The Laplacian for domains in hyperbolic space and limit sets of Kleinian groups

by

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1. Introduction and statement of results

Let X^{n+1} denote the real hyperbolic space of dimension $n+1$. We will make use of both the ball and upper half space models of X^{n+1}. The ball model is $B^{n+1}=\{x\in\mathbf{R}^{n+1};\ |x|<1\}$ with the line element $ds^2=4dx^2/(1-|x|^2)$. The upper half space model is $H^{n+1}=\{(x,y);\ x\in\mathbf{R}^n,\ y>0\}$ with the line element $ds^2=(dx^2+dy^2)/y^2$. When we write Δ, ∇ or dV, we are referring to the Laplacian, gradient and volume element, all with respect to the hyperbolic metric. For example in the H^{n+1} coordinates

$$dV=\frac{dx\,dy}{y^{n+1}} \quad\text{and}\quad -\Delta=y^2\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x_1^2}+\ldots+\frac{\partial^2}{\partial x_n^2}\right)-(n-1)\,y\frac{\partial}{\partial y}.$$

Let Ω be an open connected subset of X^{n+1}; we denote by $W^1(\Omega)$ the space of functions

$$W^1(\Omega)=\{f\in L^2(\Omega);\ \nabla f\in L^2(\Omega)\}. \tag{1.1}$$

The quadratic forms H and D on $W^1(\Omega)$ are defined as

$$\begin{aligned} H(f,g)&=\int_\Omega f\bar{g}\,dV,\\ D(f,g)&=\int_\Omega \langle\nabla f,\overline{\nabla g}\rangle\,dV. \end{aligned}\tag{1.2}$$

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The domain of the Neumann or free Laplacian Δ on Ω is determined by the fact that it is the unique selfadjoint operator on $L^2(\Omega)$ whose quadratic form is D. If Ω has a 'nice' boundary, then functions in the domain of the Neumann Laplacian have vanishing normal derivatives on the boundary. Moreover for a domain with a nice boundary a core domain for the Neumann Laplacian consists of smooth functions f with compact support in $\bar{\Omega}$ which satisfy $\partial f/\partial n=0$ on $\partial\Omega$, $\partial/\partial n$ being the unit outer normal derivative.

The spectrum for Δ on $L^2(\Omega)$ is denoted by $\sigma(\Omega)$. We are interested in the dependence of $\sigma(\Omega)$ on Ω. It is clear from (1.2) that $\sigma(\Omega)\subset[0,\infty)$. The bottom of the spectrum, denoted by $\lambda_0(\Omega)$, can be described variationally as follows:

$$\lambda_0(\Omega)=\inf[D(u);\ u\in W^1(\Omega),\ H(u)=1]. \tag{1.3}$$

This formulation of $\lambda_0(\Omega)$ plays a central role in our study. A detailed discussion of forms, the domain of the Neumann Laplacian, etc., as needed in this paper, is given in Section 2.

We are primarily intrested in domains Ω which are convex and bounded by geodesic hyperplanes. Most of the time we will be looking at such domains which have only a finite number of bounding sides, i.e. a convex polyhedron. Such domains are said to be geometrically finite. The hyperplanes are most easily described in the H^{n+1} model. In this case they are either hemispheres of the form $|x-a|^2+y^2=r^2$, $y>0$, or vertical Euclidean hyperplanes. We denote by $\mathbf{F}_m$ the family of nonempty domains Ω bounded by exactly m hyperplanes. Since we are basically interested only in the geometry of the domains, we do not distinguish between domains Ω and Ω' if Ω and Ω' are related by some global isometry of X^{n+1}, that is by a transformation in $G=O(n+1,1)$. This group is generated by inversions in the hyperplanes of X^{n+1}. In working with domains in $\mathbf{F}_m$, it must be kept in mind that quantities such as $\sigma(\Omega)$ and $\lambda_0(\Omega)$ are invariant under the action of G as it acts on $\mathbf{F}_m$.

The nature of the spectrum $\sigma(\Omega)$ for Ω in $\mathbf{F}_m$ (any $m<\infty$) is described in Theorems 2.1 and 2.4, which are slight modifications of Theorem 4.4 in Lax-Phillips [12]. These may be summarized as

THEOREM 1.1. *If* $\Omega\in\mathbf{F}_m$ *and* $\operatorname{vol}(\Omega)=\infty$, *then*

(i) $\sigma(\Omega)$ *is discrete in* $[0,(n/2)^2)$;
(ii) $\sigma(\Omega)$ *is continuous in* $[(n/2)^2,\infty)$.

The condition that Ω have only a finite number of sides in (i) cannot in general be dropped. See, for example, the 'cylinder' in H^{n+1} discussed in Proposition 3.10 or the thesis of C. Epstein [11], which treats the Laplacian for a class of finitely generated groups discovered by T. Jorgensen. Sullivan [21] raised the question of whether the converse to (i) was true; that is, if $\sigma(\Omega)$ is discrete and nonempty in $[0,(n/2)^2)$, then does Ω necessarily have a finite number of sides. In Section 6 we give examples of domains (corresponding to discrete groups) for which the canonical polyhedron has infinitely many sides and for which (i) still holds. In this case the group turns out not to be finitely generated.

From the variational formulation of $\lambda_0(\Omega)$ it is very easy to prove (Proposition 2.12) that the discrete eigenvalues $\lambda_j(\Omega)$ of Δ vary monotonically with Ω. However, contrary to what one might expect, we find that $\lambda_j(\Omega) \geqslant \lambda_j(\Omega')$ when $\Omega \supset \Omega'$.

We call a domain Ω free if $\lambda_0(\Omega)=(n/2)^2$. In terms of the form

$$E=D-(n/2)^2 H, \tag{1.4}$$

defined on $W^1(\Omega)$, it is clear that Ω is free iff $E \geqslant 0$. It follows from Theorem 1.1 that $\Omega \in \mathbf{F}_m$ is free iff $\sigma(\Omega)$ has no discrete spectrum—the name free corresponds to the fact that Δ is free of L^2 eigenfunctions. Free domains are the basic building blocks in this paper. The following result, proved in Section 3, plays a central role—it asserts that a domain is free if the number of its bounding sides is sufficiently small.

THEOREM 3.7 (and Proposition 3.5). *If Ω in X^{n+1} belongs to $\mathbf{F}_m$ with $m \leqslant [(n+4)/2]$, then Ω is free, while if $m > [n+4)/2]$ then Ω need not be free. Here $[c]$ denotes the greatest integer in c.*

There are other measures besides the number of sides which ensure that a domain is free. These show that no matter how large m, there are still many Ω in $\mathbf{F}_m$ which are free. An example of such a measure is $\tau(\Omega)$ in Theorem 5.6.

When Ω is free and hence $E \geqslant 0$, we introduce new forms K and G in $W^1(\Omega)$ (see also Lax-Phillips [12]) defined as follows:

$$K(f,g)=\int_S f\bar{g}\, dV, \tag{1.5}$$

where S is any compact subset of Ω, and

$$G=E+K. \tag{1.6}$$

We denote the completion of $W^1(\Omega)$ with respect to the G form by $\mathbf{H}_G$. It is possible for

$\Delta' = \Delta - (n/2)^2$ on $\mathbf{H}_G$ to have 0 in its discrete spectrum. The corresponding eigenfunction v is called a null vector and satisfies the condition $E(v)=0$; however it cannot lie in $W^1(\Omega)$.

Examples of free domains with null vectors are:

(i) If P is a finite sided bounded Euclidean polyhedron in $\mathbf{R}^n$ and $\Omega = \{(x, y); x \in P\}$, then Ω is free and has a null vector $v = y^{n/2}$.

(ii) Let C_1, C_2, C_3 be three mutually tangent hemispheres, each in the exterior of the other two, and let Ω be the domain in H^3 which is exterior to these hemispheres. Then Ω has a null vector (see Corollary 3.3).

A free domain with a null vector is very close to having an L^2 eigenfunction. More precisely, if Ω is such a domain and Ω' is obtained from Ω by excising a small sphere at infinity, then (as proved in Theorem 2.10) Ω' is no longer free and hence has an L^2 eigenfunction.

In order to tie this study in with the Hausdorff dimension of limit sets of Kleinian groups, we need to recall some recent work of Sullivan. Let Γ be a discrete subgroup of G. It has a discontinuous action on X^{n+1}; suppose that Ω is the fundamental domain for this action. The Laplacian leaves invariant the space of Γ-automorphic functions, i.e. the space of functions on X^{n+1} satisfying $f(\gamma w) = f(w)$ for all $\gamma \in \Gamma$ and $w \in X^{n+1}$. It also defines a selfadjoint operator on the Hilbert space $L^2(X^{n+1}/\Gamma)$. We denote by $\lambda_0(\Gamma) \leq \lambda_1(\Gamma) \leq \ldots$ the discrete spectrum (if it exists, otherwise we use the variational notation (1.3) for $\lambda_0(\Gamma)$) of this operator. It is easily seen from the variational definition of $\lambda_j(\Omega)$ (which corresponds to free boundary conditions) that

$$\lambda_j(\Omega) \leq \lambda_j(\Gamma), \tag{1.7}$$

where here Ω is a fundamental domain for Γ (see Proposition 5.1).

If the domain Ω is such that the reflections in its bounding hyperplanes generate a discrete group Γ, then we call Ω a reflection domain and Γ a reflection group. In this case Ω is a fundamental domain for Γ and $\lambda_j(\Omega) = \lambda_j(\Gamma)$. See Section 5 for a more detailed discussion of these points. If Ω is bounded by nonoverlapping hyperplanes, then the reflections form a discrete group, in this case we call Ω a Schottky domain.

Next suppose that Γ is a discrete group acting on X^{n+1}. The limit set $\Lambda(\Gamma)$ is defined to be the set of limit points in $B = \partial(X^{n+1})$ of any given orbit of Γ, i.e., of $\{\gamma w: \gamma \in \Gamma\}$, w some fixed point in X^{n+1}, see for example Thurston [23]. Thus $\Lambda(\Gamma)$ is a closed subset of B. Associated to Γ, we introduce two numbers: the exponent of convergence of the Poincaré series, $\delta(\Gamma)$, and the Hausdorff dimensions of the limit set $d(\Lambda)$. The first $\delta(\Gamma)$, is the exponent of convergence of the series

$$\sum_{\gamma\in\Gamma} \exp(-s(z,\gamma w)) \tag{1.8}$$

where z, w are fixed points and (a, b) is the hyperbolic distance from a to b. The group Γ is said to be geometrically finite if Γ has a fundamental domain with a finite number of sides.

The following theorem provides the connection between the quantities $\lambda_0(\Gamma)$, $\delta(\Gamma)$ and $d(\Lambda)$. There are a number of authors involved in proving various aspects and special cases, see Elstrodt [9], Akaza [3], Patterson [15, 16] and Sullivan [21, 22]. Patterson obtained the result quite generally but with certain restrictions on $\delta(\Gamma)$, while Sullivan in the papers quoted above has proved the result in general. We refer to the theorem as the Patterson-Sullivan theorem.

THEOREM (Patterson-Sullivan). (i) *If* $\delta(\Gamma)\geq n/2$ *then* $\lambda_0(\Gamma)=\delta(n-\delta)$.
(ii) *If* Γ *is geometrically finite then* $\delta(\Gamma)=d(\Lambda)$.

Returning to the concept of a null vector the following is proved in Section 5.

THEOREM 5.7. *If* Ω *is a free Schottky domain without cusps, then* Ω *has a null vector iff* $\delta(\Gamma)=n/2$ *where* Γ *is the corresponding reflection group.*

The first part of Section 4 is devoted to the study of the continuity of the discrete spectrum under small perturbations of the domain. It is shown (Theorem 4.2) that in dimension $n=1$ the discrete spectrum is upper semi-continuous under movements of the bounding sides. In Corollary 4.5 we show that the discrete spectrum is also continuous under what we call simple degenerations. Essentially, in such a degeneration we allow sides to degenerate in clusters of no more than $[(n+2)/2]$ sides. Examples are presented of noncontinuity when the degeneration is not simple.

In Sections 5 and 6 we present applications of the theory developed in Sections 2, 3 and 4. In Proposition 5.5 we show that the function $\max\{\delta(\Gamma), n/2\}$ is continuous under simple degenerations of reflection groups. It should be noted that $\delta(\Gamma)$ itself is not continuous under these degenerations.

We call a discrete group Γ a Schottky group if it has a fundamental domain which is a Schottky domain. The main result of Section 5 is the following:

THEOREM 5.4. *For* $n\geq 3$ *there is a number* $d_n<n$ *such that for any Schottky group* Γ *in* H^{n+1}

$$\delta(\Gamma)\leq d_n. \tag{1.9}$$

In particular if Γ is also geometrically finite, then the Hausdorff dimension of $\Lambda(\Gamma)$ satisfies $d(\Lambda(\Gamma))\leq d_n$. Explicit expressions for d_n are derived in Proposition 3.10 where the key lower bound for $\lambda_0(\Gamma)$, which corresponds to (1.9), is derived. The relation (1.9) answers a question raised by Beardon [5] and shows, when $n\geq 3$, that the Hausdorff dimension of the limit set of a group of motions of $\mathbf{R}^n$, generated by inversions in a finite number of disjoint spheres, cannot be made arbitrarily close to n.

In dimension $n=2$ we do not know if $\delta(\Gamma)$ has an upper bound less than 2. At the other end of the range, we know by Theorem 3.7 that $\delta(\Gamma)\leq 1$ for a Schottky group whose domain has three or fewer sides. It is possible with four sides to make the dimension greater than one. This was first proved by Akaza [2], but it also follows easily from our results on null vectors and the excision property (see remark following Corollary 3.4). Beardon [5] has shown that there exist constants $e(m)<2$ such that for any Schottky group of inversions on at most m hemispheres, the Hausdorff dimension of the limit set is at most $e(m)$. Unfortunately his $e(m)$ approaches 2 as m becomes infinite.

At the end of Section 6 we give some numerical calculations of the dimensions of the limit sets, for various Schottky groups. This is done for the groups generated by inversions in the circles of Figure 6.5. These results suggest that for a Schottky group of inversions on four hemispheres $\delta(\Gamma)\leq 1.31$, and for five hemispheres it is ≤ 1.40. The numerical results also suggest that for Schottky groups with fewer than 14 circles $\delta(\Gamma)\leq 1.60$. However, we can show rigorously that for certain examples, with a very large number of circles, one can make $\delta(\Gamma)\geq 1.75$, see Sarnak [19]. Previously Akaza [4] has given examples where $\delta(\Gamma)\geq 1.5$. Nevertheless it seems to us that $\delta(\Gamma)$ cannot be made arbitrarily close to 2 (when $n=2$).

