with λ.

Now consider $\theta_1\alpha - j(v)$. We have, on X_1,

$$\begin{aligned}\theta_1\alpha - jv &= d_{A_1}u_1 - d_{\tilde{A}_1}((1-\beta)v) + Q_1\{d^+_{A_1}(\theta_1\alpha)\}\\ &= d_{\tilde{A}_1}[u_1-(1-\beta)v] + R,\end{aligned}$$

where

$$\|R\|_{\hat{L}^4(X)} \lesssim \frac{1}{10}\|\alpha\|_{\hat{L}^4(X)},$$

say, for small r and large K. Put

$$\tilde{u}_1 = \gamma_1(U_1 - (1-\beta)v)$$

so

$$d_A\tilde{u}_1 = \gamma_1 d_{\tilde{A}_1}[u_1 - (1-\beta)v)] + (d\gamma_1)(u_1 - (1-\beta)v).$$

Now, and this is the key point,

$$|u_1 - (1-\beta)v_1| \leq \varepsilon(\lambda)\,\|\alpha\|_{\hat{L}^4}$$

on the support of $d\gamma_1$, and $\|d\gamma_1\|_{\hat{L}^4}$ is independent of λ ((5.13)). So by making λ small, for any fixed r and K, we can make

$$\|(d\gamma_1)(u_1 - (1-\beta)v_1)\|_{\hat{L}^4(X)} \leq \frac{1}{10}\|\alpha\|_{\hat{L}^4(X)},$$

hence

$$\|\theta_1\alpha - jv_1 - d_A\tilde{u}_1\|_{\hat{L}^4(X)} \leq \frac{2}{10}\|\alpha\|_{\hat{L}^4(X)}.$$

Similarly,

$$\|QF^+_A(\tilde{u}_1)\|_{\hat{L}^4(X)} \lesssim \|F^+_A\|_{\hat{L}^2}\,\|\tilde{u}\|_{C^0},$$

so by making r small we can get

$$\|QF^+_A(u_1)\|_{\hat{L}^4(X)} \leq \frac{1}{10}\|\alpha\|_{\hat{L}^4(X)},$$

whence

$$\|\theta_1\alpha - jv_1 - (d_A - QF^+_A)u\|_{\hat{L}^4} \leq \frac{3}{10}\|\alpha\|_{\hat{L}^4}.$$

We treat $\theta_2\alpha$ in just the same way, (using $(1-\beta)$ in place of β) to get v_2, $\bar{u}_2$ with

$$\|\theta_2\alpha-jv_2-(d_A-QF_A^+)\,\bar{u}_2\| \leq \frac{3}{10}\|\alpha\|_{\hat{L}^4}.$$

It remains to deal with the term $\psi\alpha$ supported on the neck. To do this we compare with the model connected sum $S^4=S_1^4\#S_2^4$ and the trivial bundles. Clearly the neck regions in $S_1^4\#S_2^4$ and $X_1\#X_2$ with the same parameters r, λ, K, can be identified. For a form f supported on the neck region in $X_1\#X_2$ we write f^* for the corresponding form on $S_1^4\#S_2^4$, and similarly we have operators Q_i^* etc.

Now we know that the index formula is correct on $S_1^4\#S_2^4$ by elementary de Rham theory (using the fact that in the flat case $d^+\alpha=0$ implies $d\alpha=0$). So we have a decomposition:

$$(\psi\alpha)^* = dw^*+Q^*(d^+(\psi\alpha)^*)$$

where $\|Q^*(d^+(\psi\alpha)^*\|_{L^4}$ is bounded by a multiple of $\|(d\psi)\alpha\|_{\hat{L}^4}$, hence by a small multiple of $\|\alpha\|_{\hat{L}^4}$, for large K. Consider the subset $S_i^4\setminus B_i^*(K^3\lambda^{1/2})$ in S_i^4 where $(\psi\alpha)^*$ is zero, using an obvious extension of (5.23) to the case when there are constant sections we get a bound on the *variation* of w^* over this region. Let w_1^*, w_2^* be the average values of w on these two regions. Changing w by an overall constant we can suppose $w_1^*+w_2^*=0$. On the other hand using j^* (the analogue of j for $S_4^1\#S_4^2$) we can write

$$(\psi\alpha)^* = d\bar{w}^*+j^*(v^*)\,Q^*(d^+(\psi\alpha)^*) \tag{5.24}$$

where the average values are: $\bar{w}_1^*=w_1^*+\frac{1}{2}v^*$, $\bar{w}_2^*=w_2^*-\frac{1}{2}v^*$. So we can suppose in (5.24) that $w_1^*=w_2^*=0$. Then we have:

$$\|\bar{w}^*\|_{C^0(S_i^4\setminus B_i^*(K^3\lambda^{1/2}))} \leq C(r,K)\,\|\alpha\|_{\hat{L}^4},$$

where $C(r, K)$ can be made arbitrarily small by making K large and r small. Now let

$$w=\gamma_1\gamma_2\bar{w}^*;$$

which we regard once again as being defined over X (supported in the neck region). We can estimate the norm of $d_Aw+j(v^*)-(\psi\alpha)$, just as before, to show that for $r\leq r_1$, $K\leq K_1$ and $\lambda\leq\lambda_1(r)$ we have:

$$\|d_Aw+j(v^*)-(\psi\alpha)\|_{\hat{L}^4} \leq \frac{3}{10}\|\alpha\|_{\hat{L}^4}$$

(note that $QF_A^+(w)=0$). Finally, then, put

$$Ta = a - j(v_1 + v_2 + v^*) - (d_A - QF_A^+)(\bar{u}_1 + \bar{u}_2 + w)$$

so that our three previous estimates give

$$\|T(a)\|_{\hat{L}^4} \leq \frac{9}{10} \|a\|_{\hat{L}^4}.$$

This completes our proof of Proposition 5.17 and as we stated above the extensions to cover the cohomology of the A_i, giving Proposition 5.22, are routine. The ideas involved are exactly parallel to those in [6], [26] where the moduli spaces of anti-self-dual connections are given similar local models. The problem here is essentially a linearisation of those moduli problems (in the case when E is a Lie algebra bundle $\mathfrak{g}_E$) and the analytical argument above is a modification of that in [6]. It is worth pointing out that one can also carry through the moduli description in the quasiconformal case—combining the proof here with that of [6]. In particular if a quasiconformal 4-manifold X has a conformal structure which is smooth in some region $\Omega \subset X$ then the "concentrated connections" over X with curvature concentrated in Ω can be described just as in [6] Theorem 5.5. However this description breaks down over the points where the conformal structure is not smooth.

(iv) Proof of (5.1)

We clearly have

$$i(E \oplus F) = i(E) + i(F)$$

so the index i gives a linear map

$$i: KO(X) \to \mathbf{Z}.$$

Now by straightforward algebraic topology

$$KO(X) \otimes Q \cong Q \oplus Q$$

with generators detected by the rank and first Pontryagin class. By the Hodge theory of (ii), (5.1) holds for the trivial bundle so to prove the formula in general it suffices to check it for any bundle E with $p_1(E) \neq 0$. Let E_1 be a non-trivial bundle over S^4 (which we can suppose carries an anti-self-dual connection cf. [2]) and E_2 a trivial bundle over X of the same rank. Then consider $X = S^4 \# X$ and the connected sum bundle $E = E_1 \# E_2$.

