REARRANGEMENTS OF C_1-SUMMABLE SERIES

BY

F. BAGEMIHL and P. ERDÖS

Institute for Advanced Study, Princeton, N.J., and University of Notre Dame, Notre Dame, Ind.

§ 1. In this paper we shall be concerned with infinite series whose terms are real numbers. Suppose that the series

$$(1) \qquad \sum_{n=1}^{\infty} a_n$$

is absolutely convergent and has the sum s. Then, as is well known, every rearrangement, $\sum_{n=1}^{\infty} a_n'$, of (1) also converges and has the same sum s. If, however, (1) converges, but not absolutely, then, according to Riemann's classical rearrangement theorem [3, p. 235, or 2, p. 318], for every real number s', there exists a rearrangement of (1) whose sum is s'.

Assume, now, that (1) is C_1-summable [1, p. 7, or 2, p. 464], and that its C_1-sum is σ. Consider the set of all C_1-summable rearrangements of (1); what is the nature of the corresponding set of C_1-sums? We are going to answer this question; the answer turns out to be somewhat more complicated than Riemann's rearrangement theorem (and also more difficult to obtain). We shall show, namely, that, *for any C_1-summable series* (1), *the rearrangement set* (*cf.* Definition 1 below) *consists either of a single number, or of all numbers of the form $\alpha + \nu\beta$ ($\nu = 0, \pm 1, \pm 2, \ldots$) for some particular real numbers $\beta \neq 0$ and α, or of all the real numbers. Moreover, given any α, there exists a C_1-summable series* (1) *whose rearrangement set consists of the single number α; and, given any $\beta \neq 0$ and α, there exists a C_1-summable series* (1) *whose rearrangement set consists of all numbers of the form $\alpha + \nu\beta$ ($\nu = 0, \pm 1, \pm 2, \ldots$).*

We introduce

Definition 1. *The set of numbers ϱ such that the C_1-sum of some rearrangement of* (1) *is ϱ, will be denoted by R and called the rearrangement set of* (1).

In case (1) converges, the answer to our question is immediate, because the C_1-sum of a series whose sum is s, is s [1, p. 100, or 2, p. 461]. Hence, if (1) converges absolutely, every rearrangement of (1) has the C_1-sum s; if (1) converges conditionally, then, for every real number σ', there exists a rearrangement of (1) whose C_1-sum is σ'. Since this case is settled, we shall assume, from now on, that (1) is not only C_1-summable, but is also divergent.

If $\lim_{n\to\infty} a_n = 0$, then the answer is again immediate: an examination of Riemann's proof of his rearrangement theorem shows that, for every real number σ', there exists a rearrangement of (1) which actually converges to σ', and hence has the C_1-sum σ'. An example of this case is the series $1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \cdots$, where the nth group of consecutive terms with the same sign contains n terms, each of which is equal to $(-1)^{n-1}/n$; this series obviously diverges, and is easily seen to have the C_1-sum $\frac{1}{2}$.

§ 2. Instead of assuming, as in the preceding case, that $\lim_{n\to\infty} a_n = 0$, let us suppose merely that $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = 0$ and $\sum_{k=0}^{\infty} |a_{n_k}|$ diverges. We shall show that, given an arbitrary real number σ', there exists a rearrangement, $\sum_{n=1}^{\infty} a'_n$, of (1), whose C_1-sum is σ'.

We shall employ the notation $s_n = a_1 + a_2 + \cdots + a_n$, $\sigma_n = (s_1 + s_2 + \cdots + s_n)/n$ $(n = 1, 2, 3, \ldots)$, and define s'_n, σ'_n analogously for $\sum_{n=1}^{\infty} a'_n$. Our problem, then, is to show that there exists a rearrangement, $\sum_{n=1}^{\infty} a'_n$, of (1), such that $\lim_{n\to\infty} \sigma'_n = \sigma'$.

Since (1) is divergent and C_1-summable, the subseries of positive terms of (1) diverges, and the subseries of negative terms of (1) diverges. Furthermore, because of our suppositions in the last paragraph but one, there is a divergent subseries of $\sum_{k=0}^{\infty} a_{n_k}$ consisting exclusively either of non-negative or non-positive terms, and there is no loss of generality in assuming that the former is the case; this subseries, in turn, contains a convergent infinite subseries. The sequence $\{a_n\}$ is thus seen to contain infinite subsequences $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ with the following properties:

(i) $b_n < 0$ $(n = 0, 1, 2, \ldots)$, and $\sum_{n=0}^{\infty} b_n$ diverges;

(ii) $c_n \geqq 0$ $(n = 0, 1, 2, \ldots)$, and $\sum_{n=0}^{\infty} c_n$ converges;

(iii) $d_n > 0$ $(n = 0, 1, 2, \ldots)$, $\lim_{n\to\infty} d_n = 0$, and $\sum_{n=0}^{\infty} d_n$ diverges.

Now let $\varepsilon_1 > 4\varepsilon_2 > \cdots > 4^{k-1}\varepsilon_k > \cdots > 0$, $\varepsilon_1 > 2|a_1 - \sigma'|$, and $\lim_{k\to\infty} \varepsilon_k = 0$. We define a rearrangement, $\sum_{n=1}^{\infty} a'_n$, of (1), by means of induction, as follows: Put $m_1 = 1$, $a'_{m_1} = a'_1 = a_1$. Then $s'_1 = \sigma'_1 = a_1$, so that $|\sigma'_1 - \sigma'| < \varepsilon_1$ and $|s'_1 - \sigma'| < \varepsilon_1/2$. Let $k \geqq 1$, and suppose that the terms

$$a'_1,\ a'_2,\ \ldots,\ a'_{m_k} \qquad (m_k \geqq 1) \tag{2}$$

have already been defined so as to constitute a finite subsequence of $\{a_n\}$ such that

$$|\sigma'_{m_k} - \sigma'| < \varepsilon_k \text{ and } |s'_{m_k} - \sigma'| < \varepsilon_k/2 \tag{3}$$

(these inequalities hold for $k=1$, according to the preceding sentence). There is a first term, call it $a_1^{(k+1)}$, of $\{a_n\}$, which has not been used in forming the sequence (2). If $s'_{m_k} + a_1^{(k+1)} \geqq \sigma' + \frac{\varepsilon_{k+1}}{2}$, then, according to (i), there exists a finite subsequence of terms $b_1^{(k+1)}, b_2^{(k+1)}, \ldots, b_u^{(k+1)}$ of $\{b_n\}$ not already singled out of $\{a_n\}$ in the course of this induction, such that

$$s'_{m_k} + a_1^{(k+1)} + b_1^{(k+1)} + b_2^{(k+1)} + \cdots + b_u^{(k+1)} < \sigma' + \frac{\varepsilon_{k+1}}{2};$$

