THE MAXIMUM MODULUS AND VALENCY OF FUNCTIONS MEROMORPHIC IN THE UNIT CIRCLE.

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Chapter III.

Converse Theorems.

1) The aim of this chapter is to prove converse theorems to the results of Chapter II. We remind the reader of the fundamental problem, which is to investigate the rate of growth of the maximum modulus of a function $f(z)$, meromorphic in $|z| < 1$, which takes none of a set E of complex values more than $p(\varrho)$ times in $|z| < \varrho$, $0 < \varrho < 1$. In this chapter we shall construct examples to show that all the results we have proved give the correct order of magnitude for $\log M[\varrho, f]$ when

$$p(\varrho) \equiv (1-\varrho)^{-a}, \quad 0 \leq a < \infty. \tag{1.1}$$

The functions $f(z)$ which we construct will be regular, nonzero so that $f(z) = f_*(z)$.

We remind the reader of the four separate problems we considered in the latter half of Chapter II as stated in paragraph 19 of that chapter.

(i) *What results hold if E contains the whole w plane?*

(ii) *What sets E have the same effect as the whole plane for a given function $p(\varrho)$?*

(iii) *What results hold if we merely assume that E contains some arbitrarily large values?*

(iv) *What results hold if we assume merely that E contains ∞ and at least two finite values or a bounded set?*

The positive theorem in case (i) was proved in Theorem VII, Corollary, of Chapter II. In the case of (1.1) above this result yields

$$\log M[\varrho, f_*(z)] = O\left\{\log \frac{1}{1-\varrho}\right\}, \quad a = 0 \tag{1.2}$$

$$\log M[\varrho, f_*(z)] = O(1-\varrho)^{-a}, \qquad a > 0. \tag{1.3}$$

Both these inequalities were shown to give the best possible order in paragraph 21 of Chapter II. This disposes of problem (i).

Problem (iv) is also fairly easy to deal with. The positive results were proved in Chapter II, Theorem V. In the case of (1.1) this theorem yields

$$\log M[\varrho, f_*(z)] = \frac{O(1)}{1-\varrho}, \quad a < 1. \tag{1.4}$$

$$\log M[\varrho, f_*(z)] = O\left(\frac{1}{1-\varrho}\log\frac{1}{1-\varrho}\right), \quad a = 1 \tag{1.5}$$

$$\log M[\varrho, f_*(z)] = O(1-\varrho)^{-a}, \quad a > 1. \tag{1.6}$$

The inequality (1.6) is the same as (1.3). This shows that in the case $a > 1$ the set $E\{0, 1, \infty\}$, has much the same effect as the whole plane, on the order of growth of $\log M[\varrho, f_*(z)]$. This also disposes of the problems (ii) and (iii) in this case and leaves us with the case of (1.1) when $0 \leq a \leq 1$.

Consider now problem (iv), when E is bounded, in this case. We need to give converse examples to (1.4) and (1.5). The functions

$$f(z) = M \exp\left(\frac{1+z}{1-z}\right) \tag{1.7}$$

provide convenient converses to (1.4). For given any bounded set E, we can choose M so that for every value w in E we have

$$|w| < M.$$

Then the functions $f(z)$ of (1.7) take no value of E in $|z| < 1$ while at the same time we have

$$\log M[\varrho, f] = \frac{1+\varrho}{1-\varrho} - \log M.$$

Thus (1.4) cannot be sharpened even when $a = 0$ and so a fortiori not when $0 < a < 1$.

2) The converse example to (1.5) is a little more intricate. We shall be able to use it later to construct the very much more recondite counterexamples in problems (ii) and (iii) where E is unbounded. We need first

Lemma 1. *Let* $Z = X + iY$ *and let*

$$\zeta = \xi + i\eta = \phi(Z) = Z \log(1 + Z). \tag{2.1}$$

Then for $X > 0$, $\phi(Z)$ *is schlicht and further if* $\xi \le 0$ *we have*

$$|\eta| < \frac{\pi}{2}\frac{X^2 + Y^2}{X}. \tag{2.2}$$

Let

$$Z_1 = X_1 + iY_1,\ Z_2 = X_2 + iY_2,\ X_1 > 0,\ X_2 > 0 \tag{2.3}$$

and suppose $Z_1 \ne Z_2$. We have to show that

$$\phi(Z_1) \ne \phi(Z_2). \tag{2.4}$$

We have

$$\arg \phi(Z) = \arg Z + \arg \log(1 + Z). \tag{2.5}$$

Both terms on the right hand side of (2.5) have the same sign as Y, when $Z = X + iY$ and $X > 0$. Thus $\phi(Z)$ is real if and only if $Y = 0$, and otherwise $\Im\{\phi(Z)\}$ has the same sign as Y. Thus (2.4) certainly holds unless Y_1, Y_2 have the same sign or are both zero. But if Y_1, Y_2 are both zero (2.4) holds trivially unless $Z_1 = Z_2$, since for real Z, $\phi(Z)$ is an increasing function of Z. Suppose now that

$$Y_1 > 0,\ Y_2 > 0. \tag{2.6}$$

Write

$$\phi'(Z) = U + iV = \log(1 + Z) + \frac{Z}{1 + Z}. \tag{2.7}$$

Then

$$\phi(Z_2) - \phi(Z_1) = \int_{Z_1}^{Z_2} (U\,dX - V\,dY) + i(V\,dX + U\,dY), \tag{2.8}$$

where the integral is taken along the straightline segment joining Z_1, Z_2. We may suppose without loss in generality that

$$X_1 \le X_2. \tag{2.9}$$

It follows from (2.6) and (2.3) that

$$U > 0,\quad V > 0$$

in (2.7). Suppose first

$$Y_1 \le Y_2.$$

Then it follows that both dX and dY are non negative and one of them is strictly positive in (2.8). Hence

$$\int_{Z_1}^{Z_2} (V\,dX + U\,dY) > 0$$

so that (2.4) holds. Similarly if

$$Y_2 \leq Y_1$$

we have $dX \geq 0$, $dY \leq 0$ in (2.8) so that

$$\int_{Z_1}^{Z_2} (U\,dX - V\,dY) > 0.$$

This completes the proof of (2.4) if (2.6) holds. The result follows when $Y_1 < 0$, $Y_2 < 0$ by taking complex conjugates. Thus in all cases (2.3) implies (2.4) unless $Z_1 = Z_2$, so that $\phi(Z)$ is schlicht in $X > 0$.

To complete the proof of lemma 1, we prove (2.2). We have

$$\xi = X \log|1 + Z| - Y \arg(1 + Z), \tag{2.10}$$

$$\eta = X \arg(1 + Z) + Y \log|1 + Z|. \tag{2.11}$$

Hence if $\xi \leq 0$ (2.10) gives

$$X \log|1 + Z| \leq Y \arg(1 + Z)$$

so that (2.11) gives

$$|\eta| \leq |\arg(1 + Z)| \left(X + \frac{Y^2}{X}\right) < \frac{\pi(X^2 + Y^2)}{2X}.$$

This proves (2.2) and completes the proof of lemma 1.

3). We can now provide a counterexample to (1.5) and thus dispose of Problem (iv) of paragraph 1. We have

Theorem I. *Suppose that* $1 < M < \infty$. *Then the function*

$$f(z) = M \exp \frac{1}{4}\left\{\frac{1+z}{1-z} \log \frac{2}{1-z}\right\} \tag{3.1}$$

is regular nonzero in $|z| < 1$ *and takes no value* w *such that* $|w| < M$ *more than* $1/(1-\varrho)$ *times in* $|z| \leq \varrho$, $0 < \varrho < 1$. *Further*

(3.2) $$\log f(\varrho) > \frac{1}{4(1-\varrho)} \log \frac{1}{1-\varrho}.$$

The inequality (3.2) is obvious. Suppose now that

$$f(z) = w.$$

Write

(3.3) $$Z = X + iY = \frac{1+z}{1-z}.$$

Then we have

$$\exp\left\{\frac{1}{4} Z \log(1+Z)\right\} = \frac{w}{M},$$

so that

(3.4) $$Z \log(1+Z) = 4 \log\left|\frac{w}{M}\right| + 8m\pi i + 4i \arg w.$$

Now it follows from (3.3) that for each z in $|z| < 1$, there exists a unique Z with $X > 0$. Also from lemma 1 the function $Z \log(1+Z)$ is schlicht in this half plane so that the equation (3.4) has at most one solution in $X > 0$ for each given w and m. Again if $|w| < M$ it follows from (2.2) that the equation

(3.5) $$Z \log(1+Z) = 4\left[\log\left|\frac{w}{M}\right| + i\eta\right]$$

only has a root for $X > 0$, if

(3.6) $$4|\eta| < \frac{\pi}{2} \frac{X^2 + Y^2}{X}.$$

Making use of (3.3) we have

$$|z|^2 = \left|\frac{Z-1}{Z+1}\right|^2 = \frac{(X-1)^2 + Y^2}{(X+1)^2 + Y^2}.$$

Hence if (3.5) holds with $|w| < M$ we have

$$1 - |z|^2 = \frac{4X}{(X+1)^2 + Y^2} < \frac{4X}{X^2 + Y^2} < \frac{\pi}{2\eta},$$

by (3.6), i.e.,

$$|\eta| < \frac{\pi}{2(1-|z|^2)} \leq \frac{\pi}{2(1-|z|)}.$$

It follows that (3.4) can have a solution in $|z| \leq \varrho$ only if

$$|\arg w + 2m\pi| \leq \frac{\pi}{2(1-\varrho)},$$

and this can hold for at most

$$\frac{1}{2(1-\varrho)}+1$$

different integers m. Thus (3.4) has less than

$$\frac{1}{2\left(1-\frac{1}{2}\right)}+1=2$$

different roots in $|z|<\frac{1}{2}$ for a given w with $|w|<M$. i.e., at most one such root in $|z|<\frac{1}{2}$, and at most

$$1+\frac{1}{2(1-\varrho)}\leq\frac{1}{1-\varrho}$$

different roots in $|z|\leq\varrho$, when $\varrho\geq\frac{1}{2}$. Thus if $|w|<M$ and $0<\varrho<1$ and z, Z are related as in (3.3), the equation (3.4) has at most $1/(1-\varrho)$ different roots in $|z|\leq\varrho$ and the same is therefore true of the equation $f(z)=w$. This completes the proof of Theorem I.

Having now disposed of the comparatively simple problems (i) and (iv) of paragraph 1, we shall spend the rest of the chapter in constructing counterexamples to the problems (ii) and (iii). These are very much more difficult and we shall have to employ rather a lot of general mapping Theory before we can even start to prove any particular Theorems. The theory we shall introduce will be stated in terms of lemmas only. Lemmas 2 to 8 are all vital to our constructions. General theory stops after lemmas 9 and 10 which adapt the preceding general theory to our particular problems. In paragraph 17 we take these problems up again. The counterexamples to problem (iii) occupy paragraphs 17 to 23. Paragraphs 24 to 31 deal with counterexamples in problem (ii).

The Principle of Harmonic Measure.

4). We start our constructions by introducing the conception of the harmonic measure of a connected portion α of the boundary of a domain D with respect to an interior point w of D. We write this as $\omega[w,\alpha;D]$ and recall the following.

Definition. *Let D be a simply connected domain in the finite w plane other than the whole plane. Let α be an arc of the frontier of D or, in the case of multiple frontier points, a connected set of prime ends on the frontier of D. Then*

$\omega[w, \alpha; D]$, *where* w *is a variable point of* D *is a bounded harmonic function of* w *such that*

$$\omega[w, \alpha; D] \to 1$$

as w *tends to an interior frontier point of the arc* α *and*

$$\omega[w, \alpha; D] \to 0$$

as w *tends to an interior frontier point of the complement of* α.

It is clear from the definition that $\omega[w, \alpha; D]$ is an additive function of the arc α, and that if α, β are complementary parts of the frontier of D, then

$$\omega[w, \alpha; D] + \omega[w, \beta; D] = 1. \tag{4.1}$$

Also since harmonic functions are invariant under conformal mapping it follows that if D_1, D_2 can be mapped conformally onto each other so that the frontier arcs α_1, α_2 and the interior points w_1, w_2 correspond, we have

$$\omega[w_1, \alpha_1; D_1] = \omega[w_2, \alpha_2; D_2]. \tag{4.2}$$

Thus we can define $\omega[w, \alpha; D]$ by mapping D 1:1 conformally onto the circle $|z| < 1$, so that the point w corresponds to $z = 0$ and the frontier arc α corresponds to an arc α' of length L on the circle $|z| = 1$. In this case

$$\omega[w, \alpha; D] = \omega[0, \alpha'; |z| < 1] = \frac{L}{2\pi}. \tag{4.3}$$

The basis of the theory is the well known lemma 2, the Principle of Harmonic measure, which is Nevanlinna's Generalization of a lemma due to Löwner.[1]

Lemma 2.[2] *Let* D_1, D_2 *be two simplyconnected domains in the* w *plane and suppose that* $f(w)$ *is regular in* D_1 *and has its values lying in* D_2. *Suppose further that* $f(w_1) = w_2$ *and that for* w *lying on a frontier arc* α_1 *of* D_1, $f(w)$ *has boundary values lying on a frontier arc* α_2 *of* D_2. *Then*

$$\omega[w_1, \alpha_1; D_1] \leq \omega[w_2, \alpha_2; D_2]. \tag{4.4}$$

Equality holds only if α_1 *consists of the whole frontier of* D_1, *or if* $f(w)$ *maps* D_1 1:1 *and conformally onto* D_2.

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[1] K. Löwner (1).

[2] R. Nevanlinna (2), p. 38.

Corollary. *If $D_1 < D_2$, w is an interior point of D_1 and α is a common frontier arc of D_1, D_2, we have*

$$\omega\,[w, \alpha;\, D_1] \leq \omega\,[w, \alpha;\, D_2].$$

Introduction of a General Class of Mapping Functions.

5). In the next three paragraphs we shall introduce a general class of functions from which we shall construct the counterexamples we require. Let ξ_n be an increasing sequence of real numbers, such that

$$(5.1) \qquad \xi_0 = -1$$

$$(5.2) \qquad \xi_1 > 0$$

$$(5.3) \qquad \xi_n < \xi_{n+1} \to \infty, \quad n = 1, 2, \ldots$$

Let η_n be a sequence of positive numbers, defined for $n = 1, 2, \ldots$ and such that

$$(5.4) \qquad \eta_n \leq \tfrac{1}{2}(\xi_{n+1} - \xi_n), \quad n = 1, 2, \ldots$$

$$(5.5) \qquad \eta_n \leq \tfrac{1}{2}(\xi_n - \xi_{n-1}), \quad n = 1, 2, \ldots$$

Let C_n be a curve given by all points ζ of the form

$$(5.6) \qquad \zeta = i\eta + \xi_n(\eta), \quad |\eta| \leq \xi_{n+1} - \xi_n,$$

where $\xi_n(\eta)$ is a real continuous function of η, satisfying the following conditions

$$(5.7) \qquad \xi_0(\eta) = \xi_0 = -1, \quad |\eta| \leq \xi_1 - \xi_0;$$

and if $n \geq 1$,

$$(5.8) \qquad \xi_n(-\eta) = \xi_n(\eta), \quad |\eta| \leq \xi_{n+1} - \xi_n,$$

$$(5.9) \qquad \xi_n(\eta) = \xi_n, \quad |\eta| \leq \eta_n \text{ aud } |\eta| = \xi_{n+1} - \xi_n,$$

and

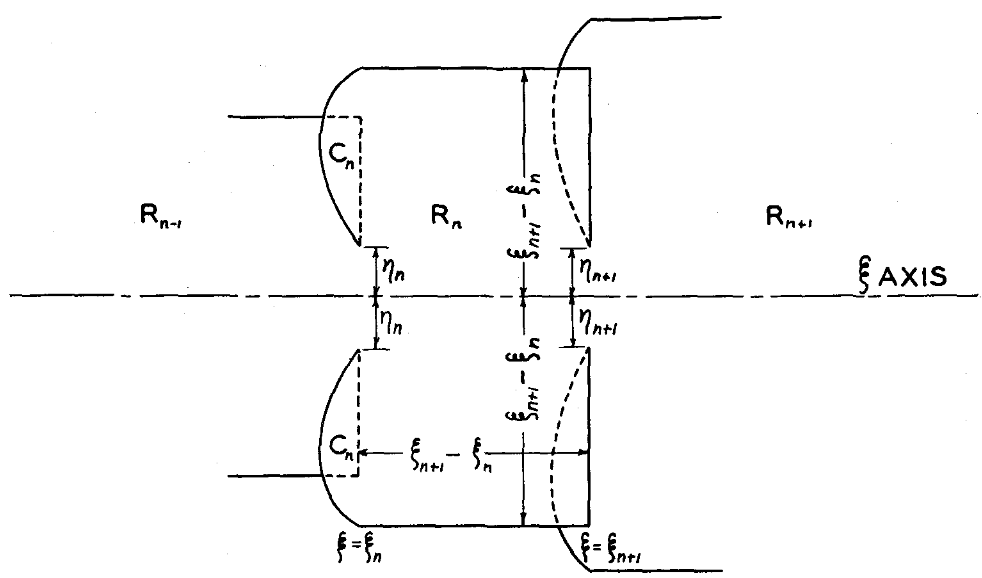
$$(5.10) \qquad 2\,\xi_n - \xi_{n+1} < \xi_n(\eta) \leq \xi_n, \quad \eta_n < |\eta| < \xi_{n+1} - \xi_n.$$

Let R_n be the domain in the ζ plane bounded by C_n and the three straight lines

$$(5.11) \qquad \xi = \xi_{n+1}, \quad \eta = \mp\, \xi_{n+1} - \xi_n.$$

We note that R_n consists of all $\zeta = \xi + i\eta$ for which both

$$(5.12) \qquad |\eta| < \xi_{n+1} - \xi_n, \text{ and } \xi_n(\eta) < \xi < \xi_{n+1}.$$



THE RIEMANN SURFACE $\mathscr{R}$

Fig. 1.

Let $\mathscr{R}$ be the Riemann surface over the ζ plane consisting of all the sheets R_n, where $n = 0, 1, 2, \ldots$ and R_n, R_{n+1} are supposed joined along their common frontier segment

$$(5.13) \qquad \xi = \xi_{n+1}, \quad |\eta| < \eta_{n+1}, \quad n = 0, 1, 2, \ldots$$

which lies on C_{n+1} by (5.6) and (5.9). Thus the points of $\mathscr{R}$ are the interior points of R_n and the segments (5.13), $n = 0, 1. 2, \ldots.$

Let

$$(5.14) \qquad \zeta = \psi(s) = c_1 s + c_2 s^2 + \cdots, \quad c_1 > 0$$

map the strip

$$(5.15) \qquad |\tau| < 1 \quad -\infty < \sigma < +\infty$$

in the $s = \sigma + i\tau$ plane symmetrically onto the Riemann surface $\mathscr{R}$ so that the positive real axes in the s and ζ planes correspond. The functions which will eventually provide our counterexamples will take the form

$$(5.16) \qquad f(z) = \exp \psi \left\{ \frac{2}{\pi} \log \frac{1+z}{1-z} \right\},$$

where $\psi(s)$ is the function of (5.14). Thus $\log f(z)$ maps the unit circle $|z| < 1$ onto the Riemann surface $\mathcal{R}$. Before considering these examples in detail, we shall need to make a general study of the function (5.14).