Also included in Section 6 are applications to the examples of Hecke groups. In $\mathbf{H}^2$ consider the groups Γ_μ generated by

$$S: z\to -1/z, \quad T_\mu: z\to z+\mu,$$

$\mu\geq 2$. We prove

THEOREM 6.1. *For $\mu>2$ there is precisely one discrete eigenvalue $\lambda_0(\mu)$ for Γ_μ. As μ ranges from* 2 *to* ∞, *$\lambda_0(\mu)$ increases continuously and strictly monotonically from* 0 *to* 1/4.

For more on the history of this problem especially in the language of Hausdorff dimension see our discussion in Section 6.

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2. Quadratic forms

We shall be mainly interested in the selfadjoint Laplacian Δ defined on $L^2(\Omega)$ with Neumann boundary condition. This means that the domain of this operator consists of the set of all functions u in $W^1(\Omega)$ with square integrable Δu (defined in the weak sense) satisfying the condition

$$H(\Delta u, v) = D(u, v) \tag{2.1}$$

for all v in $W^1(\Omega)$. For smooth functions u, an integration by parts shows that condition (2.1) is equivalent to the vanishing of the normal derivative of u on $\partial\Omega$.

We denote by B the boundary of X^{n+1}. In the ball model B consists of the unit sphere while in the upper half space model B consists of the points $\{(x, y);\ y=0\} \cup \infty$. The following result is implied by Theorem 4.8 of [12].

THEOREM 2.1. *If Ω contains a neighborhood of a point in B, then $[(n/2)^2, \infty)$ belongs to the continuous spectrum of Δ and contains no discrete spectrum of Δ.*

It is clear from (2.1) that the spectrum of Δ is contained in the half-line $\mathbf{R}_+$. The nature of the spectrum in the interval $[0, (n/2)^2)$ is not well understood in general. However if Ω has the finite geometric property, then the spectrum is discrete in this interval. Our proof of this fact, sketched below, follows the argument used by Lax and Phillips (Section 3 of [12]) in their proof of this property for the Laplacian acting on automorphic functions.

With this in mind we introduce the energy form:

$$E(u) = D(u) - (n/2)^2 H(u), \tag{2.2}$$

defined, to begin with, on functions in $W^1(\Omega)$ which vanish near B. As explained in the introduction, Ω is free if and only if $E \geqslant 0$. It is essential for our purposes to define E on a somewhat larger class of functions than $W^1(\Omega)$. If E were positive on $W^1(\Omega)$, we could obtain this extended class of functions by completion with respect to E. Unfortunately E can be indefinite on $W^1(\Omega)$. To compensate for this we construct an auxiliary form K of the kind:

$$K(u) = \int k(x)\, |u(x)|^2\, dV, \tag{2.3}$$

with $k(x) \geqslant 0$ and having the properties:

(1) $G=E+K$ is locally positive definite;

(2) K is compact with respect to G.

We then complete $W^1(\Omega)$ with respect to G, obtaining the space $\mathbf{H}_G$. Functions in $\mathbf{H}_G$ need not be square integrable and the resulting augmented Laplacian (again with Neumann boundary conditions)

$$\Delta' = \Delta - (n/2)^2 \tag{2.4}$$

can have null vectors in $\mathbf{H}_G$. This is not ruled out by Theorem 2.1 since such null vectors do not belong to the domain of Δ as defined above. The null vectors of Δ' play a very useful role in our theory.

We shall make use of the upper half space model H^{n+1} and treat only domains Ω which are bounded by a finite number of (geodesic) hyperplanes. We cover Ω with a finite number of open sets: $U_0, U_1, \ldots, U_m$. These open sets are divided into four classes: (1) U_0 which is bounded away from B and the sides of Ω; (2) U_j's which contain a single cusp of Ω but portions of no sides not bounding this cusp; (3) If one or more sides meet along a geodesic starting on B, then a U_j of this kind will contain a part of the one side or, if two or more meet, a part of the geodesic near B but no portions of sides which do not contain this geodesic; (4) U_j's which have compact closures in H^{n+1} and which contain portions of sides.

Let (φ_j) be a finite partition of unity subordinate to the U's. We may suppose that all of the φ_j's are either identically zero or identically one near a cusp and near ∞. We now set

$$E_j(u) = \int_\Omega \varphi_j \left[y^2 |\partial u|^2 - \left(\frac{n}{2}\right)^2 |u|^2 \right] \frac{dx\,dy}{y^{n+1}}. \tag{2.5}$$

Clearly

$$E = \sum_{j=0}^{m} E_j. \tag{2.6}$$

E_0 can be brought into a more convenient, but no longer invariant, form by an integration by parts. Since this device is used repeatedly throughout the paper, we shall refer to it as

PROPOSITION 2.2. *If $\partial\Omega$ is parallel to the y-axis, then Ω is free. More generally if Ω is bounded above by $\partial_1 \Omega$ and below by $\partial_2 \Omega$, then*

$$\int_\Omega \varphi \frac{|\partial_y u|^2}{y^{n-1}}\,dx\,dy = \int_\Omega \varphi\left[y\left|\partial_y\left(\frac{u}{y^{n/2}}\right)\right|^2+\left(\frac{n}{2}\right)^2\frac{|u|^2}{y^{n+1}}\right]dx\,dy$$
$$-\frac{n}{2}\int_\Omega \varphi_y\frac{|u|^2}{y^n}\,dx\,dy+\frac{n}{2}\int_{\partial\Omega_1}\varphi\frac{|u|^2}{y^n}\,dx-\frac{n}{2}\int_{\partial\Omega_2}\varphi\frac{|u|^2}{y^n}\,dx. \quad (2.7)$$

Proof. Note that

$$\int \varphi y\left|\partial_y\left(\frac{u}{y^{n/2}}\right)\right|^2 dy = \int \varphi\left[\frac{|\partial_y u|^2}{y^{n-1}}+\left(\frac{n}{2}\right)^2\frac{|u|^2}{y^{n+1}}-\frac{n}{2}\frac{\partial_y|u|^2}{y^n}\right]dy. \quad (2.8)$$

An integration by parts gives

$$\int \varphi\frac{\partial_y|u|^2}{y^n}\,dy = \varphi\frac{|u|^2}{y^n}-\int\left[\varphi_y\frac{|u|^2}{y^n}-n\varphi\frac{|u|^2}{y^{n+1}}\right]dy. \quad (2.8)'$$

Combining (2.8) and (2.8)$'$, and integrating with respect to x we get (2.7). If $\partial\Omega$ is parallel to the y-axis, then the boundary integrals in (2.7) vanish. Setting $\varphi\equiv 1$, we get

$$D(u)=\int_\Omega\left[y\left|\partial_y\left(\frac{u}{y^{n/2}}\right)\right|^2+\frac{|\partial_x u|^2}{y^{n-1}}\right]dx\,dy+\left(\frac{n}{2}\right)^2 H(u)\geqslant\left(\frac{n}{2}\right)^2 H(u), \quad (2.9)$$

so that Ω is free.

We apply this proposition to E_0, as given in (2.5). Since u vanishes near B and since φ_0 vanishes on the bounding sides of Ω, the boundary integral disappears and we get

$$E_0(u)=\int \varphi_0\left[y\left|\partial_y\left(\frac{u}{y^{n/2}}\right)\right|^2+\frac{|\partial_x u|^2}{y^{n-1}}\right]dx\,dy-K_0(u), \quad (2.10)$$

where

$$K_0(u)=\frac{n}{2}\int y(\partial_y\varphi_0)\,|u|^2\,dV. \quad (2.10)'$$

In order to treat the E_j's associated with cusps, we first map the cusp into ∞ so that its sides are parallel to the y-axis. We then proceed as above; in the transformed coordinates E_j looks exactly like the right side of (2.10) with φ_0 replaced by φ_j. Similarly for E_j's of type (3) where the support of φ_j contains only portions of sides with a common geodesic, we map the geodesic into a vertical line. All of the sides in the support of φ_j become parallel to the y-axis and we can proceed as before. Finally for E of type (4) where the support of φ_j is compact, we simply set

$$K_j(u) = n^2 \int \varphi_j |u|^2 \, dV. \tag{2.11}$$

It is clear from (2.10), (2.10)′ and (2.11) that G is locally positive. We improve on this by replacing the integrands in (2.10)′ by their absolute values and adjoining to K the integral of u over a compact subset of Ω. We now define

$$K = \sum_{j=0}^{m} k_j. \tag{2.12}$$

It is clear from this construction that convergence in the G norm implies convergence in $W^1_{\text{loc}}(\Omega)$.

It can now be shown, exactly as in Section 3 of [12], that

(1) The form K is compact with respect to G;

(2) Any two partitions of unity of the above kind result in equivalent G forms over $\mathbf{H}_G$;

(3) If to begin with $E \geq 0$ over $W^1(\Omega)$, then the above G form is equivalent over $\mathbf{H}_G$ to

$$G'(u) = E(u) + \int_S |u|^2 \, dV, \tag{2.13}$$

where S is any compact subset of Ω with a nonempty interior.

Properties (2) and (3) are direct consequences of (1).

The next result follows easily from the compactness of K (see Theorem 3.6 of [12]).

LEMMA 2.3. *If K is compact with respect to G, then there is a closed subspace of $\mathbf{H}_G$ of finite codimension on which E is positive.*

THEOREM 2.4. *If Ω has the finite geometric property, then the Laplacian Δ has a discrete spectrum in the interval $[0, (n/2)^2)$ which is nonempty if and only if E takes on negative values.*

Proof. The domain of Δ is contained in $W^1(\Omega)$ which, in turn, is contained in $\mathbf{H}_G$. Hence for u in the domain of Δ we see by (2.1) and (2.2) that

$$E(u) = D(u) - (n/2)^2 H(u) = H(\Delta u, u) - (n/2)^2 H(u). \tag{2.14}$$

It follows that $E(u)<0$ on any eigenspace of Δ in the interval $[0,(n/2)^2)$. Further any subspace of infinite dimensions will have vectors in common with a closed subspace of finite codimension. Since by Lemma 2.3, E will be positive on such a subspace, we see that Δ can have only a finite dimensional (discrete) spectrum in $[0,(n/2)^2)$ and this will be empty if $E\geqslant 0$.

Conversely if Δ has no point spectrum, then according to the first part of this proof, the spectrum of Δ must lie in the interval $[(n/2)^2,\infty)$; that is $H(\Delta u, u)\geqslant(n/2)^2H(u)$ for all u in the domain of Δ. Thus by (2.14), $E(u)\geqslant 0$ on the domain of Δ. Since the domain of Δ is dense in $W^1(\Omega)$ and $W^1(\Omega)$ is dense in $\mathbf{H}_G$, it follows that $E\geqslant 0$ on $\mathbf{H}_G$.

We prove by a similar argument

COROLLARY 2.5. *Suppose that Ω can be written as the union of two disjoint domains, Ω' and Ω'', such that Ω' is free and Ω'' has the finite geometric property. Then the Laplacian over Ω has a discrete spectrum in the interval* $[0,(n/2)^2)$.

Proof. We denote the energy forms for Ω' and Ω'' by E' and E'', respectively. Then

$$E=E'+E''$$

and since by assumption $E'\geqslant 0$, it follows that

$$E''(u|_{\Omega''})\leqslant E(u). \tag{2.15}$$

Note also that the restriction of $W^1(\Omega)$ to Ω'' is contained in $W^1(\Omega'')$. Thus if the eigenspace of Δ in the interval $[0,(n/2)^2)$ were infinite dimensional, then by (2.15) E'' would be strictly negative on the restriction of this subspace to Ω''; i.e. $E''(u|_{\Omega''})<0$ for all nonzero u in this subspace. It is easy to see from this that the restriction of this eigenspace to Ω'' is an infinite dimensional negative subspace. As in the proof of the theorem, this is contrary to the assertion of Lemma 2.3.

We are now in a position to study the null vectors of Δ'.

LEMMA 2.6. *Suppose $E\geqslant 0$ on $\mathbf{H}_G$. Then u is a null vector of Δ' if and only if* $E(u)=0$.

Proof. If $E(u)=0$ then since $E\geqslant 0$, we deduce from the Schwarz inequality that $E(u,v)=0$ for all v in $\mathbf{H}_G$. In particular for v in $C_0^\infty(\Omega)$ we see that

$$E(u,v)=D(u,v)-(n/2)^2H(u,v)=H(u,\Delta'v).$$

Consequenctly $\Delta'u=0$ in the weak sense. Any v in $W^1(\Omega)$, vanishing near B, can be approximated with respect to the H form by $C_0(\Omega)$ functions. Hence we can write

$$H(\Delta' u, v) = 0 = E(u, v)$$

and comparing the extreme members of this relation we see for such v that $H(\Delta u, v) = D(u, v)$. Since this is the weak form of the Neumann boundary condition, this proves that u is a null vector for Δ'. To establish the converse, we reverse the above steps, concluding from $\Delta' u = 0$ that $E(u, v) = 0$ for all v vanishing near B. Since such functions are dense in $\mathbf{H}_G$, this shows that $E(u) = 0$.

We show in Section 5 that null vectors of Δ' of the kind described in Lemma 2.6 are quite common. The next result is well known, but we include a proof for the sake of completeness.

LEMMA 2.7. *If $E \geq 0$ on $\mathbf{H}_G$ and u is a null vector of Δ', then $u > 0$ on Ω.*

Proof. It is clear that if $E(u) = 0$ then the same is true of the absolute value of u; i.e. $E(|u|) = 0$. According to Lemma 2.6, this implies $\Delta'|u| = 0$. But since Δ' is elliptic with real analytic coefficients, $|u|$ would have to be real analytic. This is impossible unless u were of one sign to begin with. Finally if u ever took on the value 0 in Ω, then $v = -u \leq 0$ would have a local maximum at this point while $\Delta v = -(n/2)^2 v \geq 0$. Thus the maximum principle applies, from which we deduce that v is identically zero, a contradiction.

Definition 2.8. A free domain will be called *strictly free* if Δ' has no null vector.

According to Lemma 2.6 when Ω is strictly free, then $E(u) > 0$ for all nonzero u in $\mathbf{H}_G$.

Definition 2.9. We shall say that a domain Ω has the *excision property* if $\lambda_0(\Omega') < \lambda_0(\Omega)$ for any subdomain Ω' obtained from Ω by removing a strictly free domain with the finite geometric property.