(if $s'_{m_k} + a_1^{(k+1)} < \sigma' + \frac{\varepsilon_{k+1}}{2}$, then simply ignore $b_1^{(k+1)}, \ldots, b_u^{(k+1)}$ wherever they occur; an analogous statement holds for $d_1^{(k+1)}, \ldots, d_v^{(k+1)}$ and $c_1^{(k+1)}, \ldots, c_{w+t}^{(k+1)}$ considered below). If

$$s'_{m_k} + a_1^{(k+1)} + b_1^{(k+1)} + \cdots + b_u^{(k+1)} \leqq \sigma' - \frac{\varepsilon_{k+1}}{2},$$

then, according to (iii), there exists a finite subsequence of terms $d_1^{(k+1)}, d_2^{(k+1)}, \ldots, d_v^{(k+1)}$ of $\{d_n\}$ not already singled out of $\{a_n\}$ in the course of this induction, such that $d_i^{(k+1)} < \varepsilon_{k+1}/2$ $(i = 1, 2, \ldots, v)$ and

$$|s'_{m_k} + a_1^{(k+1)} + b_1^{(k+1)} + \cdots + b_u^{(k+1)} + d_1^{(k+1)} + \cdots + d_v^{(k+1)} - \sigma'| < \frac{\varepsilon_{k+1}}{2}.$$

According to (ii), there exists a finite subsequence of terms $c_1^{(k+1)}, c_2^{(k+1)}, \ldots, c_w^{(k+1)}$ of $\{c_n\}$ not already singled out of $\{a_n\}$ in the course of this induction, the number, w, of these terms being as large as we please, such that

$$c_1^{(k+1)} + c_2^{(k+1)} + \cdots + c_w^{(k+1)} < \sigma' + \frac{\varepsilon_{k+1}}{2} - (s'_{m_k} + a_1^{(k+1)} + b_1^{(k+1)} + \cdots + b_u^{(k+1)} + d_1^{(k+1)} + \cdots + d_v^{(k+1)}). \tag{4}$$

If we put

$$a'_{m_k+1}=c_1^{(k+1)},\ \dots,\ a'_{m_k+w}=c_w^{(k+1)},\ a'_{m_k+w+1}=a_1^{(k+1)},\ a'_{m_k+w+2}=b_1^{(k+1)},\ \dots,$$

$$a'_{m_k+w+u+1}=b_u^{(k+1)},\ a'_{m_k+w+u+2}=d_1^{(k+1)},\ \dots,\ a'_{m_k+w+u+v+1}=d_v^{(k+1)},$$

then it is evident, from (3), (4), and the definition of σ'_n as the centroid of the system of points $s'_1, s'_2, \dots, s'_n$, that, by taking w large enough, we shall have

$$(5)\qquad |\sigma'_i-\sigma'|<\varepsilon_k \qquad (m_k\leqq i\leqq m_k+w+u+v+1).$$

On account of (4),

$$|s'_{m_k+w+u+v+1}-\sigma'|<\frac{\varepsilon_{k+1}}{2}.$$

As before, we can obtain a finite subsequence of unused terms $c_{w+1}^{(k+1)}, c_{w+2}^{(k+1)}, \dots, c_{w+t}^{(k+1)}$ of $\{c_n\}$, with t as large as we please, such that

$$(6)\qquad c_{w+1}^{(k+1)}+\cdots+c_{w+t}^{(k+1)}<\sigma'+\frac{\varepsilon_{k+1}}{2}-s'_{m_k+w+u+v+1}.$$

If we put

$$a'_{m_k+w+u+v+2}=c_{w+1}^{(k+1)},\ \dots,\ a'_{m_k+w+u+v+t+1}=c_{w+t}^{(k+1)},\ m_{k+1}=m_k+w+u+v+t+1,$$

(so that $m_k<m_{k+1}$) and bear in mind again the definition of σ'_n, it is evident from (5) and (6) that

$$(7)\qquad |\sigma'_i-\sigma'|<\varepsilon_k \qquad (m_k+w+u+v+1\leqq i\leqq m_{k+1}),$$

and that, if t is taken sufficiently large,

$$(8)\qquad |\sigma'_{m_{k+1}}-\sigma'|<\varepsilon_{k+1}.$$

Moreover, on account of (6),

$$|s'_{m_{k+1}}-\sigma'|<\frac{\varepsilon_{k+1}}{2}.$$

This completes the induction. The series $\sum_{n=1}^{\infty} a'_n$ is obviously a rearrangement of $\sum_{n=1}^{\infty} a_n$; and because of (5), (7), (8), and the fact that $\lim_{k\to\infty}\varepsilon_k=0$, we have $\lim_{n\to\infty}\sigma'_n=\sigma'$, q.e.d.

An example of this case, in which $\lim_{n\to\infty} a_n \neq 0$, can be obtained from the example given at the end of § 1 by inserting the terms $+1$ and -1 after each group of negative terms:

$$1-\tfrac{1}{2}-\tfrac{1}{2}+1-1+\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3}-\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{4}+1-1+\cdots;$$

the C_1-sum of this series is evidently also $\frac{1}{2}$.

§ 3. The next case to be considered is that in which (1), in addition to being divergent and C_1-summable to σ, has the property that, if $\{a_{n_k}\}$ is the subsequence of non-zero terms of $\{a_n\}$, then

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1 \tag{9}$$

and

$$0 < \delta < |a_{n_k}| \qquad (k=0, 1, 2, \ldots) \tag{10}$$

for some fixed constant δ independent of k.

We shall show that, under these conditions, given an arbitrary real number $\sigma' \neq \sigma$, there exists a rearrangement of (1) whose C_1-sum is σ'. We may assume, without loss of generality, that $\sigma < \sigma'$. For suppose that $\sigma' < \sigma$, so that $-\sigma < -\sigma'$. The series $\sum_{n=1}^{\infty}(-a_n)$, which also satisfies (9) and (10), has the C_1-sum $-\sigma$ [1, p. 8, or 2, p. 476], and if a rearrangement, say $\sum_{n=1}^{\infty}(-a_n')$, of this series has the C_1-sum $-\sigma'$, then the rearrangement $\sum_{n=1}^{\infty} a_n'$ of (1) has the C_1-sum σ'. We shall also assume that

$$0 < \sigma' - \sigma < \delta. \tag{11}$$

We shall obtain a rearrangement, $\sum_{n=1}^{\infty} a_n'$, of (1), whose C_1-sum is σ', and which has the property that, if $\{a'_{n'_k}\}$ is the subsequence of non-zero terms of $\{a_n'\}$, then $\lim_{k\to\infty} n'_{k+1}/n'_k = 1$ and $\delta < |a'_{n'_k}|$ $(k=0, 1, 2, \ldots)$, so that the analogues of conditions (9) and (10) are satisfied by this rearrangement of (1). Consequently, the procedure for obtaining this rearrangement can be applied successively a finite number of times, if necessary, so as to yield, finally, a rearrangement of (1) whose C_1-sum is an arbitrary $\sigma' > \sigma$ (σ' not necessarily satisfying (11)). Thus (11), which at first appears to be a serious restriction, entails no loss of generality either.

