THE MINIMUM OF A FACTORIZABLE BILINEAR FORM.

By

E. S. BARNES

Trinity College, Cambridge.

1. Let

$$B(x, y, z, t) = (\alpha x + \beta y)(\gamma z + \delta t) \tag{1.1}$$

be a factorizable bilinear form, where $\alpha, \beta, \gamma, \delta$ are real, and x, y, z, t take all integral values subject to

$$xt - yz = \pm 1. \tag{1.2}$$

We suppose that $\Delta = \alpha\delta - \beta\gamma \neq 0$, and that B does not represent zero. Denoting the lower bound of $|B(x, y, z, t)|$ by $M(B)$, we have the following theorem, which is due to Davenport and Heilbronn[1]:

Theorem.

(i)
$$M(B) \leq \frac{3-\sqrt{5}}{2\sqrt{5}} |\Delta|, \tag{1.3}$$

and equality occurs if and only if B is equivalent to a multiple of

$$B_1 = \left(x + \frac{1+\sqrt{5}}{2} y\right)\left(z + \frac{1-\sqrt{5}}{2} t\right), \tag{1.4}$$

in which case the lower bound is attained.

(ii) *For all forms not equivalent to a multiple of B,*

$$M(B) \leq \frac{2-\sqrt{2}}{4} |\Delta|, \tag{1.5}$$

and equality occurs if and only if B is equivalent to a multiple of

[1] *Quarterly Journal* 18 (1947), 107—123.

$$B_2 = (x - \sqrt{2}\, y)(z + \sqrt{2}\, t) \tag{1.6}$$

in which case the lower bound is attained.

(iii) *For all forms not equivalent to a multiple of B_1 or B_2,*

$$M(B) \leq \frac{\sqrt{2}-1}{3} |\Delta|, \tag{1.7}$$

and equality occurs if and only if B is equivalent to a multiple of

$$B_3 = (x - \sqrt{2}\, y)\{z + (3 - \sqrt{2})t\}, \tag{1.8}$$

in which case the lower bound is attained.

(iv) *For any $\delta > 0$, there exists a set of forms B for which*

$$M(B) > \left(\frac{\sqrt{2}-1}{3} - \delta\right) |\Delta|, \tag{1.9}$$

and the set has the cardinal number of the continuum.

They proved these results by obtaining relations between the values assumed by B and those assumed by the associated quadratic form

$$Q(x, y) = (\alpha x + \beta y)(\gamma x + \delta y) \tag{1.10}$$

for coprime integers x, y. I give here an alternative proof; the method is essentially the same as that which I used in a recent paper[1] on the analogous problem of determining the lower bound of $|Q(x, y)\, Q(z, t)|$ for integral x, y, z, t subject to (1.2).

The proof exhibits the dependence of the result (iv) on the existence of a constant $\eta > 0$ such that $|B_3(x, y, z, t)| > \left(\frac{\sqrt{2}-1}{3} + \eta\right)|\Delta|$ for all but a finite number of values of x, y, z, t.

2. The associated quadratic form Q, defined by (1.10), has discriminant $D = (\alpha\delta - \beta\gamma)^2 = \Delta^2 > 0$, and so is indefinite. Also Q does not represent zero, since B does not. Let then[2]

$$\ldots, \varphi_{-2}, \varphi_{-1}, \varphi_0, \varphi_1, \varphi_2, \ldots \tag{2.1}$$

---

[1] "The minimum of the product of two values of a quadratic form" (II), *Proc. London Math. Soc.* (3) 1 (1950).

[2] For these results, see I. Schur, *Sitz.-Ber. K. Preuss. Akad. Berlin* (1913), 212—231 (214—216), whose notation I have adopted.

be the chain of reduced forms equivalent to Q, where

$$\varphi_\nu = \varphi_\nu(x, y) = (-1)^{\nu-1} a_\nu x^2 + b_\nu xy + (-1)^\nu a_{\nu+1} y^2 \qquad (\nu = 0, \pm 1, \pm 2, \ldots), \tag{2.2}$$

so that, supposing $a_0 > 0$, all the numbers a_ν, b_ν are positive; further

$$\frac{b_\nu + b_{\nu+1}}{2\, a_{\nu+1}} = k_\nu \tag{2.3}$$

is a positive integer. We set

$$r_\nu = \frac{\sqrt{D} + b_\nu}{2\, a_{\nu+1}}, \quad s_\nu = \frac{\sqrt{D} - b_\nu}{2\, a_{\nu+1}}, \tag{2.4}$$

whence

$$\frac{a_\nu}{a_{\nu+1}} = r_\nu s_\nu, \quad \frac{b_\nu}{a_{\nu+1}} = r_\nu - s_\nu, \quad \frac{\sqrt{D}}{a_{\nu+1}} = r_\nu + s_\nu. \tag{2.5}$$