6). We need the following lemma on harmonic measure

Lemma 3. *Let γ be a Jordan arc in the $s = \sigma + i\tau$ plane, which lies entirely in the region $|\tau| \leq 1$, is symmetrical about the real axis and has its endpoints on $\tau = 1$ and $\tau = -1$, respectively. Suppose that γ does not contain the real point $s = \sigma$, but has at least one point on the line $\Re s = \sigma$. Let Δ_0 be the component containing $s = \sigma$ of the complement of γ in $|\tau| < 1$, and let γ_0 be the part of γ which lies on the frontier of Δ_0. Then we have*

$$(6.1) \qquad \omega[\sigma, \gamma_0; \Delta_0] \geq \tfrac{1}{2}.$$

We may suppose without loss in generality that $\sigma = 0$. Suppose first that γ contains the points $s = \mp i$. Let α_0 be the complement of γ_0 in the frontier of Δ_0. Then α_0, γ_0 are clearly connected, so that α_0 is either contained entirely in the region $\sigma \leq 0$ or $\sigma \geq 0$. Hence we have

$$(6.2) \qquad \omega[0, \alpha_0; \Delta_0] \leq \omega[0, \alpha_0; |\tau| \leq 1] \leq \tfrac{1}{2},$$

using lemma 2, corollary, since by symmetry the harmonic measure of each of the pairs of segments

$$(6.3) \qquad \begin{aligned} \tau &= \mp 1, \quad \sigma \leq 0, \\ \tau &= \mp 1, \quad \sigma \geq 0, \end{aligned}$$

at the origin with respect to the strip (5.15) is just $\frac{1}{2}$. Thus (6.2) shows that (6.1) holds in this case.

Suppose next that γ does not contain the points $\mp i$. It is clear that γ_0 contains exactly one point on the real axis $s = \sigma_0$ say. Suppose e.g. that

$$(6.4) \qquad \sigma_0 > 0.$$

Then Δ_0 contains the origin and since σ_0 is the only point of γ_0 on the real axis it follows that Δ_0 contains the halfline

$$\sigma < \sigma_0, \quad \tau = 0.$$

Let $\sigma' \mp i$ be the end points of γ_0. Then α_0, the part of the frontier of Δ_0 other than γ_0, consists of the two halflines

(6.5) $$\sigma < \sigma', \quad \tau = \mp i.$$

We now distinguish two cases. If

$$\sigma' < 0$$

then α_0 is contained again in the pair of segments (6.3) so that (6.2) and hence (6.1) follows. Suppose next that

$$\sigma' > 0.$$

Then γ_0 intersects the line $\Re s = 0$ by hypothesis. Let τ_1 be the greatest real number such that $0 < \tau_1 < 1$ and $\mp i \tau_1$ lie on γ_0. Since we are assuming that $\mp i$ do not lie on γ_0, τ_1 exists. Let γ_1 be the subarc of γ_0 whose endpoints are $\mp i \tau_1$. Since γ_0 contains no point on either of the two segments

(6.6) $$\sigma = 0, \quad |\tau_1| < |\tau| < 1,$$

which each have an endpoint on α_0, these two segments lie in Δ_0. Let Δ_1 be the subdomain of Δ_0 containing $s = 0$, obtained by cutting along the two segments (6.6) and let α_1 be the part of the frontier of Δ_1 consisting of the two segments (6.6) and the two halflines (6.3). Then since Δ_1 is contained in Δ_0 and still has γ_1 as part of its frontier, we have from lemma 2, corollary

(6.7) $$\omega\,[0, \gamma_1;\, \Delta_0] > \omega\,[0, \gamma_1;\, \Delta_1].$$

Again let Δ_2 consist of the strip $|\tau| < 1$ cut along the segments (6.6). Then Δ_2 contains Δ_1, and α_1, consisting of the two halflines (6.3) and the two segments (6.6) each described once, forms part of the frontier of both Δ_1 and Δ_2. The remainder of the frontier of Δ_2 consists of the reflection of α_1 in the imaginary axis, so that we have clearly from symmetry,

$$\omega\,[0, \alpha_1;\, \Delta_2] = \tfrac{1}{2}$$

and hence, since Δ_2 contains Δ_1,

(6.8) $$\omega\,[0, \alpha_1;\, \Delta_1] \leq \tfrac{1}{2}.$$

Since α_1, γ_1 make up the frontier of Δ_1, we deduce from (6.7) and (6.8)

$$\omega\,[0, \gamma_1;\, \Delta_0] > \tfrac{1}{2},$$

and since γ_1 is a subarc of γ_0, lemma 3 follows, if (6.4) holds. The proof is similar if $\sigma_0 < 0$ so that lemma 3 is always true.

7). We need lemma 3 to prove the following result concerning our mappings.

Lemma 4. *Suppose that*

$$\xi_{n-1} \le \xi' \le \xi_n - \eta_n,$$

where ξ_n, η_n *are the quantities of paragraph 5. Let* c *be the segment*

$$\xi = \xi_n, \quad |\eta| \le \xi_n - \xi' \tag{7.1}$$

on the frontier of R_{n-1}. *Suppose that* $s = \sigma'$ *corresponds to* $\zeta = \xi'$ *by the function of* (5.14) *and that* $s = \sigma + i\tau$ *corresponds to any point on* c *considered as a frontier-point of* R_{n-1}. *Then we have*

$$\sigma > \sigma'. \tag{7.2}$$

The inequality (7.2) *holds a fortiori if* $\sigma + i\tau$ *corresponds to an interior or frontier point of* R_m, *where* $m \ge n$.

Let $\mathscr{R}'$ be the subsurface of $\mathscr{R}$ consisting of the sheets $R_0, R_1 \ldots, R_{n-1}$ and their common frontier segments. Let Δ_0 be the subdomain of the strip (5.15) which maps onto $\mathscr{R}'$ by (5.14). Then it is sufficient to prove the lemma in the case where $\sigma + i\tau$ is a frontier point of Δ_0 for otherwise there exists $\sigma'' + i\tau$ where $\sigma'' < \sigma$, corresponding to a point on c by (5.14).

The segment c consists of frontier points of the domain R_{n-1}, and its endpoints are also frontier points of $\mathscr{R}$, by (5.13). Hence c corresponds by (5.14) to a Jordan curve γ_0 having its endpoints on $\tau = 1$, $\tau = -1$, respectively, and forming part of the boundary of Δ_0. Also the real point $s = \sigma$ on γ_0 corresponds to $\zeta = \xi_n > \xi'$ and so satisfies (7.2). Thus to complete the proof of the lemma it is sufficient to show that γ_0 does not meet the line $\Re s = \sigma'$, and to do this it is sufficient to show that

$$\omega\,[\sigma', \gamma_0;\, \Delta_0] < \tfrac{1}{2} \tag{7.3}$$

by lemma 3. We note that the function $\zeta = \psi(s)$ of (5.14) gives a mapping of the domain Δ_0 into (and not onto) the half plane D_1, $\Re\zeta < \xi_n$, by which γ_0 corresponds to the segment c on the boundary of D_1. Hence lemma 2 yields

$$\omega\,[\sigma', \gamma_0;\, \Delta_0] < \omega\,[\xi', c;\, D_1]. \tag{7.4}$$

Now $\omega\,[\xi', c;\, D_1]$ is equal to $1/\pi$ times the angle subtended at ξ' by the segment c. It follows from (7.1) that this angle is $\pi/2$ so that (7.4) yields (7.3) and hence (7.2). This completes the proof of lemma 4.

We shall need also the following form of Ahlfors' Theorem, (Ahlfors (1)), involving the mappings of strip-like domains into strips, and other domains.

Lemma 5. *Let Ω be an open set in the $w = u + iv$ plane, which meets any line $\Re w = u$ at most in a finite segment θ_u of length $\theta(u)$. We write*

$$I = \int_{u_1}^{u_2} \frac{du}{\theta(u)}. \tag{7.5}$$

Suppose also that Ω is mapped 1:1 conformally onto an open set lying in a simplyconnected domain D in such a way that the maps g_u of the segments θ_u all separate two points s_1, s_2 in D for $u_1 < u < u_2$. Then

(i) *we have*

$$d[s_1, s_2; D] > \frac{\pi}{2} I - \log 2. \tag{7.6}$$

(ii) *Suppose further that $I \geq 1$, that D is the strip*

$$|\tau| < 1, -\infty < \sigma < +\infty$$

in the $s = \sigma + i\tau$ plane and that the g_u join $\tau = 1$, $\tau = -1$. Then if

$$s_j = \sigma_j + i\tau_j, \quad j = 1, 2$$

we have

$$|\sigma_2 - \sigma_1| > 2(I - 1). \tag{7.7}$$

This lemma was proved in (Hayman 4). With a slight difference in the notation of the variables (7.6) follows from Theorem IV and Theorem V (3.8) and (7.7) above from Theorem I and (3.5) of that paper.

Here and subsequently we shall freely use the notion of hyperbolic distances $d[s_1, s_2; D]$ of two points s_1, s_2 with respect to a domain D. We shall need their form in the following two cases.

(i) If D is the strip $|\tau| < a$ and s_1, s_2 are real we have

$$d[s_1, s_2; D] = \frac{4}{\pi a}|s_1 - s_2|.$$

(ii) If D is the circle $|s - s_1| < R$, we have

$$d[s_1, s_2; D] = \tfrac{1}{2} \log \frac{R + |s_2 - s_1|}{R - |s_2 - s_1|}.$$

These identities are easily verified by mapping D onto the circle $|z| < 1$ and making use of Hayman (1), (3.1), (3.3) and (3.4).

8). We can now prove the following important independence principle, which shows that the behavior of the mapping function of (5.14) is for points ζ lying well inside the sheet R_n largely independent of the nature of the sheets R_ν for $\nu > n$.

Lemma 6. *Suppose that the quantities ξ_ν, η_ν and the curves C_ν have been fixed for $\nu = 1$ to n, that ξ_{n+1} is also fixed and that $\zeta = \xi'$ is a point such that*

$$0 < \xi' < \xi_{n+1}, \quad n \geq 0. \tag{8.1}$$

Suppose that the remaining ξ_ν, η_ν and C_ν are left variable subject to the conditions of paragraph 5 and also

$$\eta_{n+1} \leq \xi_{n+1} - \xi'. \tag{8.2}$$

Then if $s = \sigma_1'$, σ_2' correspond to $\zeta = \xi'$ by the mapping function (5.14) for two different Riemann surfaces $\mathcal{R}_1$, $\mathcal{R}_2$ satisfying the above conditions, we have

$$|\sigma_1' - \sigma_2'| < 1. \tag{8.3}$$

Let $\mathcal{R}'$ be the part of $\mathcal{R}_1$ consisting of the sheets R_1 to R_n and their common frontier segments. From our hypotheses these coincide for $\mathcal{R}_1$ and $\mathcal{R}_2$. Then lemma 4 (7.2) shows that by the mapping (5.14) of $|\tau| < 1$ onto $\mathcal{R}_1$ the segment θ_σ given by

$$\Re s = \sigma, \quad |\tau| \leq 1, \quad 0 < \sigma < \sigma_1',$$

corresponds in the ζ plane to a Jordan arc γ_σ whose interior points lie inside $\mathcal{R}'$ and whose endpoints do not lie on the segment

$$\Re \zeta = \xi_{n+1}, \quad |\Im \zeta| \leq \xi_{n+1} - \xi'. \tag{8.4}$$

Hence by the mapping (5.14) of $|\tau| < 1$ onto $\mathcal{R}_2$, γ_σ corresponds to an arc g_σ lying in $|\tau| < 1$ and joining $\tau = -1$ to $\tau = +1$. For the interior points of γ_σ lie in $\mathcal{R}'$ which is contained in $\mathcal{R}_2$ by hypothesis. Also the endpoints of γ_σ are frontier points of $\mathcal{R}'$ but do not lie on the segment (8.4) and so by (8.2) they must be frontier points of $\mathcal{R}_2$ and therefore correspond to points $\tau = -1$ and $\tau = +1$.

Further since the segments θ_σ separate $s = 0, \sigma_1'$ in $|\tau| < 1$, the curves g_σ separate $s = 0, \sigma_2'$ in $|\tau| < 1$. By combining the two mappings of $|\tau| < 1$ onto $\mathcal{R}_1$, $\mathcal{R}_2$, we thus obtain a mapping of the rectangle

$$0 < \sigma < \sigma_1', \quad |\tau| < 1,$$

into the strip $|\tau| < 1$, in which the segments θ_σ for $0 < \sigma < \sigma_1'$ correspond to arcs g_σ separating $s = 0$, σ_2'. This is the situation envisaged by lemma 5 (i) and we deduce from (7.6)

$$d[0, \sigma_2'; |\tau| < 1] > \frac{\pi}{2} \int_0^{\sigma_1'} \frac{d\sigma}{2} - \log 2$$

i.e.

$$\frac{\pi}{4} \sigma_2' > \frac{\pi}{4} \sigma_1' - \log 2$$

or

$$\sigma_2' > \sigma_1' - \frac{4}{\pi} \log 2 > \sigma_1' - 1.$$

Similarly we have

$$\sigma_1' > \sigma_2' - 1$$

so that (8.3) holds. This proves lemma 5. The lemma will permit us to choose ξ_n, η_n, R_n inductively so that certain inequalities are satisfied by the mapping function (5.14) inside each sheet R_n. It will follow that we can do this for each sheet R_n more or less independently of the nature of the subsequent sheets. This completes the first main stage of our argument.

Bounds for Hyperbolic Distances in $\mathcal{R}$.

9). In this and the following paragraph we shall be engaged in obtaining bounds for hyperbolic distances of two points ζ_1, ζ_2 in $\mathcal{R}$, or, what is the same thing, bounds for the hyperbolic distances in the strip $|\tau| < 1$ of the points s_1, s_2 which correspond to ζ_1, ζ_2 by (5.14).

We prove first lemma 7 below, which will itself have a certain importance in the sequel. The main result will be lemma 8, which we can prove from lemmas 5 and 7, and which like lemma 6 is an independence principle, showing that the mapping of the sheet R_n depends in the main only on this sheet and not on the other sheets. It is this result essentially, which allows us to invert the arguments of repeated application by means of which we proved the results of Chapter II. If a function exists growing at a certain rate and not taking any of a set E of values more than $p(\varrho)$ times in $|z| < \varrho$, then the function may grow at any point nearly as rapidly as if it only took zero and another value of E (depending on the point) $p(\varrho)$ times in $|z| < \varrho$, without reference to the other values of E.

Lemma 7. *Suppose that* $\sigma_1 < \sigma_2$ *and that* Δ *is a domain lying in the strip* $|\tau| < 1$ *in the* $s = \sigma + i\tau$ *plane and containing the rectangle*

$$\sigma_1 - 1 < \sigma < \sigma_2 + 1, \quad |\tau| < 1.$$

Let $s_1 = \sigma_1$ and let $s_2 = \sigma_2 + i\tau_2$, $|\tau_2| < 1$. *Then we have*

$$\frac{\pi}{4}(\sigma_2 - \sigma_1) + \frac{1}{2}\log\frac{1}{1-|\tau_2|} - 1 < d[s_1, s_2; \Delta] < \frac{\pi}{4}(\sigma_2 - \sigma_1) + \frac{1}{2}\log\frac{1}{1-|\tau_2|} + 4.$$

To prove the first inequality of lemma 7 we may suppose without loss in generality that Δ is the strip $|\tau| < 1$ since hyperbolic distances are decreased by increasing the domain. Further we may suppose $\sigma_1 = 0.$, since this may be achieved by a translation, which leaves the terms in the inequality of lemma 7 invariant.

The function

$$s = \phi(z) = \frac{2}{\pi}\log\frac{1+z}{1-z} \tag{9.1}$$

maps $|z| < 1$ onto Δ, so that $z = 0$ corresponds to $s = 0$ and

$$z_2 = \frac{e^{\pi s_2/2} - 1}{e^{\pi s_2/2} + 1}$$

corresponds to s_2. Thus we have

$$|z_2|^2 = \frac{e^{\pi\sigma_2} - 2e^{\pi\sigma_2/2}\cos\pi\tau_2/2 + 1}{e^{\pi\sigma_2} + 2e^{\pi\sigma_2/2}\cos\pi\tau_2/2 + 1},$$

$$1 - |z_2|^2 = \frac{4\cos\frac{\pi\tau_2}{2}}{e^{\pi\sigma_2/2} + e^{-\pi\sigma_2/2} + 2\cos\frac{\pi}{2}\tau_2} < \frac{4\cos\frac{\pi\tau_2}{2}}{e^{\pi\sigma_2/2}}$$

and so

$$d[0, s_2; \Delta] = d[0, z_2; |z| < 1] = \frac{1}{2}\log\frac{1+|z_2|}{1-|z_2|}$$

$$> \frac{1}{2}\log\frac{1}{1-|z_2|^2} > \frac{\pi}{4}\sigma_2 + \frac{1}{2}\log\sec\pi\tau_2/2 - \log 2$$

$$= \frac{\pi}{4}\sigma_2 + \frac{1}{2}\log\operatorname{cosec}\frac{\pi}{2}(1 - |\tau_2|) - \log 2$$

$$> \frac{\pi}{4}\sigma_2 + \frac{1}{2}\log\frac{2}{\pi(1-|\tau_2|)} - \log 2.$$

This proves the first inequality of lemma 7, since

$$\frac{1}{2}\log\frac{2}{\pi} - \log 2 = -\frac{1}{2}\log 2\pi > -1.$$

To prove the second inequality note that Δ contains the circle $|s - \sigma_2| < 1$ so that

$$d[\sigma_2, \sigma_2 + i\tau_2; \Delta] < \frac{1}{2}\log\frac{1 + |\tau_2|}{1 - |\tau_2|} < \frac{1}{2}\log\frac{1}{1 - |\tau_2|} + \frac{1}{2}\log 2. \tag{9.2}$$

Also

$$d[s_1, s_2; \Delta] \leq d[\sigma_1, \sigma_2; \Delta] + d[\sigma_2, \sigma_2 + i\tau_2; \Delta],$$

so that (9.2) gives

$$d[s_1, s_2; \Delta] \leq d[\sigma_1, \sigma_2; \Delta] + \frac{1}{2}\log\frac{1}{1 - |\tau_2|} + \frac{1}{2}\log 2. \tag{9.3}$$

To complete the proof it is sufficient to obtain a bound for $d[\sigma_1, \sigma_2; \Delta]$. To do this we may suppose without loss in generality that

$$\sigma_2 = -\sigma_1 = \sigma', \text{ say.}$$

The function $s = \phi(z)$ of (9.1) satisfies

$$-\frac{2}{\pi}\log\frac{1+r}{1-r} \leq \Re\,\phi(z) \leq \frac{2}{\pi}\log\frac{1+r}{1-r}, \quad |z| \leq r.$$

Hence if r is so chosen that

$$\frac{2}{\pi}\log\frac{1+r}{1-r} = \sigma' + 1, \tag{9.4}$$

the function

$$s = \phi(rz)$$

maps $|z| < 1$ into Δ, since by hypothesis Δ contains the rectangle

$$-\sigma' - 1 < \Re s < \sigma' + 1.$$

It follows that

$$d[0, \sigma'; \Delta] < \frac{1}{2}\log\frac{1+\varrho}{1-\varrho} \tag{9.5}$$

where ϱ is a point such that $\phi(r\varrho) = \sigma'$ so that

$$\frac{2}{\pi}\log\frac{1+\varrho r}{1-\varrho r} = \sigma'. \tag{9.6}$$

From (9.4) and (9.6) we have

$$\log \frac{1+r}{1-r} \cdot \frac{1-\varrho r}{1+\varrho r} = \frac{\pi}{2}$$

i.e.

$$\log \left\{ 1 + \frac{r(1-\varrho)}{1-r^2 \varrho} \Big/ \left[1 - \frac{r(1-\varrho)}{1-r^2\varrho} \right] \right\} = \frac{\pi}{2}$$

whence

$$\frac{r(1-\varrho)}{1-r^2\varrho} = \frac{e^{\pi/2}-1}{e^{\pi/2}+1} > \frac{1}{2},$$

so that

$$1-\varrho > \frac{1}{2r}(1-r^2\varrho) > \frac{1}{2}(1-r),$$

and hence

(9.7) $$\frac{1+\varrho}{1-\varrho} < \frac{4(1+r)}{1-r}.$$

Then (9.4), (9.5), (9.7) give

$$d[0, \sigma'; \Delta] < \frac{\pi}{4}\sigma' + \frac{\pi}{4} + \log 2.$$

Similarly we have

$$d[0, -\sigma'; \Delta] < \frac{\pi}{4}\sigma' + \frac{\pi}{4} + \log 2.$$

Thus

$$d[-\sigma_1, \sigma_2; \Delta] = d[-\sigma', \sigma'; \Delta] < \frac{\pi}{2}\sigma' + \frac{\pi}{2} + 2\log 2$$

$$= \frac{\pi}{4}(\sigma_2 - \sigma_1) + \frac{\pi}{2} + 2\log 2.$$

Combining this with (9.3) we have

$$d[\sigma_1, \sigma_2 + i\tau_2; \Delta] < \frac{\pi}{4}(\sigma_2-\sigma_1) + \frac{1}{2}\log\frac{1}{1-|\tau_2|} + \frac{\pi}{2} + \frac{5}{2}\log 2,$$

which yields lemma 7.

