CONFORMAL INVARIANTS AND FUNCTION-THEORETIC NULL-SETS.

BY

LARS AHLFORS and ARNE BEURLING

of CAMBRIDGE, MASS. of UPPSALA.

§ 1. Introduction.

The most useful conformal invariants are obtained by solving conformally invariant extremal problems. Their usefulness derives from the fact that they must automatically satisfy a principle of majorization. There is a rich variety of such problems, and if we would aim at completeness this paper would assume forbidding proportions. We shall therefore limit ourselves to a few particularly simple invariants and study their properties and interrelations in considerable detail.

Each class of invariants is connected with a category of null-sets, which by this very fact enter naturally in function-theoretic considerations. A null-set is the complement of a region for which a certain conformal invariant degenerates. Inequalities between invariants lead to inclusion relations between the corresponding classes of null-sets.

Throughout this paper Ω will denote an open region in the extended z plane, and z_0 will be a distinguished point in Ω. Most results will be formulated for the case $z_0 \neq \infty$, but the transition to $z_0 = \infty$ is always trivial. In some instances the latter case offers formal advantages.

We shall consider classes of functions $f(z)$ which are analytic and single-valued in some region Ω. For a general class $\mathfrak{F}$ the region Ω is allowed to vary with f, but the subclass of functions in a fixed region Ω will be denoted by $\mathfrak{F}(\Omega)$. For $z_0 \in \Omega$ we introduce the quantity

$$M_{\mathfrak{F}}(z_0, \Omega) = \sup_{f \in \mathfrak{F}(\Omega)} |f'(z_0)|. \tag{1}$$

The abbreviations $M_{\mathfrak{F}}$, $M_{\mathfrak{F}}(z_0)$ or $M_{\mathfrak{F}}(\Omega)$ will be used when no misunderstanding can result. It will be assumed that $\mathfrak{F}(\Omega)$ is not empty.

The class $\mathfrak{F}$ is said to be *monotonic* if $\Omega' < \Omega$ implies $\mathfrak{F}(\Omega) < \mathfrak{F}(\Omega')$. By (1) we have then

$$M_{\mathfrak{F}}(z_0, \Omega) \leqq M_{\mathfrak{F}}(z_0, \Omega'). \tag{2}$$

Suppose now that $z' = h(z)$ defines a one to one conformal mapping of Ω onto a region Ω', and set $z'_0 = h(z_0)$. We shall say that the class $\mathfrak{F}$ is *conformally invariant* if $f(z') \in \mathfrak{F}(\Omega')$ implies $f(h(z)) \in \mathfrak{F}(\Omega)$ for all such mappings. For a conformally invariant class we have evidently

$$M_{\mathfrak{F}}(z_0, \Omega) = M_{\mathfrak{F}}(z'_0, \Omega') \cdot | h'(z_0) |. \tag{3}$$

This can be written in the more symmetric form

$$M_{\mathfrak{F}}(z_0, \Omega) \, | d z_0 | = M_{\mathfrak{F}}(z'_0, \Omega) \, | d z'_0 |, \tag{4}$$

and it is seen that the differential

$$M_{\mathfrak{F}}(z, \Omega) \, | d z | \tag{5}$$

defines a conformally invariant metric in Ω. $M_{\mathfrak{F}}$ is itself a relative conformal invariant, and this is the type of invariant we shall be mainly concerned with. Absolute invariants can be introduced either as the quotient of two relative invariants or by forming the curvature of the metric (5).

If $\mathfrak{F}$ is both monotonic and conformally invariant we can combine (2) and (4) to obtain

$$M_{\mathfrak{F}}(z'_0, \Omega') \cdot | d z'_0 | \leqq M_{\mathfrak{F}}(z_0, \Omega) \cdot | d z_0 | \tag{6}$$

whenever $z' = h(z)$ maps Ω conformally and one to one onto a subregion of Ω'. We shall refer to (6) as the *weak monotonic property* of $M_{\mathfrak{F}}$.

A stronger result is obtained if $\mathfrak{F}$ is *analytically invariant*. By this we mean that $f(z') \in \mathfrak{F}(\Omega')$ implies $f(h(z)) \in \mathfrak{F}(\Omega)$ whenever $h(z')$ is single-valued and analytic in Ω with values in Ω', regardless of whether $h(z)$ is univalent or not. Since analytic invariance implies conformal invariance the metric (5) will have the same invariance property as before. An analytically invariant class is *eo ipso* monotonic. Hence (6) is valid, but the stronger assumption implies that (6) holds not only for one to one mappings, but for arbitrary analytic mappings of Ω into Ω'. In this case we shall say that $M_{\mathfrak{F}}$ has the *strong monotonic property*.

A class $\mathfrak{F}$ is said to be *compact* if the following is true: Given any increasing sequence of regions Ω_n and functions $f_n \in \mathfrak{F}(\Omega_n)$ there exists a subsequence f_{n_k} which converges to a limit function $f \in \mathfrak{F}(\Omega)$, $\Omega = \sum_1^\infty \Omega_n$, uniformly on every compact subset of Ω. For a compact class there is a function in $\mathfrak{F}(\Omega)$ which makes $|f'(z_0)|$ a maximum.

Theorem 1. *For a monotonic, conformally invariant and compact class $\mathfrak{F}$ the following holds:*

i) *if Ω_n tends increasingly to Ω, then*

$$\lim_{n \to \infty} M_{\mathfrak{F}}(z_0, \Omega_n) = M_{\mathfrak{F}}(z_0, \Omega); \tag{7}$$

ii) *$M_{\mathfrak{F}}(z, \Omega)$ is a continuous function of z;*

iii) *$\log M_{\mathfrak{F}}(z, \Omega)$ is subharmonic or $\equiv -\infty$.*

By (2) $\lim\limits_{n \to \infty} M_{\mathfrak{F}}(z_0, \Omega_n)$ exists and is $\geqq M_{\mathfrak{F}}(z_0, \Omega)$. On the other hand, if f_n is an extremal function in $\mathfrak{F}(\Omega_n)$, the compactness implies

$$M_{\mathfrak{F}}(z_0, \Omega) \geqq \lim_{n \to \infty} |f_n'(z_0)| = \lim_{n \to \infty} M_{\mathfrak{F}}(z_0, \Omega_n)$$

and (7) is proved.

To prove the continuity, let f be extremal in $\mathfrak{F}(\Omega)$ for the point z_0. Let z_0' be another point in Ω such that the circle $|z - z_0| < 2|z_0' - z_0|$ is contained in Ω. We have

$$M_{\mathfrak{F}}(z_0', \Omega) \geqq |f(z_0')|$$

and hence

$$\varliminf_{z_0' \to z_0} M_{\mathfrak{F}}(z_0', \Omega) \geqq M_{\mathfrak{F}}(z_0, \Omega). \tag{8}$$

Let Ω' be the subset of Ω consisting of all points whose distance from the boundary is $> |z_0' - z_0|$, and let Ω'' be obtained from Ω' by the parallel translation which takes z_0 to z_0'. Then

$$M_{\mathfrak{F}}(z_0', \Omega) \leqq M_{\mathfrak{F}}(z_0', \Omega'') = M_{\mathfrak{F}}(z_0, \Omega')$$

and as $z_0' \to z_0$ we obtain by (7)

$$\varlimsup_{z_0' \to z_0} M_{\mathfrak{F}}(z_0', \Omega) \leqq M_{\mathfrak{F}}(z_0, \Omega). \tag{9}$$

The inequalities (8) and (9) show that $M_{\mathfrak{F}}(z_0, \Omega)$ is continuous.

Since $\log M_{\mathfrak{F}}(z, \Omega)$ is defined as the maximum in a family of subharmonic functions $\log |f'(z)|$ it must itself be subharmonic.

In all cases that we shall treat it will be seen that $M_{\mathfrak{F}}(z, \Omega)$ cannot vanish at a single point unless it vanishes identically. It seems difficult, however, to formulate a simple general property from which this would follow.

Our attention will be focussed on three basic classes, together with a subclass of each. The first two are the class $\mathfrak{B}$ of bounded functions and the class $\mathfrak{D}$ of functions with a bounded Dirichlet integral. The third class $\mathfrak{E}$ has a more complicated characterization, but it will be shown to be related to the classes $\mathfrak{B}$ and $\mathfrak{D}$ in a very symmetric manner.