From (2.3), (2.4), we deduce that

$$r_\nu = k_\nu + \frac{1}{r_{\nu+1}}, \quad s_\nu = \frac{1}{s_{\nu+1}} - k_\nu, \tag{2.6}$$

and so

$$r_\nu = (k_\nu, k_{\nu+1}, k_{\nu+2}, \ldots), \quad s_\nu = (0, k_{\nu-1}, k_{\nu-2}, \ldots) \tag{2.7}$$

in the usual notation for continued fractions. We also write

$$r'_\nu = r_\nu - k_\nu = (0, k_{\nu+1}, k_{\nu+2}, \ldots), \tag{2.8}$$

so that

$$0 < r'_\nu, \; s_\nu < 1, \; r_\nu > k_\nu \geq 1. \tag{2.9}$$

We denote by (K) the infinite sequence of positive integers

$$(K): \qquad \ldots, k_{-2}, k_{-1}, k_0, k_1, k_2, \ldots \tag{2.10}$$

K) is then determined by Q, and hence also by B; conversely, (K) determines, to within an arbitrary multiple, the class of forms equivalent to Q, and hence also the class of bilinear forms equivalent[1] to B.

Finally, we define

$$A_\nu^{(1)} = \frac{r'_\nu}{r_r + s_\nu}; \quad A_\nu^{(2)} = \frac{1 - r'_\nu}{r_\nu + s_\nu}; \quad A_\nu^{(3)} = \frac{s_\nu}{r_\nu + s_\nu}; \quad A_\nu^{(4)} = \frac{1 - s_\nu}{r_\nu + s_\nu}; \quad (\nu = 0, \pm 1, \pm 2, \ldots). \tag{2.11}$$

Clearly, by (2.9),

$$A_\nu^{(i)} > 0. \tag{2.12}$$

[1] Two bilinear forms are said to be equivalent if the corresponding quadratic forms are equivalent under integral unimodular transformation. It is easily seen that equivalent forms assume the same set of values for integral x, y, z, t satisfying (1.2).

We now establish the following lemma, which constitutes the basis of the proof of the theorem:

Lemma 1. (i) *Suppose that* $\lambda > 0$, *and that*

$$|B(x, y, z, t)| \geq \frac{|\Delta|}{\lambda} \tag{2.13}$$

for all integral x, y, z, t *subject to* (1.2). *Then the inequalities*

$$A_\nu^{(1)} \geq \frac{1}{\lambda}, \text{ i.e. } \quad k_\nu + s_\nu \leq (\lambda - 1) r'_\nu; \tag{2.14}$$

$$A_\nu^{(2)} \geq \frac{1}{\lambda}, \text{ i.e. } \quad k_\nu + (\lambda + 1) r'_\nu + s_\nu \leq \lambda; \tag{2.15}$$

$$A_\nu^{(3)} \geq \frac{1}{\lambda}, \text{ i.e. } \quad k_\nu + r'_\nu \leq (\lambda - 1) s_\nu; \tag{2.16}$$

$$A_\nu^{(4)} \geq \frac{1}{\lambda}, \text{ i.e. } \quad k_\nu + r'_\nu + (\lambda + 1) s_\nu \leq \lambda; \tag{2.17}$$

hold for all $\nu = 0, \pm 1, \pm 2, \ldots$

(ii) *Suppose that* $\lambda \geq 2(\sqrt{2} + 1)$. *Then if the inequalities* (2.14)—(2.17) *hold for all* ν, (2.13) *is true for all integral* x, y, z, t *subject to* (1.2).

Proof. (i) Suppose that (2.13) holds. Then, since equivalent forms assume the same set of values,

$$|B_\nu| \geq \frac{|\Delta|}{\lambda} \quad (\nu = 0, \pm 1, \pm 2, \ldots), \tag{2.18}$$

where B_ν is a bilinear form corresponding to the reduced quadratic form φ_ν. From (2.2) and the relations (2.5), we have

$$\varphi_\nu = \begin{cases} -\, a_{\nu+1}(r_\nu x + y)(s_\nu x - y) & \text{if } \nu \text{ is even} \\ \, a_{\nu+1}(r_\nu x - y)(s_\nu x + y) & \text{if } \nu \text{ is odd,} \end{cases} \tag{2.19}$$

and so

$$B \sim B_\nu \sim \pm\, a_{\nu+1}(r_\nu x + y)(s_\nu z - t), \tag{2.20}$$

$$|\Delta| = |\Delta(B_\nu)| = a_{\nu+1}(r_\nu + s_\nu). \tag{2.21}$$

(2.18), (2.20), (2.21) now give

$$\frac{|(r_\nu x + y)(s_\nu z - t)|}{r_\nu + s_\nu} \geq \frac{1}{\lambda} \quad (\nu = 0, \pm 1, \pm 2, \ldots). \tag{2.22}$$

The inequalities (2.14)—(2.17) now follow at once, since they are the particular cases of (2.22) corresponding to $(x, y, z, t) = (1, -k_\nu, 0, 1)$, $(1, -k_\nu - 1, 0, 1)$, $(0, 1, 1, 0)$, $(0, 1, 1, 1)$ respectively.

