NON-HOMOGENEOUS BINARY QUADRATIC FORMS.[1]

II. The second minimum of $(x+x_0)^2-7(y+y_0)^2$.

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Introduction.

1. In a previous paper[2] we showed that for all real x_0, y_0, there exist integers x, y for which

$$|f(x+x_0,\ y+y_0)| = |(x+x_0)^2-7(y+y_0)^2| \leq \frac{9}{14}.$$

We further showed that for $(x_0, y_0) \not\equiv \left(\frac{1}{2}, \pm\frac{5}{14}\right)$, we can make $|f(x+x_0, y+y_0)|$ less than $\frac{1}{1.56}$. In this paper we modify our method with the help of a lemma, due to Dr. J. W. S. Cassels, to prove that the exact value of the second minimum is $\frac{1}{2}$. Our argument is purely geometrical and, as Dr. Cassels pointed out to the author, this seems to be the first time a purely geometrical argument has been applied to the study of the second minimum of a non-homogeneous form.

Our result can be stated as

Theorem: *Let $f(x, y) = x^2-7y^2$. Then, for any pair of real numbers x_0, y_0, there exist numbers x, y such that*

$$(1) \qquad x \equiv x_0 \ (\text{mod. } 1) \quad y \equiv y_0 \ (\text{mod. } 1)$$

and

$$(2) \qquad |f(x, y)| \leq \frac{9}{14}.$$

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[1] This paper forms a part of author's thesis: Some Results in the Geometry of Numbers: approved for the degree of Ph.D. at the University of Cambridge.

[2] Non-homogeneous Binary Quadratic Forms (I): Acta mathematica, this vol. p. 1. We shall refer to this paper as NHF. For references also see NHF.

The equality in (2) *is needed if and only if*

$$(x_0,\ y_0) \equiv \left(\frac{1}{2},\ \pm\frac{5}{14}\right) \pmod{1}. \tag{3}$$

For all x_0, y_0, *not satisfying* (3) *we can replace* (2) *by*

$$|f(x, y)| \leq \frac{1}{2}, \tag{4}$$

the equality in (4) *being necessary if and only if*

$$(x_0,\ y_0) \equiv \left(\frac{1}{2},\ \frac{1}{2}\right) \pmod{1}. \tag{5}$$

Proof of the Theorem.

2. We first prove two lemmas about (x_0, y_0) satisfying relations (3) and (5).

Lemma 1: Let $(x_0, y_0) \equiv \left(\frac{1}{2},\ \pm\frac{5}{14}\right)$ (mod. 1). Then for all $(x, y) \equiv (x_0, y_0)$ (mod. 1), $|x^2 - 7y^2| \geq \frac{9}{14}$. For some of these (x, y), for example $\left(\frac{1}{2},\ \pm\frac{5}{14}\right)$, the result holds with the equality sign.

Proof: This is lemma 1.1 of NHF.

Lemma 2: Let $(x_0, y_0) \equiv \left(\frac{1}{2},\ \frac{1}{2}\right)$ (mod. 1). Then for all $(x, y) \equiv (x_0, y_0)$ (mod. 1), $|x^2 - 7y^2| \geq \frac{1}{2}$. For some of these (x, y), for example $\left(\frac{3}{2},\ \frac{1}{2}\right)$, the result holds with the equality sign.

Proof: All $(x, y) \equiv (x_0, y_0)$ (mod. 1) are given by $x = a + \frac{1}{2}$, $y = b + \frac{1}{2}$, where a and b are integers.

For these x, y we have

$$|x^2 - 7y^2| = \left|a^2 + a - 7b^2 - 7b - \frac{3}{2}\right| \geq \frac{1}{2}, \tag{6}$$

since $a^2 + a - 7b^2 - 7b$ is an integer.

The sign of equality arises in (6), when, for example, $a = 1$, $b = 0$.

This proves the Lemma.

3. *In the rest of the proof we suppose x_0, y_0 is a pair of real numbers such that for all*

(7) $$(x, y) \equiv (x_0, y_0) \pmod{1}, \ |x^2 - 7y^2| \geq \frac{1}{2}.$$

As in NHF we can easily prove that there exist unique numbers (x_1, y_1) and (x_2, y_2) such that

(8) $$-\frac{1}{2} < y_1 = y_2 \leq \frac{1}{2},$$

(9) $$(x_i, y_i) \equiv (x_0, y_0) \pmod{1}, \quad i = 1, 2,$$

(10) $$x_1^2 - 7y_1^2 < \frac{1}{2} \leq (x_1 + 1)^2 - 7y_1^2,$$

(11) $$x_2^2 - 7y_2^2 < \frac{1}{2} \leq (x_2 - 1)^2 - 7y_2^2.$$

By (7), (9), (10) and (11), we also have

(12) $$x_1^2 - 7y_1^2 \leq -\frac{1}{2}, \ x_2^2 - 7y_2^2 \leq -\frac{1}{2}.$$

4. Now let us represent a pair (x, y) of real numbers by the point P with co-ordinates (x, y) in the $x-y$ plane.

We denote by $\mathcal{H}$ the set of all points, which have no congruent points in the region $\mathcal{T}$, defined by

$$|x^2 - 7y^2| < \frac{1}{2}.$$

Clearly it will be sufficient for the proof of the theorem to show that $\mathcal{H}$ consists of points congruent to $\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \pm\frac{5}{14}\right)$ only.

We shall refer to a member of $\mathcal{H}$ as an $\mathcal{H}$-point. In particular (x_0, y_0) is an $\mathcal{H}$-point.

5. We first define congruence of regions and points, and the translation $\mathcal{T}_{m,n}$ as in NHF. Then, defining the point (x_1, y_1), as in § 3, we shall complete the proof in three stages, outlined below.

Stage I: With the help of translations $\mathcal{T}_{m,0}$ we shall show that, if $y_1 \geq 0$, (x_1, y_1) must lie in one of two regions $\mathcal{R}_1$ and $\mathcal{R}_2$, to be defined later.

Stage II: We shall apply translations $\mathcal{T}_{3,-2}$, $\mathcal{T}_{8,-3}$ and $\mathcal{T}_{45,-17}$ in succession to prove that $\mathcal{R}_2$ cannot contain any $\mathcal{H}$-point. Then we shall use $\mathcal{T}_{1,-1}$ to show that a large part of $\mathcal{R}_1$ is free of $\mathcal{H}$-points. From this we shall deduce that all $(x_0, y_0) \not\equiv \left(\frac{1}{2}, \frac{1}{2}\right)$ lie in regions congruent to two small regions, Σ_1, Σ_2 (to be defined later), situated about the points $\left(\frac{1}{2}, \frac{5}{14}\right)$ and $\left(-\frac{1}{2}, -\frac{5}{14}\right)$ respectively.

Stage III: We shall then apply Lemma B, kindly shown to the author by Dr. J. W. S. Cassels, to complete the proof that $\mathcal{H}$ consists only of points congruent to $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \pm\frac{5}{14}\right)$.

For the sake of completeness we end this work with Cassels' proof of his Lemma.

We may remark here that the proof follows the lines of NHF up to the end of Stage II. But in stage three there is complete divergence and the argument now is purely geometrical. The method of this proof could also be applied to the theorems of NHF, but this would require much heavier details.

Stage I.

6. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be the arcs of the hyperbolas $x^2 - 7y^2 = \pm\frac{1}{2}$, as shown in fig. 1. Then $\mathcal{T}$ is the open region included between these arcs.

Let the line $y = \frac{1}{2}$ meet these arcs in the points A, B, $\bar{B}$ and $\bar{A}$ as shown in the figure. Move the part of $\mathcal{A}$, lying between $A\bar{A}$ and the x-axis, through a distance -1 parallel to the x-axis. Let it take up the position $D\,C\,V$ with the points D, C, V on $A\,\bar{A}$, $\mathcal{C}$ and the x-axis respectively. Clearly the equation of $D\,C\,V$ is $(x+1)^2 - 7y^2 = \frac{1}{2}$.

Also move the part of $\mathcal{B}$ between $A\,\bar{A}$ and the x-axis through a distance 1 parallel to the x-axis to take up the position $\bar{D}\,\bar{C}\,\bar{V}$, as shown in the figure. The equation of $\bar{D}\,\bar{C}\,\bar{V}$ is $(x-1)^2 - 7y^2 = \frac{1}{2}$.

Denote the closed curvilinear triangles $B\,C\,D$ and $\bar{B}\,\bar{C}\,\bar{D}$ by $\mathcal{R}$ and $\bar{\mathcal{R}}$ respectively.

Suppose that y_1, as defined in § 3, is non-negative, i.e. $0 \le y_1 \le \frac{1}{2}$.

Then the relations (8), (9), (10) and (12) mean that there exists a unique point $P_1 \equiv P_0(x_0, y_0)$ in $\mathcal{R}$, while relations (8), (9), (11) and (12), mean that there is just one point $P_2 \equiv P_0$ in $\bar{\mathcal{R}}$.