10). We can now combine lemmas 5 and 7 to prove.

Lemma 8. *Let D be a domain in the ζ plane given by the totality of points ζ for which*

(10.1) $$|\zeta - \zeta_0| = r, \quad r_1 < r < r_2$$

and

(10.2) $$\arg|\zeta - \zeta_0| \leq \theta(r), \text{ where } 0 < \theta(r) < \pi.$$

The points which satisfy (10.1) *and* (10.2) *for a fixed r we denote by θ_r. Suppose also that D is mapped symmetrically and* 1 : 1 *conformally onto a domain Δ in the*

$s = \sigma + i\tau$ plane lying in the strip $|\tau| < 1$, in such a way that the endpoints $re^{i\theta(r)}$, $re^{-i\theta(r)}$ of the circular arcs θ_r correspond to points on $\tau = 1$, $\tau = -1$, respectively, and so that the real axes in the s and ζ planes correspond and $\sigma(r)$ increases for $r_1 < r < r_2$. Let

$$r_1 e^{2\pi} \leq r_1' < r_2' \leq r_2 e^{-2\pi} \tag{10.3}$$

$$r_2' \geq r_1' e^{2\pi}. \tag{10.4}$$

Let $\zeta_1 = \zeta_0 + r_1'$ and $\zeta_2 = \zeta_0 + r_2' e^{i\theta_2}$, where $|\theta_2| < \theta(r_2')$, and suppose that $s_1 = \sigma_1$, $s_2 = \sigma_2 + i\tau_2$ correspond to ζ_1, ζ_2 in the mapping. Then we have

$$\sigma_2 > \sigma_1 \tag{10.5}$$

and

$$\frac{\pi}{4}(\sigma_2 - \sigma_1) + \frac{1}{2}\log\frac{1}{1-|\tau_2|} - 1 < d[\zeta_1, \zeta_2; D] < \frac{\pi}{4}(\sigma_2 - \sigma_1) + \frac{1}{2}\log\frac{1}{1-|\tau_2|} + 4. \tag{10.6}$$

The point of the lemma is to show that the precise form of the mapping has little effect on the relative position of s_1, s_2 when r_2/r_1 is sufficient large and ζ_1, ζ_2 are well inside D.

We have firstly

$$d[\zeta_1, \zeta_2; D] = d[s_1, s_2; \Delta], \tag{10.7}$$

since D is mapped $1:1$ and conformally onto Δ. The result now follows from lemma 7 provided that we can prove that s_1, s_2, Δ satisfy the conditions of that lemma.

The domain Δ is bounded by the curves $g_{(1)}$, $g_{(2)}$ which are the maps of the arcs θ_{r_1}, θ_{r_2}, and by two segments of $\tau = 1$, $\tau = -1$[1], respectively.

We write

$$w = u + iv = \log(\zeta - \zeta_0) \tag{10.8}$$

and consider the mapping of the w plane into the s plane. The segments θ_r correspond to straight line segments

$$|v| < \theta(e^u), \log r_1 < u < \log r_2$$

in the w plane. Also since the mapping is symmetrical, the upper half of each of these segments, given by

[1] This is a consequence of our hypotheses when $\theta(r)$ is continuous. Otherwise we make it an additional assumption.

$$(10.9) \qquad 0 < v < \theta(e^u), \ \log r_1 < u < \log r_2$$

corresponds to a curve g_u joining $\tau = 0$ to $\tau = 1$ and lying in the strip $0 < \tau < 1$. The length of each of these half segments is $\theta(e^u) < \pi$. Thus we may apply lemma 5 (ii) to the mapping of the segments (10.9) for $u_1 < u < u_2$ into the strip $0 < \tau < 1$ and we deduce that if $u_1 + i v_1$, $u_2 + i v_2$, where $v_1 \geq 0$, $v_2 \geq 0$, correspond to $\sigma_{(1)} + i\tau_{(1)}$, $\sigma_{(2)} + i\tau_{(2)}$ in the s plane then we have

$$(10.10) \qquad \sigma_{(2)} - \sigma_{(1)} > \left(\int_{u_1}^{u_2} \frac{du}{\theta(e^u)} - 1\right) > \frac{u_2 - u_1}{\pi} - 1 > 1,$$

if

$$(10.11) \qquad u_2 - u_1 > 2\pi.$$

Also the conditions $v_1 \geq 0$, $v_2 \geq 0$ may clearly be omitted since the mapping is symmetrical.

Taking $u_1 = \log r_1$, $u_2 = \log r'_1$, $v_2 = 0$, we deduce from (10.3) that (10.11) is satisfied. Hence (10.10) shows that if $\sigma + i\tau$ is any point on $g_{(1)}$, the image of θ_{r_1} in the s plane, then we have

$$\sigma \leq \sigma_1 - 1.$$

Similarly taking $u_1 = \log r'_2$, $v_2 = \log r_2$, (10.11) again holds by (10.3) and we deduce that in the notation of lemma 8

$$\sigma \geq \sigma_2 + 1$$

for any point $\sigma + i\tau$ on $g_{(2)}$. Thus Δ contains the rectangle

$$(10.12) \qquad \sigma_1 - 1 < \sigma < \sigma_2 + 1.$$

Again taking $u_1 = \log r'_1$, $u_2 = \log r'_2$, (10.11) holds by (10.4). Thus

$$(10.13) \qquad \sigma_2 - \sigma_1 > 1 > 0.$$

by (10.10). This gives (10.5). Also Δ contains the rectangle (10.12) and so Δ, s_1, s_2 satisfy the conditions of lemma 7. Thus (10.6) now follows from (10.7) and lemma 7. This completes the proof of lemma 8.

Specialization of the Curves C_n.

11). We shall be able to construct all the functions we require by giving to the curves C_n of paragraph 5 one of two forms. The simplest form, which would suffice for all applications where $p(\varrho)$ is given by (1.1) with $a < 1$, is that in which C_n is simply the segment

$$\zeta = \xi_n + i\eta, \quad |\eta| \leq \xi_{n+1} - \xi_n. \tag{11.1}$$

The conditions (5.6) to (5.10) are clearly satisfied in this case. The required results can be put together in

Lemma 9. *Suppose that C_n is given by the segment (11.1) for some $n > 0$. Let*

$$\zeta_{(1)} = \xi_n - \eta_n$$

$$\zeta_{(2)} = \xi_{(2)}, \quad \xi_n \leq \xi_{(2)} \leq \tfrac{1}{2}[\xi_n + \xi_{n+1}]$$

and let

$$\zeta_{(3)} = \xi_{(3)} + i\eta_{(3)}, \quad \xi_n \leq \xi_{(3)} \leq \xi_{n+1},$$

be a point of R_n. Suppose also that

$$s_{(1)} = \sigma_n,^{1} \; s_{(2)} = \sigma_{(2)}, \; s_{(3)} = \sigma_{(3)} + i\tau_{(3)}$$

correspond to $\zeta_{(1)}, \zeta_{(2)}, \zeta_{(3)}$ respectively by the mapping of (5.14). Then we have

$$\frac{\pi}{2}(\sigma_{(2)} - \sigma_n) < \log^+\left(\frac{\xi_{(2)} - \xi_n}{\eta_n}\right) + A \tag{11.2}$$

$$\frac{\pi}{2}(\sigma_{(3)} - \sigma_n) + \log\frac{1}{1 - |\tau_{(3)}|} > \log^+\left|\frac{\zeta_{(3)} - \xi_n}{\eta_n}\right| - A. \tag{11.3}$$

We define

$$\zeta_{(4)} = \xi_{(4)} = \xi_n - \tfrac{1}{2}\eta_n.$$

Then it follows from (5.5), (5.11), (5.12), that R_{n-1} contains the circle C_4[2]

$$|\zeta - \xi_{(1)}| < \eta_n,$$

which is mapped into the strip $|\tau| < 1$. Hence if $s = \sigma_{(4)}$ corresponds to $\zeta = \xi_{(4)}$, we have

$$\frac{\pi}{4}(\sigma_{(4)} - \sigma_n) = d[\sigma_n, \sigma_{(4)}; |\tau| < 1] < d[\zeta_{(1)}, \zeta_{(4)}; C_4] = \frac{1}{2}\log\frac{\eta_n + \frac{1}{2}\eta_n}{\eta_n - \frac{1}{2}\eta_n} = A,$$

so that

$$\sigma_{(4)} - \sigma_n < A. \tag{11.4}$$

Similarly if $\xi_{(5)} = \xi_n + \frac{1}{2}\eta_n$ corresponds to $s = \sigma_{(5)}$, we deduce by considering the

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[1] This will be standard notation in the sequel.

[2] Compare fig. 1 in section 5.

mapping of the circle C_5, $|\zeta - \xi_n| < \eta_n$, which lies in $\mathscr{R}$ by (5.4) (5.5) and (5.12) (5.13), that

(11.5) $$\sigma_{(5)} - \sigma_{(4)} < A.$$

Thus (11.2) follows from (11.4) and (11.5) if $\xi_{(2)} \leq \xi_n + \frac{1}{2}\eta_n$, so that $\sigma_{(2)} \leq \sigma_{(5)}$. Suppose next that

$$\xi_{(5)} \leq \xi_{(2)} \leq \tfrac{1}{2}(\xi_n + \xi_{n+1}).$$

Then the circle C given by

$$|\zeta - \zeta_{(2)}| = \xi_{(2)} - \xi_n$$

lies in R_n and is mapped into the strip $|\tau| < 1$ so that $\zeta_{(5)}$, $\zeta_{(2)}$ become $\sigma_{(5)}, \sigma_{(2)}$. Thus

$$d[\sigma_{(5)}, \sigma_{(2)}; |\tau| < 1] < d[\xi_{(5)}, \xi_{(2)}; C],$$

$$\frac{\pi}{4}(\sigma_{(2)} - \sigma_{(5)}) < \frac{1}{2}\log\frac{\xi_{(2)} - \xi_n + \xi_{(2)} - \xi_{(5)}}{\xi_{(2)} - \xi_n - \xi_{(2)} + \xi_{(5)}} < \frac{1}{2}\log\left|\frac{2(\xi_{(2)} - \xi_n)}{\frac{1}{2}\eta_n}\right|$$

which yields (11.2), on using (11.4), (11.5). Thus (11.2) is generally true.

12). To prove (11.3) suppose first

(12.1) $$\eta_n < e^{-6\pi}(\xi_{n+1} - \xi_n).$$

Then we can apply lemma 8, taking for D the domain

(12.2) $$|\zeta - \xi_n| = r, \quad \eta_n < r < \xi_{n+1} - \xi_n,$$

(12.3) $$|\arg(\zeta - \xi_n)| \leq \theta(r) = \frac{\pi}{2}.$$

This domain is mapped into the strip $|\tau| < 1$ in such a way that the semicircles (12.2), (12.3) become curves joining $\tau = -1$, $\tau = +1$. Hence if

(12.4) $$\zeta_{(6)} = \xi_n + e^{2\pi}\eta_n,$$

and

(12.5) $$e^{4\pi}\eta_n \leq |\zeta_{(3)} - \xi_n| \leq e^{-2\pi}(\xi_{n+1} - \xi_n),$$

we deduce from lemma 8, that if $s = \sigma_{(6)}$ corresponds to $\zeta = \xi_{(6)}$ we have

(12.6) $$\frac{\pi}{4}(\sigma_{(3)} - \sigma_{(6)}) + \frac{1}{2}\log\frac{1}{1 - |\tau_3|} + 4 > d[\zeta_{(6)}, \zeta_{(3)}; D].$$

Also D is contained in the half plane D' given by

$$|\arg(\zeta - \xi_n)| \leq \frac{\pi}{2}$$

so that

$$d[\zeta_{(6)}, \zeta_{(3)}; D[\geq d[\zeta_{(6)}, \zeta_{(3)}; D'] = \frac{1}{2}\log\frac{|\zeta_{(6)} + \zeta_{(3)} - 2\,\xi_n| + |\zeta_{(3)} - \zeta_{(6)}|}{|\zeta_{(6)} + \zeta_{(3)} - 2\,\xi_n| - |\zeta_{(3)} - \zeta_{(6)}|}$$

$$\geq \frac{1}{2}\log\left|\frac{\zeta_{(3)} - \xi_n}{\zeta_{(6)} - \xi_n}\right|;$$

$$(12.7) \qquad d[\zeta_{(6)}, \zeta_{(3)}; D] \geq \frac{1}{2}\log^+\left(\frac{\zeta_{(3)} - \xi_n}{e^{2\pi}\,\eta_n}\right).$$

Combining this with (12.6) and noting that $\sigma_{(6)} > \sigma_n$ we see that (11.3) holds provided that (12.5) is true. Again if (12.1) holds and

$$|\zeta_{(3)} - \xi_n| = e^{-2\pi}(\xi_{n+1} - \xi_n)$$

it follows that (12.6), (12.7) hold so that in this case

$$(12.8) \qquad \frac{\pi}{2}(\sigma_{(3)} - \sigma_n) + \log\frac{1}{1 - |\tau_{(3)}|} > \frac{1}{2}\log^+\left(\frac{\xi_{n+1} - \xi_n}{\eta_n}\right) - A.$$

This latter inequality still holds if

$$(12.9) \qquad |\zeta_{(3)} - \xi_n| \geq e^{-2\pi}(\xi_{n+1} - \xi_n),$$

since if $s_{(3)}$ corresponds to $\zeta_{(3)}$ and we consider the straight line joining $s_{(3)}$ to σ_n, this corresponds to a curve joining $\zeta_{(3)}$ to $\zeta_{(1)}$ in the ζ plane and if (12.9) holds, somewhere on this curve we must have a point $\zeta_{(7)}$ with

$$|\zeta_{(7)} - \xi_n| = e^{-2\pi}(\xi_{n+1} - \xi_n),$$

so that (12.8) holds with $\sigma_{(7)}$, $\tau_{(7)}$ instead of $\sigma_{(3)}$, $\tau_{(3)}$ and so a fortiori with $\sigma_{(3)}$, $\tau_{(3)}$. It follows that if (12.1) holds, (11.3) is true provided that

$$(12.10) \qquad e^{4\pi}\eta_n \leq |\zeta_{(3)} - \xi_n| \leq (\xi_{n+1} - \xi_n)\sqrt{2}.$$

Again it follows from lemma 4 that we have always

$$(12.11) \qquad \sigma_{(3)} - \sigma_n > 0.$$

Thus (11.3) holds generally provided that

$$(12.12) \qquad |\zeta_{(3)} - \xi_n| \leq (\xi_{n+1} - \xi_n)\sqrt{2},$$

and (12.1) holds.

Again (12.12) is satisfied by condition (5.12) for every point $\zeta_{(3)} = \xi_{(3)} + i\,\eta_{(3)}$ in

R_n. Thus (11.3) always holds if (12.1) is satisfied. Also if (12.1) is false it follows from (12.12) that

$$|\zeta_{(3)} - \xi_n| \leq A\,\eta_n,$$

and hence from (12.11) that (11.3) holds. Thus (11.3) is generally true. This completes the proof of lemma 9.

A Second Form for the Curves C_n.

13). We shall now introduce the other form of the curves C_n which we shall need in the sequel. The investigation in this case needs the full strength of our preceding theory.

Suppose that

$$\xi_{n+1} - \xi_n > 3\, e^{6\pi}\,\eta_n, \quad n > 0. \tag{13.1}$$

We define C_n as given parametrically by the equation

$$\zeta = \zeta_n(t) \tag{13.2}$$

where

$$\zeta_n(t) = \xi_n + i\,\eta_n\,t, \quad 0 \leq t \leq 1, \tag{13.3}$$

$$\zeta_n(t) = \xi_n + \frac{2\,\eta_n}{\log 2}\left[i\,t \log(1 + i\,t) + \frac{\pi}{4}\right], \quad 1 \leq t \leq t_n, \tag{13.4}$$

and t_n is the positive root of the equation

$$\frac{2\,\eta_n}{\log 2}\left|i\,t_n \log(1 + i\,t_n) + \frac{\pi}{4}\right| = e^{-2\pi}(\xi_{n+1} - \xi_n) = r_n, \text{ say.} \tag{13.5}$$

The equation (13.5) evidently has a unique solution for t_n in $1 \leq t_n < \infty$, since both $|\Re\,\zeta_n(t) - \xi_n|$ and $|\Im\,\zeta_n(t)|$ and so $|\zeta_n(t) - \xi_n|$ are monotone increasing functions of t in the range defined by (13.4), and since (13.1) holds. For $t \geq t_n$ we define $\zeta_n(t)$ as follows. Let

$$\zeta_n(t_n) = \xi_n + r_n\,e^{i\theta_n}. \tag{13.6}$$

We write

$$\zeta_n(t) = \xi_n + r_n\,e^{i(\theta_n + t_n - t)}, \quad t_n \leq t \leq t_n + \theta_n - \frac{\pi}{2}, \tag{13.7}$$

$$\zeta_n(t) = \xi_n + i(a\,t + b), \quad t \geq t_n + \theta_n - \frac{\pi}{2}, \tag{13.8}$$

where $a\,t + b$ increases from r_n to $\xi_{n+1} - \xi_n$ as t increases from $t_n + \theta_n - \frac{\pi}{2}$ by 1.

Thus C_n is completed after the range defined by (13.4) by an arc of a circle centre ξ_n, and then the straight line segment joining $\zeta = \xi_n + i r_n$ to $\zeta = \xi_n + i(\xi_{n+1} - \xi_n)$. Also for negative t we define $\zeta_n(t)$ by the symmetry relation

$$(13.9) \qquad \zeta_n(-t) = \overline{\zeta_n(t)}, \quad 0 \le t \le t_n + \theta_n - \frac{\pi}{2} + 1,$$

so that C_n is symmetrical about the real axis. The important part of C_n is the arc given by (13.4). This will insure that the mapping of the region R_n bounded by C_n and the lines

$$(13.10) \qquad \Re\zeta = \xi_{n+1}, \quad \Im\zeta = \mp(\xi_{n+1} - \xi_n),$$

into the s plane, given by the function $\psi(s)$ of (5.14) will behave locally like the mapping obtained by combining

$$(13.11) \qquad \zeta = \xi_n + \frac{2\eta_n}{\log 2}\left[Z \log(1 + Z) + \frac{\pi}{4}\right]$$

with

$$s = \frac{2}{\pi} \log Z + \text{cons.}$$

The mapping (13.11) has already been studied in lemma 1. It led to the solution of our fundamental problem when E is bounded and

$$p(\varrho) = \frac{1}{1 - \varrho}.$$

The theory which we have been building up allows us to extend this to the case when E is unbounded.