More precisely, the classes $\mathfrak{B}(\Omega)$ and $\mathfrak{D}(\Omega)$ consist of all single-valued analytic functions $f(z)$ in Ω which satisfy the conditions $|f(z)| \leqq 1$ and

$$\iint_{\Omega} |f'(z)|^2 \, dx \, dy \leqq \pi$$

respectively.

The class $\mathfrak{E}(\Omega)$ is defined only with respect to a point z_0, and consists of all single-valued analytic functions $f(z)$ in Ω with the property that $(f(z)-f(z_0))^{-1}$ omits a set of values of area $\geqq \pi$.

The corresponding invariants are denoted by $M_{\mathfrak{B}}$, $M_{\mathfrak{D}}$ and $M_{\mathfrak{E}}$. As far as these invariants are concerned we can replace $\mathfrak{B}$, $\mathfrak{D}$ and $\mathfrak{E}$ by the subclasses $\mathfrak{B}_0$, $\mathfrak{D}_0$ and $\mathfrak{E}_0$ of functions which vanish at z_0. This is obvious for the classes $\mathfrak{D}$ and $\mathfrak{E}$, and for a function $f(z) \in \mathfrak{B}$ we need only observe that

$$\frac{f(z) - \overline{f(z_0)}}{1 - \overline{f(z_0)} f(z)}$$

is in $\mathfrak{B}_0$ while its derivative at z_0 is of absolute value $\geqq |f'(z)|$.

In addition we shall consider the subclasses $\mathfrak{S}\mathfrak{B}$, $\mathfrak{S}\mathfrak{D}$ and $\mathfrak{S}\mathfrak{E}$, formed by all univalent (schlicht) functions in $\mathfrak{B}$, $\mathfrak{D}$ and $\mathfrak{E}$. In order to be sure that these classes are not empty, and in order to make the corresponding classes $\mathfrak{S}\mathfrak{B}_0$, $\mathfrak{S}\mathfrak{D}_0$ and $\mathfrak{S}\mathfrak{E}_0$ compact, we agree in this connection to consider constant functions as univalent. The invariants $M_{\mathfrak{S}\mathfrak{B}}$, $M_{\mathfrak{S}\mathfrak{D}}$ and $M_{\mathfrak{S}\mathfrak{E}}$ are then well defined.

It is easy to verify that all six classes are monotonic and conformally invariant. The classes $\mathfrak{B}$ and $\mathfrak{E}$ are also analytically invariant. Hence $M_{\mathfrak{B}}$ and $M_{\mathfrak{E}}$ have the strong monotonic property while the others have only the weak monotonic property. The classes $\mathfrak{B}_0$, $\mathfrak{D}_0$, $\mathfrak{E}_0$ and $\mathfrak{S}\mathfrak{B}_0$, $\mathfrak{S}\mathfrak{D}_0$, $\mathfrak{S}\mathfrak{E}_0$ are compact. We are thus in a position to apply Theorem 1 to all our invariants.

In this paper we shall prove the interesting relations

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\begin{aligned}
M_{\mathfrak{B}} &= M_{\mathfrak{E}} \\
M_{\mathfrak{D}} &= M_{\mathfrak{S}\mathfrak{E}} \\
M_{\mathfrak{S}\mathfrak{B}} &= M_{\mathfrak{S}\mathfrak{D}}.
\end{aligned} \tag{10}
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Since $\mathfrak{S}\mathfrak{E} < \mathfrak{E}$ and $\mathfrak{S}\mathfrak{D} < \mathfrak{D}$ it will follow that the three distinct invariants satisfy the inequality

$$M_{\mathfrak{S}\mathfrak{B}} \leqq M_{\mathfrak{D}} \leqq M_{\mathfrak{B}}. \tag{11}$$

The quantity $M_{\mathfrak{S}\mathfrak{B}} = M_{\mathfrak{S}\mathfrak{D}}$ will also be identified with the maximum of an invariant $\mu\ (z_0, p)$ of different nature, defined by means of extremal lengths.

The complement of a region Ω will be denoted by E. Conversely, if a closed set E and a point z_0 outside of E are given, the complement of E has a unique component Ω which contains z_0. We shall say that E is a *null-set* of class $N_{\mathfrak{F}}$ if $M_{\mathfrak{F}}(z_0, \Omega)$ is identically zero. To this definition we observe that for all classes considered above $M_{\mathfrak{F}}$ vanishes identically as soon as it vanishes at a point. This is trivial for the classes of univalent functions, for then the vanishing of $M_{\mathfrak{F}}$ at any point means that the class $\mathfrak{F}(\Omega)$ contains only constants. In view of (10) the property will thus need verification only for the class $\mathfrak{B}$.

The inequality (11) implies the inclusion relations

$$N_{\mathfrak{S}\mathfrak{B}} > N_{\mathfrak{D}} > N_{\mathfrak{B}}, \tag{12}$$

and it will be shown by examples that these inclusions are proper. It follows from (10) that the three types of null-sets have a double characterization, and such information is of course apt to be valuable.

We close this introduction on the remark that a greater degree of generality can be attained by introducing classes of multiple-valued functions. As examples we could consider either the whole class of functions $f(z)$ which can be continued along all paths in Ω and take only values of modulus $\leqq 1$, or the subclass for which $|f(z)|$ is single-valued and $\leqq 1$. The first choice leads to the hyperbolic metric with constant negative curvature on Ω provided that E has at least three points. The second choice leads to an invariant which for $z_0 = \infty$ reduces to the *capacity* of E. The capacity is hence a majorant of $M_{\mathfrak{B}}$, and it follows that all our classes of null-sets contain the sets of capacity zeros. The properties of capacitary null-sets are comparatively well known, and this case will not be discussed further.

There are also important intermediate metrics, for instance the one which arises from the class of Abelian integrals. It is merely for the sake of concentration that we have decided to leave this and similar cases out of consideration.

§ 2. The Invariants $M_{\mathfrak{B}}$ and $M_{\mathfrak{E}}$.

This section is devoted to the proof of the relation $M_{\mathfrak{B}} = M_{\mathfrak{E}}$. It is evident that every function in $\mathfrak{B}_0$ belongs to the class $\mathfrak{E}_0$. The relation $M_{\mathfrak{B}} \leqq M_{\mathfrak{E}}$ is hence trivial, and only the opposite inequality need be proved.

Assume that $f(z) \in \mathfrak{E}_0(\Omega)$ and denote by A the set of values which $\frac{1}{f}$ does not take in Ω. A is a closed set, and its area $I(A)$ is by hypothesis $\geqq \pi$. We form the function

$$F(z) = \frac{1}{I(A)} \iint_A \frac{du\,dv}{\frac{1}{f(z)} - w} \qquad (w = u + iv). \tag{13}$$

This function is clearly analytic in Ω, and its derivative at z_0 is

$$F'(z_0) = \frac{f'(z_0)}{I(A)} \iint_A du\,dv = f'(z_0).$$

If we can show that $|F(z)| \leqq 1$ in Ω the inequality $M_{\mathfrak{B}} \geqq M_{\mathfrak{E}}$ will follow.

It is sufficient to prove that

$$\left| \iint_A \frac{du\,dv}{w-a} \right| \leqq I(A)$$

for all complex a. An auxiliary congruence transformation is obviously allowed, and hence we may take $a = 0$ and assume that

$$\iint_A \frac{du\,dv}{w}$$

is real and positive.