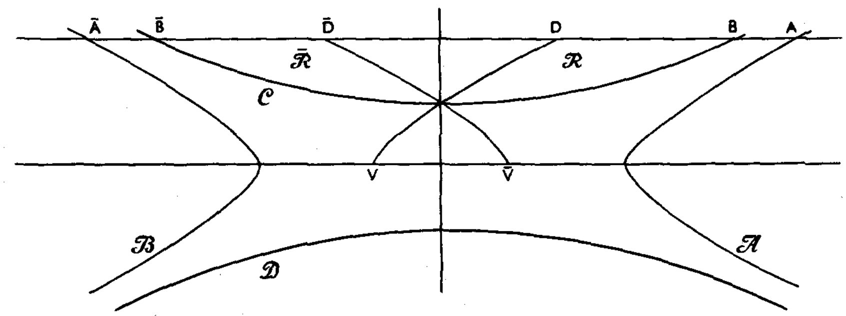

Fig. 1

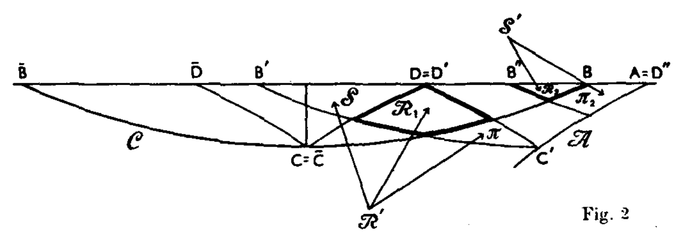

Fig. 2

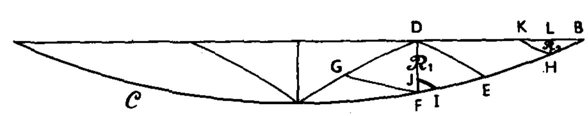

Fig. 3

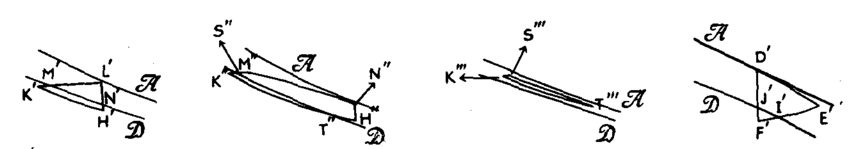

Fig. 4 Fig. 5 Fig. 6 Fig. 7

Figs. 1—7.

For convenience of reference, we tabulate the co-ordinates of the vertices of $\mathcal{R}$ and $\bar{\mathcal{R}}$. $\bar{\mathcal{R}}$ is obviously the image of $\mathcal{R}$ in the y-axis.

7. We first observe that P_1 cannot coincide with C. For, the point $C + (2, -1) = \left(2, \sqrt{\frac{1}{14}} - 1\right)$ lies in $\mathcal{T}$, since

$$\left|2^2 - 7\left(\sqrt{\frac{1}{14}} - 1\right)^2\right| = \sqrt{14} - 3.5 < \frac{1}{2}.$$

Consequently in the rest of this section we shall suppose $C = \bar{C}$ to be excluded from the regions we consider.

Table I.

Point	Curves on which it lies	Co-ordinates
B	$y = \frac{1}{2},\ x^2 - 7y^2 = -\frac{1}{2}$	$\left(\sqrt{\frac{5}{4}},\ \frac{1}{2}\right) = (1.118\ldots,\ .5)$
D	$y = \frac{1}{2},\ (x+1)^2 - 7y^2 = \frac{1}{2}$	$\left(\frac{1}{2},\ \frac{1}{2}\right) = (.5,\ .5)$
C	$x^2 - 7y^2 = -\frac{1}{2},\ (x+1)^2 - 7y^2 = \frac{1}{2}$	$\left(0,\ \sqrt{\frac{1}{14}}\right) = (0,\ .26\ldots)$.
	Image in the y-axis of	
$\bar{B}$	B	$(-1.118\ldots,\ .5)$
$\bar{D}$	D	$(-.5,\ .5)$
$\bar{C}$	C	$(0,\ .26\ldots)$.

Let the translation $\mathcal{T}_{1,0}$ change $\bar{\mathcal{R}}$ into $\mathcal{R}'$. (See fig. 2). The vertex $\bar{B}$ $\left(-1.118\ \ldots,\ \frac{1}{2}\right)$ changes into B' $(-.118,\ .5)$, $\bar{D}$ into $D' = D\,(.5,\ .5)$ and $\bar{C}$ into C' $(1,\ .26\ldots)$ on $\mathcal{A}$, i.e. the vertices of $\mathcal{R}'$ are as shown in fig. 2. Now we assert that $\mathcal{R}'$ consists of three parts.

i) π, which lies in $\mathcal{T}$,

ii) the closed curvilinear quadrilateral $\mathcal{R}_1$, which lies in $\mathcal{R}$, and

iii) the region $\mathcal{S}$, which lies outside $\mathcal{T}$ as well as $\mathcal{R}$.

The above assertion will clearly follow if we can show that the intersections of the boundary arcs of $\mathcal{R}'$ and $\mathcal{R}$ are as shown in the figure. This, in turn, is a clear consequence of lemma A of NHF.

Now $\mathcal{R}'$ is congruent to $\bar{\mathcal{R}}$. Therefore $\mathcal{R}'$ contains a point $Q \equiv P_2 \equiv P_0$. As Q cannot lie in $\mathcal{T}$, it must lie either in $\mathcal{R}_1$ or in $\mathcal{S}$. We include the common boundary of $\mathcal{R}_1$ and $\mathcal{S}$ in $\mathcal{R}_1$ only.

Now let the translation $\mathcal{T}_{1,0}$ change $\mathcal{S}$ into $\mathcal{S}'$. The arc joining $D' = D$ to the lower vertex of $\mathcal{S}$ will change into a part of $\mathcal{A}$, while B' $(-.118\ \ldots,\ .5)$ will change into B'' $(.881\ \ldots,\ .5)$ as shown in the figure. Then, because of

Lemma A of NHF, we can assert that the position of $\mathcal{S}'$ is as shown in the figure, i.e. $\mathcal{S}'$ consists of two parts. i) the region π_2 lying in $\mathcal{T}$, and ii) the closed curvilinear triangle $\mathcal{R}_2$ lying in $\mathcal{R}$.

Now if the point Q lies in $\mathcal{S}$, a point $Q' \equiv Q \equiv P_0$ will lie in $\mathcal{R}'$. As Q' cannot lie in $\mathcal{T}$, it will lie in $\mathcal{R}_2$. Hence a point $\equiv P_0$ lies in $\mathcal{R}_1$ or $\mathcal{R}_2$.

As both $\mathcal{R}_1$ and $\mathcal{R}_2$ lie in $\mathcal{R}$ and $\mathcal{R}$ contains just one point $P_1 \equiv P_0$, P_1 *must lie in* $\mathcal{R}_1$ *or* $\mathcal{R}_2$.

Let the vertices of $\mathcal{R}_1$ be D, E, F and G and let those of $\mathcal{R}_2$ be H, K and B as shown in fig. 3. We give the co-ordinates of these points in Table II.

Table II.

Point	Curves through it	Equations of the curves	co-ordinates of the point
D	$DE,\ DGC$	$(x-2)^2-7y^2=\frac{1}{2},\ (x+1)^2-7y^2=\frac{1}{2}$	$\left(\frac{1}{2}, \frac{1}{2}\right) = (.5, .5)$
E	$DE,\ EF$	$(x-2)^2-7y^2=\frac{1}{2},\ x^2-7y^2=-\frac{1}{2}$	$\left(\frac{3}{4}, \sqrt{\frac{17}{112}}\right) = (.75, .389\ldots)$
F	$FG,\ EF$	$(x-1)^2-7y^2=-\frac{1}{2},\ x^2-7y^2=-\frac{1}{2}$	$\left(\frac{1}{2}, \sqrt{\frac{3}{28}}\right) = (.5, .327\ldots)$
G	$FG,\ DGC$	$(x-1)^2-7y^2=-\frac{1}{2},\ (x+1)^2-7y^2=\frac{1}{2}$	$\left(\frac{1}{4}, \sqrt{\frac{17}{112}}\right) = (.25, .389\ldots)$
H	$HB,\ HK$	$x^2-7y^2=-\frac{1}{2},\ (x-2)^2-7y^2=-\frac{1}{2}$	$\left(1, \sqrt{\frac{3}{14}}\right) = (1, .462\ldots)$
K	$KB,\ HK$	$y=\frac{1}{2},\ (x-2)^2-7y^2=-\frac{1}{2}$	$\left(2-\sqrt{\frac{5}{4}}, \frac{1}{2}\right) = (.881\ldots, .5$
B	$KB,\ BH$	$y=\frac{1}{2},\ x^2-7y^2=-\frac{1}{2}$	$\left(\sqrt{\frac{5}{4}}, \frac{1}{2}\right) = (1.118\ldots, .5)$

Let L be the point $\left(1, \frac{1}{2}\right)$.