(ii) Suppose now that $\lambda \geq 2(\sqrt{2} + 1)$, and that

$$|B| < \frac{|\Delta|}{\lambda} \tag{2.23}$$

for some integral x_1, y_1, z_1, t_1 satisfying (1.2). Lemma 1 (ii) will follow if we then prove that at least one of the inequalities (2.14)—(2.17) is false for some ν.

We apply to B the integral unimodular transformation $\begin{pmatrix} x_1 & z_1 \\ y_1 & t_1 \end{pmatrix}$ and so obtain an equivalent form

$$B' = (\alpha X + \beta Y)(\gamma Z + \delta T), \tag{2.24}$$

say, which satisfies (2.23) with $(X, Y, Z, T) = (1, 0, 0, 1)$, i.e.

$$|\alpha \delta| < \frac{|\Delta|}{\lambda}. \tag{2.25}$$

The quadratic form associated with B' is

$$Q' = (\alpha X + \beta Y)(\gamma X + \delta Y) \equiv a X^2 + b X Y + c Y^2, \text{ say.} \tag{2.26}$$

Since $|\Delta| = |\alpha\delta - \beta\gamma|$, $|b| = |\alpha\delta + \beta\gamma| = |(\alpha\delta - \beta\gamma) - 2\alpha\delta|$, we deduce from (2.25) that

$$\left(1 - \frac{2}{\lambda}\right)|\Delta| < |b| < \left(1 + \frac{2}{\lambda}\right)|\Delta|.$$

Squaring $\left(\text{since } 1 - \frac{2}{\lambda} > 0\right)$ and using the relation $b^2 - 4ac = D(Q') = \Delta^2$, we find

$$\left(1 - \frac{2}{\lambda}\right)^2 - 1 < \frac{4ac}{D} < \left(1 + \frac{2}{\lambda}\right)^2 - 1. \tag{2.27}$$

By hypothesis, $\lambda \geq 2(\sqrt{2} + 1)$, $1 + \frac{2}{\lambda} \leq \sqrt{2}$, and so (2.27) gives $|ac| < \frac{D}{4}$. Thus either $|a| < \frac{1}{2}\sqrt{D}$ or $|c| < \frac{1}{2}\sqrt{D}$.

Suppose firstly that $|c| < \frac{1}{2}\sqrt{D}$. If we apply to Q' the parallel transformation

$$X = x,\ Y = -px + y \ (p \text{ integral}), \tag{2.28}$$

we obtain a form $a'x^2 + b'xy + cy^2$, say, equivalent to Q', for which $b' = b - 2pc$. We choose p so that $0 \leq -b + \sqrt{D} + 2pc < 2|c|$; then, since $b' = \sqrt{D}$ implies that Q represents zero, we have

$$0 < \sqrt{D} - b' < 2|c|. \tag{2.29}$$

This gives $b' > \sqrt{D} - 2|c| > 0$, whence also

$$0 < \sqrt{D} - b' < 2|c| < \sqrt{D} + b'. \tag{2.30}$$

Now (2.30) is just the condition that the form $a'x^2 + b'xy + cy^2$ be reduced. Thus

$$Q'(X, Y) = \varphi_\nu(x, y)$$

for some ν, and so, by (2.19) and (2.21),

$$\frac{|B'(X, Y, Z, T)|}{|\varDelta|} = \frac{|(r_\nu x \pm y)(s_\nu z \mp t)|}{r_\nu + s_\nu} \text{ or } \frac{|(r_\nu z \pm t)(s_\nu x \mp y)|}{r_\nu + s_\nu}, \tag{2.31}$$

where, by (2.28),

$$X = x,\ Y = -px + y,\ Z = z,\ T = -pz + t. \tag{2.32}$$

Suppose next that $|a| < \frac{1}{2}\sqrt{D}$. Then a precisely similar argument shows that we can reduce Q' by a transformation $X = x - py$, $Y = y$ (p integral), and so (2.31) still holds for some ν, where now

$$X = x - py,\ Y = y,\ Z = z - pt,\ T = t. \tag{2.33}$$

By hypothesis $|B'| < \frac{|\varDelta|}{\lambda}$ for $(X, Y, Z, T) = (1, 0, 0, 1)$, i.e. for

$$(x, y, z, t) = (1, p, 0, 1) \text{ or } (1, 0, p, 1), \tag{2.34}$$

according as (2.32) or (2.33) holds. On substituting (2.34) in (2.31), we see that, for some integer p and some $\nu = 0, \pm 1, \pm 2, \ldots$, one of the four following inequalities must be true:

$$\frac{|r_\nu - p|}{r_\nu + s_\nu} < \frac{1}{\lambda}, \qquad \frac{|s_\nu - p|}{r_\nu + s_\nu} < \frac{1}{\lambda}, \tag{2.35}$$

$$\frac{r_\nu |p s_\nu - 1|}{r_\nu + s_\nu} < \frac{1}{\lambda}, \quad \frac{s_\nu |p r_\nu - 1|}{r_\nu + s_\nu} < \frac{1}{\lambda}. \tag{2.36}$$