We know by the smooth theory, that (5.1) holds for E_1 and then Proposition 5.4 shows that the formula for $i(E)$ is correct.

§ 6. Regularity

(i) Coulomb gauges

The local regularity theory for anti-self-dual solutions is based on the fundamental theorem of Uhlenbeck on the existence of local "Coulomb gauges". Let D be the unit ball in $\mathbf{R}^4$, equipped with the standard Euclidean metric.

PROPOSITION 6.1 (Uhlenbeck [30]). *There are constants $K, c_q>0$ such that if A is an L_1^p connection matrix over D $(p>2)$ with $\|F_A\|_{L^2}<K$ then there is an L_2^p gauge transformation $\mathfrak{g}$ such that $\tilde{A}=g(A)$ satisfies:*

(i) $d^*\tilde{A}=0$,

(ii) $\|\tilde{A}\|_{L_1^q}\leq C_q\|F_A\|_{L^q}$, $2\leq q\leq p$.

The same statement holds true if A is only an L_1^2 connection, but one must take care that the gauge transformation g need not then be continuous. Uhlenbeck deduces this limiting case by an approximation argument ([30]). We want a version of this theorem which applies to $\hat{B}$ connections. These are not covered by the statement in Uhlenbeck's paper but exactly the same argument applies. If A is a $\hat{B}$ connection over D with $\|F_A\|_{L^2}<K$ we can approximate A in $\hat{B}^1$ norm by smooth connections A_ε, $\varepsilon\to 0$, and $\|F_{A_\varepsilon}\|_{L^2}<K$ for small ε. The gauge transforms A_ε given by Proposition 6.1 have:

$$d^*\tilde{A}_\varepsilon=0$$
$$d\tilde{A}_\varepsilon+\tilde{A}_\varepsilon\wedge\tilde{A}_\varepsilon=\tilde{F}_\varepsilon\in\hat{L}^2.$$

For simplicity consider an interior domain $D'\subset\subset D$. Let β be a cut-off function equal to 1 on D', then by our $\hat{L}^p$ elliptic theory, as in § 3,

$$\begin{aligned}\|\beta\tilde{A}_\varepsilon\|_{\hat{L}^4} &< \|(d^*+d)(\beta\tilde{A}_\varepsilon)\|_{\hat{L}^2}\\ &< \|(\nabla\beta)\tilde{A}_\varepsilon\|_{\hat{L}^2}+\|\tilde{F}_\varepsilon\|_{\hat{L}^2}+\|\beta\tilde{A}_\varepsilon\wedge\tilde{A}_\varepsilon\|_{\hat{L}^2}\\ &< \|\tilde{A}_\varepsilon\|_{\hat{L}^4}(1+\|\beta\tilde{A}_\varepsilon\|_{\hat{L}^4})\|\tilde{F}_\varepsilon\|_{\hat{L}^2}.\end{aligned}$$

Now combining the Sobolev embedding $L_1^2\hookrightarrow L^4$ with Proposition 6.1(ii) we see that, if K and hence $\|A_\varepsilon\|_{\hat{L}^4}$ is sufficiently small, the above inequality yields a uniform bound:

$$\|\beta\tilde{A}_\varepsilon\|_{\hat{L}^4} \lesssim \|\tilde{F}_\varepsilon\|_{\hat{L}^2}.$$

Substituting back gives a bound on $\|d(\beta A_\varepsilon)\|_{\hat{L}^2}$. So the βA_ε are bounded in $\hat{B}^1$ and we can find a weakly convergent subsequence:

$$\beta\tilde{A}_{\varepsilon_i} \to \tilde{A} \quad \text{in } \hat{B}^1.$$

It then follows easily that $\tilde{A}$ represents a $\hat{B}^1$ connection matrix over D', B^0-gauge equivalent to A, and with:

$$d^*\tilde{A} = 0$$
$$\|\tilde{A}\|_{\hat{B}^1(D')} \leqslant C\|F_A\|_{\hat{L}^2(D)}.$$

To sum up, we have:

PROPOSITION 6.2 (Uhlenbeck). *There are $k, C>0$ such that if A is a $\hat{B}^1$ connection matrix over D with $\|F_A\|_{L^2}<K$ then there is a $\hat{B}^0$ equivalent $\tilde{A}$ over D' with*:

(i) $d^*\tilde{A} = 0$.

(ii) $\|\tilde{A}\|_{\hat{B}^1(D')} \leqslant C\|F_A\|_{\hat{L}^2(D)}$.

(iii) $\|A\|_{L^2_1(D')} \leqslant C\|F_A\|_{L^2(D)}$.

Remark 6.3. A consequence of this is that a Frobenius theorem holds for $\hat{B}^1$ connections—a connection with curvature zero is locally trivial. In the usual way the flat $\hat{B}^1$ connections, in general, can be identified with representations of the fundamental group.

(ii) Regularity and compactness

Let μ represent a bounded conformal structure on D, with $|\mu|<c$ and A be a μ-anti-self-dual connection matrix over D with $d^*A=0$. We emphasise that the d^* operator here is that defined by the Euclidean metric, not of a metric compatible with μ.

LEMMA 6.4. *If $A \in L^4(D)$, $dA \in L^2(D)$ and if $\|A\|_{L^4}$ is sufficiently small then A is in $L^p_1(D')$ for some $p>2$, and $\|A\|_{L^p_1(D)} \lesssim \|F_A\|_{L^2(D)}$.*

Proof (Compare [13], and Proposition 8.3). Take a cut-off β as above then the anti-self-dual equation gives:

$$(d^* \oplus d^+_\mu)(\beta A) = \{\nabla\beta, A\} + \{\beta A, A\}$$

Now consider the linear operator:

$$T: L_1^p(\Omega^1) \to L^p(\Omega^+) \oplus L^p(\Omega^0)$$

$$T(\psi) = (d^* \oplus d_\mu^+)(\psi) + \{\psi, A\}.$$

The operator norm of the algebraic term is $O(\|A\|_{L^4})$. We transform to a compact manifold as in § 2—then the leading term $d^* \oplus d_\mu^+$ is invertible for p close to 2 and this gives that, if $\|A\|_{L^4}$ is small,

$$\beta A = T^{-1}(\{\nabla\beta, A\})$$

lies in L_1^p for some $p>2$.

One consequence of Lemma 6.4 is that our $\hat{B}$ anti-self-dual solutions in Coulomb gauge are in $L_{1,\mathrm{loc}}^{2+\varepsilon}$ for some $\varepsilon>0$. So the moduli spaces of B^+ and $\hat{B}$ solutions are identical.

