On the regularity of the minima of variational integrals

by

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0. Introduction

The problem of the regularity of functions $u(x)$ minimizing a variational integral

$$F(u;\Omega)=\int_{\Omega} f(x,u,Du)\,dx \tag{0.1}$$

has been one of the main questions since the introduction of direct methods has allowed to pove the existence of minima in suitable classes of generalized functions. It would be impossible to list all the significant contributions since the pioneering work of E. De Giorgi [4]; and we refer to the nowadays classical books by O. A. Ladyženskaya and N. N. Ural'ceva [17] and C. B. Morrey [20].

With extremely few exceptions, all the papers concerned with the regularity problem have as a common starting point the Euler equation of the functional F and therefore require at least some smoothness of the function f and suitable growth conditions for its partial derivatives f_u and f_p.

It goes without saying that the smoothness of f is necessary if one wants to prove the differentiability of the minima; on the other hand, if we look only at the continuity of the solution such assumptions seem superfluous, and it would be preferable to derive it directly from the minimizing property of u.

In addition, it is clear that results obtained from the Euler equation do not distinguish between true minima and simple extremals, and therefore it is sometimes necessary to introduce as hypotheses properties—as for instance the boundedness of the solution—which might hold for minima but are in general false for extremals.

The aim of the paper is to investigate the continuity (in the sense of Hölder) of the minima, directly working with the functional F instead of working with its Euler equation. In fact we shall not suppose any differentiability, but only that the function $f: \Omega \times \mathbf{R}^N \times \mathbf{R}^{nN} \to \mathbf{R}$ satisfies:

(i) f is a Carathéodory function; i.e. measurable in x for each $(u,p) \in \mathbf{R}^N \times \mathbf{R}^{nN}$, and continuous in (u,p) for almost every $x \in \Omega$.

(ii) There exist positive constants a and k and a real number $m>1$ such that

$$|p|^m - k \leq f(x,u,p) \leq a|p|^m + k. \tag{0.2}$$

We note that (i) and the first part of inequality (0.2) are the usual assumptions in the existence theory; we remark however that the main hypothesis for existence, the convexity of f with respect to p, is not needed when the regularity is concerned.

We shall restrict ourselves to the case $1<m<n$. In fact, when $m>n$, every function in $W^{1,m}$ is trivially Hölder-continuous by the Sobolev theorem. The borderline case $m=n$ has been treated in his book [20] by C. B. Morrey, who proved the Hölder-continuity of functions minimizing F, for every $N \geq 1$ (Theorem 4.3.1; see Corollary 4.2 below).

When $1<m<n$, we cannot expect in general that minima for F are Hölder-continuous (or even bounded) if $N>1$; a well-known example due to E. De Giorgi [5] shows that there are linear elliptic systems with unbounded solutions in dimension $n \geq 3$, and we are led to the usual distinction between the scalar ($N=1$) and the vector case ($N>1$) corresponding respectively to a single equation or to a system.

When $N=1$ we prove the Hölder-continuity of functions $u \in W^{1,m}_{\mathrm{loc}}(\Omega)$, minimizing locally the functional F. This is done in sections 2 and 3. For the sake of completeness we mention that a result of this type was obtained in [7] by J. Frehse, under very strong assumptions on the function f.

The last two sections deal with the general case $N \geq 1$. Here we prove in section 4 that minimizing functions have derivatives in $L^{m+\sigma}$, for some $\sigma>0$, whereas the last section 5 is devoted to the study of partial regularity for minima of quadratic functionals.

1. A simple, but fundamental lemma

It is the following:

LEMMA 1.1. *Let $f(t)$ be a non-negative bounded function defined for $0 \leq T_0 \leq t \leq T_1$. Suppose that for $T_0 \leq t < s \leq T_1$ we have*

$$f(t) \leq A(s-t)^{-\alpha} + B + \theta f(s) \tag{1.1}$$

where A, B, α, θ are non-negative constants, and $\theta<1$. Then there exists a constant c, depending only on α and θ such that for every ϱ, R, $T_0\leq\varrho<R\leq T_1$ we have

$$f(\varrho)\leq c[A(R-\varrho)^{-\alpha}+B]. \tag{1.2}$$

Proof. Consider the sequence $\{t_i\}$ defined by

$$t_0=\varrho;\quad t_{i+1}-t_i=(1-\tau)\,\tau^i(R-\varrho)$$

with $0<\tau<1$. By iteration

$$f(t_0)\leq\theta^k f(t_k)+\left[\frac{A}{(1-\tau)^\alpha}(R-\varrho)^{-\alpha}+B\right]\sum_{i=0}^{k-1}\theta^i\tau^{-i\alpha}.$$

We choose now τ such that $\tau^{-\alpha}\theta<1$ and let $k\to\infty$, getting (1.2) with $c=(1-\tau)^{-\alpha}(1-\theta\tau^{-\alpha})^{-1}$. Q.E.D.