Since, by (2.7), $k_\nu < r_\nu < k_\nu + 1$, $0 < s_\nu < 1$, the least value of $|r_\nu - p|$ is given by $p = k_\nu$ or $k_\nu + 1$, and the least value of $|s_\nu - p|$ is given by $p = 0$ or 1, the corresponding values being $|r_\nu - p| = r'_\nu$ or $1 - r'_\nu$, $|s_\nu - p| = s_\nu$ or $1 - s_\nu$. Hence, if either of the inequalities (2.35) holds for some p, it follows that one of the four numbers $A_\nu^{(i)}$ is $<\frac{1}{\lambda}$, i.e. that one of the inequalities (2.14)—(2.17) is false.

Finally, (2.6) gives $r_{\nu-1}=k_{\nu-1}+\frac{1}{r_\nu}$, $s_{\nu-1}=\frac{1}{s_\nu}-k_{\nu-1}$, and the inequalities (2.36) can be written as

$$\frac{|s_{\nu-1}-p+k_{\nu-1}|}{r_{\nu-1}+s_{\nu-1}}<\frac{1}{\lambda}, \quad \frac{|r_{\nu-1}-p-k_{\nu-1}|}{r_{\nu-1}+s_{\nu-1}}<\frac{1}{\lambda}.$$

But these are of the same form as (2.35), with $\nu-1$ in place of ν, and p replaced by $p\pm k_{\nu-1}$. It therefore follows, as in the preceding paragraph, that if either of the inequalities (2.36) holds, then one of (2.14)—(2.17) (with $\nu-1$ in place of ν) must be false.

3. We now take

$$\lambda=3(\sqrt{2}+1)=7.24264\ldots \tag{3.1}$$

and show that the inequalities (2.14)—(2.17) can be true for all ν only if (K) is one of three sequences[1]

$$(K_1): \qquad {}_\infty[1]_\infty \tag{3.2}$$

$$(K_2): \qquad {}_\infty[2]_\infty \tag{3.3}$$

$$(K_3): \qquad {}_\infty[2],\ 1,\ 1,\ [2]_\infty. \tag{3.4}$$

Applying Lemma 1 (i), we shall then have the following result:

Lemma 2. *The inequality*

$$|B|<\frac{|\varDelta|}{3(\sqrt{2}+1)}=\frac{\sqrt{2}-1}{3}|\varDelta| \tag{3.5}$$

can be satisfied for all forms B other than those corresponding to the sequences (K_1), (K_2), (K_3).

In this section we therefore suppose that (2.14)—(2.17) are true for all ν, where $\lambda=3(\sqrt{2}+1)$.

Lemma 3. *Every k_ν is 1 or 2.*

---

[1] The notation ${}_\infty[a]_\infty$ is used for the infinite sequence each of whose elements is a; ${}_\infty[a]$ and $[a]_\infty$ are the semi-infinite sequences with every element a, written to the left and to the right respectively. ${}_n[a]$ or $[a]_n$, where n is a positive integer, will denote a sequence of n elements a.

Proof. Suppose firstly that some $k_\nu \geq 4$. Then, by (2.9), $r_\nu + s_\nu > k_\nu \geq 4$, and one of r'_ν, $1 - r'_\nu$ must be $\leq \frac{1}{2}$. Hence either $A^{(1)}_\nu$ or $A^{(2)}_\nu$ is $< \frac{1}{8} < \frac{1}{\lambda}$, contradicting either (2.14) or (2.15).

Thus every $k_\nu \leq 3$. Suppose that some $k_\nu = 3$. Then

$$s_\nu = (0,\ k_{\nu-1},\ \ldots) > \frac{1}{k_{\nu-1} + 1} \geq \tfrac{1}{4},$$

and either $r'_\nu \leq \frac{1}{2}$ or $1 - r'_\nu \leq \frac{1}{2}$.

If $r'_\nu \leq \frac{1}{2}$, (2.14) gives

$$3 + \tfrac{1}{4} < (\lambda - 1)\tfrac{1}{2},$$

or $\lambda > 7.5$, which is false. If $1 - r'_\nu \leq \frac{1}{2}$, $r'_\nu \geq \frac{1}{2}$ and (2.15) gives

$$3 + \tfrac{1}{2}(\lambda + 1) + \tfrac{1}{4} < \lambda,$$

or $\lambda > 7.5$, which is again a contradiction.