Since (1) is C_1-summable, we have [1, p. 101, or 2, p. 484]

$$s_m = o(m) \tag{12}$$

and

$$a_m = o(m). \tag{13}$$

The fact that the C_1-sum of (1) is σ is equivalent to the assertion that

$$\frac{1}{m}\sum_{k=1}^{m} s_k = \sigma + o(1). \tag{14}$$

Suppose that t_m is a natural number ($m=1, 2, 3, \ldots$). Then (14) implies that

$$\sigma+o(1)=\frac{1}{m+t_m}\sum_{k=1}^{m+t_m} s_k=\left(1+\frac{t_m}{m}\right)^{-1}\left\{\frac{1}{m}\sum_{k=1}^{m} s_k+\frac{1}{m}\sum_{k=m+1}^{m+t_m} s_k\right\}$$
$$=\left(1+\frac{t_m}{m}\right)^{-1}\left\{\sigma+o(1)+\frac{1}{m}\sum_{k=1}^{t_m} s_{m+k}\right\},$$

so that

$$\sum_{k=1}^{t_m} s_{m+k}=t_m\,\sigma+\{m\cdot o(1)+t_m\cdot o(1)\}.$$

If $\{m/t_m\}$ is a bounded sequence, or if $m/t_m\to\infty$ sufficiently slowly, then $m\cdot o(1)+$ $+t_m\cdot o(1)=o(t_m)$, and hence

$$\sum_{k=1}^{t_m} s_{m+k}=t_m\,\sigma+o(t_m). \tag{15}$$

An immediate consequence of (12) is that if $\{m/t_m\}$ is a bounded sequence, or if $m/t_m\to\infty$ sufficiently slowly, then

$$s_{m+t_m}=o(t_m). \tag{16}$$

Now let $\{a_{p_m}\}$ be the subsequence of positive terms of $\{a_n\}$. Since (1) is C_1-summable and divergent, $\{a_{p_m}\}$ is an infinite sequence. Furthermore, we have

$$\lim_{m\to\infty}\frac{p_{m+1}}{p_m}=1. \tag{17}$$

For if (17) were false, there would be an infinite subsequence $\{p_{m_h}\}$ of $\{p_m\}$ such that, for some fixed constant $c>0$,

$$p_{m_h+1}/p_{m_h}>1+c \qquad (h=1, 2, 3, \ldots). \tag{18}$$

For all sufficiently large values of h, and for every natural number $i\leq\left[\frac{c}{2}p_{m_h}\right]$ (where $[x]$ denotes the greatest integer in x), let $\nu(h, i)$ be the number of terms

$$a_j \quad \left(j=p_{m_h}+i,\ p_{m_h}+1+i,\ p_{m_h}+2+i,\ \ldots,\ \left[\left(1+\frac{c}{2}\right)p_{m_h}\right]+i\right)$$

that are negative, and set

$$N_h=\min_{i\leq\left[\frac{c}{2}p_{m_h}\right]}\nu(h, i).$$

Then relations (9) and (18) imply that

$$\lim_{h\to\infty} N_h = +\infty. \tag{19}$$

Bearing in mind (10) and (18), we see that

$$s_{p_{m_h}+i} - s_{[(1+\frac{c}{2})p_{m_h}]+i} \geqq \nu(h, i)\cdot\delta \qquad \left(i \leqq \left[\frac{c}{2}p_{m_h}\right]\right),$$

and hence

$$\sum_{i=1}^{[\frac{c}{2}p_{m_h}]} \left(s_{p_{m_h}+i} - s_{[(1+\frac{c}{2})p_{m_h}]+i}\right) \geqq \left[\frac{c}{2}p_{m_h}\right]\cdot N_h\cdot\delta. \tag{20}$$

It follows from (15), however, since $\left\{p_{m_h}\Big/\left[\frac{c}{2}p_{m_h}\right]\right\}$ and $\left\{p_{m_h}\Big/\left[\left(1+\frac{c}{2}\right)p_{m_h}\right]\right\}$ are bounded sequences, that

$$\sum_{i=1}^{[\frac{c}{2}p_{m_h}]} s_{p_{m_h}+i} = \left[\frac{c}{2}p_{m_h}\right]\sigma + o(p_{m_h})$$

and

$$\sum_{i=1}^{[\frac{c}{2}p_{m_h}]} s_{[(1+\frac{c}{2})p_{m_h}]+i} = \left[\frac{c}{2}p_{m_h}\right]\sigma + o(p_{m_h}),$$

so that

$$\sum_{i=1}^{[\frac{c}{2}p_{m_h}]} \left(s_{p_{m_h}+i} - s_{[(1+\frac{c}{2})p_{m_h}]+i}\right) = o(p_{m_h}).$$

This, in view of (19), contradicts (20). Therefore (17) must be true.

Because of (17), we can choose an infinite subsequence, $\{q_m\}$, of $\{p_m\}$, such that, as $m\to\infty$, $q_{m+1}/q_m \to 1$ as slowly as desired; let us do this in such a way that the following conditions are satisfied:

$$s_{q_m+1} + s_{q_m+2} + \cdots + s_{q_{m+1}} = (q_{m+1} - q_m)\cdot\sigma + o(q_{m+1} - q_m) \tag{21}$$

(this is (15) with t_m replaced by $q_{m+1} - q_m$ and $m+k$ replaced by $q_m + k$);

$$s_{q_{m+1}} = o(q_{m+1} - q_m) \tag{22}$$

(this is (16) with t_m replaced by $q_{m+1} - q_m$);

(23) $\lim_{m\to\infty} \frac{s_{v_m}}{v_m}\cdot\frac{q_{m+1}}{q_{m+1}-q_m} = 0$, if, for every m, v_m is an integer such that $q_m < v_m < q_{m+1}$

(this condition can be satisfied because, according to (12), as $m \to \infty$, $s_m/m \to 0$ at a certain fixed rate);

$$\lim_{m\to\infty} \frac{a_{q_{m+1}}}{q_{m+1}} \cdot \frac{q_{m+1}}{q_{m+1}-q_m} = 0 \tag{24}$$

(this condition can be satisfied because, according to (13), as $m \to \infty$, $a_m/m \to 0$ at a certain fixed rate).