THEOREM 2.10. *Suppose that Ω is free, geometrically finite, and that Δ' has a null vector in $\mathbf{H}_G$. Then Ω has the excision property.*

Proof. We write $\Omega = \Omega' \cup \Omega''$, where Ω'' is the excised strictly free domain, and set

$$E = E' + E'',$$

where E' and E'' denote the E forms for Ω' and Ω'', respectively. According to Theorem 2.4, $\lambda_0(\Omega') < (n/2)^2$ if E' takes on negative values. Suppose that u is a null vector for Δ' in $\mathbf{H}_G(\Omega)$. We see by Lemma 2.7 that u does not vanish on Ω''. It is easy to see that the restriction of u to Ω'' belongs to $\mathbf{H}_G(\Omega'')$; in fact, in the construction of G we need only choose a partition of unity for Ω which is compatible with the requirements for a partition of unity for Ω''. In this case if a sequence in $W^1(\Omega)$ approximates u in $\mathbf{H}_G(\Omega)$, then its restriction to Ω'' will approximate the restriction of u to Ω'' in $\mathbf{H}_G(\Omega'')$. Since Ω'' is strictly free, we conclude that $E''(u)>0$. Consequently

$$E'(u) = E(u)-E''(u) = -E''(u) < 0$$

and it follows that $\lambda_0(\Omega')<(n/2)^2$.

PROPOSITION 2.11. *The hemispherical domain and the domain between two concentric hemispheres are both strictly free.*

Proof. (1) The hemispherical domain. Making use of Proposition 2.2, we can write E as

$$E(u) = \int_\Omega \left[y\left|\partial_y\left(\frac{u}{y^{n/2}}\right)\right|^2 + \frac{|\partial_x u|^2}{y^{n-1}}\right] dx\,dy + \frac{n}{2}\int_{\partial\Omega} \frac{|u|^2}{y^n}\,dx, \tag{2.16}$$

and since E is obviously nonnegative, G can be of the form (2.13). Suppose that Δ' had a null vector u on Ω. Then $E(u)=0$ and hence all of the terms on the right in (2.16) vanish. The vanishing of the first term implies that $u=cy^{n/2}$. However the surface integral in the second term does not vanish for u of this form unless $c=0$.

(2) The domain between two concentric hemispheres. It is convenient to use spherical coordinates $(\varrho, \theta, \varphi)$, described in (3.2) with $k=0$. Using the analysis following (3.2) and setting $\alpha=n/2$ and $u=v\sin^{n/2}\theta$, we obtain from (3.7) the expression

$$E(u) = D(u)-(n/2)^2H(u) = \int_\Omega \left[|v_\theta|^2 \sin^2\theta + \text{sum of squares}\right] dV + \left(\frac{n}{2}\right)^2 \int_\Omega |v|^2 \sin\theta\, dV.$$

Obviously $E\geqslant 0$. If $E(u)=0$, then, as before, u has to vanish and hence Ω is strictly free.

Next we prove a simple monotonicity property for λ_0 with respect to domains.

PROPOSITION 2.12. *Suppose Ω_0 and Ω_1 are two domains with $\bar{\Omega}_1\subset\Omega_0$ and set $\Omega_2=\Omega_0\setminus\bar{\Omega}_1$. If Ω_2 is free, then $\lambda_j(\Omega_0)\geqslant\lambda_j(\Omega_1)$ for all j. Furthermore if Ω_0 has the finite geometric property and Ω_1 is not free, then $\lambda_0(\Omega_0)>\lambda_0(\Omega_1)$.*

Proof. Let β_j denote the union of the eigenvalues of Δ over both Ω_1 and Ω_2, that is of the $\lambda_j(\Omega_1)$ and the $\lambda_j(\Omega_2)$, ordered by magnitude. Since $\Omega_0=\Omega_1\cup\Omega_2$, $W^1(\Omega_0)\subset W^1(\Omega_1)+W^1(\Omega_2)$. It therefore follows from the minimax characterization of the discrete Neumann spectrum (see Courant-Hilbert [7]), p. 408, that

$$\lambda_j(\Omega_0) \geqslant \beta_j. \tag{2.17}$$

By assumption Ω_2 is free so that $\lambda_j(\Omega_2)\geq(n/2)^2$. We conclude from this and (2.17) that

$$\lambda_j(\Omega_0)\geq\lambda_j(\Omega_1)$$

for all of the eigenvalues in the interval $[0,(n/2)^2)$. Since this interval contains the entire discrete spectrum (by Theorem 2.1), the first part of the proposition is proved.

To prove the second part, notice that if Ω_1 is not free, then $\lambda_0(\Omega_1)<(n/2)^2$. If $\lambda_0(\Omega_0)=(n/2)^2$, the assertion is obvious; so we may as well assume that $\lambda_0(\Omega_0)<(n/2)^2$. Now

$$\lambda_0(\Omega_0)=\inf[D_0(\varphi)/H_0(\varphi),\ \varphi\in W^1(\Omega_0)]. \tag{2.18}$$

Since Ω_0 is geometrically finite, we may choose φ in (2.18) to be the square integrable eigenfunction of Δ corresponding to $\lambda_0(\Omega_0)$, which exists by Theorem 2.4. In this case $\lambda_0(\Omega_0)=D_0(\varphi)/H_0(\varphi)$. Clearly φ belongs to both $W^1(\Omega_1)$ and $W^1(\Omega_2)$. Consequently

$$a_j=D_j(\varphi)=\int_{\Omega_j}|\nabla\varphi|^2dV \quad\text{and}\quad b_j=H_j(\varphi)=\int_{\Omega_j}|\varphi|^2dV,\quad j=1,2,$$

are well defined. We have

$$\lambda_0(\Omega_0)=\frac{a_1+a_2}{b_1+b_2}.$$

Since φ is real analytic, it cannot vanish on any open set; in particular $b_2>0$. Since Ω_2 is free, $a_2/b_2\geq(n/2)^2$, and hence

$$\left(\frac{n}{2}\right)^2>\lambda_0(\Omega_0)=\frac{a_1+a_2}{b_1+b_2}\geq\frac{a_1+(n/2)^2b_2}{b_1+b_2}. \tag{2.19}$$

Using the extreme members of this relation, we see that

$$(b_1+b_2)\left(\frac{n}{2}\right)^2>a_1+\left(\frac{n}{2}\right)^2b_2.$$

Hence

$$\frac{a_1}{b_1}<\left(\frac{n}{2}\right)^2 \quad\text{and}\quad a_1b_1+a_1b_2<a_1b_1+\left(\frac{n}{2}\right)^2b_1b_2.$$

It follows that

$$\lambda_0(\Omega_0)\geq\frac{a_1+(n/2)^2b_2}{b_1+b_2}>\frac{a_1}{b_1}.$$

By the analogue of (2.18) for Ω_1 instead of Ω_0, we conclude that $\lambda_0(\Omega_0)>\lambda_0(\Omega_1)$. This completes the proof of Proposition 2.12.

A similar argument proves

COROLLARY 2.13. *A disjoint union of free domains is free.*

The above monotonicity argument shows that when we increase a domain by a free domain then the smallest eigenvalue for the new domain is greater than or equal to that of the original domain. Still another variant of this idea is contained in the following:

PROPOSITION 2.14. *If a domain Ω can be subdivided into disjoint parts $\Omega_1, \Omega_2, \ldots$ such that $\lambda_0(\Omega_i)\geqslant c$ for all i, then $\lambda_0(\Omega)\geqslant c$.*

Proof. For any u in $W^1(\Omega)$, we set

$$a_i=\int_{\Omega_i}|\nabla u|^2\,dV \quad\text{and}\quad b_i=\int_{\Omega_i}|u|^2\,dV.$$

By hypothesis $a_i\geqslant cb_i$ so that

$$\frac{D(u)}{H(u)}=\frac{\sum a_i}{\sum b_i}\geqslant c.$$

Since $\lambda_0(\Omega)=\inf D(u)/H(u)$, taken over all u in $W^1(\Omega)$, the assertion of the proposition follows.

3. Lower bounds for $\lambda_0(\Omega)$

In this section we find lower bounds for the smallest eigenvalue of Δ for a variety of domains. In particular we characterize large classes of free domains, some of which have the excision property described in Definition 2.9.

Again we work with the upper half-space model H^{n+1} and with domains Ω having the finite geometric property. The boundary of Ω is made up of a finite number of geodesic hyperplanes; these are either hemispheres with their centers in B or hyperplanes parallel to the y-axis.

Two intersecting subspaces in $\mathbf{R}^n$ will be called orthogonal if any two vectors, one taken from each subspace, which are orthogonal to the intersection are also orthogonal to each other. We call the translate of a k-dimensional subspace a k-flat. A k-flat intersects a sphere orthogonally if and only if it contains the center of the sphere.

Definition 3.1. A set of spheres and hyperplanes in $\mathbf{R}^n$ is said to be k-coplanar if there is a k-flat which intersects all of them orthogonally, or if the set can be brought into such a configuration by an inversion in $\mathbf{R}^n$. We shall also say that a set of geodesic hyperplanes in H^{n+1} are k-coplanar if their intersection with B is a k-coplanar set of spheres and hyperplanes in B.

Suppose now that Ω is the fundamental domain of a discrete subgroup Γ of isometries generated by the set of reflections about the sides of Ω and suppose further that these sides are k-coplanar. Since the action of a reflection leaves invariant any line containing the center of the reflection, each of these reflections leaves invariant any $(k+1)$-dimensional flat intersecting B in the k-flat orthogonal to the sides. It follows that the limit set Λ of Γ is contained in the k-flat and hence that the Hausdorff dimension d of Λ is $\leqslant k$. According to the theorem of Patterson and Sullivan, mentioned in the introduction,

$$\lambda_0(\Omega)=\lambda_0(\Gamma)=\delta(n-\delta)\geqslant k(n-k), \tag{3.1}$$

provided that $\delta \geqslant n/2$.

We now show that the relation (3.1) holds for any domain bounded by a k-coplanar set of hyperplanes. As a by-product we are able to give an explicit construction for a class of domains for which Δ' has a null vector; according to Theorem 2.10 these domains have the excision property. Another consequence of this result is that all domains bounded by $[(n+4)/2]$ or fewer sides are necessarily free; here $[c]$ denotes the greatest integer in c.

THEOREM 3.2. *If Ω is bounded by a k-coplanar set of hyperplanes, then*

$$\lambda_0(\Omega)\geqslant k(n-k). \tag{3.1$'$}$$

However Ω is free if $k=[n/2]$.

Proof. We may suppose that the k-flat orthogonal to the sides of Ω is spanned by the first k coordinate axes. In this case the sides of Ω are either of the form

$$\sum_{i=1}^{k} a_i x_i = b,$$

or hemispheres with centers in the k-flat. In either case these hyperplane sides possess cylindrical symmetry about the k-flat; and the domain Ω can be obtained by rotating a $(k+1)$-dimensional cross-section A about this flat. This suggests that we replace the remaining coordinates by spherical coordinates:

$$\begin{aligned} y &= \varrho \sin\theta, \\ x_n &= \varrho \cos\theta \sin\varphi_1, \\ x_{n-1} &= \varrho \cos\theta \cos\varphi_1 \sin\varphi_2, \\ &\dots \\ x_{k+1} &= \varrho \cos\theta \cos\varphi_1 \dots \cos\varphi_{n-k-1}; \end{aligned} \tag{3.2}$$

the range of θ is $[0, \pi/2]$ if $n-k>1$ and $[0, \pi]$ if $n-k=1$. The parameters for A are $x_1, \dots, x_k$ and ϱ; the parameters for the rotation of A are θ and the φ's. The estimate (3.1)$'$ is obtained from an integration by parts with respect to the θ variable. We set $x' = (x_1, \dots, x_k)$.

We note that the non-Euclidean volume element in terms of the coordinates (3.2) is given by

$$dV = \cos^{n-k-1}\theta \cdot \cos^{n-k-2}\varphi_1 \dots \cos\varphi_{n-k-1} \frac{dx'\, d\varrho\, d\varphi_1 \dots d\varphi_{n-k-1}\, d\theta}{\varrho^{k+1} \sin^{n+1}\theta} \tag{3.3}$$

the H and D forms become

$$H(u) = \int |u|^2 dV, \tag{3.4}$$

$$D(u) = \int [|u_\theta|^2 \sin^2\theta + \text{sums of squares of other derivatives}]\, dV.$$

The essential θ integrations in H and D are

$$\begin{aligned} H &\sim \int |u|^2 \frac{\cos^{n-k-1}\theta}{\sin^{n+1}\theta} d\theta \\ D &\sim \int |u_\theta|^2 \frac{\cos^{n-k-1}\theta}{\sin^{n-1}\theta} d\theta. \end{aligned} \tag{3.5}$$

Setting

$$u = v \sin^\alpha \theta, \tag{3.6}$$

these become

$$H\sim\int|v|^2\frac{\cos^{n-k-1}\theta}{\sin^{n+1-2\alpha}\theta}\,d\theta$$
$$D\sim\int\left[|v_\theta|^2\frac{\cos^{n-k-1}\theta}{\sin^{n-1-2\alpha}\theta}+\alpha^2|v|^2\frac{\cos^{n-k+1}\theta}{\sin^{n+1-2\alpha}\theta}+\alpha\partial_\theta|v|^2\frac{\cos^{n-k}\theta}{\sin^{n-2\alpha}\theta}\right]d\theta. \tag{3.5$'$}$$

We need only consider a dense set of u's; so we may assume that u vanishes near $\theta=0$ (and for $n-k=1$ also near $\theta=\pi$). Integrating the last term in the integrand for D by parts, we get

$$D\sim\int\left[|v_\theta|^2\frac{\cos^{n-k-1}\theta}{\sin^{n-1-2\alpha}\theta}+|v|^2\left(\alpha(n-\alpha)\frac{\cos^{n-k+1}\theta}{\sin^{n+1-2\alpha}\theta}+\alpha(n-k)\frac{\cos^{n-k-1}\theta}{\sin^{n-1-2\alpha}\theta}\right)\right]d\theta. \tag{3.7}$$

Setting $\alpha=k$, D becomes

$$\begin{aligned}D&=\int\left[|v_\theta|^2\sin^{2+2k}\theta+\text{sum of squares of derivatives}\right]dV+k(n-k)H\\&\geqslant k(n-k)H,\end{aligned}\tag{3.8}$$

from which (3.1)$'$ follows. In particular for n even and $k=n/2$, $\lambda_0(\Omega)\geqslant(n/2)^2$ and Ω is free. For n odd and $k=(n-1)/2$ we obtain a slightly better estimate from (3.7) by setting $\alpha=n/2$, namely

$$\begin{aligned}D&=\int\left[|v_\theta|^2\sin^{n+2}\theta+\text{sum of squares}\right]dV+(n/2)^2H+n/4\int|v|^2\sin^{n+2}\theta\,dV\\&\geqslant(n/2)^2H,\end{aligned}\tag{3.8$'$}$$

from which it follows that Ω is free when $k=(n-1)/2$.