Thus $k_\nu \neq 3$ for any ν, and the lemma is proved.

Lemma 4. *If* $k_\nu = 2$, *then*

$$r'_\nu \leq 2 - \sqrt{2} = (0,\ 1,\ 1,\ 2_\infty). \tag{3.6}$$

Further, if $r'_\nu = 2 - \sqrt{2}$, *then* $s_\nu = \sqrt{2} - 1 = (\dot{0},\ 2_\infty)$.

Proof. We obtain from (2.16) and (2.15) in turn

$$k_\nu + r'_\nu \leq (\lambda - 1)s_\nu \leq \lambda(\lambda - 1) - (\lambda - 1)k_\nu - (\lambda^2 - 1)r'_\nu, \tag{3.7}$$

whence

$$0 \leq \lambda(\lambda - 1) - \lambda k_\nu - \lambda^2 r'_\nu,$$
$$\lambda r'_\nu \leq \lambda - 1 - k_\nu.$$

On substituting $k_\nu = 2$, $\lambda = 3(\sqrt{2} + 1)$, this gives

$$r'_\nu \leq 1 - \frac{3}{\lambda} = 2 - \sqrt{2}.$$

If now $r'_\nu = 2 - \sqrt{2}$, there is equality throughout in (3.7), whence

$$(\lambda - 1)s_\nu = k_\nu + r'_\nu,$$
$$(3\sqrt{2} + 2)s_\nu = 4 - \sqrt{2},$$
$$s_\nu = \sqrt{2} - 1.$$

Lemma 5. *A subsequence* $\ldots\ 2,\ 1,\ \ldots$ *of* (K) *must form part of a subsequence*

$$\ldots,\ 2,\ 1,\ 1,\ 2,\ \ldots.$$

Proof. By hypothesis, $k_\nu = 2$, $k_{\nu+1} = 1$ for some ν. Then, by Lemma 4,

$$r'_\nu = (0, 1, k_{\nu+2}, k_{\nu+3}, \ldots) \leq 2 - \sqrt{2} = (0, 1, 1, 2, 2, \ldots).$$

Comparing partial quotients in this inequality, we find $k_{\nu+2} \leqq 1$, whence $k_{\nu+2} = 1$; then $k_{\nu+3} \geq 2$, whence, by Lemma 3, $k_{\nu+3} = 2$.

We now exclude the special sequences (K_1), (K_2) defined in (3.2), (3.3). From Lemma 3, we then see that each of the numbers 1 and 2 must occur in (K), and so also a subsequence $\ldots 1, 2, \ldots$ or $\ldots 2, 1, \ldots$. By the symmetry in r'_ν, s_ν of (2.14)—(2.17), we may suppose that (K) contains a subsequence $\ldots 2, 1, \ldots$. Lemma 5' now establishes the existence of a subsequence

$$\ldots 2_{(\nu)}, 1, 1, 2, \ldots \tag{3.8}$$

of (K).

If now (3.8) extends to the right as $\ldots 2_{(\nu)}, 1, 1, [2]_\infty$, we have

$$r'_\nu = (0, 1, 1, 2_\infty) = 2 - \sqrt{2},$$

whence, by Lemma 4, $s_\nu = (0, 2_\infty)$, and so (K) is (K_3) ((3.4)). By symmetry, if (3.8) extends to the left as $_\infty[2], 1, 1, 2, \ldots$, (K) is again (K_3). Thus if we now exclude also the special sequence (K_3), there can only be a finite number of elements 2 immediately to the right and to the left of (3.8). Hence (K) must contain a subsequence

$$\ldots, 1, {}_n[2], 1, 1, [2]_m, 1, \ldots,$$

and so, using Lemma 5, a subsequence

$$\ldots 2, 1, 1, {}_n[2], 1, 1, [2]_m, 1, 1, 2, \ldots \tag{3.9}$$

where m, n are finite integers ≥ 1.

We now complete the proof of Lemma 2 by showing that the existence of a subsequence (3.9) of (K) leads to a contradiction with the inequalities (2.14)—(2.17).

Suppose firstly that m is even. Then, taking $k_\nu = 2$ as the last 2 of the block $_n[2]$, we have

$$\begin{aligned} r'_\nu &= (0, 1, 1, [2]_m, 1, 1, 2, \ldots) \\ &> (0, 1, 1, [2]_m, 1, 1), \text{ an even convergent,} \\ &= (0, 1, 1, [2]_{m+1}) \\ &> (0, 1, 1, 2_\infty), \end{aligned}$$

contradicting (3.6).

Thus we can suppose that m is odd and, similarly, that n is odd; by symmetry, we may further suppose that $m \geq n$.