For every natural number m, consider the expression

$$u_m' = (q_{m+1}-q_m) \frac{(\sigma'-\sigma) + \dfrac{a_{q_{m+1}}}{q_{m+1}-q_m}}{a_{q_{m+1}}} - 1, \tag{25}$$

and set $[u_m'] = u_m$. Because of (24), (11), (10), and the definition of the sequence $\{a_{p_m}\}$, we have, for all sufficiently large values of m, say for $m \geqq m^*$, $1 < u_m' < q_{m+1} - q_m - 1$, and hence

$$1 \leqq u_m < q_{m+1} - q_m - 1. \tag{26}$$

It follows from (25), (24), and the definition of u_m, that

$$(u_m + 1)\, a_{q_{m+1}} = (q_{m+1} - q_m)(\sigma' - \sigma) + o\,(q_{m+1} - q_m). \tag{27}$$

Now, for every $m \geqq m^*$, put

$$\begin{aligned} a_k' &= a_k \quad (q_m + 1 \leqq k \leqq q_{m+1} - (u_m + 1)), \quad a'_{q_{m+1}-u_m} = a_{q_{m+1}}, \\ a_k' &= a_{k-1} \quad (q_{m+1} - (u_m - 1) \leqq k \leqq q_{m+1}), \end{aligned} \tag{28}$$

and let $a_k' = a_k$ for every natural number $k \leqq q_{m^*}$. Then $\sum_{k=1}^{\infty} a_k'$ is obviously a rearrangement of (1), and we are going to show that the C_1-sum of this rearrangement is σ'.

According to (28), for every $m \geqq m^*$,

$$\begin{aligned} s'_{q_m+1} + s'_{q_m+2} + \cdots + s'_{q_{m+1}} &= (s_{q_m+1} + s_{q_m+2} + \cdots + s_{q_{m+1}-1}) + s_{q_{m+1}-(u_m+1)} + (u_m+1)\, a_{q_{m+1}} \\ &= ((q_{m+1}-q_m)\cdot\sigma + o\,(q_{m+1}-q_m)) + o\,(q_{m+1}-q_m) + ((q_{m+1}-q_m)(\sigma'-\sigma) + o\,(q_{m+1}-q_m)) \\ &= (q_{m+1}-q_m)\cdot\sigma' + o\,(q_{m+1}-q_m); \end{aligned} \tag{29}$$

the second equality is obtained by making use of (21) and (22), (23) with $v_m = q_{m+1} - (u_m + 1)$ (bearing in mind (26)), and (27).

We shall now show that

$$\sum_{k=1}^{n} s'_k = n\,\sigma' + o(n), \tag{30}$$

which immediately implies that the C_1-sum of $\sum_{k=1}^{\infty} a'_k$ is σ'. If n is a sufficiently large natural number, then there exists an $m \geqq m^*$ such that

$$q_m < n \leqq q_{m+1}. \tag{31}$$

We have

$$\sum_{k=1}^{n} s'_k = \sum_{k=1}^{q_m} s'_k + \sum_{k=q_m+1}^{n} s'_k. \tag{32}$$

Because of (29),

$$\begin{aligned}
\sum_{k=1}^{q_m} s'_k &= \sum_{k=1}^{q_{m^*}} s'_k + \left(\sum_{k=q_{m^*}+1}^{q_{m^*+1}} s'_k + \sum_{k=q_{m^*+1}+1}^{q_{m^*+2}} s'_k + \cdots + \sum_{k=q_{m-1}+1}^{q_m} s'_k \right) \\
&= o(q_m) + \sum_{k=m^*+1}^{m} ((q_k - q_{k-1}) \cdot \sigma' + o(q_k - q_{k-1})) \\
&= o(q_m) + (q_m - q_{m^*}) \cdot \sigma' + o(q_m - q_{m^*}) \\
&= q_m \cdot \sigma' + o(q_m) \\
&= n \cdot \sigma' + ((q_m - n) \cdot \sigma' + o(q_m)) \\
&= n \cdot \sigma' + o(n),
\end{aligned} \tag{33}$$

the third equality resulting from [2, p. 77, 4], and the last equality being a consequence of (31) and the fact that $q_m/q_{m+1} \to 1$ as $m \to \infty$. On account of (28), there exists an integer r satisfying the relation

$$0 \leqq r \leqq u_m + 1, \tag{34}$$

such that

$$\sum_{k=q_m+1}^{n} s'_k = (s_{q_m+1} + s_{q_m+2} + \cdots + s_{n-1}) + s_{n-r} + r\,a_{q_m+1}. \tag{35}$$

If we make use of (14), (31), and the fact that $q_m/q_{m+1} \to 1$ as $m \to \infty$, we see that

$$s_{q_m+1} + s_{q_m+2} + \cdots + s_{n-1} = o(n); \tag{36}$$

with the aid of (12), (34), (31), and (26), we obtain

$$s_{n-r} = o(n); \tag{37}$$

and (34), (31), and (27) yield

(38) $$r\, a_{q_{m+1}} = o(n).$$

Combining (35), (36), (37), and (38), it follows that

(39) $$\sum_{k=q_m+1}^{n} s'_k = o(n).$$

Relation (30) is now a consequence of (32), (33), and (39), and if we bear in mind (28), it is evident, finally, that the assertion following (11) is true.

An example of this case is the familiar series $1-1+1-1+-\cdots$, whose C_1-sum is $\frac{1}{2}$.

§ 4. The case to be considered in this section, in contrast to those treated in the foregoing sections, exhibits a departure from the Riemann rearrangement theorem.

We shall assume that (1) is divergent and C_1-summable, and satisfies the following condition:

(40) $$\begin{cases} \text{For every } C_1\text{-summable rearrangement, } \sum_{n=1}^{\infty} a'_n, \text{ of (1), if } \{a'_{n_k}\} \\ \text{is the sequence of non-zero terms of } \{a'_n\}, \text{ then } \overline{\lim_{k\to\infty}} \frac{n_{k+1}}{n_k} > 1. \end{cases}$$

We shall also suppose that

(41) the C_1-sum of (1) is 0.

This entails no loss of generality. For if b is a real number, and the C_1-sum of $a_1+a_2+\cdots+a_n+a_{n+1}+\cdots$ is σ, then the C_1-sum of $a_1+a_2+\cdots+a_n+b+a_{n+1}+\cdots$ is $\sigma+b$, and conversely [1, p. 102]. Hence, if the C_1-sum of $a_1+a_2+\cdots+a_n+\cdots$ is σ, then a rearrangement of this series is C_1-summable to σ' if, and only if, a rearrangement of the series $-\sigma+a_1+a_2+\cdots+a_n+\cdots$ is C_1-summable to $\sigma'-\sigma$, and the addition of a single new term to our original series (1) does not invalidate (40) or the assumptions just preceding it.