COROLLARY 3.3. *If the* $(k+1)$*-dimensional cross sectional area of* A *is finite, then*

$$u=\sin^k\theta \tag{3.9}$$

is the eigenfunction of Δ *corresponding to* $\lambda_0(\Omega)=k(n-k)$ *when* $k>n/2$, *and the null vector for* Δ' *when* $k=n/2$ *and* n *is even.*

Proof. For u defined as in (3.9), the function v, defined in (3.6) with $\alpha=k$, is a constant. It follows from (3.8) that

$$D(u)=k(n-k)\,H(u);$$

and, if $H(u)$ is finite, that u minimizes the ratio D/H. For $k>n/2$, it is easy to see from (3.5)′ that u belongs to $W^1(\Omega)$; so it follows that in this case u is the eigenfunction for Δ corresponding to $\lambda_0(\Omega)=k(n-k)$.

When n is even and $k=n/2$, u is no longer in $W^1(\Omega)$; in fact both $H(u)$ and $D(u)$ are infinite. However Theorem 3.2 does show that $E\geq 0$. According to Lemma 2.6, in order to show that u is a null vector of Δ', it suffices to prove that u belongs to $\mathbf{H}_G$ and that $E(u)=0$. To this end we construct a sequence (u_j) of smooth functions, vanishing near B, such that

(1) $u_j \to u$ in the G-norm,

(2) $E(u_j)\to 0$.

Choose χ in $C^\infty(\mathbf{R})$ so that

$$\chi(s)=\begin{cases}1 & \text{for } s>-1\\ 0 & \text{for } s<-2\end{cases}$$

and set

$$\psi_j(\theta)=\begin{cases}\chi\left(\dfrac{\log\theta}{j}\right) & \text{for } n>2\\[2ex] \chi\left(\dfrac{\log\theta}{j}\right)\chi\left(\dfrac{\log(\pi-\theta)}{j}\right) & \text{for } n=2.\end{cases}$$

The desired approximating sequence is

$$u_j=\psi_j(\theta)\sin^{n/2}\theta.$$

Clearly this sequence belongs to $\mathbf{H}_G$ and if it converges in $\mathbf{H}_G$ then it must converge to u. Since $E\geq 0$ we can choose G as in (2.13). In this case $G(u_j-u_l)=E(u_j-u_l)\leq 2[E(u_j)+E(u_l)]$ for j and l sufficiently large. From (3.8) we see that

$$\begin{aligned}E(u_j)&=|A|\int|\partial_\theta\psi_j|^2\cos^{(n-2)/2}\theta\sin\theta\,d\theta\\ &\leq\frac{c}{j^2}\int\frac{\sin\theta}{\theta^2}d\theta\leq O\left(\frac{1}{j}\right),\end{aligned}$$

since the range of integration is only over the interval $(\exp(-2j),\exp(-j))$. This implies (1) and (2) above.

If we now apply the excision Theorem 2.10, we get

COROLLARY 3.4. *For n even, suppose that the boundary hyperplanes of Ω are $n/2$-coplanar and that the $(n/2+1)$-dimensional cross-sectional area of A is finite. Then* *no subdomain Ω' of Ω, which is obtained by removing an arbitrarily small hemisphere at infinity, is free.*

For example suppose $n=2$ and that Ω lies between two parallel vertical planes and exterior to a hemisphere tangent to both of these planes. In the notation of Theorem 3.2, A can be described as $-1<x_1<1$ and $\varrho>\sqrt{1-x_1^2}$. In this case the bounding hyperplanes of Ω are clearly 1-coplanar and the area of the cross section A is finite. It follows from Corollaries 3.3 and 3.4 that Δ' has a null vector and that Ω has the excision property. This result was first proved by Akaza [2] by estimating the Hausdorff measure of the limit set of the associated Schottky group.

More generally we have

PROPOSITION 3.5. *There exist domains with* $[(n+6)/2]$ *sides which are not free.*

Proof. We begin by constructing a domain satisfying the conditions of Corollary 3.3 with $k=[(n+1)/2]$. Let S denote a $(k+1)$-sided simplex in the unit ball of $\mathbf{R}^k$. Denoting the coordinates of $\mathbf{R}^k$ by $x'=(x_1,\dots,x_k)$, the desired Ω can be described in terms of x' and the coordinates (3.2) as

$$x' \text{ in } S \quad\text{and}\quad |x'|^2+\varrho^2>2. \tag{3.10}$$

Obviously Ω has $(k+2)$ sides. The cross section A is characterized for any fixed θ and φ's by (3.10). It is clear from (3.3) and (3.10) that the area of A is finite.

When n is odd, it follows from Corollary 3.3 that u, defined as in (3.9), is the eigenfunction of Δ corresponding to the eigenvalue

$$\lambda_0(\Omega)=\frac{n+1}{2}\left(n-\frac{n+1}{2}\right)=\frac{n^2-1}{4}.$$

Since $\lambda_0(\Omega)<(n/2)^2$, Ω is not free. When n is even, $k=n/2$ and it follows from Corollary 3.3 that u, defined in (3.9), is a null vector for Δ' in Ω and hence by Corollary 3.4 that Ω has the excision property. If we excise any small hemisphere with center in B we obtain a subdomain with $(n+6)/2$ sides which is no longer free.

We show below that any domain with $[(n+4)/2]$ or fewer sides is free. For this purpose we need a characterization of k-coplanar collections of spheres.

THEOREM 3.6. *For $k+2$ spheres in $\mathbf{R}^{k+1}$, only the following (not mutually exclusive) configurations occur:*

(1) *The spheres are k-coplanar,*

(2) *The spheres have exactly one point in common,*

(3) *The interiors of the spheres have a nonempty intersection.*

So as not to interrupt the flow of ideas we apply this result to prove

THEOREM 3.7. *If Ω is a domain in H^{n+1} with at most* $[(n+4)/2]$ *sides, then Ω is free.*

Proof. We may suppose that the geodesic hyperplanes do not contain ∞. In this case Ω consists of the common part of the exteriors of at most $[(n+4)/2]$ hemispheres, whose intersections with B are S^{n-1} spheres. If these spheres are $[n/2]$-coplanar, then the result follows from Theorem 3.2. This is trivially the case if there are less than $[(n+4)/2]$ spheres. If they are not $[n/2]$-coplanar then, according to Theorem 3.6, they either meet in a single point or their interiors have a nonempty common part.

If the spheres meet in a single point, we map this point, by an inversion, into ∞; the sides of Ω are transformed into hyperplanes parallel to the y-axis. To show that the transformed Ω is free, we proceed as before with an integration by parts; this time we use Proposition 2.2 with $\varphi \equiv 1$. The resulting expression is

$$\begin{aligned} D(u) &= \int_\Omega \left[y \left| \partial_y \left(\frac{u}{y^{n/2}} \right) \right|^2 + \frac{|\partial_x u|^2}{y^{n-1}} \right] dx\, dy + \left(\frac{n}{2} \right)^2 \int_\Omega |u|^2 dV \\ &\geqslant (n/2)^2 H(u). \end{aligned} \tag{3.11}$$

If the interiors of the S^{n-1} spheres in B have a common point, we map this point into ∞ by an inversion. In this case Ω goes into the common part of the interiors of $[(n+4)/2]$ hemispheres in H^{n+1}. Note that if (x, y) belongs to the interior of a hemisphere, then so does $(x, \beta y)$ for all β in the interval $[0, 1]$. Consequently the transformed Ω also has this property. We now perform the same integration by parts as in (3.11); this time the boundary term in (2.7) does not vanish. The result is

$$\begin{aligned} D(u) &= \int_\Omega \left[y \left| \partial_y \left(\frac{u}{y^{n/2}} \right) \right|^2 + \frac{|\partial_x u|^2}{y^{n-1}} \right] dx\, dy + \left(\frac{n}{2} \right)^2 \int_\Omega |u|^2 dV + \frac{n}{2} \int_{\partial\Omega} \frac{|u|^2}{y^n} dx \\ &\geqslant (n/2)^2 H(u); \end{aligned} \tag{3.11$'$}$$

and again Ω is free.

Proof of Theorem 3.6. In the case $k=1$, the theorem deals with three circles in $\mathbf{R}^2$. If two of these circles have no common point, then they can be mapped by an inversion into two concentric circles. It is now clear that the centers of all three of the transformed circles lie on a line and hence that they are 1-coplanar. Next suppose that two of the circles are tangent to each other. Mapping the point of tangency into ∞ (again by inversion), the two tangent circles become parallel lines. If the third circle is transformed into a line, then all three meet at ∞ and only there; this is case (2). If not, the perpendicular to the parallel lines through the center of the third transformed circle establishes the 1-coplanarity of the three circles.

Finally if two of the circles intersect in two distinct points, we map one of these points into ∞. The two intersecting circles become two intersecting lines, which meet at the point Q. Suppose that the third circle transforms into a line. If this line contains Q, then any circle with center at Q intersects the three lines orthogonally and the set is 1-coplanar. If this line does not contain Q, then the three lines meet only at ∞ and the configuration is of type (2). Otherwise the third circle transforms into a circle which either (i) goes through Q, (ii) contains Q in its interior, or (iii) has Q in its exterior. It is clear that (i) corresponds to case (2) and (ii) to case (3). If (iii) occurs, we construct a fourth circle C centered at Q which intersects the transformed third circle orthogonally. Since C obviously meets the intersecting lines orthogonally, this establishes the 1-coplanarity of the given three circles.

We now proceed by induction. Suppose that the result is true for $(k+1)$ spheres in $\mathbf{R}^k$. Then given $(k+2)$ spheres, $S_1^k, \ldots, S_{k+2}^k$, in $\mathbf{R}^{k+1}$ we begin by considering the first $(k+1)$ of them. Their centers span a k-flat F. We set

$$S_i^{k-1} = S_i^k \cap F.$$

If the S_i^{k-1}, $i \leqslant k+1$, are $(k-1)$-coplanar in F, then the S_i^k, $i \leqslant k+1$, will also be $(k-1)$-coplanar in $\mathbf{R}^{k+1}$; this follows from the fact that an inversion about a point in F leaves F invariant. Hence we may suppose that the S_i^k, $i \leqslant k+1$, intersect a $(k-1)$-flat orthogonally. If we now take the flat spanned by this $(k-1)$-flat and the center of S_{k+2}^k, we obtain a k-flat which intersects all $(k+2)$ spheres orthogonally; i.e. the original set of spheres is k-coplanar.

Suppose next that the S_i^{k-1}, $i \leqslant k+1$, are not $(k-1)$-coplanar. Then by the induction hypothesis either (2) or (3) holds. If (2), then the S_i^{k-1}, $i \leqslant k+1$, have exactly one point, say Q, in common. Since the centers of these spheres lie in a k-flat F, the S_i^k, $i \leqslant k+1$, will also have only the point Q in common. If S_{k+2}^k also contains Q, then we are again in case (2) for the entire set of spheres. Otherwise we argue as follows: Take for F the k-dimensional subspace $(x_{k+1}=0)$ of $\mathbf{R}^{k+1}$. Mapping Q into ∞ by an inversion, the S_i^k,

$i \leq k+1$, are transformed into planes P_i, $i \leq k+1$, parallel to the x_{k+1}-axis. Let S_{k+2} denote the transformed S^k_{k+2} and denote its center by $(c_1, \ldots, c_{k+1})$. Then the hyperplane $x_{k+1}=c_{k+1}$ intersects all of the P_i, $i \leq k+1$, as well as S_{k+2} orthogonally and it follows that the S^k_i, $i \leq k+2$, are k-coplanar.

Finally suppose that condition (3) holds for S^{k-1}_i, $i \leq k+1$, but that (1) and (2) do not hold. Then their interiors have a nonempty intersection. The same holds true for the interiors of the S^k_i, $i \leq k+1$. In fact even more is true; we show below that the S^k_i, $i \leq k+1$, intersect in some S^l of dimension $l \geq 0$. Now if S^k_{k+2} meets S^l in a single point then we are in case (2). If it meets S^l at two points, then the line segment joining these two points is interior to all $(k+2)$ spheres so we are in case (3).

If S^k_{k+2} does not meet S^l, we proceed as follows: Map one point of S^l into ∞ by an inversion and let Q denote the transform of another point of S^l. The S^k_i, $i \leq k+1$, map into hyperplanes P_i, $i \leq k+1$, which intersect at Q and S^k_{k+2} maps into a sphere S_{k+2} which does not contain Q. Since one of the sectors eminating from Q is common to the interiors of the P_i's, if S_{k+2} contains Q in its interior we see that the interiors of all of the original $(k+2)$ spheres have a nonempty intersection; so we are in case (3). If Q is exterior to S_{k+2}, then join Q to the center of S_{k+2} by a line L. It is easy to see that all points on L exterior to S_{k+2} are centers for k-spheres intersecting S_{k+2} orthogonally. In particular the k-sphere with center at Q will intersect S_{k+2} and all of the hyperplanes P_i, $i \leq k+1$, orthogonally; so in this case the original set of spheres are k-coplanar. This completes the proof of Theorem 3.6 modulo the following lemma.

LEMMA 3.8. *Suppose the spheres S^k_i, $i \leq k+1$, in $\mathbf{R}^{k+1}$ which are neither $(k-1)$-coplanar nor have exactly one point in common, have interiors which have a nonempty intersection; then they have an S^l, $l \geq 0$, in common.*

Proof. The assertion is obvious for $k=1$ where two such intersecting circles have an S^0 intersection. Suppose it is also true for $k-1$. Let F denote the k-flat containing the centers of all of the S^k spheres. Then the interiors (relative to F) of the lower dimensional spheres

$$S^{k-1}_i = S^k_i \cap F, \quad i \leq k$$

also have a nonempty intersection. If they were $(k-2)$-coplanar, then, as we have seen above, the entire set of $k+1$ spheres would have been $(k-1)$-coplanar. A similar conclusion can be reached if the S^{k-1}_i, $i \leq k$, have exactly one point, say Q, in common.

Since Q will also be the unique point that the S_i^k, $i \leqslant k$, have in common, it follows by assumption that Q does not lie in S_{k+1}^{k-1}. Let F_0 denote the $(k-1)$-flat containing the centers for the S_i^{k-1}, $i \leqslant k$. We may set $F = \mathbf{R}^k$ and $F_0 = (x_k = 0)$. Mapping Q into ∞, the S_i^{k-1}, $i \leqslant k$, go into hyperplanes parallel to the x_k-axis and S_{k+1}^{k-1} goes into another sphere with center, say at $(c_1, \ldots, c_k)$. The hyperplane $x_k = c_k$ is a $(k-1)$-flat orthogonal to all of the S_i^{k-1}, $i \leqslant k+1$. But this again means that the S_i^k, $i \leqslant k+1$, are $(k-1)$-coplanar, contrary to the hypothesis.