We define the numbers u_i for $i = 0, 1, 2, \ldots$ by

$$u_0 = 0,\ u_1 = 1, \ldots,\ u_{i+2} = 2u_{i+1} + u_i\ (i \geq 0). \tag{3.10}$$

Then clearly $\omega_i = \frac{u_i}{u_{i+1}}$ is the i^{th} convergent to the continued fraction $(0, 2_\infty) = \sqrt{2} - 1$, and so

$$0 = \omega_0 < \omega_2 < \omega_4 < \cdots < \sqrt{2} - 1 < \cdots < \omega_5 < \omega_3 < \omega_1 = \tfrac{1}{2}. \tag{3.11}$$

Taking $k_\nu = 2$ as above, we then have

$$r'_\nu = (0, 1, 1, [2]_m, 1, 1, 2, \ldots)$$
$$> (0, 1, 1, [2]_m, 1, 1, 2), \text{ an even convergent};$$

a simple calculation shows that the value of this continued fraction is $\frac{8u_{m+1} + 2u_m}{13u_{m+1} + 5u_m}$, whence

$$r'_\nu > \frac{8 + 2\omega_m}{13 + 5\omega_m}. \tag{3.12}$$

Also,

$$s_\nu = (0, [2]_{n-1}, 1, 1, 2, \ldots)$$
$$> (0, [2]_{n-1}, 1, 1), \text{ an even convergent},$$
$$= (0, [2]_n) = \omega_n;$$

since $m \geq n$, (3.11) shows that $\omega_m \leq \omega_n$, whence

$$s_\nu > \omega_m. \tag{3.13}$$

(2.15), with (3.12) and (3.13), now gives

$$2 + (\lambda + 1)\frac{8 + 2\omega_m}{13 + 5\omega_m} + \omega_m < \lambda.$$

or

$$\lambda > \frac{5\omega_m^2 + 25\omega_m + 34}{3\omega_m + 5}. \tag{3.14}$$

The r.h.s. of (3.14) is an increasing function of ω_m for $\omega_m \geq 0$, since its derivative is a positive multiple of $15\omega_m^2 + 50\omega_m + 23 > 0$. Hence, since, by (3.11), $\omega_m > \sqrt{2} - 1$, we deduce from (3.14) that

$$\lambda > \frac{5(\sqrt{2} - 1)^2 + 25(\sqrt{2} - 1) + 34}{3(\sqrt{2} - 1) + 5} = \frac{24 + 15\sqrt{2}}{3\sqrt{2} + 2} = 3(\sqrt{2} + 1) = \lambda.$$

This contradiction shows that a sequence (3.9) cannot occur in (K); the proof of Lemma 2 is therefore complete.

4. We now examine the special sequences (K_1), (K_2), (K_3) and the corresponding classes of bilinear forms.

(i) If (K) is (K_1): $\infty[1]_\infty$, $k_\nu = 1$ and $r'_\nu = s_\nu = (0, 1_\infty) = \frac{\sqrt{5}-1}{2}$, $r_\nu = \frac{\sqrt{5}+1}{2}$ for all ν. By (2.5), $\frac{a_\nu}{a_{\nu+1}} = r_\nu s_\nu = 1$, $\frac{b_\nu}{a_{\nu+1}} = r_\nu - s_\nu = 1$, so that

$$\varphi_\nu = a(x^2 + xy - y^2) = a\left(x + \frac{1+\sqrt{5}}{2}y\right)\left(x + \frac{1-\sqrt{5}}{2}y\right).$$

Thus B is equivalent to a multiple of $B_1 = \left(x + \frac{1+\sqrt{5}}{2}y\right)\left(z + \frac{1-\sqrt{5}}{2}t\right)$.

Also, since $r'_\nu, s_\nu > \frac{1}{2}$,

$$\frac{r'_\nu}{r_\nu + s_\nu} > \frac{1 - r'_\nu}{r_\nu + s_\nu} = \frac{3-\sqrt{5}}{2\sqrt{5}}, \quad \frac{s_\nu}{r_\nu + s_\nu} > \frac{1 - s_\nu}{r_\nu + s_\nu} = \frac{3-\sqrt{5}}{2\sqrt{5}}, \text{ for all } \nu.$$

Hence, by (2.11), $A_\nu^{(i)} \geq \frac{3-\sqrt{5}}{2\sqrt{5}}$, and so, by Lemma 1 (ii),

$$|B_1| \geq \frac{3-\sqrt{5}}{2\sqrt{5}}|\Delta|.$$

Since $B_1(1, 0, 1, 1) = \frac{3-\sqrt{5}}{2}$, $|\Delta(B_1)| = \sqrt{5}$, it follows that

$$M(B_1) = \frac{3-\sqrt{5}}{2\sqrt{5}}|\Delta|, \tag{4.1}$$

and that the lower bound is attained.