2. The scalar case: local boundedness

In this section we shall consider local minimum points for the functional F, i.e. functions $u\in W^{1,m}_{\mathrm{loc}}(\Omega)$ such that for every $\varphi\in W^{1,m}(\Omega)$ with $\operatorname{supp}\varphi\subset\subset\Omega$ we have

$$F(u;\operatorname{supp}\varphi)\leq F(u+\varphi;\operatorname{supp}\varphi). \tag{2.1}$$

The function $f(x,u,p)$ satisfies hypotheses which are slightly more general than those described in the introduction: we shall suppose that

$$|p|^m-b(|u|^\alpha+1)\leq f(x,u,p)\leq a|p|^m+b(|u|^\alpha+1) \tag{2.2}$$

where a and b are positive constants, and $m\leq\alpha<m^*=mn/(n-m)$. In [17] it is proved the boundness of a function which minimizes $F(u;\Omega)$ among all functions taking prescribed values $z(x)$ at $\partial\Omega$, provided $z(x)$ is bounded: here we shall consider the problem of local boundedness of minima, in the sense of the definition above, independently of the boundary data.

THEOREM 2.1. *Let* (2.2) *hold, and let $u\in W^{1,m}_{\mathrm{loc}}(\Omega)$ be a local minimum for the functional F. Then u is locally bounded in Ω.*

Proof. We can suppose Ω bounded and $u\in W^{1,m}(\Omega)$. Let $x_0\in\Omega$, and denote by B_s the ball of radius s centred at x_0. For $k>0$ let

$$A_k = \{x \in \Omega : u(x) > k\} \tag{2.3}$$

and $A_{k,s} = A_k \cap B_s$. Let $w = \max(u-k, 0)$ and let $\eta(x)$ be a C^∞ function with $\operatorname{supp} \eta \subset B_s$, $0 \leq \eta \leq 1$, $\eta = 1$ on B_t, $|\nabla \eta| \leq 2(s-t)^{-1}$. If $v = u - \eta w$, we have using the minimality of u and (2.2),

$$\int_{A_{k,s}} |Du|^m \, dx \leq \gamma_1 \left\{ \int_{A_{k,s}} (1-\eta)^m |Du|^m \, dx + \int_{A_{k,s}} w^m |D\eta|^m \, dx + \int_{A_{k,s}} w^\alpha \, dx + (1+k^\alpha) \left| A_{k,s} \right| \right\}. \tag{2.4}$$

We observe now that if $w \in W^{1,m}(B_s)$ and $|\operatorname{supp} w| \leq \frac{1}{2}|B_s|$ we have the Sobolev inequality

$$\left(\int_{B_s} w^{m^*} \, dx \right)^{m/m^*} \leq c_1(n, m) \int_{B_s} |Dw|^m \, dx \tag{2.5}$$

and therefore, if $m \leq \alpha < m^*$,

$$\begin{aligned} \int_{B_s} w^\alpha \, dx &\leq \|w\|_{m^*}^{\alpha - m} \left| B_s \right|^{1-\alpha/m^*} \left(\int_{B_s} w^{m^*} \, dx \right)^{m/m^*} \\ &\leq c_1 \|w\|_{m^*}^{\alpha - m} \left| B_s \right|^{1-\alpha/m^*} \int_{B_s} |Dw|^m \, dx. \end{aligned} \tag{2.6}$$

We can choose T so small that for $s \leq T$ we get

$$c_1 \|u\|_{m^*}^{\alpha - m} |B_s|^{1-\alpha/m^*} \leq \frac{1}{2\gamma_1}. \tag{2.7}$$

On the other hand, we have

$$k^{m^*} |A_k| \leq \|u\|_{m^*}^{m^*}$$

and therefore for $k \geq k_0$ we have

$$|A_k| < \tfrac{1}{2} |B_{T/2}|.$$

For such values of k we then have $|\operatorname{supp} w| < \frac{1}{2}|B_{T/2}|$ and therefore, if $T/2 \leq s \leq T$,

$$\int_{B_s} w^\alpha \, dx \leq \frac{1}{2\gamma_1} \int_{A_{k,s}} |Du|^m \tag{2.8}$$

since $\|u\|_{m^*} \geq \|w\|_{m^*}$.