It remains to show that the definitions (13.2) to (13.9) define C_n in accordance with the conditions we laid down in (5.6) to (5.10). The condition (5.6) will be realized provided that C_n cuts each line

$$\Im\zeta = \eta, \quad |\eta| \le \xi_{n+1} - \xi_n,$$

in exactly one point. This it easily verified since $\Im\zeta_n(t)$ is a continuous increasing function of t in the range of definition, as we can see from (13.3) to (13.9). Also the endpoints of C_n are the points given by

$$\zeta = \zeta_n(t) = \xi_n \mp i(\xi_{n+1} - \xi_n).$$

Again (5.8) follows from (13.9), (5.9) follows from (13.3) and (13.8). Lastly $\Re\zeta_n(t)$ clearly satisfies the inequalities

$$(13.12) \qquad \Re\zeta_n(t_n) \le \Re\zeta_n(t) \le \xi_n.$$

Also

$$\xi_n - \Re\,\zeta_n(t_n) \le |\zeta_n(t_n) - \xi_n|$$
$$= e^{-2\pi}(\xi_{n+1} - \xi_n) < \xi_{n+1} - \xi_n,$$

by (13.5). Using (13.12) we have (5.10). Thus C_n defined by (13.2) to (13.9) satisfies all the conditions required for the curves C_n.

14). We now investigate the behavior of our mapping function $\zeta = \psi(s)$ of (5.14) when ζ lies in the sheet R_n and C_n is defined by (13.2) to (13.9). The results we need are contained in

Lemma 10. *Suppose that for some $n > 0$, (13.1) holds and that C_n is defined by (13.2) to (13.9). Let R_n be the domain bounded by C_n and the lines (13.10), and let the conditions of paragraph 5 be satisfied. Let*

(14.1) $$\zeta_{(1)} = \xi_n - \eta_n,$$

(14.2) $$\zeta_{(2)} = \xi_{(2)}, \quad \xi_n \le \xi_{(2)} \le \tfrac{1}{2}(\xi_n + \xi_{n+1}),$$

and let

(14.3) $$\zeta_{(3)} = \xi_{(3)} + i\,\eta_{(3)}, \quad \xi_{(3)} \le \xi_n,$$

be a point of R_n. Suppose that $s = \sigma_n$, $\sigma_{(2)}$, $\sigma_{(3)} + i\,\tau_{(3)}$ respectively correspond to $\zeta = \zeta_{(1)}$, $\zeta_{(2)}$, $\zeta_{(3)}$ by the mapping (5.14) of the strip $|\tau| < 1$ in the $s = \sigma + i\tau$ plane onto the Riemann surface $\mathcal{R}$. Then we have

(14.4) $$\left|\frac{\pi}{2}(\sigma_{(2)} - \sigma_n) - \log^+\left(\frac{\xi_{(2)} - \xi_n}{\eta_n}\right) + \log^+\log^+\left(\frac{\xi_{(2)} - \xi_n}{\eta_n}\right)\right| < A,$$

(14.5) $$\frac{\pi}{2}(\sigma_{(3)} - \sigma_n) + \log\frac{1}{1 - |\tau_{(3)}|} > \log^+\left|\frac{\eta_{(3)}}{\eta_n}\right| - A.$$

Let D_n be the subdomain of R_n for which

(14.6) $$3\,\eta_n < |\zeta - \xi_n| < \xi_{n+1} - \xi_n.$$

It follows from the definitions (13.2) to (13.10) that each circle

(14.7) $$|\zeta - \xi_n| = r, \quad 3\,\eta_n < r < \xi_{n+1} - \xi_n,$$

intersects R_n and so D_n in a segment of a circle

(14.8) $$|\arg(\zeta - \xi_n)| \le \theta(r), \quad |\zeta - \xi_n| = r$$

where

$$\frac{\pi}{2} \le \theta(r) < \pi.$$

Hence we can apply lemma 8 with D_n for D, $\zeta_0 = \xi_n$ and

$$r_1 = 3\,\eta_n, \quad r_2 = \xi_{n+1} - \xi_n. \tag{14.9}$$

Hence if

$$\zeta_{(4)} = \xi_{(4)} = \xi_n + 3\,\eta_n\,e^{2\pi}, \tag{14.10}$$

and

$$|\eta_{(3)}| > 3\,\eta_n\,e^{4\pi}, \tag{14.11}$$

we apply lemma 8 with $\zeta_{(4)}$ instead of ζ_1 and $\zeta_{(3)}$ instead of ζ_2 to the mapping of D_n into the strip $|\tau| < 1$. In fact the endpoints of the arcs of circle (14.8) correspond to points on $\tau = 1$, $\tau = -1$, respectively, since

$$\eta_n < r < \xi_{n+1} - \xi_n.$$

Also from (13.5), the point $\zeta_{(3)}$, where $\xi_{(3)} \leq \xi_n$ can only lie in R_n if

$$r_2' = |\zeta_{(3)} - \xi_n| \leq e^{-2\pi}(\xi_{n+1} - \xi_n)$$

so that we have

$$r_2' \leq e^{-2\pi}\,r_2$$

using (14.9), as required in (10.3). Also

$$r_1' = |\zeta_{(4)} - \xi_n| = r_1\,e^{2\pi}$$

by (14.9) and (14.10) so that (10.3) is satisfied. Again

$$r_2' \geq e^{2\pi}\,r_1'$$

by (14.10) and (14.11), so that (10.4) is also satisfied. Then if $s = \sigma_{(4)}$ corresponds to $\zeta = \zeta_{(4)}$ given by (14.10), lemma 8 gives

$$\frac{\pi}{4}(\sigma_{(3)} - \sigma_{(4)}) + \frac{1}{2}\log\frac{1}{1 - |\tau_{(3)}|} > d\,[\zeta_{(4)}, \zeta_{(3)};\, D_n] - 4, \tag{14.12}$$

provided that (14.11) holds.

Again if

$$e^{2\pi}(\xi_{(4)} - \xi_n) \leq (\zeta_{(2)} - \xi_n) \leq e^{-2\pi}(\xi_{n+1} - \xi_n), \tag{14.13}$$

we can apply the inequalities of (10.6) with $\zeta_{(4)}$ for $\zeta_{(1)}$ and $\zeta_{(2)}$ for ζ_2 and obtain

$$\left|\frac{\pi}{4}(\sigma_{(2)} - \sigma_{(4)}) - d\,[\zeta_{(4)}, \zeta_{(2)};\, D_n]\right| < 4. \tag{14.14}$$

We shall deduce lemma 10 from (14.12) and (14.14).

15) In order to obtain bounds for the quantities $d\,[\zeta_{(i)}, \zeta_{(j)};\, D_n]$ which appear in (14.12) and (14.14) we consider a different mapping of D_n into the strip

$|\tau| < 1$. We put $Z = X + iY$ and

$$\zeta = \xi_n + \frac{2\eta_n}{\log 2}\left[Z \log(1 + Z) + \frac{\pi}{4}\right], \tag{15.1}$$

$$s = \frac{2}{\pi} \log Z. \tag{15.2}$$

Using lemma 1 we see that this gives a schlicht mapping of the strip $|\tau| < 1$ in the $s = \sigma + i\tau$ plane onto the region bounded by the curve

$$\zeta = \xi_n + \frac{2\eta_n}{\log 2}\left[iY \log(1 + iY) + \frac{\pi}{4}\right], \quad -\infty < Y < +\infty, \tag{15.3}$$

and lying to the right of the curve when it is described in the direction of increasing Y.

The curve (15.3) coincides with C_n for the arc given by (13.4) taking $t = Y$. Also C_n lies to the right of the curve (15.3) for the points of C_n given by $|t| \geq t_n$ as we see from (13.7), (13.8) and (13.9). Lastly if ζ is a point on the curve (15.3) which corresponds to $|Y| < 1$, we have

$$|\zeta - \xi_n| < \frac{2\eta_n}{\log 2}\left|\frac{\pi}{4} + \frac{i}{2}\log 2\right| < 3\eta_n.$$

Hence no points lying in the region R_n and satisfying

$$|\zeta - \xi_n| > 3\eta_n,$$

can lie on the curve (15.3) and so in particular the whole of the region D_n defined by (14.6) lies to the right of the curve (15.3) and so is mapped into the strip $|\tau| < 1$ by the mapping given by (15.1), (15.2).

Again it follows from lemma 1, (2.2) that if

$$Z = X + iY = \exp \frac{\pi}{2}(\sigma_3 + i\tau_3)$$

corresponds to $\zeta_{(3)} = \xi_{(3)} + i\eta_{(3)}$ by (15.1), (15.2), where $\xi_{(3)} \leq \xi_n$, then we have

$$\left|\frac{\eta_{(3)} \log 2}{2\eta_n}\right| < \frac{\pi}{2}\frac{X^2 + Y^2}{X} = \frac{\pi e^{(\pi\sigma_3/2)}}{2\cos(\pi\tau_3/2)}.$$

Hence

$$\frac{\pi}{2}\sigma_3 + \log\frac{1}{1 - |\tau_3|} > \log\left|\frac{\eta_{(3)}}{\eta_n}\right| - A.$$

Also if $s = \sigma_4$ corresponds to $\zeta = \xi_4$ defined by (14.10), we have clearly $\sigma_4 = A$, so that

$$\frac{\pi}{2}(\sigma_3-\sigma_4)+\log\frac{1}{1-|\tau_3|}>\log\left|\frac{\eta_{(3)}}{\eta_n}\right|-A. \tag{15.4}$$

Now the mapping of (15.1), (15.2) maps the region D_n into the strip $|\tau|<1$, as we showed above. Hence if s_3, s_4 correspond to $\zeta_{(3)}$, $\zeta_{(4)}$ we have

$$d[\zeta_{(3)},\zeta_{(4)};D_n]\geq d[s_3,s_4;|\tau|<1]>\frac{\pi}{4}|\sigma_3-\sigma_4|+\frac{1}{2}\log\frac{1}{1-|\tau_3|}-1,$$

making use of lemma 7 with the strip $|\tau|<1$ for $\varDelta$, and σ_4 for σ_1, $\sigma_3+i\tau_3$, for $\sigma_2+i\tau_2$. Combining this with (15.4) we have

$$d[\zeta_{(3)},\zeta_{(4)};D_n]>\frac{1}{2}\log\frac{|\eta_{(3)}|}{\eta_n}-A. \tag{15.5}$$

We next note that in the mapping defined above by (15.1), (15.2) the curve (15.3) corresponds to $|\tau|=1$. Also this curve coincides with C_n for the range of values $1\leq Y\leq t_n$, where t_n is defined as in (13.5). Thus it follows that in the mapping of D_n into the strip $|\tau|<1$ defined by (15.1), (15.2) the endpoints of the arcs in which the circles

$$|\zeta-\xi_n|=r \tag{15.6}$$

intersect D_n, lie on the curve (15.3), and so correspond to points on $\tau=-1$, $\tau=+1$, respectively, provided that

$$3\eta_n<r<r_n=e^{-2\pi}(\xi_{n+1}-\xi_n). \tag{15.7}$$

We denote by D'_n the subdomain of D_n consisting of all points satisfying (15.6), (15.7). Then if we take D'_n for D_0, the mapping of D'_n into the strip $|\tau|<1$ satisfies the conditions of lemma 8, provided that (10.3) and (10.4) are satisfied.

We take $\zeta_1=\xi_{(4)}$ in lemma 8, where $\xi_{(4)}$ is defined in (14.10), and $\zeta_2=\zeta_{(2)}$, where $\zeta_{(2)}$ is defined in (14.2). The conditions (10.3), (10.4) become

$$e^{2\pi}(\xi_4-\xi_n)\leq(\xi_{(2)}-\xi_n)\leq e^{-2\pi}r_n=e^{-4\pi}(\xi_{n+1}-\xi_n). \tag{15.8}$$

If these are satisfied and σ_2, σ_4 correspond to $\zeta_{(2)}$, $\zeta_{(4)}$ in the mapping as defined by (15.1), (15.2), we obtain from (10.6)

$$\frac{\pi}{4}(\sigma_2-\sigma_4)+4>d[\zeta_{(2)},\zeta_{(4)};D'_n]>d[\zeta_{(2)},\zeta_{(4)};D_n], \tag{15.9}$$

since D'_n is contained in D_n. Also we have from (15.1), (15.2),

$$\xi_{(i)}-\xi_n=\frac{2\eta_n}{\log 2}\left[e^{\pi\sigma_i/2}\log\left(1+e^{\pi\sigma_i/2}\right)+\frac{\pi}{4}\right],\quad i=2,4. \tag{15.10}$$

We put

$$\frac{\xi_{(i)} - \xi_n}{2\,\eta_n} \log 2 - \frac{\pi}{4} = u_i, \tag{15.11}$$

$$e^{\pi \sigma_i/2} = v_i \tag{15.12}$$

so that (15.10) gives

$$u_i = v_i \log (1 + v_i).$$

Since (15.8), (14.10) hold, we have

$$u_i > 1, \quad v_i > 1, \quad i = 2, 4,$$

so that

$$v_i \log 2 < u_i < v_i^2$$

and hence

$$v_i \log (1 + \sqrt{u_i}) < u_i < v_i \log \left(1 + \frac{u_i}{\log 2}\right).$$

Thus

$$\log u_i - \log\log \left(1 + \frac{u_i}{\log 2}\right) < \log v_i < \log u_i - \log\log (1 + \sqrt{u_i}),$$

and hence

$$|\log^+ v_i - \log^+ u_i + \log^+ \log^+ u_i| < A, \quad i = 2, 4. \tag{15.13}$$

Also from (14.10), $u_4 = A$ so that $v_4 = A$ by (15.13). Hence using (15.11), (15.12) and (15.13) we have

$$\left|\frac{\pi}{2}(\sigma_2 - \sigma_4) - \log^+ \left(\frac{\xi_{(2)} - \xi_n}{\eta_n}\right) + \log^+ \log^+ \left(\frac{\xi_{(2)} - \xi_n}{\eta_n}\right)\right| < A. \tag{15.14}$$

Also since the mapping of (15.1), (15.2) maps D_n into the strip $|\tau| < 1$ so that $\zeta = \xi_{(4)}, \xi_{(2)}$ correspond to $s = \sigma_{(4)}, \sigma_{(2)}$ we have

$$d\,[\zeta_{(2)}, \zeta_{(4)}; D_n] > d\,[\sigma_2, \sigma_4; |\tau| < 1] = \frac{\pi}{4}(\sigma_2 - \sigma_4).$$

Combining this with (15.9) we have]

$$\left|\frac{\pi}{4}(\sigma_2 - \sigma_4) - d\,[\zeta_{(2)}, \zeta_{(4)}; D_n]\right| < A.$$

Combining this with (15.14) we have

$$\left|d\,[\zeta_{(2)}, \zeta_{(4)}; D_n] - \frac{1}{2} \log^+ \left(\frac{\xi_{(2)} - \xi_n}{\eta_n}\right) + \frac{1}{2} \log^+ \log^+ \left(\frac{\xi_{(2)} - \xi_n}{\eta_n}\right)\right| < A, \tag{15.15}$$

provided that (15.8) holds.

16) We can now complete the proof of lemma 10. Having obtained the inequalities (15.5) and (15.15), we go back to the original mapping of lemma 10.

We have from (14.12), (15.5)

$$(16.1) \qquad \frac{\pi}{4}(\sigma_{(3)} - \sigma_{(4)}) + \frac{1}{2}\log\frac{1}{1-|\tau_3|} > \frac{1}{2}\log\left|\frac{\eta_{(3)}}{\eta_n}\right| - A$$

provided that $\zeta_{(4)} = \xi_{(4)}$ is defined as in (14.10), $s = \sigma_{(4)}$ is the image of $\zeta = \xi_{(4)}$ in the s plane and (14.11) holds. Also $\sigma_{(4)} > \sigma_n$ since on the real axis σ is an increasing function of ξ. Thus (16.1) gives a fortiori

$$(16.2) \qquad \frac{\pi}{4}(\sigma_{(3)} - \sigma_n) + \frac{1}{2}\log\frac{1}{1-|\tau_{(3)}|} > \frac{1}{2}\log^+\left|\frac{\eta_{(3)}}{\eta_n}\right| - A,$$

provided that (14.11) holds. Also (16.2) is true if (14.11) is false since by lemma 4 we always have $\sigma_{(3)} > \sigma_n$. Thus (16.2) is always true, which proves (14.5).

Next suppose that $\xi_{(i)}$, $i = 5, 6, \ldots$ are real numbers to be defined and that $\sigma_{(i)}$ corresponds to $\xi_{(i)}$ in the mapping of (5.14). It follows from (14.1) and (5.6), (5.11), that R_{n-1} contains the circle C_1

$$|\zeta - \zeta_1| < \eta_n$$

which is mapped into the strip $|\tau| < 1$. Thus if

$$\xi_{(5)} = \xi_n - \tfrac{1}{2}\eta_n$$

we have

$$\frac{\pi}{4}(\sigma_{(5)} - \sigma_n) = d[\sigma_n, \sigma_{(5)}; |\tau| < 1] < d[\zeta_{(1)}, \xi_{(5)}; C_1] = \frac{1}{2}\log\frac{\eta_n + \frac{1}{2}\eta_n}{\eta_n - \frac{1}{2}\eta_n}$$

$$(16.3) \qquad \sigma_{(5)} - \sigma_n < A.$$

Similarly $\mathscr{R}$ contains the circle C_2, $|\zeta - \xi_n| < \eta_n$ as we see from (5.4), (5.5) so that if

$$\xi_{(6)} = \xi_n + \tfrac{1}{2}\eta_n,$$

we have

$$(16.4) \qquad \sigma_{(6)} - \sigma_{(5)} < A.$$

Next if $\xi_{(7)}$ satisfies

$$(16.5) \qquad \xi_n + \tfrac{1}{2}\eta_n \leq \xi_{(7)} \leq \xi_n + A\,\eta_n \leq \tfrac{1}{2}(\xi_n + \xi_{n+1})$$

we deduce similarly

$$(16.6) \qquad \sigma_{(7)} - \sigma_{(6)} < A$$

on considering the mapping of the circle C_7,

$$|\zeta - \xi_{(7)}| < \xi_{(7)} - \xi_n,$$

which is contained in R_n. We deduce from (16.6), that (14.4) holds if

$$(16.7) \qquad \xi_{n+1} - \xi_n < 3\,\eta_n e^{8\pi}.$$

In fact in this case we have from (14.2)

$$\xi_{(2)} < \xi_n + A\,\eta_n.$$

so that we may take $\xi_{(2)} = \xi_{(7)}$ in (16.5), and combining (16.3), (16.4), (16.6), we have

$$0 < \sigma_{(2)} - \sigma_n < A,$$

which yields (14.4).

Suppose next that (16.7) is false. Then if $\xi_{(4)}$ is defined as in (14.10) and $\sigma_{(4)}$ corresponds to $\xi_{(4)}$, we have from (16.5), (16.6), applied with $\xi_{(7)} = \xi_{(4)}$, and from (16.3), (16.4)

(16.8) $$0 < \sigma_{(4)} - \sigma_n < A.$$

Further if

$$\xi_{(4)} = \xi_n + 3\,\eta_n e^{2\pi} \le \xi_{(2)} \le \xi_n + 3\,\eta_n e^{4\pi},$$

(16.3) to (16.6) shows that

$$\sigma_{(2)} - \sigma_{(4)} < A,$$

so that again (14.4) follows. Suppose next that

(16.9) $$\xi_n + 3\,\eta_n e^{4\pi} \le \xi_{(2)} \le \xi_n + e^{-4\pi}(\xi_{n+1} - \xi_n).$$

This implies (14.13) and (15.8) and so we can apply (15.15) and (14.14). This yields

(16.10) $$\left|\frac{\pi}{4}(\sigma_{(2)} - \sigma_{(4)}) - \frac{1}{2}\log^+\left(\frac{\xi_{(2)} - \xi_n}{\eta_n}\right) + \frac{1}{2}\log^+\log^+\left(\frac{\xi_{(2)} - \xi_n}{\eta_n}\right)\right| < A,$$

which combined with (16.8) yields (14.4). Suppose lastly that

(16.11) $$\xi_n + e^{-4\pi}(\xi_{n+1} - \xi_n) \le \xi_{(2)} \le \tfrac{1}{2}(\xi_n + \xi_{n+1}).$$

Then we write

$$\xi_{(8)} = \xi_n + e^{-4\pi}(\xi_{n+1} - \xi_n),$$

and see that (16.10) holds with $\sigma_{(8)}$, $\xi_{(8)}$ instead $\sigma_{(2)}$, $\xi_{(2)}$. Also if (16.11) holds, $\xi_{(8)}$, $\xi_{(2)}$ are contained in the circle C_8,

$$|\zeta - \xi_{(2)}| < \xi_{(2)} - \xi_n$$

itself contained in R_n so that C_8 is mapped into $|\tau| < 1$ by the mapping of (5.14) and hence we deduce again from (16.11)

$$\frac{\pi}{4}[\sigma_{(2)} - \sigma_{(8)}] < d\,[\xi_{(8)}, \xi_{(2)};\ C_8] = \frac{1}{2}\log\left|\frac{\xi_{(2)} - \xi_n + \xi_{(2)} - \xi_{(8)}}{\xi_{(8)} - \xi_n}\right| < A.$$

Thus (16.10) holds also if (16.11) holds. Thus (16.10) holds whenever (16.7) is false and (16.9) or (16.11) holds, and so (14.4) follows using (16.8). We have already proved (14.4) in all other cases, when we have in fact

$$0 < \sigma_{(2)} - \sigma_n < A.$$

This completes the proof of (14.4) and of lemma 10.