(ii) If (K) is (K_2): $\infty[2]_\infty$, $k_\nu = 2$ and $r'_\nu = s_\nu = (0, 2_\infty) = \sqrt{2} - 1$, $r_\nu = \sqrt{2} + 1$ for all ν. By (2.5), $\frac{a_\nu}{a_{\nu+1}} = r_\nu s_\nu = 1$, $\frac{b_\nu}{a_{\nu+1}} = r_\nu - s_\nu = 2$, so that $\varphi_\nu = a(x^2 + 2xy - y^2) \sim$ $\sim a(x^2 - 2y^2) = a(x - \sqrt{2}y)(x + \sqrt{2}y)$. Thus B is equivalent to a multiple of $B_2 = (x - \sqrt{2}y)(z + \sqrt{2}t)$.

Also, since $r'_\nu, s_\nu < \frac{1}{2}$, the least of the expressions $A_\nu^{(i)}$ is

$$A_\nu^{(1)} = A_\nu^{(3)} = \frac{\sqrt{2}-1}{2\sqrt{2}} = \frac{2-\sqrt{2}}{4},$$

whence, by Lemma 1 (ii),

$$|B_2| \geq \frac{2-\sqrt{2}}{4}|\Delta|.$$

Since $B_2(1, 0, -1, 1)=\sqrt{2}-1$, $|\Delta(B_2)|=2\sqrt{2}$, it follows that

$$M(B_2)=\frac{2-\sqrt{2}}{4}|\Delta|, \tag{4.2}$$

and that the lower bound is attained.

(iii) Suppose finally that (K) is (K_3): $_\infty[2], 1, 1, [2]_\infty$. Taking $k_1=2$ as the last 2 of the block $_\infty[2]$, we have $r_1'=(0, 1, 1, 2_\infty)=2-\sqrt{2}$,

$$s_1=(0, 2_\infty)=\sqrt{2}-1,\ r_1=2+r_1'=4-\sqrt{2},$$

whence, by (2.19),

$$\varphi_1=a\{(4-\sqrt{2})x-y\}\{(\sqrt{2}-1)x+y\} \sim a(x-\sqrt{2}y)\{x+(3-\sqrt{2})y\}.$$

Thus B is equivalent to a multiple of $B_3=(x-\sqrt{2}y)\{z+(3-\sqrt{2})t\}$.

In order to show that the (attained) lower bound of $|B_3|$ is $\frac{\sqrt{2}-1}{3}|\Delta|$, we need the following lemma:

Lemma 6. *If, in any sequence* (K), $k_{\nu-2}=k_{\nu-1}=k_\nu=k_{\nu+1}=k_{\nu+2}=2$, *then* $A_\nu^{(i)} \geq \frac{1}{7.1}$ $(i=1, 2, 3, 4)$.

Proof. By symmetry, it suffices to show that (2.14) and (2.15) are satisfied with $\lambda=7.1$; and since $r_\nu'=(0, k_{\nu+1}, \ldots)=(0, 2, \ldots)<\frac{1}{2}$, $1-r_\nu'>r_\nu'$, and so it is sufficient to show that $A_\nu^{(1)} \geq 7.1$, i.e. that (2.14) is true. Now

$$r_\nu'=(0, 2, 2, k_{\nu+3}, \ldots)>(0, 2, 2)=\tfrac{2}{5};\ s_\nu=(0, 2, 2, k_{\nu-3}, \ldots)<(0, 2, 2, k_{\nu-3}) \leq (0, 2, 2, 1)=\tfrac{3}{7}.$$

Hence (2.14) is certainly true if

$$2+\tfrac{3}{7} \leq (\lambda-1)\tfrac{2}{5},$$

i.e. if $\lambda \geq \frac{99}{14}=7.07\ldots$, and so in particular if $\lambda=7.1$.

Our object, after Lemma 1 (ii), is to prove that the inequalities (2.14)—(2.17) hold for all ν if (K) is (K_3) and $\lambda=3(\sqrt{2}+1)=7.24\ldots$ Suppose first that k_ν occurs in the subsequence

$$2_{(0)},\ 2_{(1)},\ 1_{(2)},\ 1,\ 2,\ 2 \tag{4.3}$$

of (K_3). By symmetry, we need consider only the values 0, 1, 2 of ν.

As was shown above, $r_1' = 2 - \sqrt{2}$, $s_1 = \sqrt{2} - 1$, $r_1 = 4 - \sqrt{2}$. Thus $1 - r_1' = s_1 = \sqrt{2} - 1 < \frac{1}{2}$, and the least of the expressions $A_1^{(i)}$ is $\frac{1 - r_1'}{r_1 + s_1} = \frac{s_1}{r_1 + s_1} = \frac{\sqrt{2} - 1}{3}$.