In conclusion, if $T/2 \leq t < s \leq T$, we have from (2.4) and (2.8),

$$\int_{A_{k,s}} |Du|^m dx \leq 2\gamma_1 \left\{ \int_{A_{k,s} \setminus A_{k,t}} |Du|^m dx + \frac{1}{(s-t)^m} \int_{A_{k,s}} w^m dx + (1+k^\alpha) |A_{k,s}| \right\}.$$

Suppose now $T/2 \leq \varrho \leq t < s \leq R \leq T$; we get

$$\int_{A_{k,t}} |Du|^m dx \leq 2\gamma_1 \int_{A_{k,s} \setminus A_{k,t}} |Du|^m dx + 2\gamma_1 \left\{ (s-t)^{-m} \int_{A_{k,R}} w^m dx + (1+k^\alpha) |A_{k,R}| \right\}.$$

Adding to both sides $2\gamma_1$ times the left-hand side we get eventually

$$\int_{A_{k,t}} |Du|^m dx \leq \frac{2\gamma_1}{2\gamma_1 + 1} \int_{A_{k,s}} |Du|^m dx + \left\{ (s-t)^{-m} \int_{A_{k,R}} w^m dx + (1+k^\alpha) |A_{k,R}| \right\}.$$

We can now apply Lemma 1.1 and conclude that

$$\int_{A_{k,\varrho}} |Du|^m dx \leq \gamma_2 \left\{ (R-\varrho)^{-m} \int_{A_{k,R}} w^m dx + (1+k^\alpha) |A_{k,R}| \right\}. \tag{2.9}$$

Finally, we estimate

$$\begin{aligned} (1+k^\alpha) |A_{k,R}| &\leq 2k^\alpha |A_{k,R}| \leq 2(k^{m^*} |A_{k,R}|)^{(\alpha-m)/m^*} k^m |A_{k,R}|^{1-(\alpha-m)/m^*} \\ &\leq 2\|u\|_{m^*}^{\alpha-m} k^m |A_{k,R}|^{1-(m/n)+(1-\alpha/m^*)}. \end{aligned}$$

Introducing the last expression into (2.9) we get for $T/2 \leq \varrho < R \leq T$,

$$\int_{A_{k,\varrho}} |Du|^m dx \leq \gamma_3 \left\{ (R-\varrho)^{-m} \int_{A_{k,R}} (u-k)^m dx + k^m |A_{k,R}|^{1-(m/n)+(1-\alpha/m^*)} \right\}. \tag{2.10}$$

Since $-u$ minimizes the functional

$$\bar{F}(v; \Omega) = \int_\Omega \bar{f}(x, v, Dv)\, dx$$

with $\bar{f}(x, v, p) = f(x, -v, -p)$ satisfying the same growth condition (2.2), inequality (2.10) holds with u replaced by $-u$. We may then apply to both u and $-u$ Lemma 5.4 of chapter II of [17] and conclude that u is bounded in $B_{T/2}$. Q.E.D.

The above result may easily be generalized; for instance one might assume that the constant b appearing in (2.2) is actually a function belonging to some suitable L^r space.

Moreover, one can assume that the minimizing function $u(x)$ belongs to $W^{1,m}(\Omega) \cap L^q(\Omega)$, for some $q \geq m^*$. In this case the conclusion of the theorem holds if

$$r > \frac{n}{m} \quad \text{and} \quad \alpha < m\frac{n+q}{n} - \frac{q}{r}.$$

3. The scalar case: Hölder continuity

An argument similar to the one above will give now the Hölder continuity of local minima for the functional $F(u)$. We suppose that the function $f(x,u,p)$ satisfies the growth condition

$$|p|^m - b(M) \leq f(x,u,p) \leq a(M)|p|^m + b(M) \tag{3.1}$$

for every $x \in \Omega$, $|u| \leq M$ and $p \in \mathbf{R}^n$.

THEOREM 3.1. *Let* (3.1) *hold and let* $u(x)$ *be a function in* $W^{1,m}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, *minimizing locally the functional* $F(u)$. *Then* u *is Hölder-continuous in* Ω.