Let A^+ be the part of A situated in the right half-plane. In polar coordinates $w = re^{i\theta}$ we have then

$$(14)\qquad \iint_A \frac{du\,dv}{w} \leqq \iint_{A^+} \cos\theta\,dr\,d\theta.$$

Denote by $l(r, \theta)$ the linear measure of the set of points $w \in A^+$ with $\arg w = \theta$ and $|w| \leqq r$. Setting $l(\infty, \theta) = l(\theta)$ we have first

$$(15)\qquad \iint_{A^+} \cos\theta\,dr\,d\theta = \int_0^{\pi} l(\theta)\cos\theta\,d\theta \leqq \left(\frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} l(\theta)^2\,d\theta\right)^{1/2}.$$

On the other hand, $l(r, \theta) \leqq r$, and integration with a fixed θ gives

$$\int r\,dr \geqq \int l(r, \theta)\,d\,l(r, \theta) = \frac{l(\theta)^2}{2}.$$

Hence

$$(16)\qquad I(A) \geqq \tfrac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} l(\theta)^2\,d\theta,$$

and by (14), (15) and (16) it follows that

$$\iint_A \frac{du\,dv}{w} \leqq (\pi I(A))^{1/2} \leqq I(A).$$

This is what we wanted to prove. We have thus shown that

$$(17)\qquad M_{\mathfrak{B}}(z_0, \Omega) = M_{\mathfrak{E}}(z_0, \Omega).$$

If $M_{\mathfrak{B}}(z_0, \Omega) = 0$, every bounded function in Ω must satisfy $f'(z_0) = 0$. But if $f(z)$ is not constant, it can be written in the form

$$f(z) = f(z_0) + c_k(z - z_0)^k + \cdots,\ c_k \neq 0$$

and then

$$\frac{f(z) - f(z_0)}{(z - z_0)^{k-1}}$$

would be bounded with a non-zero derivative. This is a contradiction, and we conclude that the class $\mathfrak{B}(\Omega)$ contains only constants. It follows that $M_{\mathfrak{B}}(z, \Omega)$ is identically zero. As already pointed out, this observation is important when we consider the identical null-classes $N_{\mathfrak{B}}$ and $N_{\mathfrak{E}}$. The former was first considered by Painlevé [7], and a set E of class $N_{\mathfrak{B}}$ will be referred to as a Painlevé null-set.

The following theorem is an immediate consequence of the strong monotonic property of $M_{\mathfrak{B}}$:

Theorem 2. *A non-constant meromorphic function, considered on the complement of a null-set of class $N_{\mathfrak{B}}$, takes all complex values with the exception of another null-set of the same class.*

We interrupt to remark that the corresponding theorem is of course valid for any strongly monotonic class. Although a direct consequence of the definitions this theorem is very important as a sharp and general characterization of omitted sets. For greater emphasis we shall give it the following striking formulation:

Theorem 2′. *Let F be any strongly monotonic class of functions, and let a compact set E be measured by $m_{\mathfrak{F}}(E) = M_{\mathfrak{F}}(\infty, \Omega)$, where Ω is the complement of E. Then any normalized meromorphic function $f(z) = z + c_0 + \frac{c_1}{z} + \cdots$ in Ω omits the values of a compact set E' with $m_{\mathfrak{F}}(E') \leqq m_{\mathfrak{F}}(E)$.*

We return now to the case of Painlevé null-sets and note the further characteristic property:

Theorem 3.[1] *Suppose that a null-set E of class $N_{\mathfrak{B}}$ is contained in a region Ω'. Then every analytic and bounded function $f(z)$ in $\Omega' - E$ can be continued to an analytic function in Ω'. Conversely, if the continuation is always possible, the set E is of class $N_{\mathfrak{B}}$.*

By a standard application of Cauchy's integral formula we can write $f(z) = f_1(z) + f_2(z)$, where $f_1(z)$ is analytic in Ω' and $f_2(z)$ is analytic in Ω, the complement of E. But then $f_2(z)$ is bounded, and if E is a null-set it must reduce to a constant, so that $f(z)$ must be analytic in Ω'. The converse is obvious.

Corollary. *The value of the invariant $M_{\mathfrak{B}}(z_0, \Omega)$ does not change if a null-set of class $N_{\mathfrak{B}}$ is removed from Ω.*

In fact, the family of competing functions remains the same.

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[1] The first precise statement of this theorem is difficult to locate, but it is implicit in the work of PAINLEVÉ [7].

§ 3. The Invariants $M_{\mathfrak{D}}$ and $M_{\mathfrak{S}\mathfrak{E}}$.

We shall now prove that $M_{\mathfrak{D}} = M_{\mathfrak{S}\mathfrak{E}}$. We wish to point out that this result and the method by which it is proved are previously known, although not exactly in the present connection. The idea of the proof goes back to Grunsky [4], and a theorem by Schiffer [10] is essentially equivalent with ours. Nevertheless, it is essential for our purposes to give a new version of the proof.

The classes $\mathfrak{D}_0$ and $\mathfrak{S}\mathfrak{E}_0$ both satisfy the conditions of Theorem 1. For this reason it is sufficient to prove the relation $M_{\mathfrak{D}} = M_{\mathfrak{S}\mathfrak{E}}$ for regions which can be used to approximate an arbitrary region from within. We are therefore allowed to assume that the region Ω under consideration is bounded by a finite number of analytic curves. The complete boundary, taken in the positive sense with respect to the region, will be denoted by Γ.

The existence of a univalent function

$$p(z) = \frac{1}{z - z_0} + a(z - z_0) + \cdots$$

which maps Ω onto a region bounded by horizontal slits is well known. Similarly, there exists a function

$$q(z) = \frac{1}{z - z_0} + b(z - z_0) + \cdots$$

which maps Ω onto a region bounded by vertical slits.

Let $f(z)$ be any regular analytic function in the closed region Ω. By a familiar formula

$$D(f, p - q) = \iint_{\Omega} f'(z)\left(\overline{p'(z)} - \overline{q'(z)}\right) dx\, dy = \frac{i}{2} \int_{\Gamma} f(d\bar{p} - d\bar{q}).$$

But $d\bar{p} = dp$ and $d\bar{q} = -dq$ on Γ. Hence

$$\int_{\Gamma} f(d\bar{p} - d\bar{q}) = \int_{\Gamma} f(dp + dq) = -4\pi i f'(z_0)$$

by the residue theorem, and we obtain

(18) $$D(f, p - q) = 2\pi f'(z_0).$$

For $f = p - q$ this formula gives

$$D(p - q) = 2\pi(a - b)$$

and we find, incidentally, that $a - b$ is real and positive.

The Schwarz inequality

$$|D(f, p-q)|^2 \leqq D(f)\, D(p-q)$$

now yields

$$4\pi^2 |f'(z_0)|^2 \leqq 2\pi(a-b)\, D(f),$$

and hence $D(f) \leqq \pi$ implies

$$|f'(z_0)| \leqq \sqrt{\frac{a-b}{2}}$$

with equality for

(19) $$f(z) = \frac{p-q}{\sqrt{2(a-b)}}.$$

It is thus proved that

(20) $$M_{\mathfrak{D}}(z_0, \Omega) = \sqrt{\frac{a-b}{2}}.$$

In fact, any standard approximation technique can be used to show that (18) remains valid when $f(z)$ is known to be analytic only in the open region Ω.

We turn now to the class $\mathfrak{S}\mathfrak{E}$. For functions $g(z)$ which are analytic in the closed region Ω except for a simple pole at z_0 we introduce the integral

$$I(g) = \frac{i}{2}\int_\Gamma g(z)\, \overline{dg(z)}.$$

If the pole is missing, $I(g)$ is equal to the Dirichlet integral $D(g)$, and in the presence of a pole it can be used as a substitute for $D(g)$. If g is univalent, $-I(g)$ is the area enclosed by the image of Γ, and if $\frac{1}{g}$ is of class $\mathfrak{S}\mathfrak{E}(\Omega)$ we have hence $I(g) \leqq -\pi$.

The corresponding bilinear integral can again be evaluated by the residue theorem. We find

(21) $$I(g, p+q) = \frac{i}{2}\int_\Gamma g(d\bar{p} + d\bar{q}) = \frac{i}{2}\int_\Gamma g(dp - dq) = -\pi c(a-b),$$

where c is the residue of g at z_0. In particular,

(22) $$I(p+q) = -2\pi(a-b).$$

From the fact that

$$I\left(g - \frac{c}{2}(p+q)\right) = D\left(g - \frac{c}{2}(p+q)\right) \geqq 0$$

we obtain by (21) and (22)

$$I(g) \geqq -\frac{\pi}{2}|c|^2(a-b),$$

and if $I(g) \leqq -\pi$ the inequality

$$\frac{1}{|c|} \leqq \sqrt{\frac{a-b}{2}}$$

follows, with equality for the function

$$g = \frac{p+q}{\sqrt{2(a-b)}}.$$

The relation

$$M_{\mathfrak{S}\mathfrak{E}}(z_0, \Omega) = \sqrt{\frac{a-b}{2}} \tag{23}$$

will hence be proved if we can show that the function $p+q$ is univalent. We shall then have found identical representations (20) and (23) of $M_{\mathfrak{D}}$ and $M_{\mathfrak{S}\mathfrak{E}}$.