Thus the induction hypothesis applies to the set S_i^{k-1}, $i \leqslant k$. They therefore have an S^l, $l \geqslant 0$, in common. The S_i^k, $i \leqslant k$, will intersect in an S^{l+1} such that $S^l = S^{l+1} \cap F$. Suppose now that S_{k+1}^k links two distinct points of S^l. Then S_{k+1}^k and S^{l+1} will intersect and, since only spheres are involved, they will intersect in some S^j, $j \geqslant 0$, as asserted in the lemma.

On the other hand if S_{k+1}^k does not link any two points S^l, then either it contains only one point of S^l or it is disjoint from S^l. Since the centers of all of the given spheres lie in F, the same is true of S_{k+1}^k and S^{l+1}. The hypothesis rules out S_{k+1}^k and S^{l+1} meeting at a single point. There remains to consider only the case where S_{k+1}^k is disjoint from S^{l+1}. This situation is analogous to one treated at the end of the proof of Theorem 3.6 and we may again conclude that the $k+1$ spheres are $(k-1)$-coplanar, which is ruled out by the hypothesis of the lemma. This completes the proof of Lemma 3.8.

According to Theorem 3.7, any domain Ω bounded by $[(n+4)/2]$ or fewer sides is free. We show in Section 5 that regardless of the number of sides, Ω will be free if its sides are 'sufficiently well separated'.

Next we establish lower bounds for $\lambda_0(\Omega)$ for Schottky domains, that is for domains bounded by nonintersecting hyperplanes. Our approach is quite straightforward. We subdivide Ω into disjoint parts $\Omega_0, \Omega_1, \Omega_2, \ldots$, for which the $\lambda_0(\Omega_i)$ have a common lower bound $c > 0$. We then apply Proposition 2.14 to obtain the inequality $\lambda_0(\Omega) \geqslant c$. As one might expect with such a crude method, the results are reasonably sharp only for rather special configurations.

Let $P_1, P_2, \ldots$ denote the geodesic hyperplanes bounding the Schottky domain Ω and set $S_i = P_i \cap B$. We may as well suppose that none of the P_i's contains ∞, in which case the S_i are all $(n-1)$-spheres. We enclose the S_i's in disjoint polyhedra or, in some cases S_i itself, denoting these n-dimensional regions by $T_1, T_2, \ldots$. For $i > 0$ we set

$$\Omega_i = \text{the region above } T_i \text{ common to } \Omega. \tag{3.12}$$

The so defined Ω_i's are disjoint. Finally we set

$$\Omega_0 = \Omega \setminus \bigcup_{i>0} \Omega_i. \tag{3.13}$$

The boundary of Ω_0 is parallel to the y-axis. It therefore follows by Proposition 2.2 that Ω_0 is free. As for the other Ω_i, it is obvious that if we choose each of the T_i equal to S_i, then the Ω_i will be disjoint. Hence if we can find a common lower bound greater than zero for the $\lambda_0(\Omega_i)$ when the T_i are spheres, then this will give us a lower bound for $\lambda_0(\Omega)$. Our next result makes use of such a bound when $n \geq 3$. Since the map $(x, y) \to (\lambda x, \lambda y)$ is an isometry, it follows that $\lambda_0(\Omega_i)$ is independent of the radius of S_i; so we can, without loss of generality, take the radius to be 1.

THEOREM 3.9. *For each $n \geq 3$ there is a $d_n > 0$ such that for any Schottky domain Ω in H^{n+1}, $\lambda_0(\Omega) \geq d_n$.*

Remarks. For $n=1$, that is for hyperbolic two space, there exist Schottky domains of finite area. An example is a triangle with zero angles at each vertex; this is a fundamental domain for a Hecke group (see Section 6). For such a domain $\lambda_0(\Omega)=0$.

For $n=2$, that is H^3, we have not been able to determine whether or not an absolute positive lower bound exists. We will discuss this question again, from a different point of view, in Section 6 where $\lambda_0(\Omega)$ is numerically computed for a number of domains.

It follows from the above discussion that Theorem 3.9 is an immediate consequence of

PROPOSITION 3.10. *Let Ω be the cylindrical domain in $\mathbf{H}^{n+1}$ lying above the unit sphere S; that is*

$$\Omega = \{(x, y);\ |x| < 1,\ |x|^2 + y^2 > 1\}.$$

Then for $n \geq 3$, $\lambda_0(\Omega) \geq d_n$, where

$$d_n = (n-2)^2 \left[3c_n \left\{ 1 + \alpha \left(\left(6 + \frac{4}{n-2} \right) c_n + \frac{12}{n-2} c_n^2 \right) \right\} \right]^{-1},$$
$$c_n = (4/3)^n \quad \text{and} \quad \alpha = \frac{n}{2} - \frac{\sqrt{n^2 - 4(n-2)^2/3c_n}}{2}. \tag{3.14}$$

For $n=2$, $\lambda_0(\Omega)=0$.

Proof. It will be convenient to use cylindrical coordinates in $\mathbf{H}^{n+1}$: (ϱ, θ, y) where $\varrho>0$, $y>0$ and θ parametrizes S. In this case

$$ds^2 = \frac{d\varrho^2+\varrho^2\,d\theta^2+dy^2}{y^2} \quad \text{and} \quad \sqrt{g} = \frac{\varrho^{n-1}}{y^{n+1}} \tag{3.15}$$

the quadratic forms H and D become

$$H(u) = \int_\Omega |u|^2 \frac{\varrho^{n-1}}{y^{n+1}}\, d\varrho\, d\theta\, dy \tag{3.16}$$

and

$$D(u) = \int_\Omega \left(|u_\varrho|^2 + \frac{1}{\varrho^2}|u_\theta|^2 + |u_y|^2\right) \frac{\varrho^{n-1}}{y^{n-1}}\, d\varrho\, d\theta\, dy. \tag{3.17}$$

We note for $n>1$ that

$$\begin{aligned} \operatorname{vol}(\Omega) &= \omega_n \int_0^1 \int_{\sqrt{1-\varrho^2}}^\infty \frac{\varrho^{n-1}}{y^{n+1}}\, dy\, d\varrho \\ &= \frac{\omega_n}{n} \int_0^1 \frac{\varrho^{n-1}}{(1-\varrho^2)^{n/2}}\, d\varrho = \infty; \end{aligned}$$

here ω_n denotes the Euclidean area of S^{n-1}. Consequently the constant function is not in L^2. Now for $n=2$, a simple calculation shows that $u_\varepsilon = y^\varepsilon$ lies in $W^1(\Omega)$ for all $\varepsilon>0$ and that

$$D(u_\varepsilon) = \varepsilon^2 H(u_\varepsilon).$$

Hence in this case $\lambda_0(\Omega)=0$. If 0 were an eigenvalue of Δ with eigenfunction φ, then $D(\varphi)=0$ implies that $\varphi=$constant. Since an eigenfunction is by definition square integrable, this is impossible. Thus when $n=2$, 0 lies in the continuous spectrum of Δ over Ω; this is in contrast with the geometrically finite case where 0 cannot be in the continuous spectrum.

Next we show that for $n\geq 3$, $\lambda_0(\Omega)$ can be effectively bounded from below. We begin by subdividing Ω into two parts:

$$\begin{aligned} \Omega_1 &= \left\{(\varrho, \theta, y) \in \Omega;\ 3/5 < \varrho < 1,\ y \leq \frac{1}{2} + \frac{\sqrt{1-4(\varrho-1)^2}}{2}\right\}, \\ \Omega_2 &= \Omega \setminus \Omega_1. \end{aligned} \tag{3.18}$$

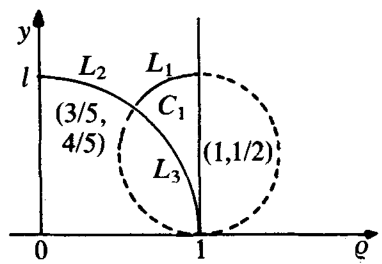


Fig. 3.1

Ω_1 is obtained by revolving in θ the relevant portion C_1 of the disk of radius 1/2 centered at $(1, 1/2)$ shown in Figure 3.1. We denote by L_1 the bounding arc of C_1 extending from $(3/5, 4/5)$ to $(1, 1)$. Next we prove an essential estimate for functions defined on C_1.

LEMMA 3.11. *For u in* $W^1(C_1)$

$$H_1'(u) \leqslant e_n D_1'(u) \tag{3.19$'$}$$

and

$$\int_{L_1} |u|^2 \frac{d\sigma}{y} \leqslant (n-2)\, e_n D_1'(u), \tag{3.20$'$}$$

where

$$\begin{aligned} H_1'(u) &= \int_{C_1} |u|^2 \frac{\varrho^{n-1}}{y^{n+1}}\, d\varrho\, dy, \\ D_1'(u) &= \int_{C_1} (|u_\rho|^2 + |u_y|^2) \frac{\varrho^{n-1}}{y^{n-1}}\, d\varrho\, dy \end{aligned} \tag{3.21$'$}$$

and

$$e_n = \frac{1}{(n-2)^2} \frac{4^n}{3^{n-1}}. \tag{3.22$'$}$$

Proof. We may assume that u vanishes near the cusp at $(1,0)$. We make a (2-dimensional) conformal change of variable:

$$z = \varrho + iy \to \zeta = \xi + i\eta = -1/(z-1);$$

mapping C_1 onto the truncated strip:

$$V = \{(\xi, \eta);\, 0 < \xi < 1/2,\, \eta > 1\}.$$

The inverse map is given by

$$\varrho=\frac{\eta^2+\xi^2-\xi}{\xi^2+\eta^2}, \quad y=\frac{\eta}{\xi^2+\eta^2}. \tag{3.23}$$

Since the transformation is conformal, it is easy to see that H_1' and D_1' become

$$H_1''(u)=\int_V |u|^2\left(\frac{\eta^2+\xi^2-\xi}{\eta}\right)^{n-1}\frac{d\xi\,d\eta}{\eta^2} \tag{3.24}$$

and

$$D_1''(u)=\int_V (|u_\xi|^2+|u_\eta|^2)\left(\frac{\eta^2+\xi^2-\xi}{\eta}\right)^{n-1} d\xi\,d\eta.$$

Fixing ξ, an integration by parts gives

$$\int_1^\infty uu_\eta\eta^{n-2}\,d\eta=-\frac{|u(1)|^2}{2}-\frac{(n-2)}{2}\int_1^\infty |u|^2\eta^{n-3}\,d\eta.$$

Applying the Schwarz inequality to the left member gives

$$\frac{|u(1)|^2}{2}+\frac{n-2}{2}\int_1^\infty |u|^2\eta^{n-3}\,d\eta\leqslant\left[\int_1^\infty |u_\eta|^2\eta^{n-1}\,d\eta\int_1^\infty |u|^2\eta^{n-3}\,d\eta\right]^{1/2}, \tag{3.25}$$

from which it follows that

$$\int_1^\infty |u|^2\eta^{n-3}\,d\eta\leqslant\left(\frac{2}{n-2}\right)^2\int_1^\infty |u_\eta|^2\eta^{n-1}\,d\eta. \tag{3.26}$$

Now for (ξ,η) in V

$$\eta\leqslant\frac{4}{3}\,\frac{\eta^2+\xi^2-\xi}{\eta} \quad\text{and}\quad \xi>\xi^2.$$

Combining this with (3.26), we get

$$\begin{aligned}\int_1^\infty |u|^2\left(\frac{\eta^2+\xi^2-\xi}{\eta}\right)^{n-1}\frac{d\eta}{\eta^2}&\leqslant\int_1^\infty |u|^2\eta^{n-3}\,d\eta\leqslant\left(\frac{2}{n-2}\right)^2\int_1^\infty |u_\eta|^2\eta^{n-1}\,d\eta\\&\leqslant\left(\frac{2}{n-2}\right)^2\left(\frac{4}{3}\right)^{n-1}\int_1^\infty |u_\eta|^2\left(\frac{\eta^2+\xi^2-\xi}{\eta}\right)^{n-1}d\eta.\end{aligned}$$

Finally replacing u_η^2 by $(u_\xi^2+u_\eta^2)$ and integrating the resulting expression with respect to ξ, we obtain

$$H_1''(u) \leqslant e_n D_1''(u), \quad e_n = \frac{3}{(n-2)^2} c_n. \tag{3.19}''$$

Starting with (3.25), an analogous string of inequalities yields

$$\int_0^{1/2} |u(\xi, 1)|^2 \, d\xi \leqslant (n-2)\, e_n D_1''(u). \tag{3.20}''$$

Since $d\xi/\eta = d\sigma/y$ (here $d\sigma^2 = d\varrho^2 + dy^2$) the inequalities (3.19)″ and (3.20)″ transform back into (3.19)′ and (3.20)″, respectively. This concludes the proof of Lemma 3.11.

We now integrate (3.19)′ and (3.20)′ with respect to θ and obtain for v in $W^1(\Omega_1)$

$$\begin{aligned} H_1(v) &\equiv \int_{\Omega_1} |v|^2 \frac{\varrho^{n-1}}{y^{n+1}} \, d\varrho \, d\theta \, dy \\ &\leqslant e_n \int_{\Omega_1} \left(|v_\varrho|^2 + \frac{1}{\varrho^2} |v_\theta|^2 + |v_y|^2 \right) \frac{\varrho^{n-1}}{y^{n-1}} \, d\varrho \, d\theta \, dy \\ &\equiv e_n D_1(v). \end{aligned} \tag{3.19}$$

Since $\varrho/y \leqslant 1$ on L_1, we also get

$$\int_{\partial_1} |v|^2 \frac{\varrho^{n-1}}{y^n} \, d\sigma \, d\theta \leqslant (n-2)\, e_n D_1(v), \tag{3.20}$$

where ∂_1 denotes the surface generated by L_1.