Proof. We take as before $v = u - \eta w$; from $F(u) \leq F(v)$ we easily deduce, using (3.1) with $M = \sup|u|$,

$$\int_{A_{k,s}} |Du|^m\,dx \leq \gamma_4 \left\{ \int_{A_{k,s}} (1-\eta)^m |Du|^m\,dx + \int_{A_{k,s}} w^m |D\eta|^m\,dx + |A_{k,s}| \right\}.$$

Observing that $\eta = 1$ on B_t and that $|D\eta| \leq 2(s-t)^{-1}$ we get therefore for $R \geq s > t$,

$$\int_{A_{k,t}} |Du|^m\,dx \leq \gamma_5 \left\{ \int_{A_{k,s} \setminus A_{k,t}} |Du|^m\,dx + (s-t)^m \int_{A_{k,R}} (u-k)^m + |A_{k,R}| \right\}.$$

Arguing again as in the above, we conclude from Lemma 1.1,

$$\int_{A_{k,\varrho}} |Du|^m\,dx \leq \gamma_6 \left\{ (R-\varrho)^{-m} \int_{A_{k,R}} (u-k)^m\,dx + |A_{k,R}| \right\}. \tag{3.2}$$

The same inequality holds with u replaced by $-u$, and therefore the function u belongs to the class $B_m(\Omega, M, \gamma_6, 1, 0)$ of [17]. Applying Theorem 6.1 of chapter II of [17] we conclude that u is Hölder-continuous in Ω. Q.E.D.

Using the same argument it is not difficult to prove regularity up to the boundary for solutions of the Dirichlet problem, provided the boundary datum is itself Hölder-continuous on $\partial\Omega$ and $\partial\Omega$ is sufficiently smooth. In fact, inequality (3.2) still hold when the ball B_R intersects $\partial\Omega$, provided the constant k is greater than $\sup_{\partial\Omega\cap B_R} u$, so that we can apply the result of [17], chapter II.7.

4. The case $N\geq 1$. Estimates for the gradient

The purpose of this section is to prove an L^q estimate for the gradient of minima, in the vector valued case.

Results of this kind were proved first by B. V. Boyarskii [2] and N. G. Meyers [19] for solutions of linear elliptic equations; besides their intrinsic interest they are an essential tool in the study of regularity of solutions of non linear elliptic systems, following the method introduced in [9] (see also [10] [11] [15]).

We shall suppose that the function f satisfies the growth condition stated in the introduction:

$$|p|^m-k\leq f(x,u,p)\leq a|p|^m+k \tag{4.1}$$

in $\Omega\times\mathbf{R}^N\times\mathbf{R}^{nN}$. For $v\in L^1(A)$ we denote by $\fint_A v\,dx$ the average of v in A: $|A|^{-1}\int_A v\,dx$. We have

THEOREM 4.1. *Let f satisfy* (4.1) *and let $u\in W^{1,m}_{\mathrm{loc}}(\Omega,\mathbf{R}^N)$ be a local minimum for the functional F. Then there exists an exponent $q>m$ such that $u\in W^{1,q}_{\mathrm{loc}}(\Omega,\mathbf{R}^N)$. Moreover for every $R<\operatorname{dist}(x_0,\partial\Omega)$ we have*

$$\left(\fint_{B_{R/2}(x_0)}(1+|Du|)^q\,dx\right)^{1/q}\leq c\left(\fint_{B_R(x_0)}(1+|Du|)^m\,dx\right)^{1/m} \tag{4.2}$$

c being a constant depending only on a, k, N, n, m.

Proof. Let $x_0\in\Omega$ and $0<t<s<R<\operatorname{dist}(x_0,\partial\Omega)$. With the usual choise of η, let $u_R=\fint_{B_R(x_0)}u\,dx$ and let $v=u-\eta(u-u_R)$. From the minimality of u and (4.1) we get as usual

$$\int_{B_s}|Du|^m\,dx\leq\gamma_7\left\{\int_{B_s\setminus B_t}|Du|^m\,dx+\frac{1}{(s-t)^m}\int_{B_s}|u-u_R|^m\,dx+|B_s|\right\} \tag{4.3}$$

and therefore, arguing as in section 2,

$$\int_{B_{R/2}}|Du|^m\,dx\leq\gamma_8\left\{R^{-m}\int_{B_R}|u-u_R|^m\,dx+|B_R|\right\}. \tag{4.4}$$

We now use the Sobolev–Poincaré inequality,

$$\int_{B_R} |u-u_R|^m \, dx \leqslant c_1(n,m) \left(\int_{B_R} |Du|^r \, dx \right)^{m/r}, \quad r = \frac{nm}{n+m}$$

and we get from (4.4)

$$\fint_{B_{R/2}} (1+|Du|)^m \, dx \leqslant \gamma_9 \left(\fint_{B_R} (1+|Du|)^r \, dx \right)^{m/r}.$$

The result now follows at once from [10], Proposition 5.1. Q.E.D.