We proceed to prove a series of lemmas.

Lemma 1. *Let $\sum_{n=1}^{\infty} a'_n$ be a rearrangement of* (1) *and have the C_1-sum α, so that $\alpha \in R$* (*cf.* Definition 1 in § 1), *and let $\{a'_{n_k}\}$ be the sequence of non-zero terms of $\{a'_n\}$. Then there exists an infinite subsequence, $\{n_{k_j}\}$, of $\{n_k\}$ such that*

(42) $$\lim_{j\to\infty} s'_{n_{k_j}} = \alpha.$$

Proof: According to (40), there exists a constant $c>1$ such that $\overline{\lim_{k\to\infty}}\,(n_{k+1}/n_k)=c$. Hence, there exists an infinite subsequence, $\{n_{k_j}\}$, of $\{n_k\}$ such that $\lim_{j\to\infty}(n_{k_j+1}/n_{k_j})=c$, so that we may write

$$n_{k_j+1}=n_{k_j}(c+\varepsilon_j) \qquad (j=1,2,3,\ldots), \tag{43}$$

where

$$\lim_{j\to\infty}\varepsilon_j=0. \tag{44}$$

It follows from the definition of the sequence $\{n_k\}$, that $a'_m=0$ if $n_{k_j}<m<n_{k_j+1}$. Using this fact as well as (43), we see that

$$\frac{1}{n_{k_j+1}}(s'_1+s'_2+\cdots+s'_{n_{k_j}}+s'_{n_{k_j}+1}+\cdots+s'_{n_{k_j+1}})=$$
$$=\frac{1}{(c+\varepsilon_j)}\cdot\frac{s'_1+\cdots+s'_{n_{k_j}}}{n_{k_j}}+\frac{n_{k_j+1}-n_{k_j}-1}{n_{k_j+1}}(s'_{n_{k_j}})+\frac{s'_{n_{k_j+1}}}{n_{k_j+1}}.$$

Solving this equation for the $s'_{n_{k_j}}$ in parentheses in the preceding line, and making use of (43), (44), the fact that the C_1-sum of $\sum_{n=1}^{\infty}a'_n$ is α, and (12), the relation (42) is obtained.

Suppose that $\sum_{k=1}^{\infty}b_k$ is an infinite series, and $b_{k_1}, b_{k_2}, \ldots, b_{k_p}$ $(k_1<k_2<\cdots<k_p)$ is a finite subsequence of the sequence $\{b_k\}$. Then we shall call $b_{k_1}+b_{k_2}+\cdots+b_{k_p}$ a subsum of $\sum_{k=1}^{\infty}b_k$.

Definition 2. *The number β is initially accessible by $\sum_{k=1}^{\infty}b_k$ provided that, for every $\varepsilon>0$ and every natural number n, there exists a subsum, $b_{k_1}+b_{k_2}+\cdots+b_{k_p}$, of $\sum_{k=1}^{\infty}b_k$ such that every b_k $(k\leqq n)$ is a term of this subsum, and $|b_{k_1}+b_{k_2}+\cdots+b_{k_p}-\beta|<\varepsilon$.*

Definition 3. *The number β is terminally accessible by $\sum_{k=1}^{\infty}b_k$ provided that, for every $\varepsilon>0$ and every natural number n, there exists a subsum, $b_{k_1}+b_{k_2}+\cdots+b_{k_p}$, of $\sum_{k=1}^{\infty}b_k$ such that no b_k $(k\leqq n)$ is a term of this subsum, and $|b_{k_1}+b_{k_2}+\cdots+b_{k_p}-\beta|<\varepsilon$.*

Lemma 2. *If $\alpha\in R$, then α is initially accessible by* (1).

Proof: Let $\sum_{k=1}^{\infty}a'_k$ have the C_1-sum α and be a rearrangement of (1). According to Lemma 1, there exists an infinite subsequence $\{s'_{k_i}\}$ of the sequence $\{s'_k\}$ of partial sums of this rearrangement, such that $\lim_{i\to\infty}s'_{k_i}=\alpha$. Now let $\varepsilon>0$ and the natural number n be given. For all sufficiently large values of i, $|s'_{k_i}-\alpha|<\varepsilon$. Moreover, since $\sum_{k=1}^{\infty} a'_k$ is a rearrangement of (1), every a_k $(k\leqq n)$ is a term of the partial sum s'_{k_i}, provided that i is sufficiently large. The truth of Lemma 2 is now evident.

Lemma 3. *If* α *is initially accessible by* (1), *then* $\alpha \in R$.

Proof: We shall employ an argument which is similar to, but simpler than, the one used in § 2. Let ε be a positive number satisfying the relation $\varepsilon>|a_1-\alpha|$. We define a rearrangement, $\sum_{n=1}^{\infty} a'_n$, of (1), by means of induction, as follows: Put $m_1=1$, $a'_{m_1}=a'_1=a_1$, so that $s'_1=\sigma'_1=a_1$ and consequently $|\sigma'_1-\alpha|<\varepsilon$ and $|s'_1-\alpha|<\varepsilon$. Let $j\geqq 1$, and suppose that the terms

$$a'_1,\ a'_2,\ \ldots,\ a'_{m_j} \qquad (m_j\geqq 1) \tag{45}$$

have already been defined so as to constitute a finite subsequence of $\{a_n\}$ such that

$$|\sigma'_{m_j}-\alpha|<\frac{\varepsilon}{j} \text{ and } |s'_{m_j}-\alpha|<\frac{\varepsilon}{j} \tag{46}$$

(note that these inequalities hold for $j=1$). Since, by hypothesis, α is initially accessible by (1), there exists a subsum, call it S_{j+1}, of (1), such that, if n_j is the largest index possessed in (1) by any term of (45), then every a_k $(k\leqq n_j)$ is a term of S_{j+1}, and

$$|S_{j+1}-\alpha|<\frac{\varepsilon}{j+1}. \tag{47}$$

If there are any terms of S_{j+1} which are not terms of (45), denote them by $a_1^{(j+1)}$, $a_2^{(j+1)}, \ldots, a_v^{(j+1)}$. On account of (40), infinitely many terms of (1) are equal to zero. Let $z_1^{(j+1)}, z_2^{(j+1)}, \ldots, z_{w+t}^{(j+1)}$ $(w\geqq 1)$ be a finite subsequence of terms, all of them equal to zero, of $\{a_n\}$, not already singled out of the latter sequence in the course of this induction. It is evident from the meaning of σ'_n, that, if w is chosen large enough, and if we put

$$a'_{m_j+1}=z_1^{(j+1)},\ a'_{m_j+2}=z_2^{(j+1)},\ \ldots,\ a'_{m_j+w}=z_w^{(j+1)},\ a'_{m_j+w+1}=a_1^{(j+1)},$$
$$a'_{m_j+w+2}=a_2^{(j+1)},\ \ldots,\ a'_{m_j+w+v}=a_v^{(j+1)},$$