In order to investigate the nature of the function $p+q$ we observe that $\frac{dq}{dp}$ is purely imaginary on Γ with two simple zeros and two simple poles on each contour. Then $\Re\frac{dq}{dp}$ cannot vanish at any interior point, for a level curve $\Re\frac{dq}{dp} = 0$ would have to pass through a pole and there are no such curves besides the contours. Since $\Re\frac{dq}{dp} = 1$ at z_0 we conclude that $\Re\frac{dq}{dp} > 0$ throughout the region. This implies that $\Im\frac{dq}{dp}$ decreases along each contour, and hence

$$\arg(dp + dq) = \operatorname{arc\,tg}\left(\Im\frac{dq}{dp}\right)$$

is also decreasing with the total variation -2π. We conclude that each contour is mapped on a convex curve, and a standard argument shows that $p+q$ is univalent. Our proof of the relation

$$M_{\mathfrak{D}}(z_0, \Omega) = M_{\mathfrak{S}\mathfrak{E}}(z_0, \Omega)$$

is now complete.

Since $M_{\mathfrak{S}\mathfrak{E}}(z_0, \Omega)$ is evidently $\leqq M_{\mathfrak{E}}(z_0, \Omega)$ we have also proved, in conjunction with (17), the inequality

$$M_{\mathfrak{D}}(z_0, \Omega) \leqq M_{\mathfrak{B}}(z_0, \Omega).$$

In the introduction we have already remarked that $M_{\mathfrak{S}\mathfrak{E}} = 0$ only if $\mathfrak{S}\mathfrak{E}$ contains only the constant zero, and this property is of course independent of z_0. We can now conclude that the class $\mathfrak{D}$ enjoys the same property, and we can introduce the identical null-classes $N_{\mathfrak{D}}$ and $N_{\mathfrak{S}\mathfrak{E}}$. The former has previously been considered by Nevanlinna [6] and Sario [9]. The identity of $N_{\mathfrak{D}}$ and $N_{\mathfrak{S}\mathfrak{E}}$ can be expressed more explicitly as follows:

Theorem 4.[1] *A set E is a null-set of class $N_{\mathfrak{D}}$ if and only if every region which is conformally equivalent with the complement of E has a complement of zero area.*

The following theorem is analogous to Theorem 3, and it is proved in the same manner.

Theorem 5.[2] *Every analytic function $f(z)$ with $D(f) < \infty$ in $\Omega' - E$ can be extended to an analytic function in Ω' if and only if E is a null-set of class $N_{\mathfrak{D}}$.*

Corollary. *The value of $M_{\mathfrak{D}}(z_0, \Omega)$ does not change if a null-set of class $N_{\mathfrak{D}}$ is removed from Ω.*

It is easy to show that the relations (20) and (23) remain valid for arbitrary regions Ω if p and q are defined as limits of the corresponding functions for an approximating sequence of regions with analytic boundary. This remark leads to the following characterization of null-sets of class $N_{\mathfrak{D}}$:

Theorem 6.[3] *A set E is a null-set of class $N_{\mathfrak{D}}$ if and only if every univalent function in the complement of E is linear.*

If E is a null-set every univalent function can be extended to a meromorphic function in the whole plane. We may in fact assume that the function has a pole outside of E, and then its Dirichlet integral over a neighbourhood of E is finite. The resulting function has a single pole and is hence linear.

Conversely, if E is not a null-set, p and q cannot both be linear, for then they would be identical and we would have $a - b = 0$.

The considerations of this section are suitably supplemented by a discussion of the quantity

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[1] The necessity was recently pointed out by MYRBERG [5]. There is no record of the sufficient condition.

[2] Stated and proved in SARIO [9].

[3] Stated in NEVANLINNA [6] and proved in SARIO [9].

$$M_{\mathfrak{D}}(z_1, z_2, \Omega) = \sup_{f \in \mathfrak{D}(\Omega)} |f(z_1) - f(z_2)|$$

defined with respect to two points z_1, z_2 in Ω. We prove first:

Theorem 7. *The vanishing of $M_{\mathfrak{D}}(z_1, z_2, \Omega)$ is equivalent with the identical vanishing of $M_{\mathfrak{D}}(z_0, \Omega)$.*

In the first place, if $M_{\mathfrak{D}}(z_0, \Omega) = 0$ the class $\mathfrak{D}(\Omega)$ contains only constant functions and $M_{\mathfrak{D}}(z_1, z_2, \Omega)$ vanishes trivially. The converse can be proved as follows: Let $f(z)$ be univalent in Ω and choose any $z_0 \in \Omega$. The function

$$\frac{f'(z_0)}{f(z) - f(z_0)} - \frac{1}{z - z_0}$$

has a finite Dirichlet integral, and if $M_{\mathfrak{D}}(z_1, z_2, \Omega) = 0$ we must consequently have

$$\frac{f'(z_0)}{f(z_1) - f(z_0)} - \frac{1}{z_1 - z_0} = \frac{f'(z_0)}{f(z_2) - f(z_0)} - \frac{1}{z_2 - z_0}.$$

With z_0 as variable this is a differential equation with linear solutions. Hence all univalent functions are linear, and by Theorem 6 this implies $M_{\mathfrak{D}}(z_0, \Omega) = 0$.

The invariant $M_{\mathfrak{D}}(z_1, z_2, \Omega)$ can be determined explicitly by a method completely analogous to the one used for deriving the relation (20). We assume again that the boundary Γ of Ω is composed by a finite number of analytic curves. It is possible to map Ω by functions $P(z)$ and $Q(z)$ onto regions bounded by concentric and radial slits respectively so that z_1 is mapped into 0 and z_2 into ∞. We may normalize the mappings so that both functions have the residue 1 at z_2, and we set $P'(z_1) = A$, $Q'(z_1) = B$.

The function $\log \frac{P}{Q}$ is analytic and single-valued in Ω. For any regular function $f(z)$ in Ω we obtain

$$D\left(f, \log \frac{P}{Q}\right) = \frac{i}{2} \int_\Gamma f \, d \, \overline{\log \frac{P}{Q}} = -\frac{i}{2} \int_\Gamma f \, d \log PQ = 2\pi (f(z_1) - f(z_2)),$$

and in particular

$$D\left(\log \frac{P}{Q}\right) = 2\pi \log \frac{A}{B}.$$

From this we derive

$$|f(z_2) - f(z_1)|^2 \leq \frac{1}{2\pi} \log \frac{A}{B} \cdot D(f),$$

and hence $D(f) \leq \pi$ implies

$$|f(z_2) - f(z_1)| \leqq \sqrt{\frac{1}{2} \log \frac{A}{B}}$$

with equality for a multiple of $\log \frac{P}{Q}$. It follows that

$$M_{\mathfrak{D}}(z_1, z_2, \Omega) = \sqrt{\frac{1}{2} \log \frac{A}{B}}. \tag{24}$$

The result remains true for an arbitrary region Ω provided we define P and Q as limits of the corresponding slit-functions for a sequence of approximating regions. We conclude that $M_{\mathfrak{D}} = 0$ if and only if the functions are identical.

It could also be proved that $\sqrt{PQ}$ is univalent and maps Ω on a region whose exterior has maximum logarithmic area.

In § 6 we shall give an interesting interpretation of the relation (24) in the case where E lies on the circle $|z| = 1$.

§ 4. The Invariants $M_{\mathfrak{S}\mathfrak{D}}$ and $M_{\mathfrak{S}\mathfrak{B}}$.

The equality of $M_{\mathfrak{S}\mathfrak{D}}$ and $M_{\mathfrak{S}\mathfrak{B}}$ will result from comparison with a third invariant, defined by means of extremal lengths. An account of the theory of extremal lengths is under preparation, but since it cannot yet be referred to we shall list below the definition and main properties of this notion.