The estimates above depend on the conformal structure only through the uniform bound c. Using the Coulomb gauges of Proposition 6.2 and the uniform bound of Lemma 6.4 we have:

COROLLARY 6.5. *Let μ_i be a sequence of conformal structures on D with $|\mu_i|<c<1$ and $D' \subset\subset D$. There are constants $k=k(c)$, $p=p(c)>2$, such that if A_i is a sequence of $\hat{B}^1$ μ_i-anti-self-dual connections over D with $\|F_{A_i}\|_{L^2(D)}<k$ then there is a subsequence $\{i'\}$, L_1^p connection matrices $\tilde{A}_{i'}$ over D' gauge equivalent to $A_{i'}$, such that the $\tilde{A}_{i'}$ are weakly convergent in L_1^p. If the μ_i converge in L^∞ to μ the $\tilde{A}_{i'}$ can be supposed to be strongly L_1^p convergent to a μ-anti-self-dual connection matrix over D'.*

The argument for extending this convergence over balls to a general manifold goes through just as in the smooth situation ([5], [13], [21]), using the fact that for an anti-self-dual connection:

$$\int_X |F_A|^2 = -\int_X \mathrm{Tr}(F_A)^2 \tag{6.6}$$

is a topological invariant. We have:

PROPOSITION 6.7. *Let X be a compact quasiconformal 4-manifold and $[g_i]$ a uniformly bounded sequence of conformal structures on X. If A_i are $[g_i]$-anti-self-dual connections on a bundle $E \to X$ we can find a gauge equivalent subsequence converging weakly in B_{loc}^{1+} on the complement of a finite set of points $\{x_1, \ldots, x_l\}$ in X. If the $[g_i]$*

converge in L^∞ to a limit $[g_\infty]$ the convergence is strong in B^{1+}_{loc} and the limiting connection A_∞ is $[g_\infty]$-anti-self-dual on $X\{x_1,\ldots,x_l\}$. Moreover:

$$\int_{X\setminus\{x_1,\ldots,x_l\}} |F_{A_\infty}|^2 \leq \int_X |F_{A_i}|^2.$$

(iii) Removability of singularities

The remaining task, completing our quasiconformal version of the local analytical theory of anti-self-dual connections, is to establish Uhlenbeck's removability of point singularities in finite energy anti-self-dual connections.

PROPOSITION 6.8. *Let μ be a bounded conformal structure on the ball $D\subset\mathbf{R}^4$ and A a μ-anti-self-dual connection over $D\setminus\{0\}$ with*

$$\int_{D\setminus\{0\}} |F_A|^2 d\mu<\infty.$$

Then there is a B^{1+} connection matrix $\tilde{A}$ over D, gauge equivalent to A over $D\setminus\{0\}$.

The original proof of this in the smooth situation [30] does not adapt very easily to the quasiconformal case. We will give a different proof, extending the ideas of [5] Appendix, [13], which reduces the analysis to the results obtained in (ii), (iii) above.

We begin by considering connections over the annulus:

$$\mathfrak{A}=\{x\in\mathbf{R}^4 \mid \tfrac{1}{2}<|x|<1\}$$

and fix a slightly smaller annulus

$$\mathfrak{A}'\subset\subset\mathfrak{A}.$$

LEMMA 6.9. *There are constants k, C such that if A is a μ-anti-self-dual connection matrix over $\mathfrak{A}$ with $|\mu|<c$ and $\int_{\mathfrak{A}}|F_A|^2\, d\mu<k$ then A is gauge equivalent over $\mathfrak{A}'$ to an $\tilde{A}$ with*

$$\|\tilde{A}\|_{\hat{B}^1(\mathfrak{A}')}\leq C\|F_A\|_{L^2(\mathfrak{A})}.$$

Proof. Write $\mathfrak{A}$ as the union of two balls D_1, D_2, whose intersection is homotopy equivalent to S^2. When $\int_{\mathfrak{A}}|F_A|^2$ is small we can apply Proposition 6.2 to each ball to get connection matrices $\tilde{A}_1$, $\tilde{A}_2$ over D_1, D_2 respectively, with $\|\tilde{A}_i\|_{\mathfrak{A}'\cap D_i}\lesssim\int_{\mathfrak{A}}|F_A|^2$. The $\tilde{A}_i$ are related by a transition function g on $D_1\cap D_2$, with $du=g\tilde{A}_1-\tilde{A}_2 g$ so dg is small in $\hat{L}^4$, and since $D_1 \cap D_2$ is *connected* g is approximately constant. By modifying one of the A_i by a constant guage transformation we can suppose that g is close to the identity, and using the composition property in Proposition 3.12, that $g = \exp(h)$ where $\|h\|_{\mathscr{G}^0} \lesssim \int_{\mathfrak{A}} |F_A|^2$. Now modify $\tilde{A}_1$ on $D_1 \cap D_2$ to $\exp(\chi h) A_1$ where χ is a cut-off function equal to 0 on a neighbourhood of $D_2 \cap \mathfrak{A}'$, and to 1 on a neighbourhood of $D_1 \cap \mathfrak{A}'$. After this modification, we can glue the two connection matrices together to get the desired small representative $\tilde{A}$ over $\mathfrak{A}$.

Now to prove Proposition 6.8 consider the family of annuli

$$\mathfrak{A}_n = \{x \mid 2^{-(n+1)} < |x| < 2^{-n}\}$$

with $\mathfrak{A}'_n \subset\subset \mathfrak{A}_n$ and the dilation map $\varphi_n : \mathfrak{A} \to \mathfrak{A}_n$. Then $\{\mu_n\}$, $\mu_n = \varphi_n^*(\mu|_{\mathfrak{A}_n})$, is a bounded sequence of structures. Now if

$$I_n = \int_{\mathfrak{A}_n} |F_A|^2 \, d\mu = \int_{\mathfrak{A}} |F_{\varphi^*(A)}|^2 \, d\mu$$

we have $I_n \to 0$, since $\Sigma_n I_n < \infty$. So by Lemma 6.9 there is, for n large, a gauge A_n for $\varphi_n^*(A)$ such that:

$$\int |\tilde{A}_n|^4 \, d\mu \lesssim I_n^{\,2}.$$

Fix a cut-off function γ, equal to 1 on the outer edge of and to 0 on the inner edge, and let A_n^* be the connection matrix

$$A_n^* = \gamma \tilde{A}_n.$$

Then

$$\begin{aligned} \|F_{A_n}^*\|_{L^2(\mathfrak{A}')}^2 &= \|\gamma F_{A_n} + d\gamma \, A_n + (\gamma^2 - \gamma) \, A_n^2\|_{L^2}^2 \\ &\lesssim I_n + \|A_n\|_{L^4}^2 \lesssim I_n. \end{aligned}$$

Now let A_n^+ be the connection formed by gluing

$$A|_{\{x \mid |x| > 2^{-n}\}}$$

to the product connection 0 on $\{x \mid |x| < 2^{-(n+1)}\}$ using $(\varphi_n^{-1})^* A_n^*$. A_n^+ is a $\hat{B}$ connection and:

$$\|F^+_{A_n^+}\|_{L^2} < I_n \to 0$$

as $n \to \infty$. Also we can suppose that $\|F_{A_n^+}\|_{L^2}$ is less than the constants k of Proposition 6.2 (shrinking the disc D). So let $\bar{A}_n$ be the connection matrix, gauge equivalent to A_n^+, in Coulomb gauge, given by Proposition 6.2. Then

$$d^*\bar{A}_n = 0$$

$$d_\mu^+\bar{A}_n + \{\bar{A}_n, \bar{A}_n\} \to 0 \quad \text{in } L^2$$

and $\|\bar{A}_n\|_{L_1^2}$ is bounded on an interior domain. So there is a weak L_1^2 convergent subsequence $\bar{A}_n \to \bar{A}_\infty$ and $F^+_{\bar{A}_\infty} = 0$. By Lemma 6.4 $\bar{A}_\infty$ lies in $\hat{B}^1$ and we conclude readily enough that it is gauge equivalent to A on $D \setminus \{0\}$.