Converse Theorems when E is Unbounded.

17) We are now ready to prove the results involving problems (ii) and (iii as stated in paragraph 1 of this chapter. We consider first problem (iii) which is a little simpler. Let

$$p(\varrho) = (1 - \varrho)^{-a}, \quad 0 \leq a < \infty. \tag{17.1}$$

What can we say about the rate of growth of $f(z)$, if $f(z)$ is meromorphic in $|z| < 1$ and takes some arbitrarily large values at most $p(\varrho)$ times in $|z| \leq \varrho$, $0 \leq \varrho < 1$. The positive results were obtained in Theorems V, VI and X of Chapter II. We showed in Theorems VI and X that if $a < 1$ in (17.1) we have

$$\overline{\lim_{\varrho \to 1}}\, (1 - \varrho) \log M[\varrho, f_*(z)] \leq 0 \tag{17.2}$$

$$\underline{\lim_{\varrho \to 1}}\, (1 - \varrho)^{\frac{1+a}{3-a}} \log M[\varrho, f_*(z)] \leq 0, \tag{17.3}$$

while if $a = 1$ it follows from Theorems V and X that

$$\overline{\lim_{\varrho \to 1}}\, \frac{(1 - \varrho) \log M[\varrho, f_*(z)]}{\log (1/(1 - \varrho))} < A. \tag{17.4}$$

$$\underline{\lim_{\varrho \to 1}}\, \frac{(1 - \varrho) \log M[\varrho, f_*(z)]}{\log \log (1/(1 - \varrho))} < A. \tag{17.5}$$

All the results (17.2) to (17.5) are best possible as we shall be able to show by examples, constructed by means of the mapping functions (5.14), which we have been studying. The inequality (17.4) represents no improvement on (1.5) the result when E contains only two finite values. This will be seen to be in accordance with facts.

We may remark here that once the functions $\zeta = \psi(s)$ of (5.14) have been introduced and studied by means of lemmas 1 to 8 it would be comparatively simple to prove more general converse theorems when $p(\varrho)$ is any sufficiently smooth function e.g.

$$p(\varrho) = (1 - \varrho)^{-a} [\log (1/(1 - \varrho))]^b \ldots,$$

by taking other forms for the curves C_n than those defined in (11.1) and (13.2) to (13.9). The simple form of (1.1) or (17.1) for $p(\varrho)$ seems, however, to cover all the essential points that arise. The whole work could have been simplified considerably if we had been prepared to exclude the case $a = 1$, which necessitates the full strength of the preceding theory. This case is, however, in many ways critical, so that its omission would be a serious gap.

The results which we shall prove are the following

Theorem II.[1] *Suppose that $p(\varrho)$ is given by (17.1) where $0 \leq a < 1$ and that $\mu(\varrho)$ is a decreasing function of ϱ for $0 \leq \varrho < 1$ such that*

$$\mu(\varrho) \to 0, \quad as \quad \varrho \to 1.$$

Then there exists $f(z)$, regular nonzero in $|z| < 1$ and taking some arbitrarily large values w at most $p(\varrho)$ times in $|z| < \varrho$, $0 < \varrho < 1$ and such that

$$\overline{\lim_{\varrho \to 1}} \frac{(1-\varrho) \log M[\varrho, f]}{\mu(\varrho)} > 0, \tag{17.6}$$

$$\underline{\lim_{\varrho \to 1}} \frac{(1-\varrho)^{\frac{1+a}{3-a}} \log M[\varrho, f]}{\mu(\varrho)} > 0. \tag{17.7}$$

Theorem III. *Suppose that $p(\varrho)$ is given by (17.1) with $a = 1$. Then there exists a function $f(z)$ regular nonzero in $|z| < 1$ and taking some arbitrarily large values at most $p(\varrho)$ times in $|z| < \varrho$, $0 < \varrho < 1$, such that*

$$\overline{\lim_{\varrho \to 1}} \frac{(1-\varrho) \log M[\varrho, f]}{\log(1/(1-\varrho))} > 0. \tag{17.8}$$

$$\underline{\lim_{\varrho \to 1}} \frac{(1-\varrho) \log M[\varrho, f[}{\log \log(1/(1-\varrho))} > 0. \tag{17.9}$$

Introduction of the Converse Functions.

18) Before proving Theorems II and III we introduce the general form of the functions which we shall investigate. This has already been done tentatively in section 5. We elaborate and recapitulate slightly. Let $\mathcal{R}$ be the Riemann surface constructed by means of the curves C_n and let

$$\zeta = \psi(s) = c_1 s + c_2 s^2 + \cdots, \quad c_1 > 0 \tag{18.1}$$

[1] This result was proved for $a = 0$ in Hayman (2) Theorem VI.

be the function of (5.14) which maps the strip $|\tau| < 1$ in the $s = \sigma + i\tau$ plane $1:1$ and conformally onto $\mathcal{R}$. The curves C_n will always be defined either in accordance with (11.1) or as in (13.2) to (13.9). If C_n is defined as in (11.1) for all n, $\mathcal{R}$ reduces to a domain D, and $\psi(s)$ maps the strip $|\tau| < 1$, $1:1$ and conformally onto D.

This case will suffice for all applications involving $p(\varrho)$ defined as in (17.1) with $a < 1$. In the case $a = 1$, $\psi(s)$ will no longer be schlicht. However, we can only have

$$\psi(s_1) = \psi(s_2)$$

for $s_1 \neq s_2$, if s_1, s_2 correspond to points in different sheets R_n of $\mathcal{R}$.

We shall put

$$s = \frac{2}{\pi} \log \frac{1 + z}{1 - z}. \tag{18.2}$$

This gives a $1:1$ mapping of the circle $|z| < 1$ onto the strip $|\tau| < 1$, in the $s = \sigma + i\tau$ plane. We also put

$$w = \exp(\zeta). \tag{18.3}$$

Then the functions we require will be

$$w = f(z) = \exp\left\{\psi\left[\frac{2}{\pi}\log\left(\frac{1+z}{1-z}\right)\right]\right\}. \tag{18.4}$$

Thus $\zeta = \log f(z)$ gives a $1:1$ mapping of the circle $|z| < 1$ onto the Riemann surface $\mathcal{R}$.

We shall also make use of the following notation. We write σ_n for the unique real and positive number such that

$$\psi(\sigma_n) = \xi_n - \eta_n, \quad n \geq 1. \tag{18.5}$$

We also write ϱ_n for the number corresponding to σ_n by (18.2), i.e.

$$\sigma_n = \frac{2}{\pi} \log \frac{1 + \varrho_n}{1 - \varrho_n}. \tag{18.6}$$

Proof of Theorem II.

19) We now commence the proof of Theorem II. We shall define all the curves C_n in accordance with (11.1). We shall define the ξ_n, η_n, $n \geq 1$ by induction.[1] We put

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[1] We shall have to define the ξ_n, η_n so as to satisfy conditions (5.1) to (5.5). Of these (5.2), (5.4), (5.5) are the only nontrivial ones. We shall see that they remain satisfied at each stage.

$$\xi_1 = 1, \tag{19.1}$$

$$\eta_1 = 1. \tag{19.2}$$

Thus (5.2) and (5.5) are satisfied for $n = 1$.

We shall also write k_n for a positive number depending only on the ξ_ν, η_ν for $\nu \leq n$ and on a, $0 \leq a < 1$, not necessarily the same, each time it occurs. Using the above notation we have first

Lemma 11. *Suppose that $n \geq 1$ and that ξ_ν, η_ν, $\nu = 1$ to n have already been defined. Then if $\xi_{n+1} > k_n$ we can define η_{n+1} such that*

$$\frac{\pi a}{2}(\sigma_{n+1} - 3) < \log \eta_{n+1} < \frac{\pi a}{2}(\sigma_{n+1} - 1), \tag{19.3}$$

however, the ξ_ν, η_ν for $\nu > n + 1$ are chosen. Moreover if (19.3) holds, $f(z)$ takes the value $w_{n+1} = \exp(-\xi_{n+1})$ at most $(1 - \varrho)^{-a}$ times in $|z| < \varrho$, $0 < \varrho < 1$.

We proceed to prove lemma 11. Since ξ_ν, η_ν for $\nu = 1$ to n have been defined, it follows from lemma 6 that, however the ξ_ν, η_ν for $\nu \geq n + 1$ are chosen, the variation of σ_n is at most 1. Thus we have

$$\sigma_n < k_n. \tag{19.4}$$

It now follows from lemma 9, (11.2) that if $s = \sigma$ corresponds to $\zeta = \xi$ for $\xi > \xi_n$, then we have

$$\frac{\pi}{2}(\sigma - \sigma_n) < \log^+ \left| \frac{\xi - \xi_n}{\eta_n} \right| + A, \ \xi_n \leq \xi \leq \frac{1}{2}(\xi_n + \xi_{n+1}),$$

i.e.

$$\frac{\pi}{2}\sigma < \log^+ (\xi - \xi_n) + k_n, \ \xi_n \leq \xi \leq \frac{1}{2}(\xi_n + \xi_{n+1}),$$

using (19.4). Since $a < 1$ it follows that we can choose k_n so large, that, however η_ν or $\xi_{\nu+1}$ for $\nu > n$ are chosen, we have

$$\frac{\pi a}{2}(\sigma - 2) < \log (\xi_n' - \xi_n) = \log (\xi_{n+1} - \xi_n') \tag{19.5}$$

when $s = \sigma$ corresponds to $\zeta = \xi_n' = \frac{1}{2}(\xi_n + \xi_{n+1})$ and

$$\xi_{n+1} > k_n. \tag{19.6}$$

We can also choose k_n in (19.6) so large that

$$2\,\eta_n < \xi_{n+1} - \xi_n,$$

which is the condition (5.4). Suppose now that ξ_ν, η_ν, $\nu > n + 1$ are chosen in some fixed way and let η_{n+1} be varied from 0 to $\frac{1}{2}(\xi_{n+1} - \xi_n)$. If η_{n+1} is very small $\zeta = \xi_{n+1}$ corresponds nearly to $\sigma = \infty$. Hence using (19.5) we see that if η_{n+1} has a fixed small value η'_{n+1}, we can find ξ' such that

$$\frac{1}{2}(\xi_n + \xi_{n+1}) < \xi' < \xi_{n+1} - \eta'_{n+1} \tag{19.7}$$

and σ' corresponds to ξ' where

$$\frac{\pi a}{2}(\sigma' - 2) = \log(\xi_{n+1} - \xi'). \tag{19.8}$$

For in this case σ is a continuous function of ξ and since (19.5) holds when $\xi = \frac{1}{2}(\xi_n + \xi_{n+1})$ and σ is large when $\xi = \xi_{n+1} - \eta'_{n+1}$, (19.8) must hold for some ξ' in the range (19.7). We now alter η_{n+1} to the value given by

$$\eta_{n+1} = \xi_{n+1} - \xi', \tag{19.9}$$

where ξ' is the value of (19.7), and at the same time leave ξ_ν, η_ν for $\nu > n + 1$ arbitrary. According to the definition (18.5), $s = \sigma_{n+1}$ now corresponds to $\zeta = \xi'$. Also it follows from lemma 6, that

$$|\sigma_{n+1} - \sigma'| < 1. \tag{19.10}$$

Making use of (19.8), (19.9), (19.10), we have

$$\frac{\pi a}{2}(\sigma_{n+1} - 3) < \log \eta_{n+1} < \frac{\pi a}{2}(\sigma_{n+1} - 1),$$

which proves (19.3). Also from (19.7) and (19.9) it follows that

$$2\eta_{n+1} < \xi_{n+1} - \xi_n$$

so that η_{n+1} satisfies the condition (5.5) and the choice of η_{n+1} is legitimate. Thus if (19.6) holds for each n the η_ν can for $\nu = 1, 2, \ldots$ be defined so that (5.4), (5,5) and (19.3) are satisfied.

It remains to prove the second statement of lemma 11. We note that if z is a point such that

$$f(z) = w_{n+1} = \exp(-\xi_{n+1}), \quad n > 0, \tag{19.11}$$

then if s is given by (18.2) and $\zeta = \psi(s)$ is the function of (18.1) we have

$$\psi(s) = \xi_{n+1} + m\pi i, \tag{19.12}$$

where m is a positive or negative odd integer. Since C_n is defined by (11.1) for all n the function $\psi(s)$ is schlicht in $|\tau| < 1$ and so (19.12) can only have one solution for each such odd integer. Also since $\mathcal{R}$ intersects the line $\Re\zeta = \xi_{n+1}$ only in the segment $|\eta| \leq \eta_{n+1}$ we must have

$$(19.13) \qquad |\pi m| \leq \eta_{n+1}$$

if (19.12) holds. If $\mp 1, \mp 3, \ldots \mp M$ are the odd integers satisfying (19.13) we have

$$2M < \pi M \leq \eta_{n+1},$$

so that the number of these integers and so the number of different solutions of (19.11) in $|z| < 1$ is at most η_{n+1}.

On the other hand, it follows from lemma 4, that if $s = \sigma + i\tau$ is any point such that (19.12) holds, then we have

$$(19.14) \qquad \sigma \geq \sigma_{n+1}$$

since $s = \sigma_{n+1}$ corresponds to $\zeta = \xi_{n+1} - \eta_{n+1}$ by (18.5). Also if $\sigma + i\tau$ corresponds to $z = \varrho e^{i\theta}$ by (18.2) we have

$$\sigma = \frac{2}{\pi} \log \left| \frac{1+z}{1-z} \right| \leq \frac{2}{\pi} \log \frac{1+\varrho}{1-\varrho}.$$

Thus if (19.11) holds for $z = \varrho e^{i\theta}$, then (19.14) yields

$$\log \frac{1+\varrho}{1-\varrho} \geq \frac{\pi}{2} \sigma_{n+1}$$

so that from (19.3)

$$a \left(\log \frac{1+\varrho}{1-\varrho} - \frac{\pi}{2} \right) > \log \eta_{n+1},$$

$$a \log \frac{1}{1-\varrho} > \log \eta_{n+1},$$

$$\eta_{n+1} < (1-\varrho)^{-a}.$$

Thus if (19.11) has roots in $|z| < \varrho$, the total number of such roots is at most η_{n+1} from (19.13) and so is less than $(1-\varrho)^{-a}$. This completes the proof of lemma 11.

20) We can now prove Theorem II. Suppose that the ξ_n, η_n are chosen in accordance with the conditions and conclusions of lemma 11, so that (19.3) is satisfied for all n. It remains to show that we can do this so that (17.7) and (17.6) are satisfied. Let $\zeta = \xi$ correspond to $s = \sigma$ by the mapping of (18.1) and take first $\xi = \xi_n' = \frac{1}{2}(\xi_n + \xi_{n+1})$. Then lemma 9 (11.2) shows that if $\sigma = \sigma_n'$ corresponds to $\xi = \xi_n'$ we have

$$\frac{\pi}{2}(\sigma_n' - \sigma_n) < \log^+\left(\frac{\xi_n' - \xi_n}{\eta_n}\right) + A,$$

so that

(20.1) $$\frac{\pi}{2}\sigma_n' < \log^+ \xi_n' + k_n$$

since

$$\sigma_n < k_n$$

by (19.4). We next take $\xi = \xi_{n+1} - \eta_{n+1}$ so that $\sigma = \sigma_{n+1}$. Then the domain D which is mapped by the inverse of (18.1) into the strip $|\tau| < 1$ contains the circle C,

$$|\zeta - \xi_n'| < \xi_{n+1} - \xi_n' = \tfrac{1}{2}(\xi_{n+1} - \xi_n).$$

Since this circle is mapped into $|\tau| < 1$ so that $s = \sigma_n', \sigma_{n+1}$ correspond to $\xi = \xi_n', \xi_{n+1} - \eta_{n+1}$, we have

$$d[\sigma_n', \sigma_{n+1}, |\tau| < 1] < d[\xi_{n+1} - \eta_{n+1}, \xi_n';\ C]$$

i.e.

$$\frac{\pi}{4}(\sigma_{n+1} - \sigma_n') < \frac{1}{2}\log\frac{\xi_{n+1} - \xi_n' + \xi_{n+1} - \eta_{n+1} - \xi_n'}{\xi_{n+1} - \xi_n' - (\xi_{n+1} - \eta_{n+1} - \xi_n')}$$

$$< \frac{1}{2}\log\frac{\xi_{n+1} - \xi_n}{\eta_{n+1}} < \frac{1}{2}\log\frac{\xi_{n+1}}{\eta_{n+1}}$$

i.e.

$$\frac{\pi}{2}(\sigma_{n+1} - \sigma_n') < \log\frac{\xi_{n+1}}{\eta_{n+1}}.$$

Combining this with (20.1) we have

$$\frac{\pi}{2}\sigma_{n+1} < 2\log\xi_{n+1} + \log\frac{1}{\eta_{n+1}} + k_n.$$

Using the first inequality of (19.3) this becomes

$$\frac{\pi}{2}\sigma_{n+1} < 2\log\xi_{n+1} - \frac{\pi a}{2}\sigma_{n+1} + k_n,$$

(20.2) $$\frac{\pi}{2}(1 + a)\sigma_{n+1} < 2\log\xi_{n+1} + k_n.$$

Suppose next that

(20.3) $$\xi_{n+1} \leq \xi \leq \tfrac{1}{2}(\xi_{n+1} + \xi_{n+2}).$$

Then (11.2) of lemma 9 gives

$$\frac{\pi}{2}(\sigma - \sigma_{n+1}) < \log^+\frac{\xi - \xi_{n+1}}{\eta_{n+1}} + A,$$

(20.4) $$\frac{\pi}{2}(\sigma - \sigma_{n+1}) < \log\xi - \log\eta_{n+1} + A,$$

since from (5.4) we have

$$\eta_{n+1} < \tfrac{1}{2}(\xi_{n+1} - \xi_n) < \tfrac{1}{2}\xi_{n+1} < \tfrac{1}{2}\xi.$$

Making use of the first inequality of (19.3), (20.4) gives

$$\frac{\pi}{2}\sigma < \log\xi + \frac{\pi}{2}(1-a)\sigma_{n+1} + A$$

$$< \log\xi + 2\frac{1-a}{1+a}\log\xi_{n+1} + k_n,$$

using (20.2), and hence we have, if (20.3) holds

$$\frac{\pi}{2}\sigma < \left[1 + \frac{2(1-a)}{1+a}\right]\log\xi + k_n$$

$$\frac{\pi}{2}\sigma < \frac{3-a}{1+a}\log\xi + k_n. \tag{20.5}$$

Clearly (20.5) also holds if $\frac{1}{2}(\xi_n + \xi_{n+1}) \leq \xi \leq \xi_{n+1}$, since σ is an increasing function of ξ and $\frac{1}{2}(\xi_n + \xi_{n+1}) > \frac{1}{2}\xi_{n+1}$. Thus (20.5) holds whenever

$$\tfrac{1}{2}(\xi_n + \xi_{n+1}) \leq \xi \leq \tfrac{1}{2}(\xi_{n+1} + \xi_{n+2}). \tag{20.6}$$

Combining the formulae (18.2) to (18.4) we see that if

$$\xi = \psi(\sigma)$$

we have

$$\log f(\varrho) = \xi$$

where

$$\frac{\pi}{2}\sigma = \log\frac{1+\varrho}{1-\varrho}.$$

Substituting these results in (20.5) we see that

$$\log f(\varrho) > k_n\left(\frac{1+\varrho}{1-\varrho}\right)^{\frac{1+a}{3-a}} \tag{20.7}$$

whenever

$$\tfrac{1}{2}(\xi_n + \xi_{n+1}) \leq \log f(\varrho) \leq \tfrac{1}{2}(\xi_{n+1} + \xi_{n+2}). \tag{20.8}$$

Similarly (20.1) yields

$$\log f(\varrho') > k_n\frac{1+\varrho'}{1-\varrho'} \tag{20.9}$$

when

$$\log f(\varrho') = \tfrac{1}{2}(\xi_n + \xi_{n+1}). \tag{20.10}$$

We may assume that the constants k_n in (20.7), (20.9) are the same. We can then choose ϱ_0 so near 1 that

$$\mu(\varrho) < k_n$$

whenever $\varrho_0 < \varrho$. We then choose ξ_{n+1} so large that $\varrho' > \varrho_0$ when (20.10) holds. It then follows that whenever (20.8) holds we have $\mu(\varrho) < k_n$, and hence

$$\log f(\varrho) > \mu(\varrho)\left(\frac{1+\varrho}{1-\varrho}\right)^{(1+\alpha)/(3-\alpha)}$$

Since this inequality holds whenever (20.8) holds for some $n > 1$ it holds whenever $\log f(\varrho) \geq \frac{1}{2}(\xi_2 + \xi_3)$. Then (17.7) follows.