Next we derive an inequality for the analogous forms H_2 and D_2 defined on Ω_2. Again we start with an integration by parts:

$$\begin{aligned} \int y^{2\alpha-n+1} \left| \partial_y \left(\frac{u}{y^\alpha} \right) \right|^2 dy &= \int \left(\frac{u_y^2}{y^{n-1}} + \alpha^2 \frac{u^2}{y^{n+1}} - \alpha \frac{(u^2)_y}{y^n} \right) dy \\ &= \int \left(\frac{u_y^2}{y^{n-1}} + (\alpha^2 - n\alpha) \frac{u^2}{y^{n+1}} \right) dy - \alpha \frac{u^2}{y^n} \Bigg|_{y \in \partial} \end{aligned}$$

here ∂ denotes the surface generated by rotating the arcs L_1 and L_2 depicted in Figure 3.1. Multiplying through by ϱ^{n-1}, rearranging terms and integrating with respect to ϱ and θ, we get

$$D_2(u) \geqslant \int_{\Omega_2} (|u_\varrho|^2 + |u_y|^2) \frac{\varrho^{n-1}}{y^{n-1}} \, d\varrho \, d\theta \, dy$$

$$= \int_{\Omega_2} \left[y^{2\alpha-n+1} \left| \partial_y \left(\frac{u}{y^\alpha} \right) \right|^2 + \frac{|u_\varrho|^2}{y^{n-1}} \right] \varrho^{n-1}\, d\varrho\, d\theta\, dy$$

$$+(n\alpha-\alpha^2) \int_{\Omega_2} |u|^2 \frac{\varrho^{n-1}}{y^{n+1}}\, d\varrho\, d\theta\, dy - \alpha \int_{\partial} |u|^2 \frac{\varrho^{n-1}}{y^n}\, d\varrho\, d\theta.$$

It follows that

$$(n\alpha-\alpha^2)\, H_2(u) \leqslant D_2(u) + \alpha I(u),$$

where

$$I(u) = \int_{\partial} |u|^2 \frac{\varrho^{n-1}}{y^n}\, d\varrho\, d\theta. \tag{3.27}$$

We now choose α so that

$$n\alpha - \alpha^2 = 1/e_n, \tag{3.28}$$

that is we take

$$\alpha = \frac{n}{2} - \frac{\sqrt{n^2-4/e_n}}{2} \approx \frac{n}{3c_n}. \tag{3.28$'$}$$

Then

$$D_2(u) \geqslant \frac{1}{e_n} H_2(u) - \alpha I(u). \tag{3.29}$$

Our aim is to find a lower bound for $\lambda_0(\Omega)$. To this end we pick an $\varepsilon > \lambda_0(\Omega)$. Then for some φ in $W^1(\Omega)$, normalized so that $H(\varphi)=1$, we will have $D(\varphi)<\varepsilon$. For such a φ,

$$\varepsilon > D_1(\varphi) + D_2(\varphi) \geqslant \frac{1}{e_n}(H_1(\varphi)+H_2(\varphi)) - \alpha I(\varphi) = \frac{1}{e_n} - \alpha I(\varphi). \tag{3.30}$$

To complete the proof we need to estimate $I(\varphi)$ from above by $D(\varphi)$. We already have in (3.20) a suitable bound for the surface integral over ∂_1. We now use this inequality to obtain a bound for the analogous surface integral over ∂_2 obtained by rotating L_2.

We begin by parameterizing L_1 by ϱ:

$$L_1 : y = \frac{1}{2} + \frac{\sqrt{1-4(\varrho-1)^2}}{2}. \tag{3.31}$$

To each point (ϱ_2, y_2) of L_2 we correspond the point on L_1 with the same y-coordinate. Then y_2 is given by (3.31) and

$$\varrho_2 = \left[\tfrac{1}{2}+(\varrho-1)^2-\frac{\sqrt{1-4(\varrho-1)^2}}{2}\right]^{1/2}. \tag{3.31$'$}$$

A tedious but straightforward calculation shows that

$$1 \leq \left|\frac{d\varrho_2}{d\varrho}\right| \leq 2 \quad \text{for } 3/5 \leq \varrho \leq 1. \tag{3.32}$$

Next we integrate $du/d\sigma$ over broken-line paths, $\Gamma(\varrho)$, with vertices: (ϱ_2, y), $(\varrho_2, 4\varrho/3)$, $(\varrho, 4\varrho/3)$, (ϱ, y); here y and ϱ_2 are given by (3.31) and (3.31)$'$, respectively. The Euclidean length of $\Gamma(\varrho)$ is less than or equal to 2 for ϱ in $[3/5, 1]$. As a result of this integration we have

$$u(\varrho_2, y) = u(\varrho, y) - \int_{\Gamma(\varrho)} \frac{du}{d\sigma}\, d\sigma,$$

where

$$\int_{\Gamma(\varrho)} \frac{du}{d\sigma}\, d\sigma = \int_y^{4\varrho/3} u_\eta(\varrho_2, \eta)\, d\eta + \int_{\varrho_2}^{\varrho} u_\xi(\xi, 4\varrho/3)\, d\xi + \int_y^{4\varrho/3} u_\eta(\varrho, \eta)\, d\eta.$$

Squaring and applying the Schwarz inequality to the line integral, we get

$$|u(\varrho_2, yu)|^2 \leq 2|u(\varrho, y)|^2 + 4\int_{\Gamma(\varrho)} \left|\frac{du}{d\sigma}\right|^2 d\sigma. \tag{3.33}$$

It is easy to see that $\varrho_2/y \leq 4\xi/3\eta$ on $\Gamma(\varrho)$. Hence multiplying through by $(\varrho_2/y)^{n-1}$ in (3.33) and integrating with respect to ϱ_2 yields

$$\int_{L_2} |u(\varrho_2, y)|^2 \left(\frac{\varrho_2}{y}\right)^{n-1} d\varrho_2 \leq 2\left(\frac{4}{3}\right)^{n-1} \int_{L_1} |u(\varrho, y)|^2 \left(\frac{\varrho}{y}\right)^{n-1} \left|\frac{d\varrho_2}{d\varrho}\right| d\varrho$$
$$+4\left(\frac{4}{3}\right)^{n-1} \iint_A (|u_\varrho|^2+|u_y|^2)\left(\frac{\varrho}{y}\right)^{n-1} \max\left(\left|\frac{d\varrho_2}{d\varrho}\right|, 1\right) d\varrho\, dy,$$

where A is the region in the (ϱ, y)-plane below $\Gamma(1)$ and above L_1 and L_2. Finally, making use of (3.32) and integrating with respect to θ gives for $\varphi(\varrho, \theta, y)$

$$\int_{\partial_2} |\varphi|^2 \left(\frac{\varrho}{y}\right)^{n-1} d\varrho\, d\theta \leq 3\left(\frac{4}{3}\right)^n \int_{\partial_1} |\varphi|^2 \left(\frac{\varrho}{y}\right)^{n-1} d\varrho\, d\theta + 6\left(\frac{4}{3}\right)^n D(\varphi).$$

Since $y>4/5$ on L, we get, on combining this with (3.20):

$$\begin{aligned} I(\varphi) &= \int_{\partial} |\varphi|^2 \left(\frac{\varrho}{y}\right)^{n-1} d\varrho\, d\theta \\ &\leq (1+3c_n)\frac{5}{4}\int_{\partial_1} |\varphi|^2 \left(\frac{\varrho}{y}\right)^{n-1} \frac{d\sigma\, d\theta}{y} + 6c_n D(\varphi) \\ &\leq (1+3c_n)\frac{5}{4}\,\frac{3}{n-2}\, c_n D_1(\varphi) + 6c_n D(\varphi) \\ &\leq \left[\left(6+\frac{4}{n-2}\right) c_n + \frac{12}{n-2} c_n^2\right] \varepsilon. \end{aligned} \tag{3.34}$$

Inserting this into (3.30) and making use of (3.27), we find that

$$\varepsilon \geq d_n \equiv (n-2)^2 \left[3c_n \left\{ 1+\alpha\left(\left(6+\frac{4}{n-2}\right) c_n + \frac{12}{n-2} c_n^2\right)\right\}\right]^{-1},$$

where c_n and α are defined by (3.14). This is the desired lower bound for $\lambda_0(\Omega)$.

Further insight into Schottky domains in H^{n+1} can be obtained by considering the Hecke domain

$$\Omega_0 = \{(x,y);\ |x_i| < 1 \text{ for } i \leq n \text{ and } |x|^2+y^2 > 1\}. \tag{3.35}$$

This is the fundamental domain of the group generated by the translations: $x_i \to x_i+2$, $i \leq n$, and the inversion through the unit sphere centered at the origin.

It is easily verified for the prism domain

$$\Omega_{00} = \{(x,y);\ |x_i| < 1 \text{ for } i \leq n\}, \tag{3.36}$$

that Ω_{00} is free (by Proposition 2.2) and that $u=y^{n/2}$ is a null vector for Δ' on Ω_{00}. Hence by Theorem 2.10, we have $\lambda_0(\Omega_0)<(n/2)^2$ and, by Theorem 2.4, $\lambda_0(\Omega_0)$ is an eigenvalue for Δ. On the other hand since $\mathrm{vol}(\Omega_0)=\infty$, we infer that $\lambda_0(\Omega_0)>0$. In Section 6 we present numerical evidence indicating that $\lambda_0(\Omega_0)$ is close to 0.66 when $n=2$.

Now for any Schottky domain Ω for which the hemispherical sides can be enclosed in disjoint prisms isometric to Ω_{00}, it follows by Proposition 2.14 that

$$\lambda_0(\Omega) \geq \lambda_0(\Omega_0). \tag{3.37}$$

In particular this will be true of domains bounded by an infinite r-lattice of hemispheres of radius 1/2, centered at the points

$$\{(n_1, \ldots, n_r, 0, \ldots, 0);\ n_i \in \mathbf{Z}\},$$

and by the hyperplanes $x_i = \pm 1/2$, $r < i \leq n$. We call such a domain an r-lattice Schottky domain.

THEOREM 3.12. *If Ω' is an r-lattice Schottky domain then*

$$\lambda_0(\Omega') = \lambda_0(\Omega_0). \tag{3.38}$$

Proof. We note that the domain Ω_0 is invariant under the reflections $x_i \to -x_i$, $i \leq n$. Any eigenfunction of Δ over Ω_0 goes into another eigenfunction under such a reflection. If we start with the base eigenfunction φ and sum over all of these reflections we obtain a nonzero eigenfunction with the corresponding symmetries. Since there is only one base eigenfunction, it must have had these symmetries to begin with. We normalize φ so that $H(\varphi)=1$ and then continue φ periodically over the r-lattice.

We obtain in this way a smooth function ψ, defined over Ω', which satisfies the equation

$$\Delta\psi = \lambda_0(\Omega_0)\,\psi.$$

To make it square integrable we multiply it by a smear function χ_R in $C^\infty(\mathbf{R}^n)$:

$$\chi_R(x) = \begin{cases} 1 & \text{for } |x| \leq R, \\ 0 & \text{for } |x| > R+1. \end{cases}$$

The function

$$\psi_R(w) = \chi_R(x)\,\psi(w)$$

satisfies the relations

$$\begin{aligned} H(\psi_R) &= c(r)\,R^r + e_1(R), \\ D(\psi_R) &= c(r)\,\lambda_0(\Omega_0)\,R^r + e_2(R), \end{aligned} \tag{3.39}$$

where $c(r) = \omega_r/r$ and

$$|e_i(R)| \leq \text{const.}\, R^{r-1} \quad \text{for } i = 1, 2.$$

Since $\lambda_0(\Omega') \leq D(\psi_R)/H(\psi_R)$, it follows that $\lambda_0(\Omega') \leq \lambda_0(\Omega_0)$ and this together with (3.37) proves the assertion of the proposition.

We apply Proposition 3.12 to obtain another proof of a result due to Beardon (Theorem 9 of [5]).

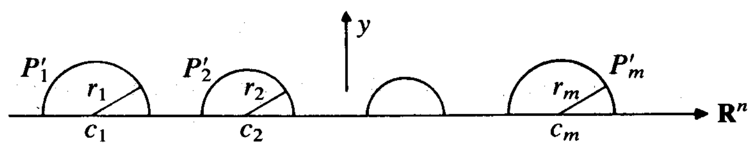


Fig. 3.2

COROLLARY 3.13. *There are nonfree Schottky domains with a finite number of sides in a hyperbolic space of any dimension.*

Proof. Suppose that instead of the r-lattice Schottky domain considered above, we had a domain Ω'' defined by the finite subset of these hemispherical sides contained in the support of χ_R. Then the relations (3.37) and (3.39) continue to hold. It follows that $\lambda_0(\Omega'')$ can be made arbitrarily close to $\lambda_0(\Omega_0)$ by enlarging the sublattice of sides.

Remark. A similar analysis applies if we replace Ω_{00} in (3.36) by any other domain whose cross-section is a fundamental domain for some crystallographic subgroup acting on $\mathbf{R}^n$. For a hexagonal lattice with deleted inscribed hemispheres, we found numerically that $\lambda_0=0.510\pm0.002$.

It is also possible to obtain a lower bound for $\lambda_0(\Omega)$ for all Schottky domains whose bounding hyperplanes are uniformly separated. To make this more precise we define for each point p in $B\cap\bar{\Omega}$ a quantity $\tau(p)$ which depends on the shape of Ω viewed from p. We use the upper half space model to define $\tau(p)$. Mapping p into ∞ by a non-Euclidean motion, the domain Ω and the faces P_j go into Ω' and P'_j, respectively. Let S'_j denote the intersection of P'_j with $\mathbf{R}^n$. Thus the P'_j are $(n-1)$-spheres whose (Euclidean) radii we call r_j (see Figure 3.2).

Let d_j be the Euclidean distance in $\mathbf{R}^n$ from the center c_j of S'_j to the nearest sphere S'_k, $k\neq j$. Although r_j and d_j depend on the choice of the mapping taking p into ∞, the ratio r_j/d_j does not; i.e. r_j/d_j is invariant under motions of H^{n+1} which fix ∞. Since the spheres S'_j are external to each other, it follows that $r_j/d_j\leqslant 1$. We define $\tau(p)$ to be

$$\tau(p) = \max\,[(r_j/d_j),\, j=1,\dots,m]. \tag{3.40}$$

Finally we set

$$\tau(\Omega) = \inf_{p\in\bar{\Omega}\cap B} \tau(p). \tag{3.41}$$

The quantity $\tau(\Omega)$ measures the minimal separation of the bounding hyperplanes of

Ω. In general $\tau(\Omega)\leqslant 1$; if the number of sides is finite then $\tau(\Omega)=1$ if and only if Ω has a cusp.