Then the equation of HL is $x = 1$. It is easily seen from the equations of HK and HB that the curvilinear triangles HBL and HKL are symmetrical to each other with respect to the line HL: $x = 1$.

Similarly the equation of the line DF is $x = \frac{1}{2}$ and the curvilinear triangles DEF, DGF are symmetrical with respect to it.

Stage II.

8. **Lemma 3**: Every point in the closed curvilinear triangle HKL has a congruent point in $\mathcal{T}$: $|x^2 - 7y^2| < \frac{1}{2}$.

Proof: Suppose HKL contains a point P_1, which has no congruent point in $\mathcal{T}$. We shall obtain a contradiction in three stages (i), (ii) and (iii) below.

(i) Let the translation $\mathcal{T}_{3,-2}$ change HKL into $H'K'L'$. (See fig. 4).

Then, the point $L' = L + (3, -2) = (4, -1.5)$ lies in $\mathcal{T}$, since $|4^2 - 7(1.5)^2| = .25 < .5$; the point $H' = H + (3, -2) = \left(4, \sqrt{\frac{3}{14}} - 2\right)$ lies below the arc $\mathcal{D}$ i.e. in that part of $x^2 - 7y^2 < -\frac{1}{2}$, where y is negative, since

$$4^2 - 7\left(\sqrt{\frac{3}{14}} - 2\right)^2 = 28\sqrt{\frac{3}{14}} - 13.5 = \sqrt{168} - 13.5 < -.5;$$

the point $K' = K + (3, -2) = \left(5 - \sqrt{\frac{5}{4}}, -\frac{3}{2}\right)$ lies below $\mathcal{D}$, since

$$\left(5 - \sqrt{\frac{5}{4}}\right)^2 - 7(-1.5)^2 = 10.5 - \sqrt{125} < -\frac{1}{2}.$$

This shows that the points H', K' and L' are situated as in the figure.

Consequently $K'L'$ and $L'H'$ meet $\mathcal{D}$ in single points M' and N', respectively. Also $K'L'$ and $L'H'$ do not intersect $\mathcal{A}$.

Now $K'H'$ arises from $\mathcal{B}$ by translation $\mathcal{T}_{2,0} + \mathcal{T}_{3,-2} = \mathcal{T}_{5,-2}$. Therefore its equation is

$$(13) \qquad (x - 5)^2 - 7(y + 2)^2 = -\frac{1}{2}.$$

The equation of $\mathcal{D}$ is $x^2 - 7y^2 = -\frac{1}{2}$.

Therefore the points of intersection, if any, of $K'H'$ and $\mathcal{D}$ satisfy the equation

$$(14) \qquad 0 = 112x^2 - (10x + 3)^2 + 56 = 12x^2 - 60x + 47 = f(x) \quad \text{(say)}.$$

Clearly $f(0) > 0$ and $f(1) < 0$. Therefore one of the roots of (14) lies between 0 and 1. As the x-co-ordinate of any point on $K'H'$ is greater than 3, $K'H'$ does not intersect $\mathfrak{D}$ in two points. Therefore by Lemma A of NHF, $K'H'$ does not intersect $\mathfrak{D}$ at all.

This shows that the position of $H'K'L'$ is as shown in the figure.

Now P_1 lies in HKL. Therefore a point $P' \equiv P_1$ lies in $H'K'L'$. *As P' cannot lie in $\mathfrak{T}$, it must lie in the closed region $K'M'N'H'K'$.*

The co-ordinates of M' and N' are

Point	Curves through it	Co-ordinates
M'	$y = -\frac{3}{2},\ x^2 - 7y^2 = -\frac{1}{2}$	$\left(\sqrt{\frac{61}{4}},\ -\frac{3}{2}\right) = (3.90\ldots,\ -1.5)$
N'	$x = 4,\ x^2 - 7y^2 = -\frac{1}{2}$	$\left(4,\ -\sqrt{\frac{33}{14}}\right) = (4,\ -1.53\ldots)$

(ii) Now apply the translation $\mathcal{T}_{8,-3}$ to $H'K'M'N'H'$. Let it change into a region $H''K''M''N''H''$. (See fig. 5.)

Then

$H'' = H' + (8,\ -3) = \left(12,\ \sqrt{\frac{3}{14}} - 5\right)$ lies in $\mathfrak{T}$, since

$$\left|(12)^2 - 7\left(\sqrt{\frac{3}{14}} - 5\right)^2\right| = \left|32.5 - \sqrt{1050}\right| < \frac{1}{2};$$

the point $N'' = N' + (8,\ -3) = \left(12,\ -3 - \sqrt{\frac{33}{14}}\right)$ lies in $\mathfrak{T}$, since

$$\left|(12)^2 - 7\left(3 + \sqrt{\frac{33}{14}}\right)^2\right| = \left|64.5 - \sqrt{4158}\right| < \frac{1}{2};$$

$M'' = M' + (8,\ -3) = \left(8 + \sqrt{\frac{61}{4}},\ -4.5\right)$ lies in $\mathfrak{T}$, since

$$\left|\left(8 + \sqrt{\frac{61}{4}}\right)^2 - 7(-4.5)^2\right| = \left|\sqrt{3904} - 62.5\right| < \frac{1}{2};$$

the point $K'' = K' + (8,\ -3) = \left(13 - \sqrt{\frac{5}{4}},\ -4.5\right)$ lies below $\mathfrak{D}$, since

$$\left(13 - \sqrt{\frac{5}{4}}\right)^2 - 7(4.5)^2 = 28.5 - \sqrt{845} < -\frac{1}{2}.$$

This shows that the points H'', K'', M'', N'' are situated relative to $\mathscr{A}$ and $\mathscr{D}$ as shown in the figure.

As $M''K''$ is parallel to the x-axis, it intersects $\mathscr{D}$ in one point S'' (say). Also it does not intersect $\mathscr{A}$ at all.

The line $H''N''$ is parallel to the y-axis and meets neither $\mathscr{A}$ nor $\mathscr{D}$.

Now $M''N''$ arises from $\mathscr{D}$ by the translation $\mathscr{T}_{8,-3}$. Therefore its equation is

$$(x-8)^2 - 7(y+3)^2 = -\frac{1}{2}. \tag{15}$$

The equation of $\mathscr{D}$ is $x^2 - 7y^2 = -\frac{1}{2}$.

Therefore the points of intersection, if any, of $M''N''$ with $\mathscr{D}$ satisfy the equation.

$$0 = -(1-42y)^2 + 1792y^2 - 128 = 28y^2 + 84y - 129,$$

which clearly possesses a positive root.

As $M''N''$ arises from $\mathscr{D}$ by translation $\mathscr{T}_{8,-3}$, the y-co-ordinate of every point on it must be less than -3. Consequently $M''N''$ cannot intersect $\mathscr{D}$ in two points. So that, by Lemma A of NHF, $M''N''$ does not intersect $\mathscr{D}$ at all.

The equation of $\mathscr{A}$ is $x^2 - 7y^2 = \frac{1}{2}$. Therefore, because of (15) the points of intersection, if any, of $M''N''$ and $\mathscr{A}$ satisfy the equation

$$0 = -(2-42y)^2 + 1792y^2 + 128 = 28y^2 + 168y + 124 = f(y) \text{ (say).} \tag{16}$$

Now $f(0) > 0$ and $f(-2) < 0$. Therefore one of the roots of (16) lies between 0 and -2. So that $M''N''$ cannot intersect $\mathscr{A}$ in two points. Therefore by Lemma A of NHF $M''N''$ has no point common with $\mathscr{A}$.

Also because of the position of $H''K''$ relative to $H''N''$ and $N''M''$, $H''K''$ does not intersect $\mathscr{A}$.

By Lemma A of NHF, $H''K''$ meets $\mathscr{D}$ in a single point T'' (say).

The above discussion shows that the region $H''K''M''N''H''$ is situated as in the figure.

Now P' lies in $H'K'M'N'H'$ and $H''K''M''N''H''$ is congruent to $H'K'M'N'H'$. Also no point congruent to $P' \equiv P_1$ lies in $\mathscr{T}$. Therefore, *a point* $P'' \equiv P' \equiv P_1$ *lies in* $K''S''T''$.

The co-cordinates of S'' are $\left(\sqrt{\frac{565}{4}}, -4.5\right)$.

We observe that

$$y(T'') > y(H'') > -5.$$

Also $y(S'') = -4.5$.

Therefore the y-co-ordinate of any point on the arc $S''T''$ lies between -4.5 and -5.