Of course it follows from this removal of singularities property that the "weak limit" connections of Proposition 6.6—initially defined on the punctured manifold—extend to $\hat{B}$ connections over all of X.

§7. Applications

(i) Transversality

We have now set up the foundations of the theory of the first order Yang-Mills equations on quasi-conformal manifolds. One could hope to go on to transfer all the differential topological results proved for smooth manifolds by means of Yang-Mills fields, to the quasi-conformal category. We shall do rather less than this and shall be content to give proofs of Theorems 1 and 2. The main ingredient which we are lacking is a firm grip on the *transversality* of the anti-self-dual equations. We would like to put ourselves in a position where the cohomology groups H_A^2 vanish for all (irreducible) anti-self-dual connections A. In that case, the moduli spaces of irreducible solutions are smooth manifolds of the "correct" dimension given by Proposition 5.4 and we can proceed to make various topological arguments using them.

In the smooth theory, there are a number of possible approaches to this transversality. One is to appeal to the result of Freed and Uhlenbeck [13] which gives the desired property for generic smooth Riemannian metrics (conformal structures) and non-trivial $SO(3)$ or $SU(2)$ bundles. Unfortunately, the proof of this does not seem to transfer to the quasiconformal case. Another approach is to use more abstract perturbations of the equations of various kinds, [5], [7], [9], [15]. In the latter case, one can distinguish two goals: the first is to find a small perturbation:

$$F_A^+ + \varepsilon(A) = 0 \tag{7.1}$$

of the equations, on a given space $\mathfrak{B} = \mathfrak{B}_E$, having transverse zeros. The second is to make a family of such perturbations for different bundles E which are compatible under the 'weak convergence' of connections discussed in §6. This second goal is considerably harder to achieve and we will not attempt to achieve it in the quasiconformal case (although we have no reason for doubting that this can be done). This gap in our theory prevents us from obtaining the most general results.

The simpler kind of perturbation $\varepsilon(A)$ on a *single* space $\mathfrak{B}_E$ can be constructed, in our set up, by abstract arguments. It is a general fact [20] that a reflexive Banach space V with V, V^* both separable admits a C^1 function $\varphi: V \to \mathbf{R}$ supported in the unit ball. This gives the existence of locally finite C^1 partitions of unity on paracompact Banach manifolds modelled on such spaces [11]. Now our function spaces have this abstract property so we construct in a standard way many C^1 perturbations of the anti-self-dual equations. (Sections of appropriate Banach bundles.) These yield C^1 perturbed moduli spaces:

$$M_E^\varepsilon = (A \mid F^+(A) + \varepsilon(A) = 0\}/\mathfrak{G} \tag{7.2}$$

which suffice for the differential topological arguments. Moreover, it is easy to arrange that if M_E is compact, say, then M_E^ε is also.

(ii) Proof of Theorem 1: topological 4-manifolds without quasi-conformal structure

We use the argument of Fintushel and Stern [12]. Suppose X is a compact simply connected topological 4-manifold with a negative definite, even intersection form which represents -2 (for example $-E_8$), such manifolds exist by the work of Freedman. We show that X does not admit a quasi-conformal structure. Assume, on the contrary, that X is quasi-conformal and, with Fintushel and Stern, consider an $SO(3)$ bundle $E \to X$ with

$$E \cong R \oplus L \tag{7.3}$$

where $c_1(L)^2 = -2$. It follows from (6.6), Proposition 6.7 and the fact that $-p_1(E) = 2 < 4$ that M_E is compact. Making a generic perturbation if necessary we achieve a perturbed moduli space M_E^ε which is a compact 1-manifold with a single end point associated to the reduction (7.3) of E. This space M_E^ε then gives the desired contradiction.

More generally we can prove that no simply connected compact 4-manifold with a non-standard definite intersection form admits a quasi-conformal structure. To do this we use the argument of [6], modifying the original proof of [5]. We sketch the argument. To begin we fix a conformal structure on such a manifold X which is *smooth* on an open set Ω, the complement of a ball in X. This is possible since, for any compact 4-manifold X, $X\setminus$point, can be smoothed [20]. We take an $SU(2)$ bundle $E\to X$ with $c_2(E)=1$ and consider a suitable perturbation M_E of the moduli space. Now any homology classes α_1, α_2 in $H_2(X)$ can be represented by smooth surfaces Σ_1, Σ_2 in $X\setminus\Omega$ and, appealing to the smooth theory, we can "cut down" M_E by codimension 2 submanifolds V_{Σ_1}, V_{Σ_2}, so that

$$M_E\cap V_{\Sigma_1}\cap V_{\Sigma_2}$$

is a 1-manifold with a number of boundary points associated to classes e in $H^2(X)$ with $e^2=-1$. Now M_E is not compact and in the smooth case one shows that its end is a collar on X (the "concentrated connections"). In the quasi-conformal case we are not able to analyse *all* of the end of M_E in this way but we can show that the same result holds for the connections concentrated over the smooth part Ω (see the remarks at the end of §5). But for elementary reasons the end of $M_E\cap V_{\Sigma_1}\cap V_{\Sigma_2}$ consists of connections concentrated over $\Sigma_1\cap\Sigma_2\subset\Omega$ so we are able to ignore the "unknown" part of the end of M_E. The same holds for the perturbed space M_E. In this way we deduce just as in [6] that

$$\alpha_1\cdot\alpha_2=-\frac{1}{2}\sum_{e^2=-1}(\alpha_1\cdot e)(\alpha_2\cdot e)$$

and the form is standard.

(iii) Proof of Theorem 2: homeomorphic 4-manifolds with distinct quasi-conformal structures (zero dimensional moduli spaces)

The simplest proof of Theorem 2 is obtained by adapting a recent result of Kotschick [19]. In general if X is a smooth simply connected 4-manifold, with $b^+(X)$ odd and at least 3 one defines diffeomorphisms invariants which are integral *polynomials* in the cohomology of X. If there is an $SO(3)$ bundle E over X such that the virtual dimension $i(E)$ is zero then the corresponding polynomial has degree 0—i.e. is just an integer. Kotschick shows how one can define such an invariant, in a special situation, where

$b^+(X)=1$. He shows that the complex "Barlow surface" is not diffeomorphic to $\mathbf{CP}^2\#8\overline{\mathbf{CP}}^2$, although by Freedman's classification they are homeomorphic.