Using (2.6) (with $\nu = 0$, $k_0 = 2$), we find that

$$r_0 = \frac{32 + \sqrt{2}}{14},\ r_0' = \frac{4 + \sqrt{2}}{14},\ s_0 = \sqrt{2} - 1.$$

The least of the expressions $A_0^{(i)}$ is therefore

$$A_0^{(1)} = \frac{r_0'}{r_0 + s_0} = \frac{4 + \sqrt{2}}{18 + 5\sqrt{2}} = \frac{\sqrt{2} - 1}{3}.$$

Again using (2.6) (with $\nu = 1$, $k_1 = 2$), we find that

$$r_2 = \frac{2 + \sqrt{2}}{2},\ r_2' = \frac{\sqrt{2}}{2},\ s_2 = \sqrt{2} - 1.$$

The least of the expressions $A_2^{(i)}$ is therefore $\frac{1 - r_2'}{r_2 + s_2} = \frac{\sqrt{2} - 1}{3}$.

If now k_ν does not belong to the subsequence (4.3) of (K_3), then clearly $k_{\nu-2} = k_{\nu-1} = k_\nu = k_{\nu+1} = 2$, whence, by Lemma 6, $A_\nu^{(i)} \geq \frac{1}{7.1} > \frac{\sqrt{2} - 1}{3}$.

We have therefore shown that $A_\nu^{(i)} \geq \frac{\sqrt{2} - 1}{3}$ for all ν, whence, by Lemma 1 (ii),

$$|B_3| \geq \frac{\sqrt{2} - 1}{3} |\Delta|.$$

Since $|B_3(1, 1, 1, 0)| = \sqrt{2} - 1$, $|\Delta(B_3)| = 3$, it follows that

$$M(B_3) = \frac{\sqrt{2} - 1}{3} |\Delta|, \tag{4.4}$$

and that the lower bound is attained.

5. Parts (i), (ii), (iii) of the theorem now follow at once from Lemma 2 and (4.1), (4.2), (4.4), since the special sequences (K_1), (K_2), (K_3) have been shown to correspond to the classes of bilinear forms which are equivalent to multiples of B_1, B_2, B_3, respectively.

It remains to establish the existence of the set of forms B satisfying (1.9) for an arbitrarily assigned $\delta > 0$. This we do by "approximating" to the sequence (K_3) by sequences of the type

$$(K^*): \qquad \ldots 1, 1, {}_{n_2}[2], 1, 1, {}_{n_1}[2], 1, 1, [2]_{m_1}, 1, 1, [2]_{m_2}, 1, 1, \ldots, \tag{5.1}$$

where $\ldots n_2, n_1, m_1, m_2, \ldots$ are sufficiently large positive integers.

If now k_ν does not belong to a subsequence

$$2, 2, 1, 1, 2, 2 \tag{5.2}$$

of (K^*), Lemma 6 shows (as in § 4 (iii)) that the corresponding

$$A_\nu^{(i)} \geq \frac{1}{7.1} > \frac{\sqrt{2}-1}{3}.$$

Suppose next that k_ν belongs to a subsequence (5.2) of (K^*). This subsequence forms part of a subsequence

$$_p[2], 1, 1, [2]_q \tag{5.3}$$

of (K^*) where p, q are some two of the integers $\ldots n_2, n_1, m_1, m_2, \ldots$ For suitably large p, q, the values of r'_ν, s_ν (for k_ν belonging to (5.2)) are as close as we please to the corresponding values of r'_ν, s_ν found above for (K_3), since they tend to these values as p and q tend to infinity. From the continuity of the expressions $A_\nu^{(i)}$ and the fact, proved above, that they are $\geq \frac{\sqrt{2}-1}{3}$ when (K) is (K_3), it follows that

$$A_\nu^{(i)} > \frac{\sqrt{2}-1}{3} - \tfrac{1}{2}\delta. \tag{5.4}$$

provided that $p, q > N = N(\delta)$.

We have therefore shown that (5.4) is true for all ν provided that the integers $\ldots n_2, n_1, m_1, m_2, \ldots$ are $> N$. Then, by Lemma 1 (ii),

$$|B^*| > \left(\frac{\sqrt{2}-1}{3} - \tfrac{1}{2}\delta\right)|\varDelta|,$$

where B^* is a bilinear form of the class corresponding to (K^*), whence

$$M(B^*) \geq \left(\frac{\sqrt{2}-1}{3} - \tfrac{1}{2}\delta\right)|\varDelta| > \left(\frac{\sqrt{2}-1}{3} - \delta\right)|\varDelta|.$$

This proves part (iv) of the theorem, since the set of sequences (K^*) with $\ldots n_2, n_1, m_1, m_2, \ldots > N$ clearly has the cardinal number of the continuum.

It was suggested to me by Professor L. J. Mordell that the methods of this paper might be extended to deal with more general classes of forms. Such an extension is in fact possible, and I hope shortly to publish some results on the minimum of a general bilinear form in four variables.