Let $\{\gamma\}$ denote a family of rectifiable curves in a region Ω. Consider the class of non-negative functions $\varrho(z)$ in Ω for which the quantities

$$L_\varrho\{\gamma\} = \inf_{\gamma} \int_{\gamma} \varrho\,|dz|$$

$$A_\varrho(\Omega) = \iint_{\Omega} \varrho^2\,dx\,dy$$

are defined and not simultaneously 0 or ∞. The least upper bound

$$\lambda\{\gamma\} = \sup_{\varrho} \frac{L_\varrho\{\gamma\}^2}{A_\varrho(\Omega)}$$

with respect to this class is called the *extremal length* of the family $\{\gamma\}$. The value of $\lambda\{\gamma\}$ does not depend on the region Ω, but very frequently the family $\{\gamma\}$ will be defined with reference to a specific Ω.

It is easy to see that $\lambda\{\gamma\}$ is a conformal invariant in the sense that any conformal one to one mapping of Ω will transform $\{\gamma\}$ into a family $\{\gamma'\}$ with $\lambda\{\gamma'\} = \lambda\{\gamma\}$.

The following properties are immediate consequences of the definition:

Lemma 1. *If two families $\{\gamma\}$ and $\{\gamma'\}$ are such that every γ contains a γ', then*

$$\lambda\{\gamma\} \geqq \lambda\{\gamma'\}.$$

Lemma 2. *If the families $\{\gamma_1\}$ and $\{\gamma_2\}$ cover disjoint pointsets, and if a third family $\{\gamma\}$ is such that every γ contains a γ_1 and a γ_2, then*

$$\lambda\{\gamma\} \geqq \lambda\{\gamma_1\} + \lambda\{\gamma_2\}.$$

Lemma 3. *If the families $\{\gamma_1\}$ and $\{\gamma_2\}$ cover disjoint pointsets, and if every γ_1 and γ_2 is contained in a curve γ of a third family $\{\gamma\}$, then*

$$\frac{1}{\lambda\{\gamma\}} \geqq \frac{1}{\lambda\{\gamma_1\}} + \frac{1}{\lambda\{\gamma_2\}}.$$

Lemma 4. *The extremal length of the family of curves which join the sides of length a in a rectangle with the sides a, b is $\frac{b}{a}$.*

Lemma 5. *The extremal length of the family of curves which separate two circles $|z| = r$ and $|z| = R > r$ is equal to $2\pi/\log\frac{R}{r}$.*

In the present connection we shall only consider extremal lengths which are defined in a very special way. Let Ω be a region, z_0 a point of Ω and E_0 a subset of the complement E of Ω. We denote by $\{\gamma\}_r$ the class of simple closed curves in Ω which separate z_0 from E_0 while maintaining a distance $\geqq r$ from z_0. The extremal length $\lambda\{\gamma\}_r$ will tend to zero with r. But if $r' < r$ it follows from Lemmas 3 and 5 that

$$\frac{1}{\lambda\{\gamma\}_{r'}} \geqq \frac{1}{\lambda\{\gamma\}_r} + \frac{1}{2\pi}\log\frac{r}{r'},$$

or

$$\frac{2\pi}{\lambda\{\gamma\}_r} + \log r \leqq \frac{2\pi}{\lambda\{\gamma\}_{r'}} + \log r'.$$

We conclude that

$$\mu(z_0, E_0) = \lim_{r\to 0}\frac{1}{r}e^{-\frac{2\pi}{\lambda\{\gamma\}_r}}$$

exists. The differential element

$$\mu(z_0, E_0)\,|dz_0|$$

is conformally invariant for a proper definition of the transform of E_0.

It follows from Lemma 1 that $\mu(z_0, E_0)$ is a non-decreasing function of the set E_0 and a non-decreasing function of Ω. We shall call $\mu(z_0, E_0)$ the *perimeter* of E_0 with respect to the region Ω and the center z_0. For a circle $|z - z_0| < R$ all subsets of the complement have the perimeter $1/R$.

It is obvious that the value of $\mu(z_0, E_0)$ depends only on the set of components of E which contain points of E_0, and not on the individual points within a component. Thus the perimeter of a single point is equal to the perimeter of the component to which it belongs. The perimeter of a point p is denoted by $\mu(z_0, p)$. For a simply connected region $\mu(z_0, p)$ has only one value, and for $z_0 = \infty$ this value equals the capacity of E. In the general case, $\mu(\infty, E) = \operatorname{cap} E$.

We shall prove:

Theorem 8. *The invariants $M_{\mathfrak{S}\mathfrak{D}}$ and $M_{\mathfrak{S}\mathfrak{B}}$ are both equal to the maximum of $\mu(z_0, p)$ for $p \in E$.*

We suppose first that Ω is bounded by a finite number of analytic contours $\Gamma_1, \ldots, \Gamma_n$. Then $\mu(z_0, p)$ has only n values, one for each component of the complement. There exists a function $f_k(z)$ which maps Ω on a region bounded by the unit circle, corresponding to Γ_k, and $n - 1$ concentric circular slits; we suppose that the center corresponds to z_0. For a region of this sort it is easily proved that the perimeter of the outer contour is exactly 1, regardless of the number and location of the slits. By conformal invariance we have hence

$$\mu(z_0, \Gamma_k) = |f_k'(z_0)| \leqq M_{\mathfrak{S}\mathfrak{B}}(z_0, \Omega)$$

and we have proved that

$$\max \mu(z_0, p) \leqq M_{\mathfrak{S}\mathfrak{B}}(z_0, \Omega).$$

Conversely, suppose that $f(z)$ maps Ω on a subregion of $|w| = 1$ and that $f(z_0) = 0$. The image of Ω has a definite outer contour which corresponds to a Γ_k, and by application of Lemma 1 and conformal invariance we obtain at once

$$|f'(z)| \leqq \mu(z_0, \Gamma_k).$$

Hence

$$M_{\mathfrak{S}\mathfrak{B}}(z_0, \Omega) \leqq \max \mu(z_0, p),$$

and we have proved that $\max \mu(z_0, p) = M_{\mathfrak{S}\mathfrak{B}}(z_0, \Omega)$.

Let us now consider a mapping of Ω by a function of class $\mathfrak{S}\mathfrak{D}$. The image region has again a definite outer contour which we suppose corresponds to Γ_k. We replace Ω by its image Ω_k under $w = f_k(z)$; Ω_k is a unit circle with concentric slits.

For $f \in \mathfrak{S}\mathfrak{D}(\Omega_k)$, set

$$L(r) = \int_{|w|=r} |f'|\,|dw|$$

whenever $|w| = r$ does not contain any slit, and

$$D(r) = \iint_{|w|<r} |f'|^2\, du\, dv \qquad (w = u + iv)$$

for all r. By the Schwarz inequality we have first

$$L(r)^2 \leqq 2\pi r D'(r)$$

for all non-exceptional r. On the other hand, since the image of $|w| < r$ will always have the image of $|w| = r$ as its outer contour, the isoperimetric inequality yields

$$L(r)^2 \geqq 4\pi D(r).$$

Hence

$$\frac{D'(r)}{D(r)} \geqq \frac{2}{r},$$

and integration from r_0 to 1 gives

$$D(r_0) \leqq D(1)\, r_0^2 \leqq \pi r_0^2.$$

Letting r_0 tend to 0 we conclude that $|f'(0)| \leqq 1$. In terms of the original region Ω it is then proved that

$$M_{\mathfrak{S}\mathfrak{D}} \leqq \max |f_k'(z_0)|,$$

and since all the functions $f_k(z)$ are of class $\mathfrak{S}\mathfrak{D}$ we find

$$M_{\mathfrak{S}\mathfrak{D}} = \max |f_k'(z_0)| = M_{\mathfrak{S}\mathfrak{B}}.$$

In the general case we approximate Ω with an increasing sequence of regions Ω_n with analytic boundaries. We write $\mu_n(z_0, p)$ when the invariant is taken with respect to Ω_n. We have trivially

$$\mu(z_0, p) \leqq \mu_n(z_0, p)$$

and hence

$$\sup \mu(z_0, p) \leqq \lim_{n \to 0} \max \mu_n(z_0, p) = M_{\mathfrak{S}\mathfrak{B}}(z_0, \Omega).$$

The opposite inequality can be proved directly. Suppose that $w = f(z)$ with $f(z_0) = 0$ maps Ω on a subregion Ω_w of $|w| < 1$. We can find a sequence of points $w_n = f(z_n)$ which tends towards the infinite component of the complement of Ω_w. Let p be a limit point of the sequence z_n. Then any curve γ which separates z_0 from p has an image which separates 0 from $|w| = 1$, and we conclude immediately that

$$|f'(z_0)| \leqq \mu(z_0, p)$$

and consequently the equation

$$M_{\mathfrak{S}\mathfrak{B}}(z_0, \Omega) = \max \mu(z_0, p)$$

holds for arbitrary Ω. The relation $M_{\mathfrak{S}\mathfrak{B}} = M_{\mathfrak{S}\mathfrak{D}}$ for arbitrary Ω follows of course directly by a limit process.