Again we deduce from (20.9) that

$$\log f(\varrho') > \mu(\varrho')\frac{1+\varrho'}{1-\varrho'},$$

and since this holds for some values of ϱ' arbitrarily near 1, (17.6) follows. This completes the proof of Theorem II.

Proof of Theorem III.

21) We proceed to prove Theorem III. The proof is similar to that of Theorem II, but is complicated by the fact that we shall have to define the curves C_n as in (13.2) to (13.9). Thus our Riemann surface will no longer be a simple domain, and $\zeta = \psi(s)$ will not be schlicht. This will make it a little harder to obtain upper bounds for $p(\varrho)$. We continue to use the notation of (18.1) to (18.6).

Let $c > 1$, and let ξ_n, $n = 1, 2, \ldots$, be any sequence of positive numbers such that

(21.1) $$\xi_1 = 1$$

(21.2) $$\xi_{n+1} \geq 2(\xi_n)^c, \quad n = 1, 2, \ldots.$$

We shall show that the conclusions of (17.8), (17.9) hold for a function $f(z)$ taking none of the values

$$w_n = -\exp \xi_n, \quad n = 1, 2, \ldots$$

more than $1/(1-\varrho)$ times in $|z| \leq \varrho$ for $0 < \varrho < 1$. We need

Lemma 12. *Suppose that the ξ_n satisfy (21.1), (21.2). Then given any real constant k, we can choose numbers η_n satisfying (5.4), (5.5) and (13.1) and the curves C_n in accordance with (13.2) to (13.9) so that with the notation of (18.5) we have*

$$\log \eta_n < \frac{\pi}{2}\sigma_n - k + 2\pi, \quad n > 0 \tag{21.3}$$

$$\log \eta_n > \frac{\pi}{2}\sigma_n - k - 2\pi, \quad n > n_0 \tag{21.4}$$

where n_0 is a positive integer.

We choose η_1 so that

$$\eta_1 = \min\left[1, e^{-k}, \tfrac{1}{3}e^{-6\pi}(\xi_2 - \xi_1)\right]. \tag{21.5}$$

Hence (5.4), (5.5), (13.1) are satisfied. Also we have

$$\xi_1 - \eta_1 \geq \xi_1 - 1 = 0$$

using (21.1) so that since $\zeta = \xi_1 - \eta_1$ corresponds to $s = \sigma_1$, we have $\sigma_1 \geq 0$ from (18.1). Then (21.3) follows for $n = 1$ from (21.5).

Next if we choose η_2 very small $\zeta = \xi_2$ corresponds nearly to $\sigma = +\infty$ by (18.1) and so does $\zeta = \xi_2 - \eta_2$. Hence we can find η_2 such that

$$\eta_2 < \tfrac{1}{2}(\xi_2 - \xi_1)$$
$$\eta_2 < \tfrac{1}{3}e^{-6\pi}(\xi_3 - \xi_2),$$

so that (5.4), (5.5), (13.1) are satisfied, and in addition we have

$$\log \eta_2 < \frac{\pi}{2}\sigma_2 - k + 1.$$

Continuing in this way, we see that we can certainly choose the η_n so that the conditions (5.4), (5.5), (13.1) and (21.3) are satisfied for $n = 1, 2, \ldots, n_1$. Next it follows from lemma 10, (14.4), that if η_n has been so chosen and

$$\xi'_n = \tfrac{1}{2}(\xi_n + \xi_{n+1}) \tag{21.6}$$

and $s = \sigma'_n$ corresponds to $\zeta = \xi'_n$ by (18.1) then we have

$$\frac{\pi}{2}(\sigma'_n - \sigma_n) < \log \frac{\xi'_n - \xi_n}{\eta_n} - \log^+ \log^+ \frac{\xi'_n - \xi_n}{\eta_n} + A. \tag{21.7}$$

Also we have from (5.5), which we assume satisfied,

$$\eta_n \leq \tfrac{1}{2}(\xi_n - \xi_{n-1}) \leq \tfrac{1}{2}\xi_n, \tag{21.8}$$

and so (21.7), (21.6) give

$$\frac{\pi}{2}(\sigma'_n - \sigma_n) < \log \frac{\xi_{n+1} - \xi'_n}{\eta_n} - \log^+ \log^+ \frac{\xi_{n+1} - \xi_n}{\xi_n} + A. \tag{21.9}$$

Also it follows from (21.2) that

$$\frac{\xi_{n+1}-\xi_n}{\xi_n} \to \infty,$$

so that if n_1 is sufficiently large (21.9) gives

$$\frac{\pi}{2}(\sigma_n' - \sigma_n) < \log \frac{\xi_{n+1}-\xi_n'}{\eta_n} - (1+\pi), \quad n \geq n_1 \tag{21.10}$$

and also that (21.8) implies

$$\eta_n \leq \tfrac{1}{3} e^{-6\pi}(\xi_{n+1}-\xi_n), \quad n \geq n_1, \tag{21.11}$$

if n_1 is suffficiently large.

Suppose that $\eta_1, \ldots, \eta_{n_1}$ have been chosen to satisfy (21.3). We proceed to define η_n for $n \geq n_1$. We do so by induction. Suppose that η_n, $n \geq n_1$, has already been defined. Let

$$\xi_n' \leq \xi' < \xi_{n+1}, \tag{21.12}$$

where ξ_n' is defined as in (21.6). Then if σ' corresponds to ξ' we can make σ' arbitrarily large for some ξ' in this range and

$$\eta_{n+1} < \xi_{n+1} - \xi'.$$

Since also (21.10) holds it follows that if we give η_{n+1} a fixed small value and $\eta_{n+2}, \eta_{n+3}, \ldots$ any fixed values, we can find σ' corresponding to ξ' in (21.12) such that

$$\frac{\pi}{2}(\sigma' - \sigma_n) = \log \frac{\xi_{n+1}-\xi'}{\eta_n} + l, \tag{21.13}$$

where l is any number such that $l \geq -(1+\pi)$. We now alter η_{n+1} so that if ξ' is the number satisfying (21.13), we have

$$\xi_{n+1} - \xi' = \eta_{n+1} \tag{21.14}$$

and we leave the numbers η_{n+2}, η_{n+3} arbitrary. It follows from lemma 6 that σ', σ_n can be varied by at most 1 as a result of this. Also in the new notation σ' becomes σ_{n+1}. Hence (21.13), (21.14) yield

$$\left| \frac{\pi}{2}\sigma_{n+1} - \log \eta_{n+1} - \left(\frac{\pi}{2}\sigma_n - \log \eta_n\right) - l \right| < \pi. \tag{21.15}$$

Now the η_n have been chosen so that (21.3) is satisfied for $\nu = 1, 2, \ldots n$, $n \geq n_1$. If possible, i.e. if $\frac{\pi}{2}\sigma_n - \log \eta_n \leq k + 1 + \pi$, we choose $l \geq -(1+\pi)$ so that (for some fixed choice of the η_ν, $\nu > n$)

$$\frac{\pi}{2}\sigma_n - \log \eta_n + l = k. \tag{21.16}$$

Then for any other choice of the η_ν, $\nu > n$, we must have

$$\left|\frac{\pi}{2}\sigma_n - \log \eta_n + l - k\right| < \frac{\pi}{2}$$

by lemma 6.

Then (21.3), (21.4) for $n + 1$ follow from (21.15). If this is impossible we choose $l = -(1 + \pi)$ and see from (21.15) that we have

$$\frac{\pi}{2}\sigma_n - \log \eta_n - 1 - 2\pi < \frac{\pi}{2}\sigma_{n+1} - \log \eta_{n+1} < \frac{\pi}{2}\sigma_n - \log \eta_n - 1,$$

while (21.3) still holds for $n + 1$. Continuing in this way we shall have after a finite number of steps for some $m > n$

$$k - (1 + \pi) < \frac{\pi}{2}\sigma_m - \log \eta_m < k + \pi,$$

however the η_ν are chosen for $\nu > m$. [Thus (21.3), (21.4) are satisfied for $n = m$]. We can then define l in accordance with (21.16), and so (21.3), (21.4) will be satisfied with $n = m + 1$. Similarly we can obtain (21.16) and hence (21.3), (21.4) for $n > m + 1$, so that (21.3), (21.4) hold for $n \geq n_0 = m$. Also (21.3) holds for all positive integers n.

Lastly it follows from (21.12), (21.14) that η_n defined in this way for $n > n_1$, satisfies (21.8) and hence (21.11), i.e. (5.5) and (13.1), which implies (5.4). Thus the choice of η_n is legitimate and so the conditions of lemma 12 have all been satisfied. This completes the proof of lemma 12.

22) We show next that if the conditions of lemma 12 hold with any constant k and $f(z)$ is defined as in (18.4) then (17.8) and (17.9) hold. Suppose that $n \geq n_0$, that

$$\xi_n' = \tfrac{1}{2}(\xi_n + \xi_{n+1}) \tag{22.1}$$

and that $s = \sigma_n'$ corresponds to $\zeta = \xi_n'$ by (18.1). Then lemma 10. (14.4) gives

$$\frac{\pi}{2}(\sigma_n' - \sigma_n) < \log \frac{\xi_{n+1} - \xi_n'}{\eta_n} - \log^+ \log^+ \frac{\xi_n' - \xi_n}{\eta_n} + A. \tag{22.2}$$

Also by (5.5)

$$\eta_n < \tfrac{1}{2}(\xi_n - \xi_{n-1}) \leq \tfrac{1}{2}\xi_n \leq \tfrac{1}{2}(\xi_{n+1})^{1/c}$$

from (21.2), so that using this and (22.2) we have

$$\log^+ \log^+ \frac{\xi_n' - \xi_n}{\eta_n} = \log^+ \log^+ \frac{\xi_{n+1} - \xi_n}{2\,\eta_n}$$

$$> \log^+ \log^+ \frac{\xi_{n+1} - \xi_n}{(\xi_{n+1})^{1/c}} > \log^+ \log^+ [(\xi_{n+1})^{1-1/c} - 1]$$

$$> \log^+ \log^+ \xi_{n+1} - A(c).$$

Hence (22.2) gives

$$\frac{\pi}{2}\sigma_n' < \log(\xi_n' - \xi_n) - \log^+ \log^+ \xi_n' + \frac{\pi}{2}\sigma_n - \log \eta_n + A(c)$$

$$< \log \xi_n' - \log^+ \log^+ \xi_n' + k + A(c)$$

using (21.4). We shall denote by k' any positive constant depending only on k in (21.4) and on c. Thus we have

$$\frac{\pi}{2}\sigma_n' < \log \xi_n' - \log^+ \log^+ \xi_n' + k', \quad n > n_0. \tag{22.3}$$

Next if

$$\zeta_{(1)} = \xi_{n+1} - \eta_{n+1},$$

the domain R_n contains the circle C, $|\zeta - \xi_n'| < \xi_{n+1} - \xi_n' = \frac{1}{2}(\xi_{n+1} - \xi_n)$, so that if $s = \sigma_{n+1}$ corresponds to $\zeta = \zeta_{(1)}$ we have

$$d[\sigma_n', \sigma_{n+1}; |\tau| < 1] = \frac{\pi}{4}(\sigma_{n+1} - \sigma_n') < d[\xi_n', \xi_1; C]$$

$$= \frac{1}{2} \log \frac{\frac{1}{2}(\xi_{n+1} - \xi_n) + (\zeta_{(1)} - \xi_n')}{\frac{1}{2}(\xi_{n+1} - \xi_n) - (\zeta_{(1)} - \xi_n')} < \frac{1}{2} \log \frac{\xi_{n+1}}{\eta_{n+1}}$$

$$< \frac{1}{2} \log \frac{2\,\xi_n'}{\eta_{n+1}}.$$

Combined with (22.3) this gives

$$\frac{\pi}{2}\sigma_{n+1} < \log \xi_n' - \log^+ \log^+ \xi_n' + \log \xi_n' - \log \eta_{n+1} + k', \quad n > n_0.$$

Using (21.4) to eliminate $\log \eta_{n+1}$, this yields

$$\frac{\pi}{2}\sigma_{n+1} < \log \xi_n' - \tfrac{1}{2} \log \log \xi_n' + k', \quad n > n_0. \tag{22.4}$$

We now suppose

$$\xi_{n+1} \leq \xi \leq \tfrac{1}{2}(\xi_{n+1} + \xi_{n+2}) \tag{22.5}$$

and apply lemma 10, (14.5) with $\zeta_{(2)} = \xi$ and $n + 1$ for n. This yields

$$\frac{\pi}{2}(\sigma - \sigma_{n+1}) < \log^+ \frac{\xi - \xi_{n+1}}{\eta_{n+1}} - \log^+ \log^+ \frac{\xi - \xi_{n+1}}{\eta_{n+1}} + A.$$

Using (21.4) with $n+1$ for n we deduce

$$\frac{\pi}{2}\sigma < \log^+(\xi-\xi_{n+1}) - \log^+\log^+\frac{\xi-\xi_{n+1}}{\eta_{n+1}} + k', \quad n > n_0 \tag{22.6}$$

or

$$\sigma < \sigma_{n+1} + A, \quad n > n_0. \tag{22.7}$$

Now it follows from (21.3) and (22.4) that

$$\log \eta_{n+1} < \log \xi_{n+1} - \tfrac{1}{2}\log\log \xi_{n+1} + k',$$

i.e.

$$\eta_{n+1} < \frac{k'\,\xi_{n+1}}{(\log \xi_{n+1})^{1/2}}.$$

Hence if

$$\xi - \xi_{n+1} > \frac{\xi}{(\log \xi)^{1/4}} \tag{22.8}$$

we have

$$\frac{\xi-\xi_{n+1}}{\eta_{n+1}} > k'\frac{\xi}{\xi_{n+1}}\cdot\left(\frac{\log \xi_{n+1}}{\log \xi}\right)^{1/2}(\log \xi)^{1/4} > k'(\log \xi)^{1/4}$$

so that (22.6) yields

$$\frac{\pi}{2}\sigma < \log \xi - \log^+\log^+\log^+ \xi + k'. \tag{22.9}$$

Again if (22.8) is false we deduce from (22.6), neglecting the second term on the right hand side,

$$\frac{\pi}{2}\sigma < \log \xi - \tfrac{1}{4}\log^+\log^+ \xi + k',$$

which also implies (22.9). Again (22.7) yields (22.9) using (22.4). Thus (22.9) holds throughout the range (22.5), and since

$$\tfrac{1}{2}(\xi_n + \xi_{n+1}) > \tfrac{1}{2}\xi_{n+1},$$

it follows that (22.9) also holds for

$$\tfrac{1}{2}(\xi_n + \xi_{n+1}) \le \xi \le \tfrac{1}{2}(\xi_{n+1} + \xi_{n+2}), \quad n > n_0. \tag{22.10}$$

Thus (22.9) holds for

$$\xi \ge \tfrac{1}{2}(\xi_{n_0} + \xi_{n_0+1}).$$

We see from (18.1) to (18.4) that if σ is real and corresponds to ϱ and ξ then we have

$$\frac{\pi}{2}\sigma = \log\frac{1+\varrho}{1-\varrho},$$

$$\xi = \log f(\varrho).$$

Thus (22.9) gives

$$\frac{1+\varrho}{1-\varrho} < \frac{k'\xi}{1+\log^+\log^+\xi}, \quad \xi \geq \tfrac{1}{2}(\xi_{n_0}+\xi_{n_0+1}).$$

Hence if $\varrho = \varrho_0$ corresponds to $\xi = \frac{1}{2}(\xi_{n_0}+\xi_{n_0+1})$, we have

$$\xi > k'\frac{1+\varrho}{1-\varrho}\log^+\log^+\frac{1}{1-\varrho}, \quad \varrho > \varrho_0.$$

$$\log f(\varrho) > k'\frac{1}{1-\varrho}\log\log\frac{1}{1-\varrho}, \quad \varrho > \varrho_0.$$

This proves (17.9). Similarly if in (22.3) $\sigma = \sigma_n'$ corresponds to $\varrho = \varrho_n'$ we have

$$\log\frac{1+\varrho_n'}{1-\varrho_n'} < \log\log f(\varrho_n') - \log\log\log f(\varrho_n') + k',$$

so that

$$\log f(\varrho_n') > k'\frac{1}{1-\varrho_n'}\log\frac{1}{1-\varrho_n'},$$

for some values $\varrho = \varrho_n'$ arbitrarily near 1. This proves (17.8).

23) To complete the proof of Theorem III it remains to show that we can choose the constant k in lemma 12 so that the function $f(z)$ takes no value

(23.1) $$w_n = -\exp.(\xi_n)$$

more than $1/(1-\varrho)$ times in $|z| < \varrho$. Consider a fixed value w_n. The roots of $f(z) = w_n$ occur when

(23.2) $$\psi(s) = \log w_n = \log \xi_n + m\pi i,$$

where m is a positive or negative odd integer.

Let σ_n, ϱ_n be defined as in (18.5), (18.6). Then if $\varrho \leq \varrho_n$ and $z = \varrho e^{i\theta}$ corresponds to $s = \sigma + i\tau$ by (18.2), we have

$$\frac{\pi}{2}\sigma = \log\left|\frac{1+\varrho e^{i\theta}}{1-\varrho e^{i\theta}}\right| \leq \log\frac{1+\varrho_n}{1-\varrho_n} = \frac{\pi}{2}\sigma_n.$$

Also it follows from lemma 4 that if $\sigma \leq \sigma_n$, the point $\sigma + i\tau$ cannot correspond to an interior or boundary point of the sheet R_n by (18.1). It follows that the equation (23.2) has no roots for $s = \sigma + i\tau$ with $\sigma \leq \sigma_n$. Hence also the equation

(23.3) $$f(z) = w_n$$

has no roots in $|z| \leq \varrho_n$, and so none of the equations (23.3) for any n have roots in $|z| \leq \varrho_1$.