As we have already noted, results of this type were obtained previously for solutions of elliptic equations and systems. Recently, H. Attouch and C. Sbordone [1] have proved a conclusion similar to our Theorem 4.1 in the special case of $f=f(x,p)$ convex in p. It is worth remarking that the above theorem does not hold for extremals of the functional F, even assuming that f is convex in p and $N=1$, see J. Frehse [7]. When $N>1$, the result is in general false for elliptic systems, even if we assume that u is bounded (see [6]), and it is necessary to suppose that u is small ([9], [10]).

It is easily seen from the proof of Proposition 5.1 of [10] that the exponent $q<m$ can be taken in an interval $(m, m+\sigma)$, with σ independent of m for m close to n. We have therefore the following.

COROLLARY 4.2. *There exists a $\sigma>0$, depending only on a and k in* (4.1), *n and N such that if $m>n-\sigma$ and the function f satisfies the growth condition* (4.1), *then every local minimum of the functional F is Hölder-continuous in Ω.*

In particular, the above corollary extends Theorem 4.3.1 of Morrey's [20]. For elliptic systems, results of this type have been proved by K. O. Widman [24] (see also [21], [23], [10]).

5. Quadratic functionals

In this section we shall prove some regularity results for minima of quadratic functionals

$$F(u) = \int_\Omega A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j \, dx \quad (A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}). \tag{5.1}$$

Here summation over repeated indices is understood, the greek indices running from 1 to n, and the latin ones from 1 to N. We assume that the coefficients $A_{ij}^{\alpha\beta}$ are bounded continuous functions in $\Omega\times\mathbf{R}^N$ and satisfy the condition

$$A_{ij}^{\alpha\beta}(x,u)\,\xi_\alpha^i\xi_\beta^j\geq\lambda|\xi|^2,\quad \forall\xi\in\mathbf{R}^{nN},\quad \lambda>0. \tag{5.2}$$

For sake of simplicity we shall assume that the coefficients $A_{ij}^{\alpha\beta}$ are uniformly continous and bounded in $\Omega\times\mathbf{R}^N$. This implies in particular that there exists a continuous, increasing, concave function ω: $\mathbf{R}^+\to\mathbf{R}^+$ satisfying $\omega(0)=0, \omega(t)\leq M$, and such that

$$\left|A_{ij}^{\alpha\beta}(x,u)-A_{ij}^{\alpha\beta}(y,v)\right|\leq\omega(|x-y|^2+|u-v|^2). \tag{5.3}$$

THEOREM 5.1. *Let the hypotheses above be satisfied, and let* $u\in W_{\mathrm{loc}}^{1,2}(\Omega,\mathbf{R}^N)$ *be a local minimum for the functional* (5.1). *Then there exists an open set* $\Omega_0\subset\Omega$ *such that* $u\in C^{0,\alpha}(\Omega_0,\mathbf{R}^N)$ *for every* $\alpha<1$. *Moreover*

$$\Omega-\Omega_0=\left\{x_0\in\Omega:\liminf_{R\to 0}R^{2-n}\int_{B_R(x_0)}|Du|^2\,dx>\varepsilon_0\right\} \tag{5.4}$$

where ε_0 *is a positive constant independent of* u. *Finally*

$$H^{n-q}(\Omega-\Omega_0)=0 \tag{5.5}$$

for some $q>2$, H^{n-q} *denoting the* $(n-q)$*-dimensional Hausdorff measure.*

Proof. We use the ideas introduced in [9]. Let $x_0\in\Omega$, $R<\frac{1}{2}\mathrm{dist}\,(x_0,\partial\Omega)$ and let v be the solution of the problem

$$\begin{gathered}\int_{B_R(x_0)}A_{ij}(x_0,u_R)\,Dv^iDv^j\,dx\to\min.\\ v-u\in W_0^{1,2}(B_R(x_0),\mathbf{R}^N).\end{gathered} \tag{5.6}$$