§ 5. Further Characterization of the Null-sets $N_{\mathfrak{D}}$.

If E_1 and E_2 are disjoint compact sets in or on the boundary of a region Ω, the *extremal distance* $\lambda_\Omega(E_1, E_2)$ between the sets with respect to the region Ω is by definition the extremal length $\lambda\{\gamma\}$ of the family of curves γ which join E_1 and E_2 within Ω.

We stated in Lemma 4 of § 4 that the extremal distance between opposite sides of a rectangle R is equal to the ratio $\frac{a}{b}$ of the sides. Suppose now that a compact set E is removed from R. Then, by Lemma 1 of § 4, the extremal distance between the sides with respect to $R - E$ is known to be $\geqq a/b$. We claim that the sign of equality will hold for all rectangles R if and only if E is a null-set of class $N_{\mathfrak{D}}$.

We may assume that the rectangle R lies symmetrically with respect to the coordinate axis, the sides of length a being parallel to the x-axis. The ratio $\frac{a}{b}$ is the extremal length between the vertical sides. As in § 3 we shall approximate the complement Ω of E by regions Ω_n with analytic boundary, and introduce the functions $p_n(z)$ for $z_0 = \infty$. If E is a null-set of class $N_{\mathfrak{D}}$ we know that $\lim_{n \to \infty} p_n(z) = z$.

For large n $p_n(z)$ will map the perimeter of R on a quadrilateral which differs very little from R. We may hence find a_n and b_n, tending to a and b, such that $|\Re p_n(z)| \leqq \frac{a_n}{2}$ on the vertical sides of R and $|\Im p_n(z)| \geqq \frac{b_n}{2}$ on the horizontal sides. Let the rectangle with sides a_n and b_n be denoted by R_n. Every curve which joins the vertical sides of R_n contains the image of a curve joining the vertical sides of R within $R-E$. By Lemma 1, § 4, we can hence conclude that the extremal distance λ_{R-E} with respect to $R-E$ satisfies

$$\lambda_{R-E} \leqq \frac{a_n}{b_n},$$

and passing to the limit we obtain $\lambda_{R-E} \leqq a/b$. This proves that the extremal distance does not change when a set E of class $N_{\mathfrak{D}}$ is removed.

To prove the converse, assume not only that $\lambda_{R-E} = \frac{a}{b}$, but also that the extremal distance $\tilde{\lambda}_{R-E}$ between the horizontal sides of R has the value $\frac{b}{a}$. Let $\zeta = s(z)$ be an arbitrary univalent mapping of Ω with a pole at ∞. It will transform the perimeter of R into a simple closed curve whose interior can be mapped in turn by a function $w = \phi(z)$ onto a rectangle R' of dimensions a', b'. Conformal invariance and Lemma 1, § 4, lead to opposite inequalities from which we conclude that

$$\frac{a'}{b'} = \frac{a}{b}.$$

Choose $\varrho = \left|\frac{dw}{dz}\right|$ in $R-E$. For every curve γ which joins the vertical sides of R within $R-E$ we shall then have

$$\int_{\gamma} \varrho\, |dz| \geqq a'.$$

By the definition of λ_{R-E} the rectangle R must hence be mapped onto an area $\geqq$

$$a'^2/\lambda_{R-E} = a'b'.$$

This means that the map of $R-E$ will fill out all of R' except for a set of measure zero, and since the derivative $|\phi'(z)|$ is bounded away from zero on the image of $R-E$, it follows that $s(z)$ must map Ω onto a region whose complement is of zero measure. This is true for an arbitrary univalent mapping, hence for the mapping $p+q$ of § 3, and hence E is of class $N_{\mathfrak{D}}$.

The proof could be modified so as to apply to arbitrary quadrilaterals, and in fact to arbitrary extremal distances. We shall therefore announce our result in the following form:

Theorem 9.[1] *A set E is a null-set of class $N_{\mathfrak{D}}$ if and only if the removal of E does not change extremal distances.*

We have assumed, so far, that E is contained in the open rectangle R. The result remains of course true when the intersection of R with an arbitrary E of class $N_{\mathfrak{D}}$ is removed, although the proof is not so trivial as it might seem. Let R' be a concentric rectangle with sides $a' > a$ and $b' < b$. Since E is totally disconnected, it is possible to find a curvilinear quadrilateral R'' which is contained in the rectangle with sides a', b and contains the rectangle with sides a, b', and whose perimeter does not meet E. It encloses a compact subset E'' of E, and we have hence $\lambda_{R''-E''} = \lambda_{R''}$. On the other hand, by two applications of Lemma 1, § 4, we obtain

$$\lambda_{R-E} \leqq \lambda_{R''-E''} = \lambda_{R''} \leqq \lambda_{R'} = a'/b'$$

and since a'/b' can be chosen arbitrarily near to a/b we find $\lambda_{R-E} \leqq a/b$ as desired.

We remark also that the other half of Theorem 8 has been proved in slightly stronger form, for we have shown that E is of class $N_{\mathfrak{D}}$ as soon as two particular extremal distances are unchanged. It can be proved in a trivial manner that $\lambda_{R-E} = a/b$ if the projection of E on the vertical sides is of mea sure zero. This accounts for the sufficient condition in

Theorem 10. *A set E is of class $N_{\mathfrak{D}}$ if its projections in two orthogonal directions are of linear measure zero. On the other hand, if E is of class $N_{\mathfrak{D}}$ any two points in the complement Ω can be joined by a curve in Ω whose length differs arbitrarily little from the distance between the points.*

The necessary condition is easily proved. If two points have a distance in Ω which is superior to their distance in the plane, it is clear that a thin rectangle R can be constructed such that the distance of two sides is greater in $R-E$ than in R. This implies $\lambda_{R-E} > \lambda_R$, and hence E cannot be of class $N_{\mathfrak{D}}$.

[1] A related theorem in different terminology and connection is found in Grötzsch [3].

§ 6. **Linear Sets.**

In this section we shall always choose $z_0 = \infty$. We can then think of the invariants $M_{\mathfrak{B}}$, $M_{\mathfrak{D}}$ and $M_{\mathfrak{S}\mathfrak{B}}$ as functions of a compact set E which does not divide the plane.

There is a classical relation between $M_{\mathfrak{B}}$ and the linear measure of the set E. More precisely, we shall denote by Λ the greatest lower bound of the total length of a system of closed curves γ which separate E from ∞, and we shall prove that

$$M_{\mathfrak{B}} \leqq \frac{1}{2\pi} \cdot \Lambda . \tag{25}$$

This is an immediate consequence of Cauchy's theorem. If $f(z) = \frac{c}{z} + \cdots$ is regular and of absolute value $\leqq 1$ in Ω we have indeed

$$|c| \leqq \frac{1}{2\pi} \int\limits_{\gamma} |f(z)|\, |dz| \leqq \frac{1}{2\pi} \int\limits_{\gamma} |dz|,$$

and the relation (25) follows at once.