We will show that these invariants are *quasiconformal* invariants of smooth 4-manifolds. (So we will not go so far as to define invariants for manifolds only known to have a quasiconformal structure.) In this section we consider a situation like Kotschick's where the invariant is an integer obtained from a bundle $E \to X$ with $i(E)=0$. Then for generic smooth metrics g on X the moduli space $M_E(g)$ is a finite set of points (representing irreducible, transverse connections) and the invariant $q=q_X$ is the total number of points, counted with suitable signs. (Strictly there is an overall choice of sign involved, see [7]). Suppose then that (X_1, g_1, E_1), (X_2, g_2, E_2), are two triples of this kind and

$$f: X_1 \to X_2$$

is a quasiconformal homeomorphism, with $f^*(E_2) \cong E_1$. We have to prove that the integers q_{X_1}, q_{X_2} are equal.

To show this we let $[\bar{g}_2]$ be the pull back of $[g_2]$ by f. So we can suppose $\bar{g}_2$ is a bounded metric on X_1. By a familiar regularisation we can find a uniformly bounded sequence of *smooth* metrics $\bar{g}_2^{(i)}$ on X_1 such that

$$\bar{g}_2^{(i)} \to \bar{g}_2 \quad \text{in } L^2(X_1)$$

(hence in any $L^N(X_1)$). Now let $[g_2^{(i)}] = (f^{-1})^*[\bar{g}_2^{(i)}]$. So the $[\bar{g}_2^{(i)}]$ are a uniformly bounded family of conformal structures on X_2. Arguing as in Lemma 2.29 we see that the $[\bar{g}_2^{(i)}]$ converge to $[g_2]$ in any $L^N(X_2)$—that is:

$$\int_{X_2} d([g_2^{(i)}], [g_2])^N \to 0.$$

If we represent the structures $g_2^{(i)}$ relative to g_2 by bundle maps μ_i then

(7.4) $$|\mu_i| < c$$

and

(7.5) $$\|\mu_i\|_{L^2(X_2)} \to 0.$$

By our theory the $\hat{B}$ anti-self-dual moduli spaces for $(X_1, [\bar{g}_2^{(i)}])$. $(X_2, [g_2^{(i)}])$, are matched up by f^*. The heart of the matter then is to prove:

PROPOSITION 7.6. *For large enough i and for each point* $[A]$ *of* $M(g_2)$ *there is a small* $\hat{B}$ *neighbourhood of* $[A]$ *in* $\mathfrak{B}^*_{E_2}$ *which contains exactly one point of* $M[g_2^{(i)}]$. *These points are transverse zeros and there are no other points in* $M[g_2^{(i)}]$.

It will be clear from our proof that the signs with which the points of $M(g_2)$, $M(g_2^{(i)})$ are counted agree and Proposition 7.6 then gives the equality of q_{X_1} and q_{X_2}. The point here is that any of the smooth metrics $\tilde{g}_2^{(i)}$ can be used to calculate q_{X_1}.

For the proof of Proposition 7.6 we simplify notation and write X, g, E for X_2, g_2, E_2. If the bundle maps μ_i (representing $g_2^{(i)}$) converged to 0 in L^∞ the result would follow straightaway from our discussion in § 4. The extension to the hypotheses (7.4), (7.5) is another application of the idea used in our fundamental Lemma 2.16.

We begin by considering the linearised problem. Let A be a g-anti-self-dual connection over X.

LEMMA 7.7. *If*

$$d_A^+ : (\operatorname{Ker} d_A^* \subset \Omega_X^1(\mathfrak{g}_E)) \to \Omega_X^+(\mathfrak{g}_E)$$

is an isomorphism then for large i so also is $d_A^+ + \mu_i d_A^-|_{\operatorname{Ker} d_A^*}$. *Moreover, if* Q_i *is the inverse map the operator norms of* $d_A^+ Q_i$ *on* L^p *are bounded for i large and p close to* 2.

Proof. Consider the Laplacian

$$\Delta = \frac{1}{2} d_A d_A^* + (d_A^+)^* d_A^+$$

on $\Omega_X^1(\mathfrak{g}_E)$. This preserves $T = \{a | d_A^* a = 0\}$ (since $F_A^+ = 0$) and on T

$$\langle \Delta a, a \rangle = \|d_A^+ a\|_{L^2}^2.$$

Now

$$\|d_A^+ a\|^2 - \|d_A^- a\|^2 = -\int_X \operatorname{Tr}(a \wedge [F, a])$$

so

$$\|d_A^+ a\|^2 = \langle (\Delta + [F_-, \ ]) a, a \rangle.$$

The algebraic operator $[F_-, \ ]$ does not preserve T so put

$$\varphi(a)=[F_-,a]-d_AG_Ad_A^*[F_-,a]$$

where G_A is the Greens operator of $d_A^*d_A$ on $\Omega_X^0(\mathfrak{g}_E)$. This is the L^2 projection of $[F^-,a]$ to T. Then the quadratic forms

$$\|d_A^+a\|^2,\quad \|d_A^-a\|^2$$

on T are represented by Δ, $\Delta+\varphi$ respectively. φ is a pseudo-differential operator of order zero and if $0\leqslant k<1$

$$\Delta_k=\Delta-k^2(\Delta+\varphi)$$

is a self adjoint, elliptic, second order operator with positive symbol. So Δ_k has a discrete spectrum, bounded below, and there is an orthogonal decomposition of $\Omega_X^1(\mathfrak{g}_E)$ into Δ_k eigenspaces. This induces a decomposition of T and in particular a splitting:

$$T=H_1\oplus B$$

into Δ_k invariant subspaces, with $\Delta_k>0$ on H_1 and B finite dimensional. Now fix $k>c$, where c is the uniform bound in (7.4). Let $H_2=d_A^+H_1$ and

$$C=H_2^\perp\subset\Omega_X^+(\mathfrak{g}_E).$$

Write p,q for the L^2 projections to H_2, C. For α in H_1:

$$\begin{aligned}\|p(\mu_id_A^-\alpha)\|^2&\leqslant\|\mu_id_A^-\alpha\|^2\\&\leqslant c^2\|d_A^-\alpha\|^2\\&\leqslant(c^2/k^2)\|p(d_A^+\alpha)\|^2\end{aligned}$$

since $\Delta_k>0$ on H_1. Then, as in §2, we can invert the operator

$$p(d_A^++\mu_id_A^-)|_{H_1}:H_1\to H_2$$

with inverse P say. With k, c fixed we have a fixed bound on the L^2 operator norms of $d_A^+P_i$, $d_A^-P_i$ (and also on the L^p operator norms for p close to 2). We claim now that if, in addition, $\|\mu_i\|_{L^2(X)}$ is sufficiently small (i.e. if i is large) then $d_A^++\mu_id_A^-$ is invertible on T. This reduces to a finite dimensional problem. Let

$$B' = (d_A^+)^{-1} C \subset T,$$

then for $h \in H_1$, $b \in B'$, $g \in H_2$, $c \in C$:

$$(d_A^+ + \mu_i d_A^-)(h+b) = (g+c)$$

if and only if:

$$h = P(g - p(d_A^+ + \mu_i d_A^-) b) \tag{7.8}$$

and

$$q(d_A^+ + \mu_i d_A^-) h = c - q(d_A^+ + \mu_i d_A^-) b. \tag{7.9}$$

Regard g, c as fixed and h as defined by (7.8). We must show that there is a unique solution to (7.9) for b in B', when $\|\mu_i\|_{L^2}$ is small. Write (7.9) as $R(b)=c$ where $R: B \to C$ is:

$$\begin{aligned} R(b) &= q(d_A^+ + \mu_i d_A^-)(b + P_i p(d_A^+ + \mu_i d_A^-) b) \\ &= d_A^+ b + q(\mu_i d_A^-(b + P_i p(d_A^+ + \mu_i d_A^-) b)). \end{aligned}$$

By construction $d_A^+: B' \to C$ is an ismorphism so it suffices to show the operator norm of the remainder is small; that is if

$$\beta = d_A^-(b + P_i(p(d_A^+ + \mu_i d_A^-) b))$$

then $\|q\mu_i(\beta)\|_{L^2} \leq \varrho \|b\|_{L^2}$, for any $\varrho > 0$, when $\|\mu_i\|_{L^2}$ is sufficiently small.