(iii) Now let the translation $\mathcal{T}_{45,-17}$ change $K''S''T''$ into $K'''S'''T'''$. (See fig. 6).

Then both

$$K''' = K'' + (45, -17) = \left(58 - \sqrt{\frac{5}{4}},\ -21.5\right)$$

and

$$S''' = S'' + (45, -17) = \left(45 + \sqrt{\frac{565}{4}},\ -21.5\right)$$

lie in $\mathcal{C}$, since

$$\left|\left(58 - \sqrt{\frac{5}{4}}\right)^2 - 7(21.5)^2\right| = \left|129.5 - \sqrt{16820}\right| < \frac{1}{2},$$

and

$$\left|\left(45 + \sqrt{\frac{565}{4}}\right)^2 - 7(21.5)^2\right| = \left|45\sqrt{565} - 1069.5\right| < \frac{1}{2}.$$

Again, the translation $\mathcal{T}_{45,-17}$ does not change the positions of $K''T''$ and $S''T''$ relative to each other, i.e. $K''T''$ is below $S''T''$. Therefore, to prove that $K'''S'''T'''$ lies entirely in $\mathcal{C}$, we have only to show that (i) $S'''T'''$ lies below $\mathcal{A}$, and (ii) $K'''T'''$ lies above $\mathcal{D}$. For this it will suffice to show that a) $S'''T'''$ does not intersect $\mathcal{A}$, and b) $K'''T'''$ has no points common with $\mathcal{D}$. These we prove below.

(a) The equation of $S'''T'''$ is

$$(x-45)^2 - 7(y+17)^2 = -\frac{1}{2};$$

and that of $\mathcal{A}$ is $x^2 - 7y^2 = \frac{1}{2}$.

Therefore, on eliminating x between these equations, we find that the points of intersection, if any, of $S'''T'''$ and $\mathcal{A}$ satisfy the relation

$$0 = -(238y - 3)^2 + 7 \times 8100y^2 + 4050$$

(17) $$= 56y^2 + 1428y + 4041 = f(y) \quad \text{(say)}.$$

Now $f(0) > 0,$

$$f(-4) = 896 - 5712 + 4041 < 0,$$

$$f(-22) = 28\,(-22)\,(-44 + 51) + 4041 = 4041 - 4312 < 0,$$

$$f(-\infty) > 0.$$

Therefore (17) has no root in the interval $(-21.5, \ -22)$. We have already seen that the y-co-ordinates of points of $S'' T''$ lie between -4.5 and -5. Therefore the y-co-ordinates of points of $S''' T'''$ lie between -21.5 and -22. So that $S''' T'''$ does not have any point common with $\mathcal{A}$ i.e. (a) is true.

(b) The arc $K''' T'''$ arises from $K'' T''$ by $\mathcal{T}_{45,-17}$, and therefore from $K' H'$ by $\mathcal{T}_{53,-20}$. Therefore, by (13), its equation is

$$(x - 58)^2 - 7\,(y + 22)^2 = -\frac{1}{2}. \tag{18}$$

The equation of $\mathcal{D}$ is

$$x^2 - 7\,y^2 = -\frac{1}{2}.$$

Therefore their points of intersection would satisfy

$$0 = 847\,x^2 - (29\,x + 6)^2 + 423.5 = 6\,x^2 - 348\,x + 387.5 = f(x) \quad \text{(say)}. \tag{19}$$

Now $f(0) > 0,$

$$f(+2) < 0,$$

$$f(56.88) = 6\,(56.88)\,(56.88 - 58) + 387.5 = 387.5 - 382.2336 > 0.$$

Therefore both the roots of (19) are less than 56.88.

But the x-co-ordinate of any point of $K''' T'''$ is greater than that of $K'' = 58 - \sqrt{\frac{5}{4}} = 56.881 \ldots$ Therefore $K''' T'''$ does not intersect $\mathcal{D}$ i.e. (b) is true. Consequently $K''' S''' T'''$ lies entirely in $\mathcal{T}$.

As $K''' S''' T'''$ is congruent to $K'' S'' T''$, a point $P''' \equiv P'' \equiv P_1$ lies in $\mathcal{T}$. This gives the required contradiction and the lemma follows.

Corollary: The closed curvilinear triangle HKL does not contain any $\mathcal{H}$-point i.e. a point which has no congruent point in $\mathcal{T}$: $|x^2 - 7\,y^2| < \frac{1}{2}$.

Lemma 4: There is no $\mathcal{H}$-point in $\mathcal{R}_2$.

Proof: We have seen that there is no $\mathcal{H}$-point in HKL. Now consider HBL. We saw at the end of stage I that HBL is the image of HKL in the line HL: $x = 1$. Therefore if HBL contains an $\mathcal{H}$-point (x, y), its image $(2 - x, y)$, is an $\mathcal{H}$-point in HKL, which is impossible. Consequently there is no $\mathcal{H}$-point in $\mathcal{R}_2$.

Lemma 5: Every point in the closed curvilinear triangle DEF, excluding the point D and a closed curvilinear triangle FIJ, has a congruent point in $\mathcal{T}$.

Proof: Let the translation $\mathcal{T}_{1,-1}$ change DEF into $D'E'F'$. (See fig. 7). Then, from table II,

$$D' = D + (1, -1) = \left(\frac{3}{2}, -\frac{1}{2}\right) \text{ lies on } \mathcal{A},$$

since

$$\left(\frac{3}{2}\right)^2 - 7\left(-\frac{1}{2}\right)^2 = \frac{1}{2},$$

also

$$E' = E + (1, -1) = \left(\frac{7}{4}, \sqrt{\frac{17}{112}} - 1\right) \text{ lies in } \mathcal{T},$$

since

$$\left|\left(\frac{7}{4}\right)^2 - 7\left(\sqrt{\frac{17}{112}} - 1\right)^2\right| = \left|\sqrt{\frac{119}{4}} - 5\right| < \frac{1}{2},$$

and $F' = F + (1, -1) = \left(\frac{3}{2}, \sqrt{\frac{3}{28}} - 1\right)$ lies below the arc $\mathcal{D}$ i.e. in that part of $x^2 - 7y^2 < -\frac{1}{2}$, where y is negative, since

$$\left(\frac{3}{2}\right)^2 - 7\left(\sqrt{\frac{3}{28}} - 1\right)^2 = \sqrt{21} - 5.5 < -.5.$$

This proves that the position of the points D', E' and F' is as shown in fig. 7.

We also observe that $D'E'$ and $E'F'$ are arcs of hyperbolas with asymptotes parallel to $x \pm \sqrt{7}\, y = 0$.

By Lemma A of NHF, $E'F'$ meets $\mathcal{D}$ in a single point I' (say). As $F'D'$ is a line parallel to the y-axis, it intersects $\mathcal{D}$ in one point J', say. The arc $D'E'$ arises from $\mathcal{B}$ by a translation $\mathcal{T}_{3,-1} = \mathcal{T}_{1,0} + \mathcal{T}_{1,0} + \mathcal{T}_{1,-1}$. Therefore its equation is

$$(x - 3)^2 - 7(y + 1)^2 = \frac{1}{2}. \tag{20}$$

The equation of $\mathscr{D}$ is

$$x^2 - 7y^2 = -\frac{1}{2}.$$

Therefore, eliminating y between (20) and the above, we find that the points of intersection, if any, of $D'E'$ and $\mathscr{D}$ satisfy the equation

$$0 = -14 - 28x^2 + (1 - 6x)^2 = 8x^2 - 12x - 13.$$

This has a negative root. Therefore, as all points on $D'E'$ have a positive abscissa, there is at most one point of intersection of $D'E'$ and $\mathscr{D}$. But, by Lemma A of NHF, the points of intersection of $D'E'$ and $\mathscr{D}$ are two or none. Therefore $D'E'$ does not intersect $\mathscr{D}$.

The equation of $\mathscr{A}$ is $x^2 - 7y^2 = \frac{1}{2}$.

From (20) and the equation of $\mathscr{A}$ it is easily seen that $D'E'$ and $\mathscr{A}$ touch each other at D'.

The effect of all this is to show that the position of $D'E'F'$ is as shown in the figure. The translation $\mathscr{T}_{-1,1}$, i.e. the translation inverse to $\mathscr{T}_{+1,-1}$, change $F'I'J'$ into the FIJ of the lemma.

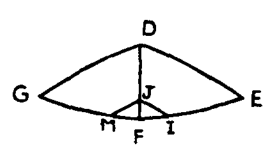

Fig. 8.

9. Let JM, the image of JI in DF, meet the arc FG in M. As in Lemma 4, we see that there is no $\mathscr{H}$-point in the whole of $\mathscr{R}_1$, excluding the point and the closed region $FIJM$. We shall denote the region $FIJM$ by Σ_1.