We shall consider separately the case where E lies on a straight line, for instance on the real axis. If the linear measure of E is L we have $\Lambda = 2L$ and (25) implies

$$M_{\mathfrak{B}} \leqq \frac{1}{\pi} L . \tag{26}$$

An inequality in the opposite direction is obtained by considering the function

$$f(z) = \int\limits_{E} \frac{dx}{z - x} = \frac{L}{z} + \cdots .$$

It is immediately seen that $|\Im f(z)| < \pi$, and the function

$$\frac{e^{\frac{f(z)}{2}} - 1}{e^{\frac{f(z)}{2}} + 1}$$

is hence of class $\mathfrak{B}$ with the first coefficient $\frac{L}{4}$. We have thus

$$M_{\mathfrak{B}} \geqq \frac{L}{4} . \tag{27}$$

In particular, we may conclude:

Theorem 11.[1] *A linear set is of class $N_{\mathfrak{B}}$ if and only if it is of linear measure zero.*

More generally, we may consider a set E on an analytic curve γ. We can still prove:

Theorem 11'. *A set E on an analytic curve is of class $N_{\mathfrak{B}}$ if and only if it is of length zero.*

This is proved by showing that every bounded function which is analytic in a region $\Omega' - E$, where Ω' is an open neighborhood of E, is analytic in Ω' if and only if E is of linear measure zero. But it is clearly sufficient to prove the corresponding local statement, which follows from the fact that every point on γ has a neighborhood which can be mapped conformally so that γ will correspond to a segment of the real axis. In order to apply Theorem 3 it is necessary to choose the neighborhood so that its boundary does not intersect E. If E is totally disconnected this is always possible, and if E contains an arc neither $M_{\mathfrak{B}}$ nor the linear measure can be zero.

Let us now find a bound for $M_{\mathfrak{D}}$ when E is a compact set on the real axis and has given length L. This problem is not quite easy and needs some preparations. Consider first a function $f(z)$ of the form

$$f(z) = \int_{-\infty}^{\infty} \frac{\varphi(t)}{z - t}\, dt$$

where $\varphi(t)$ is of summable square and vanishes outside E. By an application of the Fourier integral and the Parseval relation, we find this relation for $D(f)$

$$D(f) = \pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\varphi(s) - \varphi(t)|^2}{(s-t)^2}\, ds\, dt \equiv H(\varphi)$$

which holds whether both sides are finite or infinite. Conversely, if $D(f)$ is finite, $f(z)$ is generated by a function φ with $H(\varphi) = D(f)$.

Since $H(\varphi) \geq H(|\varphi|)$, we may conclude that the extremal function f of the class $\mathfrak{D}(\Omega)$ is generated by a φ which is real and ≥ 0 on E. Let now φ^* be

[1] This theorem is due to DENJOY [2].

the even, symmetrically decreasing and equi-measurable function to φ, and set $k_n(s) = \mathrm{Min}\,(\pi s^{-2}, n)$. Thus $H(\varphi)$ is the limit as $n \to \infty$ of the expression

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}((\varphi^2(s) + \varphi^2(t))\,k_n(s-t)\,ds\,dt - 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(s)\,\varphi(t)\,k_n(s-t)\,ds\,dt = A_n(\varphi) - B_n(\varphi).$$

Obviously $A_n(\varphi) = A_n(\varphi^*)$, while $B_n(\varphi) \leq B_n(\varphi^*)$ according to a rearrangement theorem due to Hardy, Littlewood and Polya (see e. g. Inequalities, Cambridge 1934, Theorem 380). Thus $H(\varphi) \geq H(\varphi^*)$ and $D(f) \geq D(f^*)$ follows, where f^* is the function generated by φ^*. Since φ^* vanishes outside the segment E^* limited by the points $\pm L/2$, the function f^* must be holomorphic outside E^* and we conclude that $M_{\mathfrak{D}}(E) \leq M_{\mathfrak{D}}(E^*) = \frac{L}{4}\cdot$

Theorem 12. *For a linear set of length L we have the string of inequalities*

$$M_{\mathfrak{S}\mathfrak{B}} \leqq M_{\mathfrak{D}} \leqq \frac{L}{4} \leqq M_{\mathfrak{B}} \leqq \frac{L}{\pi}\cdot \tag{28}$$

It is interesting to note that $M_{\mathfrak{B}}$ is smallest while $M_{\mathfrak{S}\mathfrak{B}}$ and $M_{\mathfrak{D}}$ are largest when E consists of a single segment. In the next section we shall show that there is no lower bound for $M_{\mathfrak{S}\mathfrak{B}}$ or $M_{\mathfrak{D}}$ in terms of L.

Theorem 13. *For linear sets $M_{\mathfrak{S}\mathfrak{B}}$ and $M_{\mathfrak{D}}$ are simultaneously positive or* $= 0$.

According to Theorem 8, the perimeter μ vanishes for every boundary point of Ω if $M_{\mathfrak{S}\mathfrak{B}} = 0$. This property is obviously invariant under schlicht mappings of Ω onto Ω' and thus implies that the complement of Ω' is always totally disconnected. If in particular E is linear, both slit functions p and q must degenerate. Thus $a = b = 0$ and $M_{\mathfrak{D}} = 0$ follows.

We shall now give a more precise characterization of linear sets of class $N_{\mathfrak{D}}$. This is most easily done for sets E which lie on the unit circle $|z| = 1$.

We begin by supposing that E consists of a finite number of closed arcs α_i. The complement of E on the circle is denoted by E' and consists of open arcs β_i. According to formula (24) of § 3 the invariant $M_{\mathfrak{D}}(0, \infty, \Omega)$ is deter mined by the functions $P(z)$ and $Q(z)$ introduced in that section. Obviously $P(z) \equiv z$, making $A = 1$, while $Q(z)$ must satisfy the relation

$$\bar{Q}\left(\frac{1}{\bar{z}}\right) = \frac{B}{Q(z)}$$

from which it follows that $|Q(z)| = B^{\frac{1}{2}}$ on the arcs β_i. On the other hand, we know also that $\frac{\partial}{\partial n} \log |Q(z)| = 0$ on α_i, and for this reason $B^{-\frac{1}{2}} Q(z)$ can be continued from $|z| > 1$ across the α_i to a function $Q_1(z)$, defined and single-valued outside of the arcs β_i, which satisfies

$$\overline{Q}_1\left(\frac{1}{\bar z}\right) = Q_1(z).$$

We conclude that the function

$$V(z) = \tfrac{1}{2}\left(-\log |z\, Q_1(z)| - \tfrac{1}{2}\log B\right)$$

is regular and harmonic outside of E' except for a logarithmic pole at ∞ which is such that $V(z) + \log |z|$ vanishes for $z = \infty$. $V(z)$ is then the equilibrium potential of E' with the constant value $-\frac{1}{4}\log B$ on the set. The capacity of E' is hence $B^{\frac{1}{4}}$ and it follows from (24) that

$$M_{\mathfrak{D}}(0, \infty, \Omega) = \sqrt{2 \log \frac{1}{\operatorname{cap} E'}}. \tag{29}$$

This result can immediately be carried over to the case of an arbitrary closed set E on the circle. The complement E' has then an *inner* capacity, defined as the least upper bound of the capacities of closed subsets of E'. It follows by a trivial limiting process that (29) remains valid, provided that cap E' is interpreted as the inner capacity.

From (29) we derive the following criterion:

Theorem 14.[1] *A closed set E on the unit circle is of class $N_{\mathfrak{D}}$ if and only if the inner capacity of its complement is equal to* 1.

It will be noted, of course, that this does not imply that the set E is of zero capacity.

There is a more general theorem whose proof we shall omit.

Theorem 14′. *A closed set on an analytic arc is of class $N_{\mathfrak{D}}$ if and only if the inner capacity of its complement is equal to the capacity of the arc.*

§ 7. Special Sets.

In order to show that the classes $N_{\mathfrak{S}\mathfrak{B}}$, $N_{\mathfrak{D}}$ and $N_{\mathfrak{B}}$ are all distinct we must exhibit a set which is in $N_{\mathfrak{S}\mathfrak{B}}$ but not in $N_{\mathfrak{D}}$ and a set in $N_{\mathfrak{D}}$ which is not in $N_{\mathfrak{B}}$.

[1] Certain results in de Possel [8] are related to this theorem.

We shall also show that there are linear sets of positive measure in $N_{\mathfrak{SB}}$ and $N_{\mathfrak{D}}$. In most of the cases the examples will be generalized Cantor sets.