then, because of (46),

$$|\sigma'_i-\alpha|<\frac{\varepsilon}{j} \qquad (m_j\leqq i\leqq m_j+w+v), \tag{48}$$

and, because of (47),

(49) $$|s'_{m_j+w+v}-\alpha|<\frac{\varepsilon}{j+1}.$$

Referring again to the meaning of σ'_n, it is clear from (48) and (49) that, if t is taken large enough, and if we put

$$a'_{m_j+w+v+1}=z^{(j+1)}_{w+1},\ a'_{m_j+w+v+2}=z^{(j+1)}_{w+2},\ \ldots,\ a'_{m_j+w+v+t}=z^{(j+1)}_{w+t},$$

and set $m_j+w+v+t=m_{j+1}$, then, since $w\geqq 1$, we have $m_{j+1}>m_j$, and

(50) $$|\sigma'_i-\alpha|<\frac{\varepsilon}{j} \qquad (m_j+w+v\leqq i\leqq m_{j+1}),$$

(51) $$|\sigma'_{m_{j+1}}-\alpha|<\frac{\varepsilon}{j+1},$$

and $|s'_{m_{j+1}}-\alpha|<\varepsilon/(j+1)$. This completes the induction. The series $\sum_{n=1}^{\infty}a'_n$ thus defined is obviously a rearrangement of (1), and it follows from (46), (48), (50), and (51) that $\lim_{i\to\infty}\sigma'_i=\alpha$, q.e.d.

Lemma 4. *If* $\alpha\in R$, *then* $-\alpha$ *is terminally accessible by* (1).

Proof: Let $\varepsilon>0$ and the natural number n be given. By hypothesis and Lemma 2, α is initially accessible by (1). Hence, there is a subsum, S, of (1) such that every a_k $(k\leqq n)$ is a term of S, and

(52) $$|S-\alpha|<\frac{\varepsilon}{2}.$$

According to (41) and Lemma 2, 0 is initially accessible by (1). Hence, there is a subsum, T, of (1) such that every term of S is a term of T, at least one term of T is not a term of S, and

(53) $$|T|<\frac{\varepsilon}{2}.$$

Let U be the subsum of (1) consisting of those terms of T that are not terms of S. Then

(54) $$U=T-S,$$

no a_k $(k\leqq n)$ is a term of U, and (52), (53), and (54) imply that $|U+\alpha|<\varepsilon$, which means that $-\alpha$ is terminally accessible by (1), q.e.d.

Lemma 5. *If* $\alpha\in R$, *then* -2α *is terminally accessible by* (1).

Proof: Let $\varepsilon > 0$ and the natural number n be given. According to Lemma 4, there exists a subsum, S, of (1) such that no a_k $(k \leqq n)$ is a term of S, and

$$|S + \alpha| < \frac{\varepsilon}{2}. \tag{55}$$

Let n' denote the largest index possessed in (1) by any term of S. Then there exists a subsum, T, of (1) such that no a_k $(k \leqq n')$ is a term of T, and

$$|T + \alpha| < \frac{\varepsilon}{2}. \tag{56}$$

Let U be the subsum of (1) consisting of the terms of S and the terms of T. Then

$$U = S + T, \tag{57}$$

no a_k $(k \leqq n)$ is a term of U, and (55), (56), and (57) imply that $|U + 2\alpha| < \varepsilon$, which means that -2α is terminally accessible by (1), q.e.d.

Lemma 6. *If $\alpha \in R$, then $-\alpha$ is initially accessible by* (1).

Proof: Let $\varepsilon > 0$ and the natural number n be given. By hypothesis and Lemma 2, there exists a subsum, S, of (1) such that every a_k $(k \leqq n)$ is a term of S, and

$$|S - \alpha| < \frac{\varepsilon}{2}. \tag{58}$$

Let n' denote the largest index possessed in (1) by any term of S. By Lemma 5, there exists a subsum, T, of (1) such that no a_k $(k \leqq n')$ is a term of T, and

$$|T + 2\alpha| < \frac{\varepsilon}{2}. \tag{59}$$

Let U be the subsum of (1) consisting of the terms of S and the terms of T. Then (57) holds, every a_k $(k \leqq n)$ is a term of U, and (58), (59), and (57) imply that $|U + \alpha| < \varepsilon$, which means that $-\alpha$ is initially accessible by (1), q.e.d.

An immediate consequence of Lemma 6 and Lemma 3 is

Corollary 1. *If $\alpha \in R$, then $-\alpha \in R$.*

Lemma 7. *If $\beta \in R$ and $\gamma \in R$, then $\beta + \gamma \in R$.*

Proof: According to Corollary 1, $-\beta \in R$, and hence, by Lemma 4, β is terminally accessible by (1). On account of Lemma 2, γ is initially accessible by (1). An argument analogous to that used in the proof of Lemma 6 now shows that $\beta + \gamma$ is initially accessible by (1), and then Lemma 3 implies that $\beta + \gamma \in R$, q.e.d.

Corollary 2. *If* $\alpha \in R$, *then* $m\alpha \in R$ $(m=0, \pm 1, \pm 2, \ldots)$.

Lemma 8. *The set* R *is closed.*

Proof: Let I be the set of numbers that are initially accessible by (1). According to Lemmas 2 and 3, $R=I$. Suppose that λ is a limit point of I; we have to show that $\lambda \in I$. Let $\varepsilon>0$ and the natural number n be given. Then there exists a number $\mu \in I$ such that

$$|\mu-\lambda|<\frac{\varepsilon}{2}. \tag{60}$$

There is a subsum, S, of (1) such that every a_k $(k \leqq n)$ is a term of S, and

$$|S-\mu|<\frac{\varepsilon}{2}. \tag{61}$$

From (60) and (61) it follows that $|S-\lambda|<\varepsilon$, and consequently $\lambda \in I$, q.e.d.

It is evident now from Corollary 2 and Lemma 8, that there are only three possibilities:

(A) $R=\{0\}$;

(B) $R=\{m\,\alpha\}_{m=0,\,\pm 1,\,\pm 2,\,\ldots}$ for some $\alpha \neq 0$;

(C) R is the set of real numbers.

We shall show, by means of examples, that each of these possibilities can actually be realized.

Example A. Let

$$a_n=\begin{cases} 2^{2^k-k} & \text{if } n=2^{2^k} \\ -2^{2^k-k} & \text{if } n=2^{2^k}+1 \\ 0 & \text{if } n \text{ is any other natural number.}\end{cases} \qquad (k=1, 2, 3, \ldots)$$

Since $\overline{\lim\limits_{n\to\infty}}\, a_n \neq 0$, (1) diverges.