Suppose next that

$$\varrho_{M+1} \leq \varrho \leq \varrho_{M+2}, \quad M \geq 0, \tag{23.4}$$

and let $p_\mu(\varrho)$ denote the total number of roots of (23.3) for which

$$\zeta = \log w_n = \xi_n + m\pi i = \log f(z)$$

lies in the sheet R_μ [as an interior or frontier point]. We must have

$$|\pi m| < \xi_{\mu+1} - \xi_\mu$$

by (5.12). Since m can take only odd integral values, we deduce from this and the fact that $\zeta = \log f(z)$ gives a schlicht map onto R_μ that

$$p_\mu(\varrho) \leq p_\mu(1) \leq \frac{2}{\pi}(\xi_{\mu+1} - \xi_\mu) < \xi_{\mu+1} - \xi_\mu.$$

Hence we have

$$\sum_{\mu=1}^{M-1} p_\mu(\varrho) < \sum_{\mu=1}^{M-1} (\xi_{\mu+1} - \xi_\mu) < \xi_M < (\xi_{M+1})^{1/c}.$$

by (21.2). Thus

$$\log \sum_{\mu=1}^{M-1} p_\mu(\varrho) < \frac{1}{c} \log \xi_{M+1}. \tag{23.5}$$

Now it follows from lemma 10, (14.4), that if

$$\xi = \xi'_M = \tfrac{1}{2}(\xi_M + \xi_{M+1}) \tag{23.6}$$

corresponds to $s = \sigma'_M$ we have

$$\frac{\pi}{2}(\sigma'_M - \sigma_M) > \log \frac{\xi'_M - \xi_M}{\eta_M} - \log^+ \log^+ \frac{\xi'_M - \xi_M}{\eta_M} - A,$$

which gives

$$\log \frac{\xi'_M - \xi_M}{\eta_M} < \frac{\pi}{2}(\sigma'_M - \sigma_M) + \log^+ (\sigma'_M - \sigma_M) + A.$$

Using (23.6) and (21.3) this gives, since $\xi_{M+1} > 2\,\xi_M$ by (21.2),

$$\log \xi_{M+1} < \frac{\pi}{2}\sigma_M + \log^+ \sigma'_M + A - k. \tag{23.7}$$

Also since

$$\tfrac{1}{2}(\xi_M + \xi_{M+1}) \leq \xi_{M+1} - \eta_{M+1}$$

we have

$$\sigma'_M \leq \sigma_{M+1} \leq \frac{2}{\pi} \log \frac{1+\varrho}{1-\varrho}$$

by (23.4) and (18.6). Thus (23.5), (23.7) give

$$\log \sum_{\mu=1}^{M-1} p_\mu(\varrho) < \frac{1}{c}\left[\log \frac{1+\varrho}{1-\varrho} + \log\log \frac{1}{1-\varrho} + A - k\right]$$

$$< \frac{1}{c}\left[c \log \frac{1}{1-\varrho} + A(c) - k\right]$$

since $c > 1$. Hence if

$$k > A(c) \tag{23.8}$$

we shall have

$$\sum_{\mu=1}^{M-1} p_\mu(\varrho) < \frac{1}{3(1-\varrho)}, \quad 0 < \varrho < 1. \tag{23.9}$$

Next consider $p_\mu(\varrho)$ for $\mu = M$ or $M+1$, i.e., the roots of (23.3) in $|z| \leq \varrho$ for which

$$\zeta = \log w_n = \xi_n + m\pi i$$

lies in the sheet R_M, R_{M+1}, respectively. If ζ lies in R_μ and corresponds to $s = \sigma + i\tau$ in the strip $|\tau| < 1$ by (18.1) we see from lemma 10, (14.5) that

$$\frac{\pi}{2}(\sigma - \sigma_\mu) + \log \frac{1}{1-|\tau|} > \log \left|\frac{m\pi}{\eta_\mu}\right| - A.$$

This gives

$$\log |m| < \frac{\pi}{2}\sigma + \log \frac{1}{1-|\tau|} + A - k \tag{23.10}$$

making use of (21.3). Also if ζ gives rise to a root of (23.3) lying in $|z| \leq \varrho$ then $\sigma + i\tau$ must correspond to $z' = \varrho' e^{i\theta}$ with $\varrho' < \varrho$. Hence

$$d[0, \sigma + i\tau; |\tau| < 1] = d[0, \varrho' e^{i\theta}; |z| < 1] = \frac{1}{2}\log \frac{1+\varrho'}{1-\varrho'}.$$

Using lemma 7 we deduce

$$\frac{1}{2}\log \frac{1+\varrho}{1-\varrho} > \frac{1}{2}\log \frac{1+\varrho'}{1-\varrho'} > \frac{\pi}{4}\sigma + \frac{1}{2}\log \frac{1}{1-|\tau|} - A.$$

Thus (23.10) gives

$$\log |m| \leq \log \frac{1}{1-\varrho} + A - k. \tag{23.11}$$

If m is the largest odd integer satisfying (23.11) we have $p_\mu(\varrho) \leq 2m$. Thus we have

$$\log p_\mu(\varrho) \leq \log \frac{1}{1-\varrho} - \log 3,$$

(23.12)

$$p_\mu(\varrho) < \frac{1}{3(1-\varrho)}$$

if $k > A$. Taking $\mu = M, M+1$ in (23.12) we have from this and (23.9)

(23.13) $$\sum_{\mu=1}^{M+1} p_\mu(\varrho) < \frac{1}{1-\varrho}.$$

Also since (23.4) holds, it follows from lemma 4, as we remarked earlier, that the circle $|z| < \varrho$ can contain no points which correspond to points ζ in the sheets $R_{M+2}, R_{M+3}, \ldots$ etc. Thus if $p(\varrho)$ denotes the total number of roots of the equation (23.3) in $|z| < \varrho$ we have from (23.13)

$$p(\varrho) = \sum_{\mu=1}^{M+1} p_\mu(\varrho) < \frac{1}{1-\varrho},$$

provided that (23.8) holds with a sufficiently large constant $A(c)$. Since we have already shown that (17.9), (17.8) hold in this case, the proof of Theorem III is complete.

Sets of Values E Having the Same Effect as the Whole Plane.

24) We now turn our attention to the last problem of this chapter, problem (ii) of paragraph 1. It has been shown in Chapter II, Theorem VII, that if $f(z)$ takes none of a sequence of values w_n which satisfy

(24.1) $$w_0 = 0,$$

(24.2) $$|w_{n+1}| < k|w_n|, \quad n = 1, 2 \ldots,$$

(24.3) $$|w_n| \to \infty, \quad \text{as} \quad n \to \infty,$$

more than $p(\varrho)$ times in $|z| < \varrho$ for $0 < \varrho < 1$, then we have

(24.4) $$\log M[\varrho, f_*(z)] = O\left\{\int_0^{\varrho_*} \frac{1 + p(t)}{1-t}\, dt\right\}$$

where

$$\varrho_* = \frac{1 + 2\varrho}{2 + \varrho}.$$

No stronger result than this holds even if $f(z)$ takes *no* value more than $p(\varrho)$ times, at least when

$$p(\varrho) = (1 - \varrho)^{-a}, \quad 0 \leq a < \infty. \tag{24.5}$$

This was shown in Chapter II, paragraph 21. Our problem is to what extent the conditions (24.1) to (24.3) can be relaxed, without weakening (24.4). We show first that we cannot greatly weaken (24.2), (24.3) when $p(\varrho) \equiv 1$, i.e., $a = 0$ in (24.5). Clearly (24.1) represents a mere normalization. In this case (24.4) gives

$$\log M[\varrho, f_*(z)] = O\left(\log \frac{1}{1-\varrho}\right).$$

As a converse we have

Theorem IV. *Suppose that r_n is a sequence of real positive numbers such that*

$$\frac{r_{n+1}}{r_n} \to \infty.$$

Then there exists $f(z)$ regular nonzero in $|z| < 1$ and taking no value w_n such that $|w_n| = r_n$ more than once and such that

$$\frac{\log |f(\varrho)|}{\log (1/(1-\varrho))} \to +\infty, \quad as \quad \varrho \to 1.$$

Although we cannot weaken (24.2), (24.3) much for all functions $p(\varrho)$ we can do so if $p(\varrho)$ grows as rapidly as in (24.5) with $a > 0$. In fact we showed in Theorem IX of Chapter II that in this case we can replace (24.2) by the weaker condition

$$|w_{n+1}| < |w_n|^k, \quad n = 1, 2, \quad k = \text{cons.} > 1. \tag{24.6}$$

This condition is best possible in an even sharper sense than that of Theorem IV. We have in fact

Theorem V. *Let E be any set of complex values which does not contain a sequence of values w_n satisfying (24.3), (24.6). Then given a, $0 < a \leq 1$ there exists $f(z)$ regular nonzero in $|z| < 1$ taking no value in E more than $(1-\varrho)^{-a}$ times in $|z| \leq \varrho$ for $0 < \varrho < 1$ and such that*

$$\overline{\lim_{\varrho \to 1}}\, (1-\varrho)^a \log M[\varrho, f] = \infty.$$

Thus the condition that E shall contain a sequence satisfying (24.3), (24.6) is necessary and sufficient in order that E shall have the effect of the whole plane when $p(\varrho)$ is given by (24.5) with $0 < a \leq 1$.

Lastly when $a > 1$ in (24.5) it follows from Theorem V of Chapter II that even the set E given by $w = 0, 1, \infty$ is sufficient to result in (24.4). Thus the proof of Theorems IV and V will dispose of problem (ii) of paragraph 1, completely when $a > 0$, and to a large extent when $a = 0$.

Proof of Theorem IV.

25) We prove first Theorem IV which is much simpler than Theorem V. Let r_n be the numbers of Theorem IV supposedly arranged in order of increasing magnitude. We may suppose without loss in generality that

$$r_{n_0} = e \tag{25.1}$$

for some integer n_0. Then we choose

$$\xi_n = \log r_{n+n_0-1}, \quad n = 1, 2, \ldots \tag{25.2}$$

Since the r_n satisfy

$$\frac{r_{n+1}}{r_n} \to \infty$$

we shall have

$$\xi_{n+1} - \xi_n \to \infty. \tag{25.3}$$

Thus the numbers $\xi_{n+1} - \xi_n$ have a positive minimum and so we can find $\eta_0 > 0$ such that

$$\eta_0 < 1$$

$$2\eta_0 < \xi_{n+1} - \xi_n, \quad n = 1, 2, \ldots$$

We then choose

$$\eta_n = \eta_0, \quad n = 1, 2, \ldots \tag{25.4}$$

and it follows that the numbers η_n, ξ_n satisfy the conditions (5.4), (5.5),

We shall define the curves C_n in accordance with (11.1), so that $\xi = \psi(s)$, defined as in (18.1) maps the strip 1 : 1 conformally onto a domain D, since the sheets R_n are non-overlapping. We define $f(z)$ by (18.1) to (18.4).

Suppose that

$$f(z) = w_n \tag{25.5}$$

where $|w_n| = r_n$. It follows that either $n < n_0$, so that

$$\xi = \log |w_n| < \xi_1 \tag{25.6}$$

by (25.1) or $n \geq n_0$ so that

$$\xi = \log |w_n| = \xi_{n-n_0+1}. \tag{25.7}$$

Also if (25.5) holds,

$$\log w_n = \xi + i \arg w_n + 2 m \pi i$$

must lie in D and in case (25.6) holds we deduce from (5.12)

$$|\arg w_n + 2 m\pi| < \xi_1 - \xi_0 = 2$$

which can hold only for at most one value of m. Also if (25.7) holds we deduce from (25.4) and (5.13) that

$$|\arg w_n + 2 m\pi| < \eta_0 < 1$$

which can again hold for at most one value of m. Thus if (25.5) holds we must have

$$\log f(z) = \log |w_n| + i (\arg w_n + 2 \pi m)$$

which can hold for at most one value of m in all cases. Since $\log f(z)$ gives a schlicht mapping of $|z| < 1$ onto the domain D, we deduce that (25.5) has at most one solution in $|z| < 1$ for each w_n.

26) It remains to show that

$$\frac{\log |f(\varrho)|}{\log \frac{1}{1-\varrho}} \to +\infty, \quad \text{as} \quad \varrho \to 1.$$

Since ξ, σ, ϱ are related as in (18.1) to (18.4) this is equivalent to proving that

$$\frac{\xi}{\sigma} \to +\infty, \quad \text{as} \quad \xi \to +\infty, \tag{26.1}$$

where $\zeta = \xi$ correspond to $s = \sigma$ in the mapping of (18.1). Suppose

$$\xi_n \leq \xi \leq \tfrac{1}{2}(\xi_n + \xi_{n+1}) \tag{26.2}$$

and that $s = \sigma$ corresponds to $\zeta = \xi$. Then it follows from (11.2), that if σ_n is defined as in (18.5) we have

$$\frac{\pi}{2}(\sigma - \sigma_n) < \log^+ \frac{\xi - \xi_n}{\eta_n} + A$$

$$\frac{\pi}{2}(\sigma - \sigma_n) < \log^+ (\xi - \xi_n) + A + \log^+ \frac{1}{\eta_0} \tag{26.3}$$

by (25.4). In particular if $s = \sigma_n'$ corresponds to

$$\xi = \xi_n' = \tfrac{1}{2}(\xi_n + \xi_{n+1})$$

we deduce

$$\frac{\pi}{2}(\sigma_n' - \sigma_n) < \log(\xi_n' - \xi_n) + A + \log^+ \frac{1}{\eta_0}. \tag{26.4}$$

Also the domain D contains the circle C,

$$|\zeta - \xi_n'| < \xi_{n+1} - \xi_n' = \tfrac{1}{2}(\xi_{n+1} - \xi_n),$$

which contains the point $\zeta = \xi_{n+1} - \eta_{n+1} = \xi_{n+1} - \eta_0$ by (25.4) and

$$d\,[\xi_{n+1} - \eta_0, \xi_n'; C] = \frac{1}{2}\log\frac{\xi_{n+1} - \xi_n - \eta_0}{\eta_0} < \frac{1}{2}\log\frac{\xi_{n+1} - \xi_n}{\eta_0}.$$

Hence, since $\sigma = \sigma_n', \sigma_{n+1}$ correspond to $\zeta = \xi_n', \xi_{n+1} - \eta_{n+1}$, we have

$$\frac{\pi}{4}(\sigma_{n+1} - \sigma_n') < \frac{1}{2}\log\frac{\xi_{n+1} - \xi_n}{\eta_0} < \frac{1}{2}\log(\xi_{n+1} - \xi_n) + \log^+\frac{1}{\eta_0}.$$

Combining this with (26.4) we have

$$\frac{\pi}{2}(\sigma_{n+1} - \sigma_n) < 2\log(\xi_{n+1} - \xi_n) + 2\log^+\frac{1}{\eta_0} + A.$$

Since (25.3) holds we deduce from this that

$$\frac{\sigma_{n+1} - \sigma_n}{\xi_{n+1} - \xi_n} \to 0, \quad \text{as} \quad n \to \infty,$$

and so that

$$\frac{\sigma_n}{\xi_n} \to 0, \quad \text{as} \quad n \to \infty.$$

Thus given $\varepsilon > 0$ we can find n_0 such that

$$\frac{\pi}{2}\sigma_n < \varepsilon\,\xi_n, \quad n > n_0.$$

Then (26.3) shows that if $n > n_0$ and (26.2) holds, then

$$\frac{\pi}{2}\sigma < \varepsilon\,\xi_n + \log^+(\xi - \xi_n) + A + \log^+\frac{1}{\eta_0}$$

$$\frac{\pi}{2}\sigma < \varepsilon\,\xi + \log^+\frac{1}{\varepsilon} + A + \log^+\frac{1}{\eta_0}. \tag{26.5}$$

Again if

(26.6) $$\tfrac{1}{2}(\xi_{n-1} + \xi_n) \leq \xi \leq \xi_n$$

we have $\xi > \frac{1}{2}\xi_n$. Since also σ is an increasing function of ξ we deduce in this case from (26.5) that

$$\frac{\pi}{2}\sigma < 2\,\varepsilon\,\xi + \log^+\frac{1}{\varepsilon} + A + \log^+\frac{1}{\eta_0}.$$

Thus in any case we see that if either (26.2) or (26.6) holds for some $n \geq n_0$ and ξ is sufficiently large we have

$$\frac{\pi}{2}\sigma < 3\,\varepsilon\,\xi$$

so that this inequality holds for all sufficiently large ξ. This proves (26.1) and completes the proof of Theorem IV.

Proof of Theorem V.

27) We now commence the proof of Theorem V. The method is similar to that already employed in the proofs of Theorems II and III. We use the definitions and notation of (18.1) to (18.6). The preliminary result analogous to lemmas 11 and 12 is as follows.

Lemma 13. *Suppose that the conditions of Theorem V are satisfied. Then given any positive constant K we can define the ξ_n, η_n and the curves C_n to satisfy the conditions of paragraph 5 and in addition the following.*

(i) (27.1) $$\xi_1 = 1,$$

(27.2) $$\xi_{2m} > 3(\xi_{2m-1})^2, \quad m = 1, 2, \ldots$$

(27.3) $$\xi_{2m+1} = A_1\, m\, \xi_{2m}, \quad m = 1, 2, \ldots$$

where A_1 is an absolute constant greater than 2. Further the set E of Theorem V contains no value w such that

(27.4) $$\xi_{2m} < \log|w| < \xi_{2m+1}, \quad m \geq 1.$$

(ii) *The quantities η_n and σ_n satisfy*

(27.5) $$\left|\left(\frac{1}{a} - 1\right)\log \xi_{n+1} - \left(\frac{\pi}{2}\sigma_n - \log \eta_n\right) + K\right| < A_2, \quad n \geq 1,$$

where A_2 is another absolute constant.

(iii) *The curve* C_n *is defined as in* (11.1) *when* n *is odd, and when* n *is even,* (13.1) *holds, and* C_n *is defined in accordance with* (13.2) *to* (13.9).

(iv) *If the conditions* (i) *to* (iii) *are satisfied and* $f(z)$ *is defined as in* (18.4) *we have*

$$\overline{\lim_{\varrho \to 1}} (1-\varrho)^a \log |f(\varrho)| = +\infty.$$

28) Suppose that ξ_{2m-1} has already been defined to satisfy the conditions of lemma 13. We leave A_1 undefined for the time being and define ξ_{2m}, ξ_{2m+1} as follows. Suppose that E contains some value w such that

$$\xi < \log |w| < A_1 m \xi \tag{28.1}$$

for any real ξ such that

$$\xi > 3 (\xi_{2m-1})^2. \tag{28.2}$$

Then we can find w_ν in E such that

$$3 (A_1 m)^\nu (\xi_{2m-1})^2 < \log |w_\nu| < 3 (A_1 m)^{\nu+1} (\xi_{2m-1})^2$$

for every $\nu \geq 1$. The sequence w_ν clearly satisfies

$$\log |w_{\nu+1}| < (A_1 m)^2 \log |w_\nu|$$

and

$$w_\nu \to \infty$$

contrary to the hypotheses of Theorem V. Thus we can find ξ satisfying (28.2), such that E contains no value w for which (28.1) holds and we then define $\xi_{2m} = \xi$, $\xi_{2m+1} = A_1 m \xi$ and we see that this inductive definition satisfies (27.1) to (27.4).