Let $\{q_i\}_1^\infty$ be a sequence of real numbers $0 < q_i < 1$ and $\{n_i\}_1^\infty$ a sequence of positive integers. We shall construct a corresponding linear Cantor set $E(\{q_i\}, \{n_i\})$ as a closed subset of the unit interval E_0: $0 \leqq t \leqq 1$. The first step is to divide E_0 in $2n_1 + 1$ subintervals, the odd ones of length $a_1 = q_1/(n_1+1)$ and the even ones of length $b_1 = (1-q_1)/n_1$; for simplicity they will be referred to as a_1-intervals and b_1-intervals, and the union of the closed a_1-intervals is denoted by E_1. In the next step each a_1-interval will be subdivided in $2n_2 + 1$ alternating a_2- and b_2-intervals of length a_2 and b_2 respectively. These lengths are chosen so that the a_2-intervals cover a proportion q_2 of the a_1-intervals, and the union of all a_2-intervals is denoted by E_2. The process is repeated and we obtain a nested sequence of sets $E_1 > E_2 > \cdots$ whose product $E = E_1 E_2 \ldots$ is the Cantor set $E(\{q_i\}, \{n_i\})$ which we set out to define. The length of E is $\prod_1^\infty q_i$. It is positive if and only if $\sum_1^\infty (1-q_i) < \infty$.

We shall first derive a sufficient condition for E to be a null-set of class $N_{\mathfrak{SB}}$. By Lemma 3 and 5 of § 4 this will be the case if each point of E can be surrounded by a sequence of disjoint annuli c_ν which do not meet E and whose decreasing radii r''_ν and r'_ν satisfy the condition

$$\sum_1^\infty \log \frac{r''_\nu}{r'_\nu} = \infty. \tag{30}$$

Let us fix our attention on a point $t \in E$. It belongs for each k to a certain a_k-interval which we shall denote by $a_k(t)$. We surround t by annuli centered at the midpoint of $a_k(t)$ which pass through the b_k-intervals contained in $a_{k-1}(t)$; some of these may intersect the real axis in only one b_k-interval. In order to make sure that the annuli do not meet E and are all disjoint we agree to include only those annuli whose inner radius is at least equal to $a_k + b_k$ while the outer radius is at most equal to b_{k-1}. It is clear that such an annulus cannot intersect any a_{k-1}-interval other than $a_{k-1}(t)$ and hence cannot meet E. Moreover, an annulus of the $k+1$:st generation cannot meet an annulus of the k:th generation, for a common point would at once be at a distance $\geqq a_k + b_k$ from the center of $a_k(t)$ and at a distance $\leqq b_k$ from the center of $a_{k+1}(t)$ which is impossible since the two centers have a mutual distance $< a_k$.

The smallest annulus of the k:th generation which can satisfy the imposed conditions has radii $\frac{3}{2} a_k + b_k$ and $\frac{3}{2} a_k + 2 b_k$, and these radii are increased by $a_k + b_k$ at each step. The number ν_k of the last permissible annulus is therefore determined by the condition

$$\frac{a_k}{2} + b_k + \nu_k (a_k + b_k) \leqq b_{k-1}.$$

But $a_k + b_k < a_{k-1}/n_k$ and $b_{k-1} > \frac{1 - q_{k-1}}{q_{k-1}} a_{k-1} > (1 - q_{k-1}) a_{k-1}$. It is therefore sufficient to take

$$\nu_k = [n_k (1 - q_{k-1})] - 1$$

whenever this number is positive. We note, moreover, that

$$\log \left(\frac{a_k}{2} + b_k + \nu (a_k + b_k)\right) - \log \left(\frac{a_k}{2} + \nu (a_k + b_k)\right)$$

$$= - \log \left(1 - \frac{b_k}{\frac{a_k}{2} + b_k + \nu (a_k + b_k)}\right) >$$

$$> b_k \Big/ \left(\frac{a_k}{2} + b_k + \nu (a_k + b_k)\right) > \frac{1}{\nu + 1} \frac{b_k}{a_k + b_k} > \frac{1 - q_k}{\nu + 1}.$$

Hence the annuli of the k:th generation contribute to the sum (30) an amount greater than

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\nu_k + 1}\right) (1 - q_k),$$

where the factor in front is $> \log \frac{\nu_k + 2}{4}$. The whole contribution is thus greater than

$$(1 - q_k) \log \frac{n_k (1 - q_{k-1})}{4}$$

and we conclude that E is of class $N_{\mathfrak{D}}$ whenever

$$\sum_1^\infty (1 - q_k) \log \frac{n_k (1 - q_{k-1})}{4} \tag{31}$$

diverges. This condition does not contradict the convergence of $\sum\limits_1^\infty (1 - q_k)$, and we have proved:

Theorem 15. *There exists a linear set of positive measure which is a null-set of class* $N_{\mathfrak{S}\mathfrak{B}}$.

The Cartesian product of two identical linear Cantor sets of positive length is a 2-dimensional Cantor set of positive area. As such it will certainly not be of class $N_{\mathfrak{D}}$. However, this set is again of class $N_{\mathfrak{S}\mathfrak{B}}$ whenever the series (31) diverges. The proof is the same as above, except that it is more convenient to replace the circular rings by quadratic frames. In (30) we let r''_ν / r'_ν be the ratio of the outer and inner dimensions of the frames and it is elementary to show that the divergence of the series (30) is still a sufficient condition for the set to be of class $N_{\mathfrak{S}\mathfrak{B}}$.

Theorem 16. *There exists a set of class* $N_{\mathfrak{S}\mathfrak{B}}$ *of positive area, and hence not of class* $N_{\mathfrak{D}}$.

It remains only to construct a linear set of positive measure which is of class $N_{\mathfrak{D}}$. Such a set cannot be of class $N_{\mathfrak{B}}$ and therefore also serves to show that $N_{\mathfrak{B}}$ is a proper subclass of $N_{\mathfrak{D}}$.

To this purpose we shall make use of Theorem 14, and our object is thus to construct a closed set E on the unit circle which is of positive length while the inner capacity of its complement E' is equal to 1.

Let us first observe that the inner capacity of a finite number of open arcs is equal to the capacity of the closed arcs. It is also wellknown that the capacity of an arc of length $4/\lambda$ and radius 1 is $\sin 1/\lambda$. Let now $u(z)$ be the equilibrium potential of the arc

$$E'_1: \qquad |\Theta - \Theta_0| < 2/\lambda \qquad (\mathrm{mod}\ 2\pi).$$

Then $\frac{1}{n} u(z^n)$ is found to be the equilibrium potential of the set

$$E'_n: \qquad |n\,\Theta - \Theta_0| < 2/\lambda \quad (\mathrm{mod}\ 2\pi),$$

from which we conclude that

$$\mathrm{cap}\ E'_n = \sqrt[n]{\sin 1/\lambda} = 1 - \frac{\log\lambda}{n} + O\left(\left(\frac{\log\lambda}{n}\right)^2\right)$$

while the length of E'_n is $4/\lambda$.

This example proves the existence of open sets with arbitrarily small length and with an inner capacity arbitrarily close to 1. Taking a sequence $\{E'_n\}_0^\infty$ of such sets with length $L_n = 4/\lambda_n$ such that

$$\lim_{n=\infty} \frac{\log \lambda_n}{n} = 0, \quad \sum_0^\infty 4/\lambda_n \leq L$$

we find that the union E' of the sequence $\{E'_n\}_0^\infty$ has a length $\leq L$ and an inner capacity $= 1$.

Theorem 17. *There exists, on the circle or on the line, a set of positive linear measure which is of type $N_{\mathfrak{D}}$.*

At the end of his thesis Sario [9] lists a number of unsolved questions. Those which concern plane regions have all been answered in this paper. The questions are stated below in our own terminology.

Can a linear or plane Cantor set of class $N_{\mathfrak{D}}$ have positive linear or areal measure? For the areal measure of plane sets the answer is certainly negative as seen by Theorem 4. For linear sets the answer is affirmative, as implied by the proof of Theorem 16, but it must be noted that we have considered more general Cantor sets.

Are there any totally disconnected pointsets which are not of class $N_{\mathfrak{D}}$? The affirmative answer is trivial for we need only consider a totally disconnected linear pointset of positive measure.

Is the total disconnectedness of a pointset invariant under conformal mappings of the complement? This is not so, for there exist totally disconnected pointsets which are not of class $N_{\mathfrak{S}\mathfrak{B}}$. An example was not given, but it suffices to take a totally disconnected set whose complement has finite area. This implies that the corresponding $M_{\mathfrak{S}\mathfrak{D}} > 0$ and the set is not of class $N_{\mathfrak{S}\mathfrak{B}}$.

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