Since the coefficients are now constant, the Euler operator is coercive and the problem has a unique solution. Moreover we have

$$\int_{B_R(x_0)}|Dv|^p\,dx\leq c_1\int_{B_R(x_0)}|Du|^p\,dx \tag{5.7}$$

and for every $\varrho<R$ (see [3]),

$$\int_{B_\varrho(x_0)} |Dv|^2\,dx \leq c_2\left(\frac{\varrho}{R}\right)^n \int_{B_R(x_0)} |Dv|^2\,dx. \tag{5.8}$$

Let now $w=u-v$; we have $w \in W_0^{1,2}(B_R, \mathbf{R}^N)$ and therefore

$$c_3 \int_{B_R} |Dw|^2\,dx \leq \int_{B_R} A_{ij}^{\alpha\beta}(x_0, u_R)\,D_\alpha w^i D_\beta w^j\,dx. \tag{5.9}$$

On the other hand

$$\int_{B_R} A_{ij}^{\alpha\beta}(x_0, u_R)\,D_\alpha v^i D_\beta w^j\,dx = 0$$

and therefore

$$\begin{aligned}\int_{B_R} A_{ij}^{\alpha\beta}(x_0, u_R)\,D_\alpha w^i D_\beta w^j\,dx &= \int_{B_R} A_{ij}^{\alpha\beta}(x_0, u_R)\,D_\alpha u^i D_\beta w^j\,dx \\ &= \int_{B_R} [A_{ij}^{\alpha\beta}(x_0, u_R) - A_{ij}^{\alpha\beta}(x, u)]\,D_\alpha(u^i+v^i)\,D_\beta w^j\,dx \\ &\quad + \int_{B_R} [A_{ij}^{\alpha\beta}(x, v) - A_{ij}^{\alpha\beta}(x, u)]\,D_\alpha v^i D_\beta v^j\,dx \\ &\quad + \int_{B_R} A_{ij}^{\alpha\beta}(x, u)\,D_\alpha u^i D_\beta u^j\,dx - \int_{B_R} A_{ij}^{\alpha\beta}(x, v)\,D_\alpha v^i D_\beta v^j\,dx.\end{aligned}$$

Since u minimizes F, and $u=v$ on ∂B_R, the sum of the last two terms is non positive. Using the Schwartz inequality $ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$ and assumption (5.3), we easily get from (5.9)

$$\int_{B_R} |Dw|^2\,dx \leq c_4 \int_{B_R} [|Du|^2+|Dv|^2][\omega^2(R^2+|u-u_R|^2)+\omega^2(R^2+|u-v|^2]\,dx. \tag{5.10}$$

Taking inte account the fact that ω is bounded, we have

$$\begin{aligned}\int_{B_R} |Du|^2\omega^2\,dx &\leq c_5\left(\int_{B_R} |Du|^q\,dx\right)^{2/q}\left(\int_{B_R} \omega\,dx\right)^{1-2/q} \\ &\leq c_6 \int_{B_{2R}} (1+|Du|)^2\,dx \cdot \left(⨍_{B_R} \omega\,dx\right)^{1-2/q}\end{aligned}$$

and using (5.7) with $p=q$

$$\fint_{B_R} |Dv|^2\omega^2\,dx \le c_6 \int_{B_{2R}} (1+|Du|)^2\,dx \left(\fint_{B_R} \omega\,dx\right)^{1-2/q}.$$

Since ω is concave, we have

$$\begin{aligned}\fint_{B_R} \omega(R^2+|u-v|^2)\,dx &\le \omega\left(R^2+\fint_{B_R} |u-v|^2\,dx\right)\\ &\le \omega\left(R^2+c_7R^{2-n}\int_{B_R} |Dw|^2\,dx\right)\\ &\le \omega\left(R^2+c_8R^{2-n}\int_{B_R} |Du|^2\,dx\right)\end{aligned}$$

and similarly

$$\fint_{B_R} \omega(R^2+|u-u_R|^2)\,dx \le \omega\left(R^2+c_8R^{2-n}\int_{B_R} |Du|^2\,dx\right).$$

In conclusion,

$$\int_{B_R} |Dw|^2\,dx \le c_9\,\omega\left(R^2+c_{10}R^{2-n}\int_{B_R} |Du|^2\,dx\right)^{1-2/q}\int_{B_{2R}} (1+|Du|)^2\,dx.$$

From (5.8) we now easily get

$$\int_{B_\varrho} (1+|Du|^2)\,dx \le c_{11}\left[\left(\frac{\varrho}{R}\right)^n+\omega\left(R^2+c_{10}R^{2-n}\int_{B_R} |Du|^2\,dx\right)^{1-2/q}\right]\int_{B_{2R}} (1+|Du|^2)\,dx \tag{5.11}$$

for every $\varrho<R<2R<\operatorname{dist}(x_0, \partial\Omega)$. The result now follows as in [9]. Q.E.D.