As any point congruent to an $\mathscr{H}$-point, is an $\mathscr{H}$-point, the above combined with Lemma 4 shows:— Let (x_0, y_0) be an $\mathscr{H}$-point and $y_0 \equiv y_1$ with $0 \leq y_1 \leq \frac{1}{2}$, then (x_0, y_0) is congruent either to $\left(\frac{1}{2}, \frac{1}{2}\right)$ or to a point (x_1, y_1) in Σ_1.

Now let Σ_2 be the image of Σ_1 in the origin, Then, by symmetry, if $-\frac{1}{2} < y_1 < 0$, (x_0, y_0) has a congruent point $(x_2, y_2) = (x_2, y_1)$ in Σ_2.

Regarding the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ as a region Σ_3 we conclude that all $\mathscr{H}$-points lie in regions congruent to Σ_1, Σ_2 and Σ_3.

We give below the equations of the boundaries of Σ_1 and the co-ordinates of its vertices.

$$FI\colon (x^2-7y^2)=-\frac{1}{2}; \qquad FM\colon (x-1)^2-7y^2=-\frac{1}{2}$$

$$IJ\colon (x+1)^2-7(y-1)^2=-\frac{1}{2}; \qquad JM\colon (x-2)^2-7(y-1)^2=-\frac{1}{2}.$$

$$F\colon \left(\frac{1}{2},\sqrt{\frac{3}{28}}\right)=(.5,\ .3273\ldots); \qquad J\colon \left(\frac{1}{2},\ 1-\sqrt{\frac{11}{28}}\right)=(.5,\ .3732\ldots)$$

$$I\colon \left(\sqrt{\frac{7}{6}}-\frac{1}{2},\ \frac{1}{2}-\sqrt{\frac{1}{42}}\right)=(.5801\ldots,\ .3456\ldots);$$

$$M\colon \left(\frac{3}{2}-\sqrt{\frac{7}{6}},\ \frac{1}{2}-\sqrt{\frac{1}{42}}\right)=(.4198\ldots,\ .3456\ldots).$$

The point $\left(\frac{1}{2},\frac{5}{14}\right)=\left(\frac{1}{2},\ .3571\ldots\right)$ obviously lies on FJ between F and J. F_2, I_2, J_2 and M_2 are the images in the origin of F, I, J and M respectively.

Stage III.

10. We now introduce some standard notation for use in the rest of the chapter.

If $\mathcal{R}_1$ and $\mathcal{R}_2$ be two sets of points, $\mathcal{R}_1 \cup \mathcal{R}_2$ will denote their union i.e. the set of points which belong to $\mathcal{R}_1$ or to $\mathcal{R}_2$ or to both.

$\bigcup\limits_{i=1}^{m} \mathcal{R}_i$ or $\bigcup\{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_m\}$ will denote the union of $\mathcal{R}_1, \mathcal{R}_2, \ldots$ and $\mathcal{R}_m$.

$\mathcal{R}_1 \cap \mathcal{R}_2$ will denote the intersection of $\mathcal{R}_1$ and $\mathcal{R}_2$ i.e. the set of points which belong to both $\mathcal{R}_1$ and $\mathcal{R}_2$.

$\bigcap\limits_{i=1}^{m} \mathcal{R}_i$ or $\bigcap\{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_m\}$ will denote the intersection of $\mathcal{R}_1, \mathcal{R}_2, \ldots$ and $\mathcal{R}_m$.

Λ will denote the lattice of points with integer co ordinates.

If P be a point with co-ordinates (a, b), $\mathcal{R}+P$ will denote the region into which $\mathcal{R}$ is changed by the translation $\mathcal{T}_{a,b}$. In particular if P be a point of Λ, $\mathcal{R}+P$ will be a region congruent to $\mathcal{R}$.

If T be any transformation $T(P)$ will denote the point into which T transforms P, while $T(\mathcal{R})$ will stand for the region into which $\mathcal{R}$ is changed by T.

T^{-1} will denote the transformation inverse to T.

11. In this notation we can express the conclusions of § 9 as follows:

Every $\mathcal{H}$-point must belong to the set Σ, where

$$\Sigma=\bigcup_{\substack{i=1\\ P\in\Lambda}}^{3}(\Sigma_i+P).$$

Consider the transformation T defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8x - 21y \\ -3x + 8y \end{pmatrix}.$$

Both T and T^{-1} are integral and uni-modular. They leave $x^2 - 7y^2$ unchanged. Consequently, both T and T^{-1} change an $\mathcal{H}$-point into an $\mathcal{H}$-point, i.e. if P is an $\mathcal{H}$-point, so also are $T(P)$ and $T^{-1}(P)$. This implies that if P is an $\mathcal{H}$-point, there exist $\mathcal{H}$-points Q and R such that

$$T(R) = T^{-1}(Q) = P.$$

As P, Q and R are members of Σ, P belongs to Σ, $T(\Sigma)$ and $T^{-1}(\Sigma)$. Continuing like this we can show P belongs to $T^2(\Sigma)$, $T^3(\Sigma), \ldots$ and $T^{-2}(\Sigma)$, $T^{-3}(\Sigma), \ldots$.

In other words, every $\mathcal{H}$-point belongs to the set

$$\gamma = \bigcap \{\ldots, T^{-2}(\Sigma), T^{-1}(\Sigma), \Sigma, T(\Sigma), T^2(\Sigma), \ldots\}.$$

12. We now state Cassels' Lemma. We shall give a proof at the end of the paper.

Cassels' Lemma.

Suppose

i) S is the transformation defined by

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

where a, b, c, d, are integers such that $ad - bc = \pm 1$, and $x^2 - x(a+d) + ad - bc = 0$ has two distinct real roots,

ii)[1] $F_1 = (\xi_1, \eta_1)$, $F_2 = (\xi_2, \eta_2), \ldots, F_t = (\xi_t, \eta_t)$ are t points incongruent mod. Λ, such that

$$S(F_i) \equiv F_i \ (\text{mod. } \Lambda),\ i = 1, 2, \ldots, t.$$

iii) $\mathcal{R}_1, \ldots, \mathcal{R}_t$ are bounded regions containing $F_1, \ldots, F_t$ respectively, such that

a) no two of the regions

$$\mathcal{R}_i + P,\ i = 1, \ldots, t,\ P \in \Lambda \quad (*)$$

intersect, and

---

[1] It is not difficult to generalise the lemma to cover the case when the $S(F_i)$ are congruent to the F_i in any permuted order.

b) for each $j = 1, 2, \ldots, t$, $S(\mathcal{R}_j)$ intersects only one region of (*), namely $\mathcal{R}_j + P$.

Then, if $\mathcal{R}$ denotes the union of all regions (*) i.e.

$$\mathcal{R} = \bigcup_{\substack{i=1\\ P \in \Lambda}}^{t} (\mathcal{R}_i + P),$$

the only points common to $\mathcal{R}$, $S(\mathcal{R})$, $S^2(\mathcal{R})$, $\ldots$, $S^{-1}(\mathcal{R})$, $S^{-2}(\mathcal{R})$, $\ldots$, are the points $F_i + P$ ($i = 1, 2, \ldots t$, $P \in \Lambda$).

13. Now take

$$S = T \text{ i.e. } S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8x - 21y \\ -3x + 8y \end{pmatrix}.$$

The condition i) is easily seen to be satisfied.

Let $F_1 = \left(\frac{1}{2}, \frac{5}{14}\right)$, $F_2 = \left(-\frac{1}{2}, -\frac{5}{14}\right)$, and $F_3 = \left(\frac{1}{2}, \frac{1}{2}\right)$. These points are obviously incongruent mod. Λ.

Also

$$T(F_1) = \left(-\frac{7}{2}, \quad \frac{9}{14}\right) \equiv F_1,$$

$$T(F_2) = \left(\quad \frac{7}{2}, -\frac{9}{14}\right) \equiv F_2,$$

$$T(F_3) = \left(-\frac{13}{2}, \quad \frac{5}{2}\right) \equiv F_3,$$

so that ii) is satisfied.

Let

$$\mathcal{R}_1 = \Sigma_1, \quad \mathcal{R}_2 = \Sigma_2, \quad \mathcal{R}_3 = \Sigma_3.$$

The regions $\Sigma_1 + P$, $\Sigma_2 + P$, $\Sigma_3 + P$ respectively contain the points $\left(\frac{1}{2} + m, \frac{5}{14} + n\right)$, $\left(\frac{1}{2} + m, -\frac{5}{14} + n\right)$, $\left(\frac{1}{2} + m, \frac{1}{2} + n\right)$, (m, n) integers. These regions lie in rows and columns as shown in fig. 9.

The regions in the rows immediately above and immediately below the lines $y = a + \frac{1}{2}$ are congruent to Σ_2 and Σ_1 respectively.