We know that

$$\|\beta\|_{L^2} \lesssim \|b\|_{L^2}.$$

The result now follows from the fact that elements of C are *smooth*. (We deduce this via elliptic regularity for the overdetermined operator $(d_A^+)^*$.) For if ψ is in C:

$$\langle \psi, \mu_i(\beta) \rangle_{L^2} \leq \sup|\psi| \, \|\mu_i\|_{L^2} \|\beta\|_{L^2}.$$

COROLLARY 7.10. *With A, μ_i as above, we can find $\eta > 0$ such that for large i there is a unique solution a in T to*

$$F_{A+a}^+ + \mu_i(F_{A+a}^-) = 0$$

with $\|a\|_{\hat{B}^1} < \eta$.

Proof. The equation to be solved is

$$(d_A^+ + \mu_i d_A^-)\, a + (a\wedge a)^+ + \mu_i (a\wedge a)^- = -\mu_i F_A^-.$$

Now F_A^- is smooth so:

$$\|\mu_i (F_A^-)\|_{L^2} \to 0$$

as $i\to\infty$. On the other hand we claim that if i is sufficiently large the equation

$$(d_A^+ + \mu_i d_A^-)\, a + (a\wedge a)^+ + \mu_i (a\wedge a)^- = \chi$$

has a unique small L_1^2 for χ small in L^2. To see this write $a = Q_i\psi$ and apply the contraction mapping principle using the uniform bound on $d_A^+ Q_i$, $d_A^- Q_i$ and the Sobolev inequality:

$$\|a\|_{L^4} \lesssim \|d_A^+ a\|_{L^2} \quad \text{for } a \text{ in } T.$$

The corresponding result for a $\hat{B}^1$ (or L_1^{2+}) neighbourhood follows just as in § 2, § 6.

To complete the proof of Proposition 7.6 we have to show that there are no additional points in $M_E(g_2^{(i)})$ for large i. It suffices to prove that any sequence $[A_i]$ of points in $M_E(g_2^{(i)})$ contains a subsequence converging to a point of $M_E(g_2)$. To see this we apply the compactness results of § 6. Those results give, first, that a subsequent converges, weakly in $\hat{B}_1$, on the complement of a finite set $S\subset X$. The next lemma shows that the weak limit is a g_2-anti-self-dual connection.

LEMMA 7.11. *Suppose $\mu_i\to 0$ in L^2 and $F_i = F_i^+ + F_i^- \in L^2$ converge weakly to F_∞ in L^2 and satisfy $F_i^+ + \mu_i F_i^- = 0$. Then $F_\infty^+ = 0$.*

Proof. If $F_\infty^+ = 0$ there is a *smooth* ζ with $\int \zeta F_\infty^+ \neq 0$. But:

$$\left| \int \zeta \mu_i F_i^- \right| \leq \|F_i^-\|_{L^2} \|\mu_i\|_{L^2} \sup|\zeta| \to 0 \quad \text{as} \quad i\to\infty,$$

and

$$\int \zeta F_\infty^+ = \lim \int \zeta F_i^+ = \lim \int \zeta \mu_i F_i^-.$$

Now, by hypothesis, $i(E)=0$ so for any bundle F with $(-p_1(F)) < (-p_1(E))$, we have $i(F)<0$. Since $w_2(E)\neq 0$ all the "lower" moduli spaces $M_F(g_2)$, where F satisfies

$$0 < -p_1(F) < -p_1(E) \quad \text{and} \quad w_2(F) = w_2(E),$$

are empty for generic smooth metrics g_2. (Actually, in Kotschick's case, the lower moduli spaces are automatically empty by Proposition 6.7.) It follows that the exceptional set S does not occur here and the global strong $\hat{B}_1$ convergence that we need follows from the following local result.

LEMMA 7.12. *Suppose A_i are connection matrices over D which satisfy:*

(i) $d^*A_i=0$.

(ii) $F^+_{A_i}+\mu_i(F^-_{A_i})=0$ *where* $\mu_i \leq c$ *and* $\mu_i \to 0$ *in* L^2.

(iii) $A_i \to A_\infty$ *weakly in* B_1.

(iv) *The* A_i *are bounded in* $L^{2+\delta}_1$ *for some* $\delta>0$.

(v) *Then for* $D' \subset\subset D$ *there is a subsequence of the* A_i *converging strongly in* $\hat{B}_1$ *to* A_∞ *over* D'.

Proof. Using (iv) we can suppose that A_i converge weakly in $L^{4+\varepsilon}$ for some $\varepsilon>0$. Then substituting into the equation:

$$d^+A_i+\mu_i(d^-A_i)$$

is strongly convergent in $L^{2+\varepsilon/2}$, while d^+A_i, d^-A_i are bounded in $L^{2+\varepsilon/2}(D')$ by our L^p theory of §2. But $\mu_i \to 0$ in L^N for all N so, taking

$$\frac{1}{N} < \frac{1}{2+\varepsilon/4} - \frac{1}{2+\varepsilon/2}$$

we get $(\mu_i d^-A_i) \to 0$ in $L^{2+\varepsilon/4}$; so finally d^+A_i is $L^{2+\varepsilon/4}$ convergent and the result follows.

(iv) General polynomial invariants

We get many more examples, as explained in [9], of manifolds with distinct quasiconformal structures by extending the general polynomial invariants of [9] to the quasiconformal setting. We will sketch how this can be done.

In the smooth theory the polynomial invariants are defined by intersecting a $2d$ dimensional moduli space M_E with d codimension 2 submanifolds V_i. The V_i are associated to surfaces $\Sigma_i \subset X$. They are the zeros of sections of line bundles

$$r^*_{\Sigma_i}(\mathscr{L}_{\Sigma_i})$$

pulled back under the restriction maps:

$$r_{\Sigma_i}: \mathcal{B}_{X,E} \to \mathcal{B}_{\Sigma_i}.$$

(There is some complication here to do with reducible connections: we shall ignore this point and refer to [6], [9] for more details.) Then one obtains compactness of the intersection

$$M_E \cap \left(\bigcap_{i=1}^{d} V_i \right)$$

under some mild restrictions on E. The important ingredients are that triple intersections $\Sigma_i \cap \Sigma_j \cap \Sigma_k$ $(i, j, k$ district) are empty and that one can find many smooth sections of the line bundles $\mathcal{L}_{\Sigma_i}$ over the Banach manifolds $\mathcal{B}^*_{\Sigma_i}$. This latter yields the generic transversality used in proving compactness (see [6], Lemma 3.16).