The case of (non uniformly) continuous coefficients needs some technical adjustments both in the statement and in the proof. We shall not discuss the details, and we limit ourselves to the remark that $\Omega-\Omega_0$ is now the union of the set given by (5.4) and of

$$\left\{x_0\in\Omega : \limsup_{R\to 0^+} \left|u_{x_0,R}\right| = +\infty\right\}.$$

Consequently, instead of (5.5) we have the weaker conclusion

$$H^{n-q+\varepsilon}(\Omega-\Omega_0)=0, \quad \varepsilon>0. \tag{5.12}$$

Let us now consider a special case of the functional (5.1), namely when the coefficients take the diagonal form

$$A_{ij}^{\alpha\beta}(x, u) = \delta_{ij} A^{\alpha\beta}(x, u) \tag{5.13}$$

so that

$$F(u) = \int_\Omega A^{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i \, dx. \tag{5.14}$$

We suppose, of course, that

$$\begin{cases} A^{\alpha\beta}(x, u) \xi_\alpha \xi_\beta \geq \lambda |\xi|^2, \quad \lambda>0; \\ |A^{\alpha\beta}(x, u)| \leq M. \end{cases} \tag{5.15}$$

Moreover, we shall assume that the coefficients are differentiable, so that every bounded local minimum u is a solution of the system of Euler equations

$$\int A^{\alpha\beta}(x, u) D_\alpha u^i D_\beta \varphi^i \, dx = \int g_i(x, u, Du) \varphi^i \, dx \tag{5.16}$$

for every φ in $W_0^{1,2}(\Omega, \mathbf{R}^N) \cap L^\infty(\Omega, \mathbf{R}^N)$, where

$$g_i(x, u, p) = -\frac{1}{2} A_{u^i}^{\alpha\beta}(x, u) p_\alpha^h p_\beta^h. \tag{5.17}$$

Systems of the type (5.16) have been considered in connection with the regularity of harmonic mappings by S. Hildebrandt and K. O. Widman [13] (see also [14]). They conjectured that every bounded solution of (5.16) was Hölder-continuous if the right-hand side g would satisfy the inequalities

$$\begin{aligned} |g(x, u, p)| &\leq a|p|^2 \\ u^i g_i(x, u, p) &\leq \lambda^* |p|^2 \end{aligned} \tag{5.18}$$

with $\lambda^* < \lambda$. The conjecture has been proved in dimension $n=2$ by M. Wiegner [26]. When $n>2$, M. Meier [18] and P. A. Ivert [16] have shown that *a-priori* estimates for the Hölder norm of solutions of (5.16) cannot exist, thus throwing serious doubts on the validity of the conjecture. Recently, M. Struwe [22] has shown with a counter-example that the result does not hold for $n \geq 3$.

The situation is completely different for minima of quadratic functionals. In fact, we have

THEOREM 5.2. *Assume that*

$$-\frac{1}{2}u^h A^{\alpha\beta}_{u^h}(x,u)\, D_\alpha u^i D_\beta u^i \leqslant \lambda^* |Du|^2 \tag{5.19}$$

with $\lambda^* < \lambda$. Then every bounded local minimum of the functional (5.14) *is Hölder-continuous.*

Proof. In view of Theorem 5.1 it is sufficient to prove that for every $x_0 \in \Omega$ we have

$$\varrho^{2-n} \int_{B_\varrho(x_0)} |Du|^2\, dx < \varepsilon_0^2 \tag{5.20}$$

for some $\varrho < \operatorname{dist}(x_0, \partial\Omega)$.