We can easily see that $\mathcal{R}_i$ does not intersect $\mathcal{R}_i + P$, $i \neq j$ or $\mathcal{R}_i + P$, $P \in \Lambda$, $P \neq 0$, $i, j = 1, 2, 3$, i.e. no two of $\mathcal{R}_i + P$, $i = 1, 2, 3$, $P \in \Lambda$, intersect.

Now we verify iii) (b).

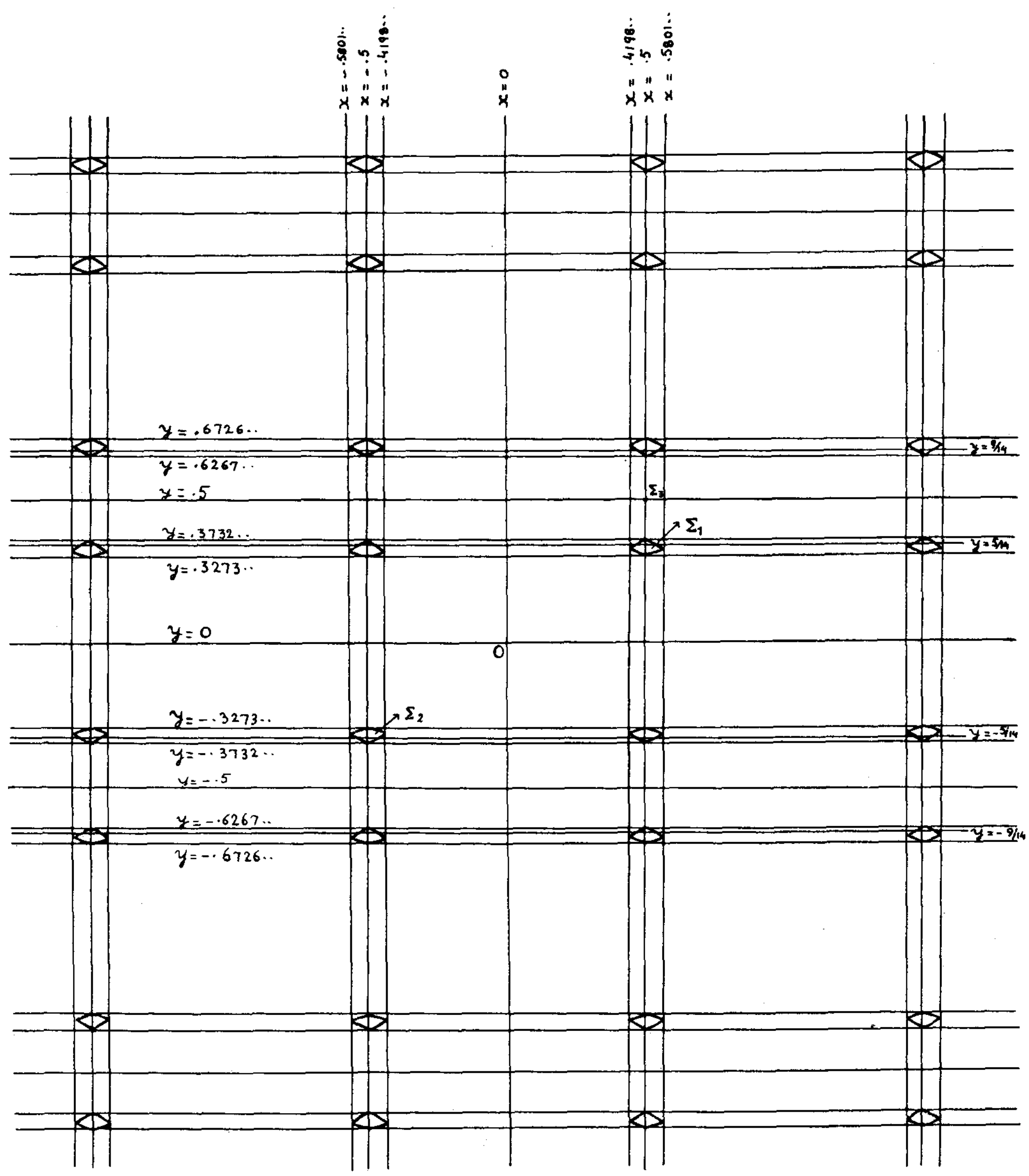

Fig. 9.

$$S(\mathcal{R}_3) = T(\Sigma_3) = \left(-\frac{13}{2}, \frac{5}{2}\right) = \Sigma_3 + (-7, 2)$$

and so is easily seen to have points common with no other region of (*).

The region Σ_1 was defined by

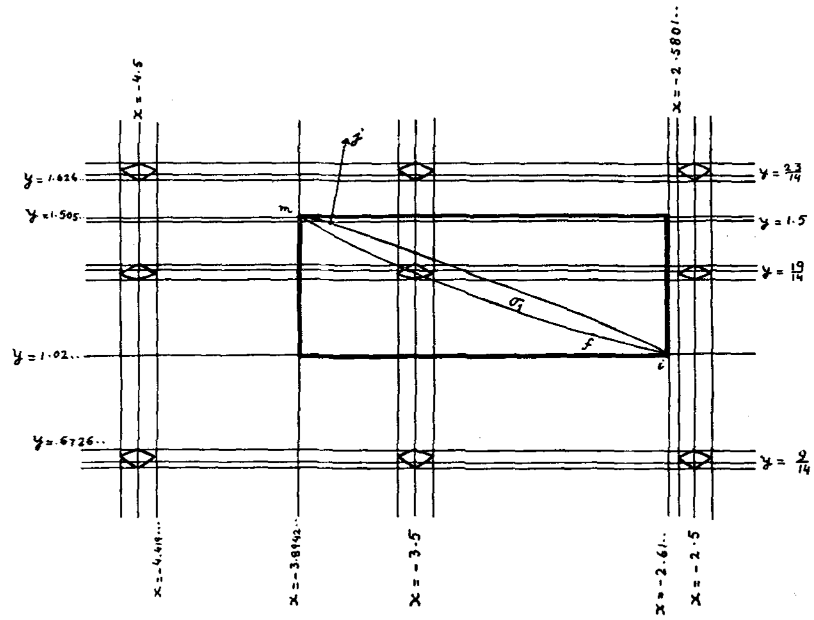

Fig. 10.

$$x^2 - 7y^2 \leq -\frac{1}{2},$$

$$(x-1)^2 - 7y^2 \leq -\frac{1}{2},$$

$$(x+1)^2 - 7(y-1)^2 \leq -\frac{1}{2},$$

$$(x-2)^2 - 7(y-1)^2 \leq -\frac{1}{2}.$$

Let $T(\Sigma_1)$ be denoted by σ_1, the points $T(F)$, $T(I)$, $T(J)$ and $T(M)$ being denoted by f, i, j and m respectively. (See fig. 10).

We easily see that T transforms $(x-a)^2 - 7(y-b)^2$ into $(x-a')^2 - 7(y-b')^2$ where

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8a - 21b \\ -3a + 8b \end{pmatrix}.$$

Therefore the equations of fi, ij, jm and mf are

$$fi = T(FI): \quad x^2 - 7y^2 = -\frac{1}{2},$$

$$f\,m = T(F\,M):\quad (x-\ \ 8)^2 - 7(y+\ \ 3)^2 = -\frac{1}{2},$$

$$i\,j\ = T(I\,J):\quad (x+29)^2 - 7(y-11)^2 = -\frac{1}{2},$$

$$j\,m = T(J\,M):\quad (x+\ \ 5)^2 - 7(y-\ \ 2)^2 = -\frac{1}{2}.$$

Consequently σ_1 can be defined by

$$\begin{aligned} x^2 - 7y^2 &\leq -\frac{1}{2}, \\ (x-\ \ 8)^2 - 7(y+\ \ 3)^2 &\leq -\frac{1}{2}, \\ (x+29)^2 - 7(y-11)^2 &\leq -\frac{1}{2}, \\ (x+\ \ 5)^2 - 7(y-\ \ 2)^2 &\leq -\frac{1}{2}. \end{aligned}$$

The co-ordinates of f, i, j and m are

$$f = T(F):\quad T\begin{pmatrix} \frac{1}{2} \\ \sqrt{\frac{3}{28}} \end{pmatrix} = \begin{pmatrix} -2.87\ldots \\ 1.11\ldots \end{pmatrix},$$

$$i = T(I):\quad T\begin{pmatrix} \sqrt{\frac{7}{6}} - \frac{1}{2} \\ \frac{1}{2} - \sqrt{\frac{1}{42}} \end{pmatrix} = \begin{pmatrix} -2.61\ldots \\ 1.02\ldots \end{pmatrix},$$

$$j = T(J):\quad T\begin{pmatrix} \frac{1}{2} \\ 1 - \sqrt{\frac{11}{28}} \end{pmatrix} = \begin{pmatrix} -3.83\ldots \\ 1.485\ldots \end{pmatrix},$$

$$m = T(M):\quad T\begin{pmatrix} \frac{3}{2} - \sqrt{\frac{7}{6}} \\ \frac{1}{2} - \sqrt{\frac{1}{42}} \end{pmatrix} = \begin{pmatrix} -3.8992\ldots \\ 1.505\ldots \end{pmatrix}.$$