Suppose that $n \geqq 4$. Then there is a $k \geqq 1$ such that $2^{2^k} \leqq n < 2^{2^{k+1}}$. We have

$$\sigma_n=\frac{1}{n}\sum_{j=1}^{k} 2^{2^j-j} \leqq \frac{1}{2^{2^k}}\cdot k\cdot 2^{2^k-k}=k\cdot 2^{-k},$$

and hence $\lim\limits_{n\to\infty} \sigma_n=0$, so that (41) holds.

Let $\sum\limits_{n=1}^{\infty} a_n'$ be a C_1-summable rearrangement of (1). Then there exists an infinite subsequence, $\{n_i\}$, of $\{n\}$ such that

(62) $$a'_j = 0 \qquad (n_i < j \leqq 2\,n_i;\ i = 1, 2, 3, \ldots).$$

For if this is not so, then, for every sufficiently large n, there is at least one j satisfying $n < j \leqq 2\,n$ and $a'_j \neq 0$; hence, for every sufficiently large k, there are at least 2^k values of j satisfying $2^{2^k} < j \leqq 2^{2^{k+1}}$ and $a'_j \neq 0$. There are precisely $2\,k+2$ values of n satisfying $1 \leqq n \leqq 2^{2^{k+1}} + 1$ and $a_n \neq 0$. Consequently, for every sufficiently large k, since $2^k > 2\,k+2$, there is at least one m_k satisfying $2^{2^k} < m_k \leqq 2^{2^{k+1}}$ and $|a'_{m_k}| \geqq 2^{2^{k+2}-k-2}$, which implies that

$$\frac{|a'_{m_k}|}{m_k} \geqq \frac{2^{2^{k+2}-k-2}}{2^{2^{k+1}}} = 2^{2^{k+1}-k-2},$$

so that $\lim\limits_{k\to\infty} |a'_{m_k}|/m_k = \infty$, contradicting the fact that $\sum\limits_{n=1}^{\infty} a'_n$ is C_1-summable (cf. (13)). An immediate consequence of (62) is that (40) holds.

Suppose that there are infinitely many values of i such that $s'_{n_i} = 0$ (throughout the rest of this paragraph, let i represent only these values). Then, because of (62), we have also $s'_j = 0$ $(n_i < j \leqq 2\,n_i)$. Hence,

$$\sigma' = \lim_{i\to\infty} \frac{s'_1 + s'_2 + \cdots + s'_{n_i}}{n_i} = \lim_{i\to\infty} \frac{s'_1 + s'_2 + \cdots + s'_{n_i} + 0}{2\,n_i} = \tfrac{1}{2}\,\sigma',$$

so that $\sigma' = 0$.

Suppose, however, that $s'_{n_i} \neq 0$ for every sufficiently large value of i. Then there is a largest value of k, call it k_i, such that one of the terms a'_j $(1 \leqq j \leqq n_i)$ is either equal to $2^{2^{k_i}-k_i}$ or to $-2^{2^{k_i}-k_i}$, but none of these terms is equal to $-2^{2^{k_i}-k_i}$, $2^{2^{k_i}-k_i}$, respectively. Since $\sum\limits_{n=1}^{\infty} a'_n$ is a rearrangement of (1), $\lim\limits_{i\to\infty} k_i = \infty$. Now

$$|s'_{n_i}| \geqq 2^{2^{k_i}-k_i} - \sum_{j=1}^{k_i-1} 2^{2^j-j} \geqq 2^{2^{k_i}-k_i} - (k_i - 1)\,2^{2^{k_i-1}-k_i+1} = 2^{2^{k_i-1}-k_i+1}\,(2^{2^{k_i-1}-1} - k_i + 1),$$

which implies that $\lim\limits_{i\to\infty} |s'_{n_i}| = \infty$. Assume that $\sigma' \neq 0$. Then, for every sufficiently large i,

(63) $$|s'_{n_i}| > 2\,|\sigma'|.$$

Because of (62), we have $s'_j = s'_{n_i}$ $(n_i < j \leqq 2\,n_i;\ i = 1, 2, 3, \ldots)$. Hence,

$$\sigma' = \lim_{i\to\infty} \frac{s'_1 + s'_2 + \cdots + s'_{n_i}}{n_i} = \lim_{i\to\infty} \frac{s'_1 + s'_2 + \cdots + s'_{n_i} + n_i\,s'_{n_i}}{2\,n_i}$$

$$= \tfrac{1}{2}\,\sigma' + \tfrac{1}{2} \lim_{i\to\infty} s'_{n_i},$$

and consequently

$$\lim_{i\to\infty} s'_{n_i} = \sigma',$$

which contradicts (63). Therefore we must have $\sigma' = 0$.

Thus we see that $R = \{0\}$.

Example B. Let

$$a_n = \begin{cases} 2^{2^k - k} & \text{if } n = 2^{2^k} \\ -2^{2^k - k} - 1 & \text{if } n = 2^{2^k} + 1 \\ 1 & \text{if } n = 2^{2^k} + 2 \\ 0 & \text{if } n \text{ is any other natural number.} \end{cases} \qquad (k = 1, 2, 3, \ldots)$$

Then it is obvious that (1) diverges.

Suppose that $n \geqq 4$. Then there is a $k \geqq 1$ such that $2^{2^k} \leqq n < 2^{2^{k+1}}$. We have

$$\sigma_n \leqq \frac{1}{n} \sum_{j=1}^{k} 2^{2^j - j},$$

and an argument analogous to one employed in connection with Example A now shows that $\lim_{n\to\infty} \sigma_n = 0$, so that (41) holds.

Let $\sum_{n=1}^{\infty} a'_n$ be a C_1-summable rearrangement of (1). Then there exists an infinite subsequence, $\{n_i\}$, of $\{n\}$ such that

$$\text{(64)} \qquad \text{either } a'_j = 0 \text{ or } a'_j = 1 \qquad (n_i < j \leqq 16\, n_i;\ i = 1, 2, 3, \ldots).$$

The proof of this is analogous to the proof of the existence, for Example A, of the sequence $\{n_i\}$ satisfying (62), and will therefore be omitted. A consequence of (64) is

$$\text{(65)} \qquad s'_{n_i} \leqq s'_{n_i+1} \leqq \cdots \leqq s'_{16\, n_i} \qquad (i = 1, 2, 3, \ldots).$$