We encounter two difficulties in proving the invariance of the intersection numbers, defined in this way, under quasi-conformal maps $f: X_1 \to X_2$ between a pair of such manifolds. First, f does not preserve the class of smoothly embedded surfaces Σ. Second, it is hard to make sense of the restriction maps r_Σ for $\hat{B}$ connections. To get round these difficulties we work with domains in the 4-manifolds in place of surfaces. For any domain $\Omega_2 \subset X_2$ there is a topological space $\mathcal{B}_{\Omega_2}$ of $\hat{B}^1_{\mathrm{loc}}$ connections modulo $\hat{B}^0_{\mathrm{loc}}$ gauge transformations, endowed with the quotient topology. We also have a continued restriction map $\mathcal{B}_{X_2} \to \mathcal{B}_{\Omega_2}$. The problem is that it is not easy to put a manifold structure on $\mathcal{B}_{\Omega_2}$. So we reduce to the case of compact manifolds by a "doubling" argument.

First consider a $\hat{B}^1_{\mathrm{loc}}$ connection A over the half-space $(x_4 > 0)$ in $\mathbf{R}^4$. Let $\delta > 0$ and

$$f: (-\infty, \infty) \to [\delta, \infty)$$

be a smooth function with

$$f(t) = \begin{cases} \delta & \text{if} \quad t < \delta/2 \\ t & \text{if} \quad t > 2\delta \end{cases}.$$

Let $p: \mathbf{R}^4 \to \{x_4 \geq \delta\}$ be the map $p(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, f(x_4))$, and $A = p^*(A)$. Then if $\chi: \mathbf{R}^4 \to \mathbf{R}^4$ is the reflection map:

$$\chi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4)$$

we have a natural isomorphism between A and $\chi^*(A)$ over the strip $(-1/2 < x_4 < \delta/2]$ and we can define a corresponding B^1_{loc} connection A over $\mathbf{R}^4$.

More generally, if $\Omega_2 \subset X_2$ has a smooth boundary we can use this construction to make:

(1) a closed manifold Y_2 diffeomorphism to the double of Ω_2 and with a common subdomain $\Omega_2^\delta \subset \Omega_2$, $\Omega_2^\delta \subset Y_2$.

(2) a continuous map

$$j_2 \colon \mathscr{B}_{\Omega_2} \to \mathscr{B}_{Y_2} \quad \text{with} \quad j_2(A)|_{\Omega_2^\delta} = A|_{\Omega_2^\delta}.$$

Now choose Ω_2 to be a tubular neighbourhood of a smooth surface $\Sigma_2 \subset Y_2$. The homology class of Σ_2 in Y_2 gives us a corresponding class

$$\mu(\Sigma_2) \in H^2(\mathscr{B}^*_{Y_2}; \mathbf{Z})$$

(see [6]) which we can represent by a line bundle $\mathscr{L}_2 \to \mathscr{B}^*_{Y_2}$. Pull this line bundle back to the moduli space M_E by the composite map:

$$M_E \to \mathscr{B}^*_{\Omega_2} \to \mathscr{B}^*_{Y_2}.$$

We can now carry out the same transversality arguments as in [6], [9] (using C^1 sections) taking the ambient smooth manifold $\mathscr{B}_{Y_2}$ in place of $\mathscr{B}_\Sigma$. We have the same compactness properties of the resulting intersections $M_E \cap V_d$ provided we take the tubular neighbourhoods thin enough for their triple intersections in X to be empty.

We can now verify that this modified construction gives invariants preserved by f. If $\Sigma_2 \subset \Omega_2 \subset X_2$ are as above we can choose a smooth surface:

$$\Sigma_1 \subset f^{-1}(\Omega_2) \subset X_1$$

homologous in $f^{-1}(\Omega_2)$ to $f^{-1}(\Sigma_2)$ and a neighbourhood Ω_1 of Σ_1 with

$$\Omega_1 \subset\subset f^{-1}(\Omega_1^\delta).$$

Put $\tilde{\Omega}_1 = \Omega_1^\delta$ and $\tilde{\Omega}_2 = f^{-1}(\Omega_2^\delta)$. We have maps:

$$\begin{tikzcd} \mathscr{B}_{\tilde{\Omega}_2} \arrow[r] \arrow[d, "(f^*)^{-1} j_2"'] & \mathscr{B}_{\tilde{\Omega}_1} \arrow[d, "j_1"] \\ \mathscr{B}_{Y_2} \arrow[ur, "\text{res}"] & \mathscr{B}_{Y_1} \end{tikzcd}$$

and it is clear that

$$(j_1 \circ \text{res})^* \mathscr{L}_1 \cong \mathscr{L}_2$$

over $\mathscr{B}^*_{Y_2}\cap(j_1\circ\mathrm{res})^{-1}\mathscr{B}^*_{Y_1}$. So we can construct a homotopy between any sections s_1, s_2 of $\mathscr{L}_1, \mathscr{L}_2$ respectively using this isomorphism and the ambient space $\mathscr{B}_{Y_2}$. Moreover, since $\mathscr{B}^*_{Y_2}$ is a manifold we can make this homotopy in general position relative to all of the moduli spaces.

Finally, then, if we start with an intersection

$$M(g_2)\cap V_1^{(2)}\cap \dots \cap V_d^{(2)}$$

constructed using X_2; we first argue (as in the simple case of (iii)) that for metrics $g_2^{(i)}$ near g_2 the points of

$$M(g_2^{(i)})\cap V_1^{(2)}\cap \dots \cap V_d^{(2)}$$

match up with those of the original intersection. We next use f to transfer to the manifold X_1 where $g_2^{(i)}$ correspond to smooth metrics $\bar{g}_2^{(i)}$. We choose generic $V_1^{(1)}, \dots, V_d^{(1)}$ using the X_1 smooth structure and thin neighbourhoods as above. The discussion of the previous paragraph allows us to find a homology between:

$$M(\bar{g}_2^i)\cap V_1^{(2)}\cap \dots \cap V_d^{(2)}$$

and

$$M(g_1^i)\cap V_1^{(1)}\cap \dots \cap V_d^{(1)}$$

so the two invariants agree. (Of course the polynomials we have defined also agree with those in the smooth theory, as one sees by considering the restriction map of smooth connections to surfaces.)

Appendix 1: Gehring's theorem in even dimensions

Let (X, g) be an oriented Riemannian manifold of even dimension $2l$. The $*$ operator on Ω^l_X depends only on the conformal class of g and satisfies $**=(-1)^l$. If we work with complexified forms (which we denote henceforth by Ω^p), we can decompose:

$$\Omega^l_X=\Omega^+_X\oplus\Omega^-_X$$

where $*=\pm 1$ or $\pm i$ on $\Omega^\pm$, as l is even or odd respectively. We then have a "half de Rham complex":

$$\Omega^0_x\xrightarrow{d}\Omega\xrightarrow{d}\dots\to\Omega^{l-1}_x\xrightarrow{d^+_g}\Omega^+_x$$

in which the metric appears only in the last term d_g^+, the projection of d to Ω_x^+. In a local coordinate system we can represent the conformal structure by a bundle map:

$$\mu\colon \Lambda^- \to \Lambda^+,$$

with $\|\mu\|<1$. Then d_g^+ is represented by:

$$d^+ + \mu d^-$$

where d^+, d^- are constant coefficient operators defined by the Euclidean metric in the coordinate system. (If one prefers compact manifolds one can work on a domain in S^{2l} with the round, conformally flat, metric.) All of this is an immediate generalisation of the four dimensional case considered in § 2. The only point to note is that the map from conformal structures to bundle homorphisms with operator norm less than 1 is injective but not surjective when $l>2$.