As all the arcs $f\,i$, $i\,j$, $j\,m$ and $m\,i$ are parts of hyperbolas with axes parallel to the co-ordinate axes, it is easily seen that σ_1 lies between the lines

$$y = 1.505\ldots,\quad y = 1.02\ldots,\quad x = -3.8992\ldots,\quad x = -2.61\ldots$$

We know that the regions $\Sigma_i + P$, $i = 1, 2, 3$, $P \in \Lambda$ are situated in **rows and** columns about the points $\left(\frac{1}{2} + m, \pm\frac{5}{14} + n\right)$, $\left(\frac{1}{2} + m, \frac{1}{2} + n\right)$ where m **and** n are integers. In order to show that σ_1 intersects no region of $\bigcup_{\substack{i=1\\P\in\Lambda}}^{3} (\Sigma_i + P)$ other than $\Sigma_1 + (-4, 1)$, it will clearly suffice to show that

1) σ_1 does not intersect the columns about the lines $x = -4.5$ **and** $x = -2.5$,

2) σ_1 does not intersect the rows about the lines $y = \frac{23}{14}$ and $y = \frac{9}{14}$, **and**

3) the point $\left(-\frac{7}{2}, \frac{3}{2}\right)$ does not lie in σ_1.

1) Consider the column about the line $x = -4.5$. The x-co-ordinates of **any** point in a region $\Sigma_i + P$ in this column is $\leq \sqrt{\frac{7}{6}} - \frac{1}{2} - 5 = -4.4198\ldots$, **and** the x-co-ordinate of any point of $\sigma_1 \geq -3.89\ldots$. Therefore σ_1 does **not** intersect this column.

Now consider the column about the line $x = -2.5$. The x-co-ordinate **of** every point in a region $\Sigma_i + P$ of this column $\geq \frac{3}{2} - \sqrt{\frac{7}{6}} - 3 = -2.5801\ldots$, and the x-co-ordinate of any point in $\sigma_1 \leq -2.61\ldots$. Therefore σ_1 does **not** intersect this column and 1) is proved.

2) Consider the row of $\Sigma_i + P$ about the line $y = \frac{23}{14}$. Every point in **this row** has an ordinate $\geq \sqrt{\frac{11}{28}} + 1 = 1.626\ldots$. As the y-co-ordinate of **every** point in $\sigma_1 \leq 1.505\ldots$, σ_1 does not intersect this row.

Similarly, since the y-co-ordinate of any point in the row about $y = \frac{9}{14}$ **is** $\leq 1 - \sqrt{\frac{3}{28}} < 1.02\ldots$, σ_1 does not intersect this row. This proves (2).

3) Finally the point $\left(-\frac{7}{2}, \frac{3}{2}\right)$ lies outside σ_1, since

$$\left(-\frac{7}{2} + 5\right)^2 - 7\left(\frac{3}{2} - 2\right)^2 = \frac{1}{2} > -\frac{1}{2}.$$

Therefore the only $\Sigma_i + P$ which can have a point common with σ_1 is **the region** $\Sigma_1 + (-4, 1)$.

By symmetry about the origin, it is easily seen that the only region $\Sigma_i + P$, **which can intersect $T(\Sigma_2)$ is $\Sigma_2 + (4, -1)$.**

This shows that the condition iii) (b) of Cassels' Lemma is also satisfied. Consequently the set

$$\gamma = \cap \{\ldots,\ T^{-2}(\Sigma),\ T^{-1}(\Sigma), \Sigma,\ T(\Sigma),\ T^{2}(\Sigma), \ldots\}$$

consists of points congruent to $\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \pm\frac{5}{14}\right)$ only.

Combined with Lemmas 1, 2 and § 11, this means that the set of $\mathcal{H}$-points consists only of points congruent to $\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \pm\frac{5}{14}\right)$. This proves the theorem.

14. We now give Cassels' proof of his lemma.

Cassels' Proof of His Lemma.

We build up the proof in a series of lemmas and corollaries.

Lemma C 1: Suppose that $t \neq \pm 1$ is real and that T is the transformation

$$(x, y) \rightarrow (t\,x,\ \pm y/t).$$

Let $\mathcal{R}$ be a bounded region of the $x-y$ plane, containing the origin. Then the only point common to

$$\mathcal{R},\ T(\mathcal{R}),\ T^{2}(\mathcal{R}), \ldots$$
$$T^{-1}(\mathcal{R}),\ T^{-2}(\mathcal{R}),\ T^{-3}(\mathcal{R}), \ldots$$

is the origin.

Proof: Suppose, for example, $|t| > 1$.

Since $\mathcal{R}$ is bounded, for n large enough the ordinate of any point of $T^{n}(\mathcal{R})$ can be made arbitrarily small in absolute value. Therefore, $\bigcap\limits_{n=0}^{\infty} T^{n}(\mathcal{R})$ cannot contain any point with non-zero ordinate. Also the origin belongs to $T^{n}(\mathcal{R})$ for all n. Therefore, $\bigcap\limits_{n=0}^{\infty} T^{n}(\mathcal{R})$ is a part of the x-axis containing the origin.

Similarly $\bigcap\limits_{-\infty}^{\infty} T^{n}(\mathcal{R})$ is a part of the y-axis containing the origin.

Therefore, $\bigcap\limits_{-\infty}^{\infty} T^{n}(\mathcal{R})$ consists of the origin alone.

Corollary: Let S be the transformation defined by

$$(x, y) \rightarrow (a\,x + b\,y,\ c\,x + d\,y),$$

where a, b, c, d are real numbers such that $ad - bc = \pm 1$ and such that $x^2 - x(a+d) + ad - bc = 0$ has two distinct real roots λ_1, λ_2. Let $\mathcal{R}$ be as before. Then the set

$$\bigcap_{-\infty}^{\infty} S^n(\mathcal{R}) = \bigcap \{\ldots, S^{-2}(\mathcal{R}), S^{-1}(\mathcal{R}), \mathcal{R}, S(\mathcal{R}), S^2(\mathcal{R}), \ldots\}$$

consists of the origin alone.

Proof: We can change the axes so that in the new co-ordinates x^1, y^1, S can be defined by

$$(x^1, y^1) \to (\lambda_1 x^1, \lambda_2 y^1).$$

Since

$$\lambda_1 \lambda_2 = \pm 1 \text{ and } \lambda_1 \neq \lambda_2, |\lambda_1| \neq 1,$$

the corollary follows from Lemma C 1.

Lemma C 2: Suppose S is a transformation as in the corollary, but that a, b, c, d are integers. Let Λ be the lattice of points with integer co-ordinates. Further, let $\mathcal{R}$ be a bounded region containing the origin, such that the regions $\mathcal{R}$, $S(\mathcal{R})$, and so also $S^{-1}(\mathcal{R})$ do not intersect any $\mathcal{R} + P$, $(P \neq 0, P \in \Lambda)$. Write

$$\gamma = \bigcup_{P \in \Lambda} (\mathcal{R} + P).$$

Then the set

$$\bigcap_{-\infty}^{\infty} S^n(\gamma) = \bigcap \{\ldots, S^{-2}(\gamma), S^{-1}(\gamma), \gamma, S(\gamma), S^2(\gamma), \ldots\}$$

consists of the points of Λ alone.

Proof: One can easily see[1] that the set $\bigcap_{-\infty}^{\infty} S^n(\gamma)$ consists of mutually disjoint sets congruent to $\bigcap_{-\infty}^{\infty} S^n(\mathcal{R})$ and containing the points P of Λ. Each of these sets consists of one point, namely the lattice point P. From this the lemma follows.

Formally: Define a set of regions $\mathcal{A}_1, \mathcal{A}_2, \ldots$ by the relations

$$\begin{aligned}
\mathcal{A}_1 &= \mathcal{R}, \\
\mathcal{A}_2 &= \mathcal{A}_1 \cap S(\mathcal{A}_1) = \mathcal{R} \cap S(\mathcal{R}), \\
\mathcal{A}_3 &= \mathcal{A}_2 \cap S(\mathcal{A}_2) = \bigcap \{\mathcal{R}, S(\mathcal{R}), S^2(\mathcal{R})\}, \\
&\vdots \\
\mathcal{A}_{r+1} &= \mathcal{A}_r \cap S(\mathcal{A}_2) = \bigcap \{\mathcal{R}, S(\mathcal{R}), \ldots, S^r(\mathcal{R})\}.
\end{aligned}$$

[1] We shall give the formal proof also.