Suppose that for every sufficiently large i and for every k satisfying $n_i \leqq k \leqq 8\, n_i$, there is at least one j satisfying $k < j \leqq 2\,k$ and $a'_j = 1$. Let σ' be the C_1-sum of $\sum_{n=1}^{\infty} a'_n$. Then there are two possibilities: either $s'_{2\, n_i} > \sigma' - 1$ for every sufficiently large i, or else there is an infinite set, I', of natural numbers such that $s'_{2\, n_i} \leqq \sigma' - 1$ for every $i \in I'$. If the second alternative holds, then, in view of (65),

$$s'_{n_i+1} + s'_{n_i+2} + \cdots + s'_{2\, n_i} \leqq n_i(\sigma' - 1)$$

for every $i \in I'$, and hence

$$\sigma' = \lim_{\substack{i\to\infty \\ i\in I'}} \frac{s_1' + \cdots + s_{n_i}' + s_{n_i+1}' + \cdots + s_{2n_i}'}{2n_i} \leqq \sigma' - \tfrac{1}{2},$$

which is absurd. If the first alternative holds, then, according to the first sentence of this paragraph, $s_{4n_i}' > \sigma'$ and $s_{8n_i}' > \sigma' + 1$ for every sufficiently large i, so that, in view of (65), $s_{8n_i+1}' + s_{8n_i+2}' + \cdots + s_{16n_i}' > 8n_i(\sigma' + 1)$, and hence

$$\sigma' = \lim_{i\to\infty} \frac{s_1' + \cdots + s_{8n_i}' + s_{8n_i+1}' + \cdots + s_{16n_i}'}{16n_i} \geqq \sigma' + \tfrac{1}{2},$$

which is also absurd. The initial supposition in this paragraph must therefore be false. Consequently, there exists an infinite set, I'', of natural numbers such that, for every $i \in I''$, there is an m_i, satisfying $n_i \leqq m_i \leqq 8n_i$, for which

(66) $$a_j' = 0 \qquad (m_i < j \leqq 2m_i;\ i \in I'')$$

An immediate consequence of (66) is that (40) holds.

According to Lemma 1, there exists an infinite subsequence, $\{n_i'\}$, of $\{n\}$ such that $\lim\limits_{i\to\infty} s_{n'_i}' = \sigma'$. Since s_n' is an integer for every n, σ' must also be an integer. This means that every number belonging to the rearrangement set of (1) is an integer.

Conversely, if ν is an integer, then $\nu \in R$. We have already seen that $0 \in R$. Suppose, then, that $\nu > 0$. Let $a_n' = 1$ $(1 \leqq n \leqq \nu)$ and $a_n' = a_{n-\nu}$ $(n = \nu+1, \nu+2, \nu+3, \ldots)$. The series $\sum\limits_{n=1}^{\infty} a_n'$ thus defined is a rearrangement of (1), because infinitely many terms of (1) are equal to 1, and, according to the second sentence following (41), the C_1-sum of this rearrangement is ν. Similarly, if $\nu < 0$, the series obtained from (1) by simply deleting the terms a_n $(n = 2^{2^k} + 2,\ k = 1, 2, \ldots, -\nu)$, is a rearrangement of (1), and the C_1-sum of this rearrangement is ν.

Thus we see that R is the set of integers.

Example C. Let

$$a_n = \begin{cases} 2^{2^{2k-1}-(2k-1)} & \text{if } n = 2^{2^{2k-1}} \\ -2^{2^{2k-1}-(2k-1)} - 1 & \text{if } n = 2^{2^{2k-1}} + 1 \\ 1 & \text{if } n = 2^{2^{2k-1}} + 2 \\ 2^{2^{2k}-2k} & \text{if } n = 2^{2^{2k}} \\ -2^{2^{2k}-2k} - \sqrt{2} & \text{if } n = 2^{2^{2k}} + 1 \\ \sqrt{2} & \text{if } n = 2^{2^{2k}} + 2 \\ 0 & \text{if } n \text{ is any other natural number.} \end{cases} \qquad (k = 1, 2, 3, \ldots)$$

Then arguments analogous to ones employed in connection with Example B show that (40) and (41) hold (and (1) is obviously divergent), and that the rearrangement set of (1) contains every number of the form $\mu+\nu\sqrt{2}$, where μ and ν are integers. It is well known that the set of all such numbers is everywhere dense in the set of real numbers, and from this fact and Lemma 8, it follows that R is the set of real numbers.

§ 5. Let us return to our original question. Suppose that (1) is C_1-summable; what is the nature of its rearrangement set R?

If (1) is convergent, the answer is given in § 1. Suppose that (1) is divergent. If 0 is not a limit point of the sequence $\{a_m\}$, then our question is answered in § 3. Assume that 0 is a limit point of $\{a_m\}$. If, for every $\varepsilon>0$, there is a non-zero limit point of $\{a_m\}$ in the interval $(-\varepsilon, \varepsilon)$, then, as is easily seen, this case can be reduced to the one treated in § 2. If, however, there exists an $\varepsilon>0$ such that 0 is the only limit point of $\{a_m\}$ in the interval $(-\varepsilon, \varepsilon)$, then the terms of (1) in this interval form an infinite subsequence, $\{a_{m_k}\}$, of $\{a_m\}$ such that $\lim\limits_{k\to\infty} a_{m_k}=0$. Now there are two possibilities: either $\sum\limits_{k=1}^{\infty}|a_{m_k}|$ diverges or it converges. If it diverges, we have the case discussed in § 2. Suppose, however, that it converges. Let $\sum\limits_{k=1}^{\infty} a_{m_k}=\alpha$. If the C_1-sum of (1) is σ, then the C_1-sum of the series obtained from (1) by setting $a_{m_k}=0$ $(k=1, 2, 3, \ldots)$ exists and is equal to $\sigma-\alpha$, and conversely. (This is very easy to prove if one considers, in addition to the series already mentioned, the series obtained from (1) by setting $a_i=0$ $(i\neq m_k;\ k=1, 2, 3, \ldots)$, and makes use of the fact that C_1-summable series may be added and subtracted term by term.) Hence, there is no loss of generality in assuming that $a_{m_k}=0$ $(k=1, 2, 3, \ldots)$. This means that if we put $\delta=\frac{\varepsilon}{2}$, and if $\{a_{n_k}\}$ is the subsequence of non-zero terms of $\{a_n\}$, then $|a_{n_k}|>\delta$ $(k=1, 2, 3, \ldots)$. If $\lim\limits_{k\to\infty} n_{k+1}/n_k=1$, then we have the case considered in § 3. If, however, this limit is not equal to 1, the discussion in § 4 applies.

Thus it is evident that the assertion made in the second paragraph of § 1 is true.

References

[1]. G. H. Hardy, *Divergent Series*, Oxford, 1949.

[2]. K. Knopp, *Theory and Application of Infinite Series*, second English edition (translated from the fourth German edition by R. C. H. Young), London and Glasgow, 1951.

[3]. B. Riemann, *Gesammelte mathematische Werke*, second edition, Leipzig, 1892.