Suppose next that η_ν, $\nu = 1$ to n have been defined so that whatever the values of η_ν are for $\nu > n$, (27.5) is satisfied for $1, 2, \ldots, n$. Suppose further, for the present, that if

$$\xi_n' = \tfrac{1}{2} (\xi_n + \xi_{n+1}) \tag{28.3}$$

and $s = \sigma_n'$ corresponds to $\xi = \xi_n'$ then

$$\left(\frac{1}{a} - 1\right) \log \xi_{n+2} - \left[\frac{\pi}{2} \sigma_n' - \log (\xi_{n+1} - \xi_n')\right] + K > 2 - A_2, \tag{28.4}$$

where A_2 is the constant of (27.5). Then it follows, by a now familiar method that we can satisfy (27.5) with $n+1$ also. In fact we choose first all the η_ν

for $\nu > n$ fixed and η_{n+1} fixed and small, and since then σ increases with ξ and becomes very large if $\xi = \xi_{n+1} - \eta_{n+1}$, we can find ξ' so that

$$\xi_n' \leq \xi' \leq \xi_{n+1}$$

and σ' corresponds to ξ' where

$$\left(\frac{1}{a} - 1\right) \log \xi_{n+2} - \left[\frac{\pi}{2}\sigma' - \log (\xi_{n+1} - \xi')\right] + K = 2 - A_2. \tag{28.5}$$

We then change η_{n+1} to the value given by

$$\eta_{n+1} = \xi_{n+1} - \xi' \tag{28.6}$$

so that σ' becomes σ_{n+1}. It follows from lemma 6 that, however the η_ν, $\nu > n + 1$ are chosen this cannnot alter σ' by more than 1 so that

$$|\sigma_{n+1} - \sigma'| < 1. \tag{28.7}$$

Using (28.5), (28.6) (28.7) we shall have

$$\left|\left(\frac{1}{a} - 1\right) \log \xi_{n+2} - \left[\frac{\pi}{2}\sigma_{n+1} - \log \eta_{n+1}\right] + K + A_2 - 2\right| < \frac{\pi}{2} < 2,$$

from which (27.5) follows for $n + 1$ provided that $A_2 > 2$ which we may assume.

Also we have clearly

$$\eta_{n+1} < \xi_{n+1} - \xi_n' = \tfrac{1}{2}(\xi_{n+1} - \xi_n) \tag{28.8}$$

so that (5.5) is satisfied. Further (27.1), (27.2) imply

$$\xi_{n+1} - \xi_n > \xi_n - \xi_{n-1}, \quad n \geq 1,$$

since $\xi_0 = -1$ by (5.1), so that it follows from (28.8) that η_{n+1} also satisfies (5.4). Further if

$$A_1 > 3\, e^{6\pi} + 1 \tag{28.9}$$

we shall have from (27.3)

$$\xi_{2m+1} - \xi_{2m} > 3\, e^{6\pi}\, \xi_{2m} > 3\, e^{6\pi} (\xi_{2m} - \xi_{2m-1}), \quad m > 0$$

so that (28.8) yields

$$\xi_{2m+1} - \xi_{2m} > 3\, e^{6\pi}\, \eta_{2m}.$$

Thus when n is even (13.1) is also satisfied provided that (28.9) holds so that in this case we can define the curves C_n in accordance with the conditions (iii). Thus the conditions (i) to (iii) of lemma 13 can all be satisfied provided that we can satisfy (28.4) for all $n > 0$.

29) Now when $n = 0$, we have from (28.3), (5.1) and (27.1)

$$\xi_0' = \tfrac{1}{2}(-1 + 1) = 0$$

so that always $\sigma_0' = 0$ and (28.4) becomes

$$\left(\frac{1}{a} - 1\right) \log \xi_2 + K > 2 - A_2,$$

which is always true, since ξ_2, K are positive and we have assumed $A_2 > 2$. Thus we can define η_1 to satisfy the conditions of lemma 13 with $A_2 = 3$.

Suppose now that η_{2m-1} has already been defined to satisfy (27.5) with a constant $A_2 = 3$. Then it follows from lemma 9, (11.2) that we have with the notation of (28.3)

$$\frac{\pi}{2}(\sigma_{2m-1}' - \sigma_{2m-1}) < \log \frac{\xi_{2m-1}' - \xi_{2m-1}}{\eta_{2m-1}} + A$$

$$= \log \frac{\xi_{2m} - \xi_{2m-1}'}{\eta_{2m-1}} + A,$$

i.e.

$$\log(\xi_{2m} - \xi_{2m-1}') - \frac{\pi}{2}\sigma_{2m-1}' + \frac{\pi}{2}\sigma_{2m-1} - \log \eta_{2m-1} + A > 0.$$

Using (27.5), which holds by hypothesis for $n = 2m - 1$, $A_2 = 3$ and (28.3) this becomes

$$\log(\xi_{2m} - \xi_{2m-1}') - \frac{\pi}{2}\sigma_{2m-1}' + \left(\frac{1}{a} - 1\right) \log \xi_{2m} + K > -A,$$

which yields (28.4) for $n = 2m - 1$, with $A_2 = A$ on noting that $\xi_{2m} < \xi_{2m+1}$.

Thus if (27.5) can be satisfied for $n = 2m - 1$ with a constant $A_2 = 3$, then (28.4) holds for $n = 2m - 1$, with $A_2 = A$ and hence (27.5) can be satisfied for $n = 2m$, with $A_2 = A$, where A is an absolute constant.

Suppose now that (27.5) holds for $n = 2m$ with $A_2 = A$. Then (14.4) of lemma 10 gives with the notation of (28.3)

$$\frac{\pi}{2}(\sigma_{2m}' - \sigma_{2m}) < \log \frac{\xi_{2m}' - \xi_{2m}}{\eta_{2m}} - \log^+ \log^+ \left(\frac{\xi_{2m}' - \xi_{2m}}{\eta_{2m}}\right) + A. \quad (29.1)$$

Now we have

$$\eta_{2m} \leq \tfrac{1}{2}(\xi_{2m} - \xi_{2m-1}) < \tfrac{1}{2}\xi_{2m}$$

so that from (28.3), (27.3) we deduce

$$\frac{\xi'_{2m}-\xi_{2m}}{\eta_{2m}} > \frac{\xi_{2m+1}-\xi_{2m}}{\xi_{2m}} > A_1 m - 1.$$

Thus (29.1) yields

$$\frac{\pi}{2}(\sigma'_{2m}-\sigma_{2m}) < \log\frac{\xi'_{2m}-\xi_{2m}}{\eta_{2m}} - \log^+\log^+ A_1 m + A \tag{29.2}$$

Assuming that (27.5) holds with a certain constant $A_2 = A$, when $n = 2m$ we deduce

$$\left(\frac{1}{a}-1\right)\log\xi_{2m+1} - \frac{\pi}{2}\left[\sigma'_{2m} - \log(\xi'_{2m}-\xi_{2m})\right] + K + A - \log^+\log^+ A_1 m > 2 - 3.$$

Since $\xi_{2m+1} < \xi_{2m+2}$ this yields (28.4) with $n = 2m$ and $A_2 = 3$, provided that the constant A_1 which has hitherto been left undetermined is so chosen that

$$\log^+\log^+ A_1 > A.$$

Thus if A_1 is a suitably large absolute constant and (27.5) holds for $n = 2m - 1$ with $A_2 = 3$, we can ensure that (27.5) holds for $n = 2m$, with $A_2 = A$ and for $n = 2m + 1$ with $A_2 = 3$. We have already shown that (27.5) can be made to hold when $n = 1$, with $A_2 = 3$. Thus we can ensure that (27.5) holds for all values of n with a suitable constant A_2, if A_1 is a large absolute constant. Hence we can satisfy conditions (i) to (iii) of lemma 13.

It remains to prove (iv). We use (29.2). This yields

$$\frac{\pi}{2}\sigma'_{2m} < \log(\xi'_{2m}-\xi_{2m}) + \frac{\pi}{2}\sigma_{2m} - \log\eta_{2m} - \log^+\log^+ m + A.$$

Using (27.5) this gives

$$\frac{\pi}{2}\sigma'_{2m} < \log(\xi'_{2m}-\xi_{2m}) + \left(\frac{1}{a}-1\right)\log\xi_{2m+1} - \log^+\log^+ m + A + K$$

$$< \frac{1}{a}\log\xi_{2m+1} - \log^+\log^+ m + A + K,$$

using (28.3); and a further use of (28.3) gives

$$\frac{\pi}{2}\sigma'_{2m} < \frac{1}{a}\log\xi'_{2m} - \log^+\log^+ m + C,$$

where C is a constant independent of m. Hence

$$\lim_{m\to\infty}\frac{\pi}{2}\sigma'_{2m} - \frac{1}{a}\log\xi'_{2m} = -\infty. \tag{29.3}$$

Also if $f(z)$ is defined by (18.1) to (18.4) and

$$\frac{\pi}{2}\sigma'_{2m} = \log\frac{1+\varrho'_{2m}}{1-\varrho'_{2m}}$$

we have

$$\log f(\varrho'_{2m}) = \xi'_{2m}.$$

Thus (29.3) gives

$$\log\frac{1+\varrho'_{2m}}{1-\varrho'_{2m}} - \frac{1}{a}\log\log f(\varrho'_{2m}) \to -\infty,$$

i.e.

$$\left(\frac{1-\varrho'_{2m}}{1+\varrho'_{2m}}\right)^a \log f(\varrho'_{2m}) \to +\infty, \quad \text{as} \quad \varrho'_{2m} \to 1.$$

This proves lemma 13 (iv) and completes the proof of lemma 13.

30) To complete the proof of Theorem V it remains to show that if $f(z)$ is defined as in lemma 13 then we can choose the constant K of that lemma so that for $0 < \varrho < 1$ $f(z)$ takes no value of E more than $(1-\varrho)^{-a}$ times in $|z| < \varrho$. Let

$$(30.1) \qquad w = \exp(\xi + i\eta)$$

be a value of E. It follows from (27.4) that we must have either

$$\xi \leq \xi_2$$

or alternatively

$$(30.2) \qquad \xi_{2m-1} \leq \xi \leq \xi_{2m}, \quad m = 2, 3, \ldots$$

Consider the equation

$$(30.3) \qquad f(z) = w.$$

It has solutions only where

$$\log f(z) = \xi + i\eta + 2\nu\pi i$$

and exactly one solution corresponding to each point

$$\xi + i\eta + 2\nu\pi i$$

lying in some sheet R_μ.

Suppose first that $|z| \leq \varrho_{n+1}$ where the ϱ_ν are defined as in (18.6). Then it follows that z corresponds to a point $\sigma + i\tau$ in the s plane by (18.2) where

$$\sigma = \frac{2}{\pi}\log\left|\frac{1+z}{1-z}\right| \leq \frac{2}{\pi}\log\frac{1+\varrho_{n+1}}{1-\varrho_{n+1}} = \sigma_{n+1}.$$

Thus it follows from lemma 4, that in the mapping of (18.1) the point $\sigma + i\tau$ cannot correspond to a point ζ lying in the sheet R_{n+1}. Thus the value of $\log f(z)$ lies inside or on the frontier of one of the sheets $R_0, R_1, R_2, \ldots, R_n$.

In particular if $|z| \leq \varrho_1$, $\log f(z)$ lies in R_0 so that we have from (5.12), (27.1)

$$|\arg f(z)| < \xi_1 - \xi_0 = 2.$$

Thus $f(z)$ is schlicht in $|z| \leq \varrho_1$ and so takes no value more than once and a fortiori not more than $(1-\varrho)^{-a}$ times in $|z| < \varrho$ if $\varrho \leq \varrho_1$.

Suppose next that

$$\varrho_n \leq \varrho \leq \varrho_{n+1}, \quad n \geq 1. \tag{30.4}$$

Let w, given by (30.1), be a value of E, consider the roots of (30.3) which lie in $|z| < \varrho$ and let $p(\varrho)$ be their total number. We divide these roots into $n+1$ groups according as the corresponding value of

$$\zeta = \log w = \log f(z)$$

lies in the sheet R_μ, $\mu = 0$ to n.[1] As we remarked above, ζ cannot lie in R_μ with $\mu > n$, if (30.4) holds. We denote the corresponding total number of roots of (30.3) in $|z| < \varrho$ by $p_\mu(\varrho)$.

If ζ lies in R_μ we have

$$\zeta = \log w + 2m\pi i$$

for some integer m. It follows from (5.12) that we have in R_μ

$$|\Im \zeta| < \xi_{\mu+1} - \xi_\mu.$$

Hence there can be at most

$$\frac{1}{\pi}(\xi_{\mu+1} - \xi_\mu) + 1$$

different values of m. Each of these gives rise to exactly one root of the equation (30.3), so that we have

$$p_\mu(\varrho) \leq \frac{1}{\pi}(\xi_{\mu+1} - \xi_\mu) + 1 < \xi_{\mu+1} - \xi_\mu,$$

making use of (27.1) to (27.4). Thus we have

$$\sum_{\mu=0}^{n-2} p_\mu(\varrho) < \sum_{\mu=0}^{n-2} (\xi_{\mu+1} - \xi_\mu) = \xi_{n-1} + 1, \quad n \geq 2. \tag{30.5}$$

---

[1] A point on the frontier segment of R_{n-1}, R_n we consider as lying in R_n.

Now if n is odd so that C_n is defined in accordance with (11.1), we have from lemma 9, (11.3)

$$\frac{\pi}{2}(\sigma - \sigma_n) > \log^+ \frac{\xi - \xi_n}{\eta_n} - A,$$

if $\xi_n \leq \xi \leq \frac{1}{2}(\xi_{n+1} + \xi_n)$ and $s = \sigma$ corresponds to $\zeta = \xi$. Choosing $\zeta = \xi_{n+1} - \eta_{n+1} \geq \geq \frac{1}{2}(\xi_n + \xi_{n+1})$, so that $\sigma = \sigma_{n+1}$, we deduce a fortiori

$$\frac{\pi}{2}(\sigma_{n+1} - \sigma_n) > \log \frac{\xi_{n+1} - \xi_n}{\eta_n} - A.$$

Also since n is odd and so

$$\xi_{n+1} > 3\,\xi_n$$

from (27.3), we deduce

$$\log \xi_{n+1} < \frac{\pi}{2}\sigma_{n+1} + \log \eta_n - \frac{\pi}{2}\sigma_n + A.$$

Making use of (27.5) this gives

$$\frac{1}{a}\log \xi_{n+1} + K < \frac{\pi}{2}\sigma_{n+1} + A,$$

$$\log \xi_{n+1} < \frac{\pi}{2} a\,[\sigma_{n+1} + A] - a\,K.$$

Hence if n is odd we have

$$\log \xi_{n+1} < \frac{\pi}{2} a\,\sigma_{n+1} - \log 20,$$

provided that

(30.6) $$K > A(a).$$

We deduce that whether n is even or odd, we have always

$$\log \xi_{n-1} < \frac{\pi a}{2}\sigma_n - \log 20, \quad n \geq 2,$$

if (30.6) holds. Using (18.6) we deduce

(30.7) $$\xi_{n-1} + 1 \leq 2\,\xi_{n-1} < \frac{1}{10}\left(\frac{1 + \varrho_n}{1 - \varrho_n}\right)^a, \quad n \geq 2$$

provided K is suitably chosen.

We deduce from this and (30.5) that

(30.8) $$\sum_{\mu=0}^{n-2} p_\mu(\varrho) < \tfrac{1}{5}(1 - \varrho_n)^{-a} \leq \tfrac{1}{5}(1 - \varrho)^{-a}, \quad n \geq 2,$$

if (30.6) holds, using (30.4). We define the left hand side of (30.8) to be zero if $n < 2$.

31) Consider now $p_\mu(\varrho)$ for $\mu = n,\ n-1$. If (30.3) holds for w in E and if

$$\zeta = \log w = \xi + i\eta + 2m\pi i \tag{31.1}$$

lies in R_μ, then we must have either $\mu = 0$, or μ odd, or μ even and $\mu > 0$ and

$$\xi \leq \xi_\mu, \tag{31.2}$$

making use of (27.4). If $\mu = 0$ (31.1) can hold for at most a single value of m as we have already seen. Suppose next μ odd. Then C_μ is defined by (11.1) and hence if $\zeta = \log w$ corresponds to $\sigma + i\tau$ in the s plane we have from lemma 9, (11.3)

$$\frac{\pi}{2}(\sigma - \sigma_\mu) + \log \frac{1}{1-|\tau|} > \log^+ \left|\frac{\eta + 2m\pi}{\eta_\mu}\right| - A. \tag{31.3}$$

Similarly if μ is even and (31.2) holds, so that C_μ is defined by (13.2) to (13.9) we have (31.3) from lemma 10, (14.5). In either case we deduce

$$\log|\eta + 2m\pi| < \frac{\pi}{2}\sigma + \log\frac{1}{1-|\tau|} + \log \eta_\mu - \frac{\pi}{2}\sigma_\mu + A. \tag{31.4}$$

Making use of (27.5), (31.4) gives for any $\mu \geq 1$

$$\log|\eta + 2m\pi| < \frac{\pi}{2}\sigma + \log\frac{1}{1-|\tau|} + \left(1 - \frac{1}{a}\right)\log \xi_{\mu+1} - K + A. \tag{31.5}$$

Also we have from (5.12)

$$|\eta + 2m\pi| < \xi_{\mu+1} - \xi_\mu < \xi_{\mu+1}, \quad \mu \geq 1,$$

if (31.1) holds. Thus (31.5) gives

$$\frac{1}{a}\log|\eta + 2m\pi| < \frac{\pi}{2}\sigma + \log\frac{1}{1-|\tau|} + A - K. \tag{31.6}$$

Now if $|z| = \varrho'$ and z is a point such that $f(z) = w$, where $\log w$ lies in R_μ and z corresponds to $\sigma + i\tau$ in the mapping of (18.2), then we have

$$\frac{1}{2}\log\frac{1+\varrho'}{1-\varrho'} > \frac{\pi}{4}\sigma + \frac{1}{2}\log\frac{1}{1-|\tau|} - A \tag{31.7}$$

making use of lemma 7. Also $p_\mu(\varrho)$ does not exceed the total number of values of m (positive, negative or zero) for which (31.1) holds with ζ corresponding by (18.1), (18.2) to a point z in $|z| < \varrho$. Thus (31.6), (31.7) give

$$\frac{1}{a}\log\left[p_\mu(\varrho)-1\right] < \log\frac{1+\varrho}{1-\varrho} + A - K.$$

We can take $A(a)$ in (30.6) so large that this gives

$$\log\left[p_\mu(\varrho)-1\right] < a\left[\log\frac{1+\varrho}{1-\varrho} - \log 10\right]$$

$$< a\log\frac{1}{1-\varrho} - \log 5,$$

(31.8) $$p_\mu(\varrho) < \tfrac{1}{5}(1-\varrho)^{-a} + 1.$$

Now ζ given by (31.1) can only be interior to R_0 if $\xi < \xi_1$ and in this case ζ cannot lie in R_1 since C_1 is given by (11.1). Hence (31.8) applied with $\mu = 1$, gives

$$p_0(\varrho) + p_1(\varrho) < 1 + \tfrac{1}{5}(1-\varrho)^{-a}.$$

Since $p_0(\varrho) + p_1(\varrho)$ is an integer, we deduce from this that

$$p_0(\varrho) + p_1(\varrho) \leq 1 \leq (1-\varrho)^{-a}$$

if $(1-\varrho)^{-} \leq 5$ and

$$p_0(\varrho) + p_1(\varrho) \leq \tfrac{2}{5}(1-\varrho)^{-a} < (1-\varrho)^{-a}$$

otherwise. Hence in any case we have

$$p_0(\varrho) + p_1(\varrho) \leq (1-\varrho)^{-a}.$$

Since $\zeta = \log f(z)$ cannot lie in R_μ with $\mu \geq 2$ if $|z| \leq \varrho_2$ this proves that $p(\varrho)$, the total number of roots of $f(z) = w$ in $|z| < \varrho$, is at most $(1-\varrho)^{-a}$, for $\varrho < \varrho_2$ and any w. Suppose next that (30.4) holds with $n \geq 2$. Then we have

$$(1-\varrho)^{-a} \geq (1-\varrho_2)^{-a} \geq 5$$

from (30.7). Hence we have from this and (30.8), (31.8)

$$\sum_{\mu=0}^{n} p_\mu(\varrho) < \sum_{\mu=0}^{n-2} p_\mu(\varrho) + p_{n-1}(\varrho) + p_n(\varrho)$$

$$< \tfrac{1}{5}(1-\varrho)^{-a} + 2\left[1 + \tfrac{1}{5}(1-\varrho)^{-a}\right] \leq \left(\tfrac{1}{5} + \tfrac{4}{5}\right)(1-\varrho)^{-a}.$$

Thus again the equation $f(z) = w$ has at most $(1-\varrho)^{-a}$ roots in $|z| < \varrho$ when w lies in E. Hence this is true in all cases and the proof of Theorem V is complete.

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