THEOREM 3.4. *For all Schottky domains* $\Omega\subset H^{n+1}$ with $\tau(\Omega)<c<1$, *there exists a constant* $e_n>0$ *such that*

$$\lambda_0(\Omega)\geqslant e_n. \tag{3.42}$$

Proof. If $\tau(\Omega)<c$, then Ω can be mapped into Ω' with faces P_j' such that for $S_j'=P_j'\cap\mathbf{R}^n$ with center at c_j, radius r_j and minimal distance d_j from c_j to S_k', $k\neq j$, we have

$$r_j/d_j<c \tag{3.43}$$

for all j. In this case

$$\begin{aligned}\operatorname{dist}(c_j, c_k)&\geqslant \tfrac{1}{2}[(r_j+d_k)+(r_k+d_j)]\\ &>\frac{r_j+r_k}{2}\left(1+\frac{1}{c}\right).\end{aligned} \tag{3.44}$$

Set

$$r_j''=\frac{r_j}{2}\left(1+\frac{1}{c}\right).$$

Then the spheres S_j'' with centers at c_j and radii r_j'' do not intersect. Hence if we use the same approach as in the previous two theorems, we see that it is enough to prove

PROPOSITION 3.14. *Denote by* Ω_1 *and* Ω *the domains*

$$\begin{aligned}\Omega_1&=\{(x,y);\ |x|<1\}\\ \Omega&=\{(x,y)\in\Omega_1;\ |x|^2+y^2>\alpha^2\}\end{aligned} \tag{3.45}$$

where $\alpha<1$. *Then there exists an* $e_n>0$ *such that*

$$\lambda_0(\Omega)\geqslant e_n.$$

Proof. Since neither Ω_1 nor Ω have the finite geometric property we cannot use the argument in Theorem 3.12 directly. However we can construct a polyhedron T in $\mathbf{R}^n$ containing $\{|x|<\alpha\}$ in its interior and which is contained in the interior of $\{|x|<1\}$. Setting

$$\Omega_{00} = \text{the region above } T$$

$$\Omega_0 = \{(x, y) \in \Omega_{00};\ |x|^2 + y^2 > \alpha^2\}.$$

These domains have properties similar to their counterparts in the proof of Theorem 3.12 and it follows as before that there is an $e_n > 0$ such that $\lambda_0(\Omega_0) \geqslant e_n$. Since the sides of $\Omega \setminus \Omega_{00}$ are parallel to the y-axis, Proposition 2.2 shows that $\Omega \setminus \Omega_{00}$ is free. Since $\Omega = (\Omega \setminus \Omega_{00}) \cup \Omega_0$, it follows from Proposition 2.12 that

$$\lambda_0(\Omega) \geqslant \lambda_0(\Omega_0) \geqslant e_n,$$

as desired.

4. Continuity

In this section we study the dependence of the spectrum of Δ on the domain Ω. Throughout we assume that Ω is geometrically finite. We shall use the following concept:

Definition 4.1. Let $\Omega \subset \mathbf{R}^m$ be a domain. We say that the two Riemannian metrics g and $\tilde{g}$ on Ω are *K-quasi-isometric* if for each $x \in \Omega$ and $\xi \in T_x(\Omega)$ we have

$$K^{-1} g_x(\xi, \xi) \leqslant \tilde{g}_x(\xi, \xi) \leqslant K g_x(\xi, \xi) \tag{4.1}$$

(uniformly in x and ξ).

It follows from this definition that if $\gamma(t)$, $a \leqslant t \leqslant b$ is a curve in Ω, then

$$K^{-1/2} l_g(\gamma) \leqslant l_{\tilde{g}}(\gamma) \leqslant K^{1/2} l_g(\gamma)$$

where $l(\gamma)$ is the length of γ. Thus locally the distance functions $d_g(x, y)$ and $d_{\tilde{g}}(x, y)$ satisfy

$$K^{-1/2} d_g(x, y) \leqslant d_{\tilde{g}}(x, y) \leqslant K^{1/2} d_g(x, y). \tag{4.2}$$

For the cases which are of interest to us (Ω, g) and $(\Omega, \tilde{g})$ are of constant negative curvature and also Ω is convex with respect to each of the metrics, so that (4.2) actually holds for all $x, y \in \Omega$.

Conversely if (4.2) holds it clearly implies the truth of (4.1).

Suppose now that (Ω, g) and (Ω', g') are as above and that $\varphi: \Omega \to \Omega'$ is a diffeomorphism of Ω on Ω'. Let $\tilde{g}$ be the metric on Ω obtained by pulling back the metric g'

to Ω. We call φ a *K-quasi-isometry* if (g, Ω) and $(\tilde{g}, \Omega)$ are K-quasi-isometric. In view of (4.1), φ is a K-quasi-isometry iff

$$K^{-1/2}d(x,y) \leqslant d'(\varphi x, \varphi y) \leqslant K^{1/2}d(x,y). \tag{4.3}$$

It is important in this definition that only the ratio d'/d need be bounded, rather than the difference, since our main interest is in noncompact regions. A scalar quantity, depending on the metric will be called *$f(K)$-quasi-invariant* if it changes by a factor of at most $f(K)$ under a K-quasi-isometry. Thus the relation (4.2) says that distances are $K^{1/2}$-quasi-invariant. We now show that the volume, the H form, D form and the Neumann spectrum are all $f(K)$-quasi-invariant for suitable f.

Notice however that the "geometry" may change drastically under a K-quasi-isometry—for example the curvature is certainly not an $f(K)$-quasi-invariant for any f.

Let $g = \det g_{\alpha\beta}$, $\tilde{g} = \det \tilde{g}_{\alpha\beta}$.

Using (4.1) and the fact that if $A \geqslant B > 0$, where A, B are real symmetric matrices, implies $\det A \geqslant \det B$ we find that

$$\sqrt{\tilde{g}}\, K^{-m/2} \leqslant \sqrt{g} \leqslant K^{m/2} \sqrt{\tilde{g}} \tag{4.4}$$

pointwise in Ω.

Hence

$$K^{-m/2}\,\mathrm{Vol}_{\tilde{g}}(A) \leqslant \mathrm{Vol}_g(A) \leqslant K^{m/2}\,\mathrm{Vol}_{\tilde{g}}(A). \tag{4.5}$$

for any $A \subset \Omega$. Thus volumes are $K^{m/2}$-quasi-invariant.

It follows from (4.4) that $L^2(\Omega, g) = L^2(\Omega, \tilde{g})$ if g and $\tilde{g}$ are K-quasi-isometric, and that

$$K^{-m/2} H_g(u) \leqslant H_{\tilde{g}}(u) \leqslant K^{m/2} H_g(u) \tag{4.6}$$

where as usual

$$H_g(u) = \int_\Omega u^2 \sqrt{g}\, dx$$

is the H form.

Concerning the D forms we proceed as follows: For positive definite symmetric matrices we have

$$A \geqslant B \quad \text{iff} \quad B^{-1} \geqslant A^{-1}.$$

Thus (4.1) is the same as

$$K^{-1}g^{\alpha\beta}\xi_\alpha\xi_\beta \leq \tilde{g}^{\alpha\beta}\xi_\alpha\xi_\beta \leq Kg^{\alpha\beta}\xi_\alpha\xi_\beta. \tag{4.7}$$

for $u \in C^\infty(\Omega)$ with $u_\alpha = \partial u/\partial x^\alpha$

$$D_g(u) = \int_\Omega g^{\alpha\beta}u_\alpha u_\beta \sqrt{g}\, dx$$

and hence

$$K^{-m/2-1}D_g(u) \leq D_{\tilde{g}}(u) \leq K^{m/2+1}D_g(u). \tag{4.8}$$

It follows that the spaces $W_g^1(\Omega)$ and $W_{\tilde{g}}^1(\Omega)$ are the same.

If we now form the quotient D/H and use the variational definition of λ_0 we learn immediately that

$$K^{-m-1}\lambda_0(\Omega, g) \leq \lambda_0(\Omega, \tilde{g}) \leq K^{m+1}\lambda_0(\Omega, g). \tag{4.9}$$

To obtain information about the higher eigenvalues we simply use minimax (see Courant-Hilbert, p. 407, [7]), from which bounds (4.9) with $\lambda_0(\Omega, g)$ replaced by $\lambda_j(\Omega, g)$ are deduced. Notice that

$$\lambda_j(\Omega, \tilde{g}) \to \lambda_j(\Omega, g) \quad \text{as } K \to 1.$$

With these notions we turn to the proof of the continuity of the Neumann spectrum. We begin with dimension two where we prove a general theorem. For higher dimensions the various 'incidence' patterns become very complicated and we will only discuss the ones which are needed elsewhere in the paper.

In dimension two, Ω is a convex polygon bounded by geodesics: $g_1, \ldots, g_m$. We assume that none of these geodesics is redundant; that is $g_j \cap \bar{\Omega}$ is a nontrivial (i.e. not just a point) subarc of g_j. With this assumption a small enough movement of the sides will deform Ω into Ω' which is still bounded by m geodesics, none of which is redundant.

THEOREM 4.2. *Suppose $\Omega' \to \Omega$ in H^2. If no cusp is broken in going from Ω to Ω' then*

$$\lim \lambda_j(\Omega') = \lambda_j(\Omega).$$

If a cusp is broken then $\overline{\lim}\, \lambda_0(\Omega') \leq \lambda_0(\Omega)$. If in addition in the last case we have $\Omega' \supset \Omega$ and $\Omega' \setminus \Omega$ is free then $\lambda_0(\Omega') \to \lambda_0(\Omega)$.

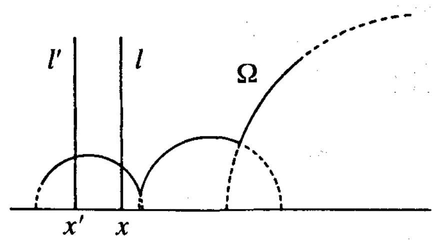


Fig. 4.1

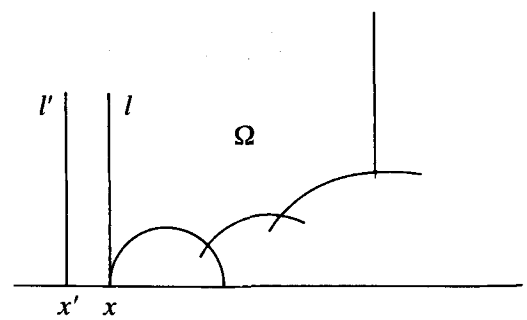


Fig. 4.2

Proof. The general motion of a geodesic is to move its endpoints. This can be achieved by moving one end point at a time. Thus in the upper half plane model the most general motion we need consider is moving a vertical geodesic to one side. There are a number of configurations which may arise depending on how l', the geodesic to be moved, meets the other sides. We distinguish between the cases. One is when the end point of l on $\mathbf{R}$, call it x, is not a 'cusp' of Ω, i.e. it is not the case that x is the end point of one of the g_j's. In this case the motion $l' \to l$, $\Omega' \to \Omega$ can be realized by a K-quasi-isometry where $K \to 1$ as $x' \to x$ (see Figure 4.1). By the discussion at the beginning of this section this would prove continuity for such variations.

The second case, when x is a cusp, does not correspond to a quasi-isometry. For example, in the domain pictured in Figure 4.2, Ω and Ω' cannot be K-quasi-isometric for any K, since Ω' has infinite area while Ω does not. For the second case (i.e. of breaking a cusp) a special argument for the continuity will be given.

We begin by proving in the first case that the two domains are related by a K-quasi-isometry. The following (Figure 4.3) depicts the various incidences possible for l with the other sides of Ω. The number of relevant sides of course remains fixed.

In order to map Ω to Ω', we leave the part of Ω to the right of the dotted lines fixed. To the left, points are moved smoothly (and with small derivative, if x' is close to x) along the curves shown, so that $l \to l'$.

For example in the case (b) this may be done explicitly as follows (we have set the dotted line in this case to be the axis $\xi=0$)

$$(y, \xi) \to (y, f(\xi))$$

where f is a C^1 smoothing of the piecewise linear function

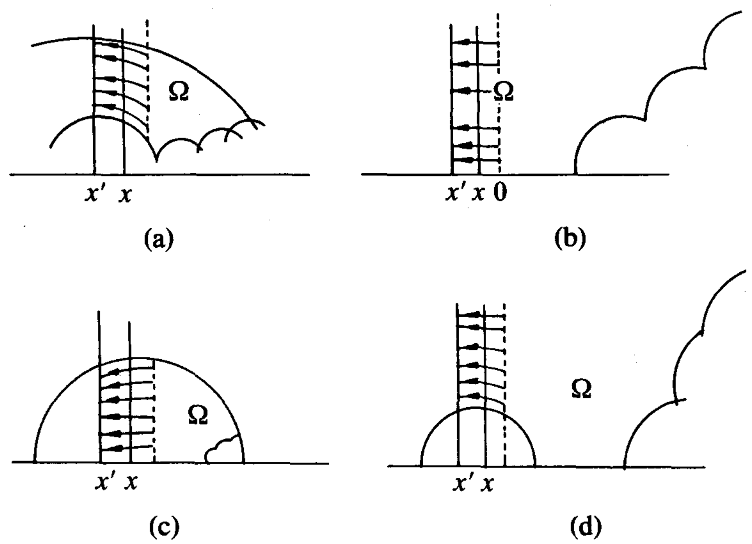


Fig. 4.3

$$\xi \to \xi \quad \text{if } \xi > 0$$
$$\xi \to \frac{x'}{x}\xi \quad \text{if } \xi < 0.$$

In the other cases one may write down explicit expressions for these vector fields which near a given geodesic follow curves concentric to the geodesic and are otherwise convex combinations (with variable weights) of such fields.

Notice that in all cases we have made the vector field horizontal for y small enough. As a consequence it is clear that the mapping of $\Omega \to \Omega'$ so constructed distorts distances boundedly. The mapping is therefore a K-quasi-isometry and it is also evident that $K \to 1$ as $x' \to x$.