Similarly we define

$$\mathscr{B}_1 = \mathscr{R},\ \mathscr{B}_{r+1} = \mathscr{B}_r \cap S^{-1}(\mathscr{B}_r) = \cap \{\mathscr{R}, S^{-1}(\mathscr{R}), \ldots, S^{-r}(\mathscr{R})\}.$$

Since $\mathscr{A}_r < \mathscr{R}$ and $S(\mathscr{R})$ does not intersect $\mathscr{R} + P$, $P \neq 0$, $P \in \Lambda$, we have

a) $S(\mathscr{A}_r)$ does not intersect $\mathscr{A}_r + P\,(P \in \Lambda,\ P \neq 0)$.

Similarly, since $\mathscr{B}_r < \mathscr{R}$ and $S^{-1}(\mathscr{R})$ does not intersect $\mathscr{R} + P$, $P \neq 0$, $P \in \Lambda$, we have

b) $S^{-1}(\mathscr{B}_r)$ does not intersect $\mathscr{B}_r + P$, $P \neq 0$, $P \in \Lambda$.

Also as $\mathscr{R} + P$ does not intersect $\mathscr{R} + Q$ when $P \neq Q$, $P, Q \in \Lambda$, we have

c) $\mathscr{B}_r + P$ does not intersect $\mathscr{A}_r + Q$ when $P \neq Q$, $P, Q \in \Lambda$.

Now we show that $\gamma \cap S(\gamma) = \bigcup_{P \in \Lambda} (\mathscr{A}_2 + P)$.

Since $\mathscr{A}_1 = \mathscr{R}$,

$$\gamma \cap S(\gamma) = \bigcup_{P, Q \in \Lambda} \{(\mathscr{A}_1 + P) \cap (S(\mathscr{A}_1) + Q)\}.$$

By (a), $\{\ \}$ vanishes except when $P = Q$. Therefore,

$$\begin{aligned}\gamma \cap S(\gamma) &= \bigcup_{P \in \Lambda} [\{\mathscr{A}_1 + P\} \cap \{S(\mathscr{A}_1) + P\}] \\ &= \bigcup_{P \in \Lambda} [\{\mathscr{A}_1 \cap S(\mathscr{A}_1)\} + P] \\ &= \bigcup_{P \in \Lambda} (\mathscr{A}_2 + P).\end{aligned}$$

By induction we can easily show that

$$\bigcap_{n=0}^{r} S^n(\gamma) = \bigcup_{P \in \Lambda} (\mathscr{A}_{r+1} + P).$$

Similarly

$$\bigcap_{n=0}^{r} S^{-n}(\gamma) = \bigcup_{Q \in \Lambda} (\mathscr{B}_{r+1} + Q).$$

Therefore,

$$\bigcap_{n=-r}^{r} S^n(\gamma) = \bigcup_{P, Q \in \Lambda} [\{\mathscr{A}_{r+1} + P\} \cap \{\mathscr{B}_{r+1} + Q\}].$$

Again, the $[\ ]$ vanishes except when $P = Q$, so that

$$\bigcap_{n=-r}^{r} S^n(\gamma) = \bigcup_{P \in \Lambda} \{(\mathscr{A}_{r+1} \cap \mathscr{B}_{r+1}) + P\}.$$

Now write

$$\mathscr{C}_r = \mathscr{A}_{r+1} \cap \mathscr{B}_{r+1} = \bigcap_{n=-r}^{r} S^n(\mathscr{R}).$$

Clearly

$$\mathscr{R} > \mathscr{C}_1 > \mathscr{C}_2 > \ldots.$$

Since

$$\mathcal{C}_r = \cap \{S^{-r}(\mathcal{R}), \ldots, S^{-1}(\mathcal{R}), \mathcal{R}, S(\mathcal{R}), \ldots, S^r(\mathcal{R})\},$$

we can apply the corollary of lemma C 1 to show that the only point common to $\mathcal{C}_1, \mathcal{C}_2, \ldots$ is the origin. Therefore

$$\lim_{r\to\infty} \bigcup_{P\in\Lambda} \{\mathcal{C}_r + P\} = \Lambda.$$

This gives the lemma.

Corollary: Suppose S is a transformation of the type

$$(x, y) \to (ax + by + l,\ cx + dy + m),$$

where a, b, c, d, l, m are integers and a, b, c, d satisfy the conditions of the corollary of lemma C 1. Then also the conclusion of lemma C 2 holds.

Corollary 2: Suppose S is as in corollary 1, and (ξ, η) is a point such that $S\begin{pmatrix}\xi\\ \eta\end{pmatrix} \equiv \begin{pmatrix}\xi\\ \eta\end{pmatrix}$ (mod. Λ). Suppose further that $\mathcal{R}$ is now a bounded region containing (ξ, η). Define γ by

$$\gamma = \bigcup_{P\in\Lambda} (\mathcal{R} + P).$$

Then the set $\bigcap_{n=-\infty}^{\infty} S^n(\gamma)$ consists of the points $(\xi, \eta) + P$ alone.

Proof: If we transfer the origin to the point (ξ, η), the previous corollary applies.

Cassels' Lemma.

Let S be as in the last corollary. Let $F_1 = (\xi_1, \eta_1)$, $F_2 = (\xi_2, \eta_2), \ldots, F_t = (\xi_t, \eta_t)$ be t points incongruent mod. Λ. Also suppose

$$S(F_j) \equiv F_j \text{ (mod. } \Lambda) \ (j = 1, 2, \ldots, t).$$

Further, suppose $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_t$ are t bounded regions containing the points $F_1, F_2, \ldots, F_t$ respectively and that

i) No two of the regions

$$\mathcal{R}_i + P;\ i = 1, 2, \ldots, t;\ P \in \Lambda, \qquad (*)$$

intersect; and

ii) for each j, $S(\mathcal{R}_j)$ intersects one and only one region of (*), namely $\mathcal{R}_j + P$, where P is a point of Λ.

Denote by γ the union of regions (*), i.e.

$$\gamma = \bigcup_{\substack{j=1\\ P\in\Lambda}}^{t} \{\mathcal{R}_j + P\}.$$

Then the only points common to γ, $S(\gamma)$, $S^2(\gamma), \ldots$, $S^{-1}, (\gamma)$, $S^{-2}(\gamma), \ldots$ are the points $F_i + P$; $i = 1, \ldots, t$, $P \in \Lambda$.

Proof: For simplicity suppose $t = 3$. Write

$$\gamma_1 = \bigcup_{P \in \Lambda} \{\mathcal{R}_1 + P\},$$

$$\gamma_2 = \bigcup_{Q \in \Lambda} \{\mathcal{R}_2 + Q\},$$

$$\gamma_3 = \bigcup_{R \in \Lambda} \{\mathcal{R}_3 + R\}.$$

We shall now prove that the set $\bigcap_{n=-\infty}^{\infty} S^n(\gamma)$ consists of three distinct sets, namely the sets $\bigcap_{n=-\infty}^{\infty} S^n(\gamma_i)$, $i = 1, 2, 3$. We first have

$$\begin{aligned} &\gamma \cap S(\gamma) \\ &= \bigcup_{\substack{P, Q, R, \\ P', Q', R' \in \Lambda}} [\{\cup(\mathcal{R}_1 + P, \mathcal{R}_2 + Q, \mathcal{R}_3 + R)\} \cap \{\cup(S(\mathcal{R}_1) + P', S(\mathcal{R}_2) + Q', S(\mathcal{R}_3) + R')\}] \\ &= \bigcup_{\substack{P, Q, R, \\ P', Q', R' \in \Lambda}} \left[\begin{array}{l} [\{\mathcal{R}_1 + P\} \cap \{S(\mathcal{R}_1) + P'\}] \cup [\{\mathcal{R}_1 + P\} \cap \{S(\mathcal{R}_2) + Q'\}] \cup \\ \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; \cup [\{\mathcal{R}_3 + R\} \cap \{S(\mathcal{R}_3) + R'\}] \end{array}\right] \\ &= \{\gamma_1 \cap S(\gamma_1)\} \cup \{\gamma_2 \cap S(\gamma_2)\} \cup \{\gamma_3 \cap S(\gamma_3)\}, \end{aligned}$$

since $\{\mathcal{R}_i + P\} \cap \{S(\mathcal{R}_j) + Q\} = 0$, if $i \neq j$. Continuing this process we can show

$$\bigcap_{n=-\infty}^{\infty} S^n(\gamma) = \bigcup_{i=1}^{3} \{\bigcap_{n=-\infty}^{\infty} S^n(\gamma_i)\}.$$

Now applying corollary 2 of Lemma C 2 we have

$$\bigcap_{n=-\infty}^{\infty} S^n(\gamma) = \bigcup_{i=1}^{3} \{\bigcap_{n=-\infty}^{\infty} S^n(\gamma_i)\} = \bigcup_{\substack{i=1 \\ P \in \Lambda}}^{3} (F_i + P).$$

This completes the proof.

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