Now as in § 2 we can consider operators $d^+ + \mu d^-$ where $\|\mu\|<1$ but μ is otherwise only assumed to be measurable. For compactly supported $l-1$ forms α we have:

$$\int |d^+\alpha|^2 - |d^-\alpha|^2 = \left\{\begin{matrix} \displaystyle\int d\alpha \wedge d\alpha \\ i\displaystyle\int d\alpha \wedge d\alpha \end{matrix}\right\} = 0$$

(as l is even or odd), so the argument of Lemma 2.7 shows that we have the usual elliptic estimates for $d^+ + \mu d^-$ on $\operatorname{Ker} d^*$ for a small range of indices p about $p=2$. Similarly a $2l$ form which is closed and "self-dual" relative to the bounded conformal structure defined by μ is locally in $L^{2+\varepsilon}$, for some $\varepsilon>0$. We can then easily deduce Gehring's theorem in dimension $2l$.

PROPOSITION (Gehring). *If D is a domain in $\mathbf{R}^{2l}$ and $\varphi\colon D \to \mathbf{R}^{2l}$ is a K quasiconformal map then the partial derivatives of φ are in $L^{2l+\eta}_{\mathrm{loc}}$ for some $\eta(K)>0$.*

Again the argument is almost identical to that in the four dimensional case of § 3. We choose a closed, nowhere-vanishing, form ω in Ω^+, for example:

$$dx_1 \ldots dx_l + dx_{l+1} \ldots dx_{2l} \quad \text{or} \quad dx_1 \ldots dx_l - i\, dx_{l+1} \ldots dx_{2l},$$

then consider $w = \varphi^*(\omega)$, which is self dual relative to the bounded structure obtained by pull-back under φ. On a suitable interior domain D we have:

$$\int_{D'} |\tilde{\omega}|^{2+\varepsilon} < \infty$$

and so:

$$\int_{D'} |\nabla\varphi|^{2l+\varepsilon l} \lesssim \int_{D'} |\omega|^{2+\varepsilon} < \infty.$$

When $l=1$ this proof is just a restatement of that given by Boyarskii. The operator d^+ is in this case the Cauchy-Riemann $\bar{\partial}$ operator and $(d^+ + \mu d^-)f = g$ is the usual Beltrami equation. Of course Gehring's proof applies equally well to even and odd dimensions. It would be interesting to look for an odd-dimensional counterpart to the argument, depending on the Calderon-Zygmund theory, given here.

Appendix 2: Index theory on quasiconformal manifolds

The approach to the index theory we have adopted in Section 4 extends to general, even dimensional quasiconformal manifolds and gives, in particular, an alternative route to the main results obtained by Teleman for Lipschitz manifolds. First, by standard Hodge theory, we have integral "homotopy operators" for the constant coefficient half-complex, that is operators:

$$\sigma_p \colon \Omega_X^p \to \Omega_X^{p-1}, \quad \sigma_l \colon \Omega_X^+ \to \Omega_X^{l-1}$$

such that

$$d\sigma_p + \sigma_p d = 1, \quad p < l$$
$$d^+ \sigma_l = 1.$$

On a compact quasiconformal $2l$ manifold X we can introduce spaces of forms, just as in § 3. For example we have a Banach space of "B-forms" B^i, with

$$\alpha \in L^{2l/i}, \quad d\alpha \in L^{2l/i+1}.$$

Let E be a complex vector bundle over X (with a $\hat{B}$ or B^+ structure). We wish to associate an integer "analytic index" invariant $i(E)$ in the same way as in § 5. To do this we choose a bounded conformal structure on X, defining subspaces Ω_X^+, Ω_X^-, and a ($\hat{B}$ or B^+) connection A on E. We have then a sequence:

$$\Omega_X^0(E) \xrightarrow{d_A} \Omega_X^1(E) \to \ldots \to \Omega_X^{l-1}(E) \xrightarrow{d_A^+} \Omega_X^+(E).$$

This can be modified to yield a complex by the procedure of § 4, extended inductively over all the last terms. First one sees that $\operatorname{Im} d_A^+$ has a finite dimensional cokernel H^l (and it does not matter whether we use the B or $\hat{B}$ framework here). We find a right inverse:

$$Q_l\colon (H^l)^\perp \to \Omega_X^{l-1}(E)$$

by starting with a parametrix constructed using σ_l in local charts. Then we put

$$\delta_{l-2} = d_A - Q_l F_A^+ \quad \text{on } \Omega_X^{l-2}(E),$$

so $d^+\delta_{l-2}=0$. Suppose inductively that we have defined operators δ_p for $q>p\geq l-2$, differing from the d_A by compact operators, and such that:

$$\delta_p \delta_{p-1} = 0.$$

(Then we can show using the arguments of § 4 that the δ_p have closed range and the cohomology groups H^p are finite dimensional.) To construct δ_q we start once again with a parametrix, an operator

$$P\colon \Omega_X^{q+2}(E) \to \Omega_X^{q+1}(E)$$

such that:

$$\delta_{q+1} P = 1 + (\text{compact}) \quad \text{on } \operatorname{Im} \delta_{q+1}$$

We can take:

$$P(\phi) = \sum \gamma_\alpha \sigma_{q+2}(\phi|_{U_\alpha}) \quad (\text{cf. } (4.8)).$$

It is then a straightforward exercise, extending the discussion of § 4, to find a right inverse Q_{q+2}, and we can put $\delta_q = d_A - Q_{q+2} F_A$. We define $i(E)$ to be the Euler characteristic of the δ_q complex:

$$i(E) = \sum_q (-1)^q \dim H^q.$$

As we explained above, the theory of Fredholm complexes shows that $i(E)$ is independent of the conformal structure on X and connection on E. The advantage of this approach, compared to Teleman's, is precisely that the underlying Banach spaces are independent of these auxiliary structures so the invariance of the index is a comparatively routine matter.

When X is a smooth manifold we can choose all our data to be smooth. The Atiyah-Singer index theory can then be used to calculate $i(E)$. The single operator:

$$D=\delta+\delta^*\colon \Omega^0+\Omega^2\ldots\to\Omega^1+\Omega^3\ldots$$

has index $i(E)$ and $D\oplus D$ is equivalent (module compact operators and direct sum with invertibles) to:

$$D_\chi+(-1)^l D_\tau$$

where D_τ is the signature operator and D_χ the Euler characteristic operator. So we have the index formula, in the smooth case:

$$i(E)=\frac{1}{2}(\text{rank } E)\chi(X)+(-1)^l\langle \text{ch}(E)\,L(X),[X]\rangle.$$

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