Let $R < \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega)$. Taking $\varphi = \eta u$ in (5.16), $\eta \in C_0^\infty(B_{2R}(x_0))$, $\eta \geqslant 0$, we get from (5.15) and (5.19)

$$\lambda \int_{B_{2R}} \eta |Du|^2\, dx \leqslant -\frac{1}{2}\int_{B_{2R}} A^{\alpha\beta} D_\alpha |u|^2 D_\beta \eta\, dx + \lambda^* \int_{B_{2R}} \eta |Du|^2\, dx \tag{5.21}$$

and since $\lambda > \lambda^*$, the function $z = M^2(2R) - |u|^2$, $M(t) = \sup_{B_t} |u|$, is a non-negative supersolution of an elliptic operator. From the weak Harnack inequality [12] we have therefore

$$R^{-n} \int_{B_{2R}} z\, dx \leqslant c_{12} \inf_{B_R} z. \tag{5.22}$$

Let now $w \in W_0^{1,2}(B_{2R})$ be the solution of the equation

$$\int_{B_{2R}} A^{\alpha\beta} D_\beta w D_\alpha \varphi\, dx = \frac{1}{R^2}\int_{B_{2R}} \varphi\, dx, \quad \forall \varphi \in W_0^{1,2}(B_{2R}).$$

Taking $\varphi = wz$ we easily get

$$\frac{1}{2}\int_{B_{2R}} A^{\alpha\beta} D_\beta w^2 D_\alpha z\, dx + \int_{B_{2R}} z A^{\alpha\beta} D_\alpha w\, D_\beta w\, dx = \frac{1}{R^2}\int_{B_{2R}} wz\, dx. \tag{5.23}$$

The second integral on the left-hand side is non-negative; moreover we have $w \leqslant \alpha_1$ in B_{2R} and from the weak Harnack inequality $w \geqslant \alpha_2 > 0$ in B_R, since w is a positive supersolution.

We note that the constants α_1 and α_2 do not depend on R. To see that, we perform the change of independent variables $x = Ry$. The function $v(y) = w(Ry)$ is a solution of the Dirichlet problem

$$\begin{cases} -D_\alpha(a^{\alpha\beta}D_\beta v)=1, & \text{in } B_2 \\ v=0, & \text{on } \partial B_2 \end{cases}$$

where $a^{\alpha\beta}(y)=A^{\alpha\beta}(Ry, u(Ry))$, and therefore

$$v(y)\leqslant \alpha_1, \quad \text{in } B_2$$
$$v(y)\geqslant \alpha_2>0, \quad \text{in } B_1$$

with a_1 and α_2 depending only on the constants λ and M in (5.15). The same constants α_1 and α_2 give an upper and lower bound for w in B_{2R} and B_R) respectively.

Let now $\eta=w^2$; we have from (5.23),

$$\int_{B_{2R}} A^{\alpha\beta}D_\alpha zD_\beta\eta\,dx\leqslant c_{13}R^{-2}\int_{B_{2R}} z\,dx$$

which together with (5.21) and (5.22) gives

$$\int_{B_R}|Du|^2\,dx\leqslant c_{14}R^{n-2}\inf_{B_R} z=c_{14}R^{n-2}[M^2(2R)-M^2(R)]. \tag{5.24}$$

On the other hand, we have

$$\sum_{k=0}^{\infty}[M^2(2^{1-k}R)-M^2(2^{-k}R)]\leqslant M^2(2R)\leqslant \sup_\Omega|u|^2 \tag{5.25}$$

and inequality (5.24) implies immediately (5.20) with $\varrho=2^{-k}R$ for some k and therefore the regularity of u. Q.E.D.

We remark that from (5.25) follows that the radius ϱ for which (5.20) holds can be estimated in terms of $\sup_\Omega|u|$ only and hence the Hölder norm of u in any compact set $K\subset\Omega$ is bounded in terms of dist $(K, \partial\Omega)$ and of $\sup_\Omega|u|$.

Finally, we observe that all the results in this paper hold for relative minima of the functional F; i.e. for functions u such that

$$F(u;\operatorname{supp}\varphi)\leqslant F(u+\varphi;\operatorname{supp}\varphi)$$

for every φ with compact support and small L^m-norm.

Moreover, the results of sections 2, 3 and 4 hold for *quasi-minima*, i.e. for functions u such that:

$$F(u,\operatorname{supp}\varphi)\leqslant A\,F(u+\varphi,\operatorname{supp}\varphi) \tag{5.26}$$

for some constant A independent of φ.

An important example of quasi-minima is that of quasi-conformal mappings (see [8] for the definition). Actually, if u is a quasi-conformal mapping it satisfies (5.26) with

$$F(x, E) = \int_E |Dv|^n \, dx,$$

and therefore by Corollary 4.2 it is Hölder-continuous; a result already proved in [8].

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