We turn to the case of a cusp. Typically the relation of l to the other geodesics is one of those shown in Figure 4.4. When $\lambda_0(\Omega) = 1/4$ the upper semi-continuity is trivially valid. Hence we may suppose that $\lambda_0(\Omega) < 1/4$ and that there is a square integrable base eigenfunction u_0 on Ω. If $\Omega \supset \Omega'$ as in Figure 4.4 (c) and (d), we use u_0 directly to prove upper semi-continuity. In this case it is clear that

$$H_{\Omega'}(u_0) \to H_\Omega(u_0) \quad \text{and} \quad D_{\Omega'}(u_0) \to D_\Omega(u_0).$$

Consequently

$$\lambda_0(\Omega) = \frac{D_\Omega(u_0)}{H_\Omega(u_0)} = \lim \frac{D_{\Omega'}(u_0)}{H_{\Omega'}(u_0)};$$

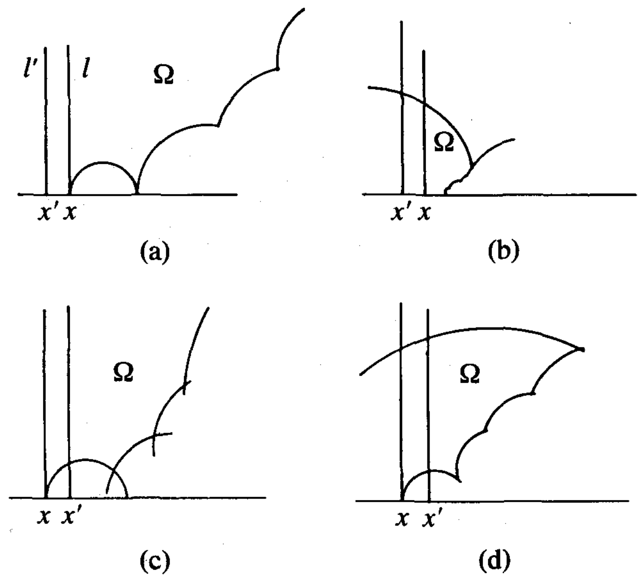


Fig. 4.4

and since by (1.3)

$$\lambda_0(\Omega') \leqslant D_{\Omega'}(u_0)/H_{\Omega'}(u_0).$$

we conclude that

$$\overline{\lim}\, \lambda_0(\Omega') \leqslant \lambda_0(\Omega).$$

When $\Omega' \supset \Omega$ and u_0 is no longer defined on all of Ω', we extend u_0 to Ω' and argue as above. To do so we first smear out u_0 near $y=0$, and by renormalizing we get a function w with $w=0$ for $y<y_\varepsilon$, $(x, y) \in \Omega$, with $\partial_n w=0$ on $\partial\Omega$ and such that

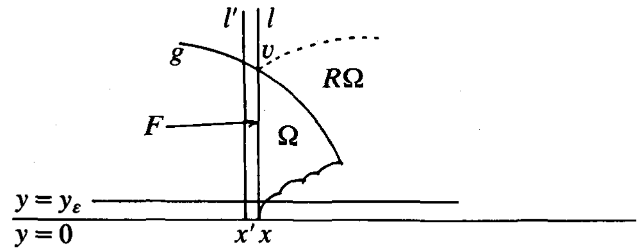


Fig. 4.4 (b)

$$H_\Omega(w) = 1, \quad D_\Omega(w) < \lambda_0(\Omega) + \varepsilon.$$

We extend w to Ω' to begin with by setting w equal to zero for $y<y_\varepsilon$ and by making w even about the line l (recall that $\partial_n w=0$ on l). If x' is close enough to x this will define w on Ω' except possibly in case (b) near the vertex v and in F (see Figure 4.4(b)). However since the normal derivative of w is zero along g as well, we may extend w by reflection to what is labeled $R\Omega$. If necessary we repeat this last reflection a finite number of times (depending on the angle between g and l) until the reflection of the so augmented domain about l contains Ω'. In this way we obtain $w \in W'(\Omega')$ and it follows that if x' is close enough to x, then $D_{\Omega'/\Omega}(w)$ and $H_{\Omega'/\Omega}(w)$ are arbitrarily small. Hence for such x'

$$D_{\Omega'}(w)/H_{\Omega'}(w) < \lambda_0(\Omega) + 2\varepsilon$$

and so $\lambda_0(\Omega')<\lambda_0(\Omega)+2\varepsilon$, as needed.

If moreover $\Omega' \setminus \Omega$ is free as in cases (a) and (b), then $\lambda_0(\Omega') \geqslant \lambda_0(\Omega)$ by Proposition 2.2 and this combined with the upper semi-continuity proves the claimed continuity. This completes the proof of Theorem 4.2.

In higher dimensions $n \geqslant 2$, the configuration of the bounding sides of a domain Ω can become exceedingly complicated and proving the analogue of Theorem 4.2 becomes rather involved. Since we will not need such a general result for our later applications we do not consider this question. It turns out when we do need a local continuity result for $n \geqslant 2$, the situation will be simple enough that the methods used in the proof of Theorem 4.2 can easily be adapted.

For applications we will also need to know that the discrete spectrum is continuous under more drastic deformations associated with certain types of degenerations. As we will see, continuity fails under other kinds of degeneration.

Definition 4.3. We say that a sequence of domains Ω_k degenerates to Ω and write $\Omega_k \overset{\text{deg}}{\to} \Omega$ if

(1) $\Omega_k \subset \Omega$ and $\Omega \setminus \Omega_k$ is free for all k,

(2) $\Omega_k \to \Omega$ in the sense that every point of Ω is eventually in the Ω_k's,

(3) there is a fixed $\tilde{\Omega}$ which is geometrically finite, whose bounding hyperplanes contain those of Ω, and such that $\tilde{\Omega} \subset \Omega_k$, while $\Omega_k \setminus \tilde{\Omega}$ is free for every k.

THEOREM 4.4. *If the Ω_k's are geometrically finite and $\Omega_k \overset{\text{deg}}{\to} \Omega$ then $\lambda_0(\Omega_k) \to \lambda_0(\Omega)$.*

Proof. First $\Omega \supset \Omega_k$ and $\Omega \setminus \Omega_k$ is free so by Proposition 2.12.

$$\lambda_0(\Omega_k) \leqslant \lambda_0(\Omega).$$

It therefore suffices to prove that

$$\varliminf_{k \to \infty} \lambda_0(\Omega_k) \geqslant \lambda_0(\Omega).$$

Suppose this were not the case, then we could find a subsequence, call it again Ω_k, such that

$$\lim_{k \to \infty} \lambda_0(\Omega_k) = A < \lambda_0(\Omega) \leqslant \left(\frac{n}{2}\right)^2. \tag{4.10}$$

To each such $\lambda_0(\Omega_k)$ there is a base L^2 eigenfunction, φ_k, which may be normalized so that

$$H_{\Omega_k}(\varphi_k) = 1 \quad \text{and} \quad \Delta\varphi_k = \lambda_0(\Omega_k)\,\varphi_k \quad \text{in } \Omega_k. \tag{4.11}$$

We extend φ_k to be defined on Ω by setting it to be zero in $\Omega \setminus \Omega_k$. Thus defined, φ_k is a sequence in $L^2(\Omega)$ with $H_\Omega(\varphi_k)=1$. It therefore has a weakly convergent subsequence, again denoted by φ_k, with a weak limit φ. It is clear that φ satisfies

$$\Delta\varphi = A\varphi$$

in the weak sense on Ω. On the other hand the restriction of the φ_k to $\tilde{\Omega}$ are bounded in $W^1(\tilde{\Omega})$ and so also have a weakly convergent subsequence, again denoted by φ_k, in $W^1(\tilde{\Omega})$ which converges to the restriction of φ to $\tilde{\Omega}$. Thus for any smooth function ψ on $\tilde{\Omega}$, vanishing near those hyperplanes of $\tilde{\Omega}$ not bounding Ω we have

$$\lim D_{\tilde{\Omega}}(\varphi_k, \psi) = D_{\tilde{\Omega}}(\varphi, \psi)$$

since $\tilde{\Omega} \subset \Omega_k \subset \Omega$ we also have

$$\begin{aligned} D_\Omega(\varphi_k, \psi) = D_{\Omega_k}(\varphi_k, \psi) &= \lambda_0(\Omega_k)\,H_{\Omega_k}(\varphi_k, \psi) \\ &= \lambda_0(\Omega_k)\,H_\Omega(\varphi_k, \psi). \end{aligned}$$

Passing to the limit gives

$$D_\Omega(\varphi, \psi) = AH_\Omega(\varphi, \psi).$$

This proves that φ satisfies the Neumann boundary conditions on $\partial\Omega$.

Being a weak solution of $\Delta\varphi=A\varphi$, φ is also a classical solution. Since φ is square integrable, if $\varphi\not\equiv 0$ it would follow that A is in the spectrum of Δ over Ω. However this would contradict (4.10). Thus Theorem 4.4 will be proven if we can show that $\varphi\not\equiv 0$.

We can write Ω_k as the disjoint union

$$\Omega_k=\tilde{\Omega}\cup V_k$$

where V_k is free. Therefore

$$D_{V_k}(\varphi_k)\geq\left(\frac{n}{2}\right)^2 H_{V_k}(\varphi_k). \tag{4.12}$$

However

$$\lambda_0(\Omega_k)=D_{\Omega_k}(\varphi_k)=\frac{D_{\tilde{\Omega}}(\varphi_k)+D_{V_k}(\varphi_k)}{H_{\tilde{\Omega}}(\varphi_k)+H_{V_k}(\varphi_k)}\leq A_1 \tag{4.13}$$

where $A_1<(n/2)^2$. By (4.12) and (4.13)

$$D_{\tilde{\Omega}}(\varphi_k)+\left(\frac{n}{2}\right)^2 H_{V_k}(\varphi_k)\leq A_1[H_{\tilde{\Omega}}(\varphi_k)+H_{V_k}(\varphi_k)]$$

or

$$D_{\tilde{\Omega}}(\varphi_k)\leq A_1 H_{\tilde{\Omega}}(\varphi_k)+\left(A_1-\left(\frac{n}{2}\right)^2\right)H_{V_k}(\varphi_k).$$

Keeping in mind that $H_{\tilde{\Omega}}(\varphi_k)+H_{V_k}(\varphi_k)=1$, we learn from the last inequality that

$$D_{\tilde{\Omega}}(\varphi_k)\leq A_1 H_{\tilde{\Omega}}(\varphi_k) \tag{4.14i}$$

$$H_{\tilde{\Omega}}(\varphi_k)\geq\frac{\left(\frac{n}{2}\right)^2-A_1}{\left(\frac{n}{2}\right)^2}>0. \tag{4.14ii}$$

Now $\tilde{\Omega}$ is geometrically finite so by Theorem 2.4 we can find $u_0,u_1,\dots,u_r$, $\lambda_0<\lambda_1\leq\lambda_2\dots\leq\lambda_r<(n/2)^2$ spanning the discrete spectrum of $\tilde{\Omega}$ (i.e. the spectrum in $[0,(n/2)^2)$). We may normalize so that $H_{\tilde{\Omega}}(u_j)=1$, $j=1,2,\dots,r$. Using these fixed functions we may expand the restriction of φ_k to $\tilde{\Omega}$ as

$$\varphi_k=a_0(k)\,u_0+a_1(k)\,u_1+\dots+a_r(k)\,u_r+g_k \tag{4.15}$$

where $g_k\perp\operatorname{span}\{u_0,u_1,\dots,u_r\}$.

It follows from the calculus of variations approach to the eigenvalue problem that

$$D_{\tilde\Omega}(u_j, g_k) = \lambda_j H_{\tilde\Omega}(u_j, g_k) = 0. \tag{4.15)$'$}$$

In terms of this expansion

$$D_{\tilde\Omega}(\varphi_k) = a_0^2(k)\,\lambda_0(\tilde\Omega)+\ldots+a_r^2(k)\,\lambda_r(\tilde\Omega)+D_{\tilde\Omega}(g_k) \tag{4.16}$$

and

$$H_{\tilde\Omega}(\varphi_k) = a_0^2(k)+\ldots+a_r^2(k)+H_{\tilde\Omega}(g_k). \tag{4.17}$$

In view of (4.15)$'$

$$D_{\tilde\Omega}(g_k) \geqslant \left(\frac{n}{2}\right)^2 H_{\tilde\Omega}(g_k). \tag{4.18}$$

Using (4.16) and (4.18) in (4.14 i) we see that

$$a_0^2(k)\,\lambda_0(\tilde\Omega)+\ldots+a_r^2(k)\,\lambda_r(\tilde\Omega)+\left(\frac{n}{2}\right)^2 H_{\tilde\Omega}(g_k) \leqslant A_1\, H_{\tilde\Omega}(\varphi_k)$$

and combining this with (4.17), we get

$$a_0^2\lambda_0(\tilde\Omega)+\ldots+a_r^2\lambda_r(\tilde\Omega)+\left(\frac{n}{2}\right)^2 [H_{\tilde\Omega}(\varphi_k)-(a_0^2+\ldots+a_r^2)] \leqslant A_1\, H_{\tilde\Omega}(\varphi_k).$$

Consequently, by (4.14 ii), we have

$$\left[\left(\frac{n}{2}\right)^2-\lambda_0(\tilde\Omega)\right]a_0^2+\left[\left(\frac{n}{2}\right)^2-\lambda_1(\tilde\Omega)\right]a_1^2+\ldots\left[\left(\frac{n}{2}\right)^2-\lambda_r(\tilde\Omega)\right]a_r^2$$
$$\geqslant \left(\left(\frac{n}{2}\right)^2-A_1\right)H_{\tilde\Omega}(\varphi_k) \geqslant \frac{\left(\left(\frac{n}{2}\right)^2-A_1\right)^2}{\left(\frac{n}{2}\right)^2}.$$

It follows from the last inequality that for some subsequence of the k's and some j, the $a_j(k)$ are bounded away from zero. Hence

$$\int_{\tilde\Omega} \varphi u_j\, dV \neq 0$$

and φ cannot vanish identically. This completes the proof of Theorem 4.4.

To show how Theorem 4.4 may be used we consider the case of Schottky domains.

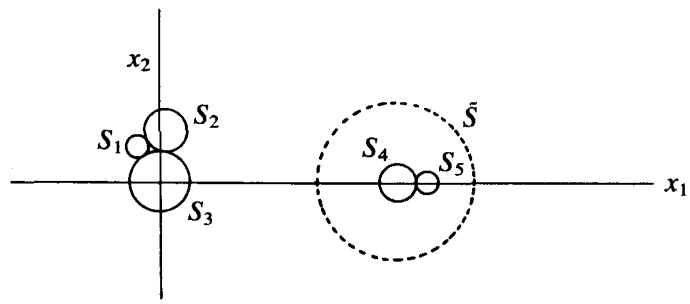


Fig. 4.5

These are easily described in the H^{n+1} model by giving the set of disjoint bounding hyperplanes $P_1, P_2, \dots, P_m$, or even more simply the $n-1$ spheres in $\mathbf{R}^n$ denoted by $S_1, S_2, \dots, S_m$, which are the intersections of $P_1, P_2, \dots, P_m$ with $\mathbf{R}^n$. The spheres S_j may be described by (p_j, r_j) where $p_j \in \mathbf{R}^n$ is the center of S_j and $r_j > 0$ its radius. We now describe what we mean by a *simple degeneration* of domains Ω_k to Ω, written as $\Omega_k \overset{\text{simp}}{\to} \Omega$.

Throughout the degeneration Ω_k should have a fixed number of sides (though Ω need not have the same number). Suppose Ω_k is described by $(p_k(1), r_k(1))$, $(p_k(2), r_k(2)), \dots, (p_k(m), r_k(m))$. The first, say, l spheres, $S_1, S_2, \dots, S_l$ are fixed and do not degenerate, while $S_{l+1}, \dots, S_m$ degenerate, i.e. $r_k(j) \to 0$ for $j = l+1, \dots, m$. We say the degeneration is *simple* if the $S_{l+1}, \dots, S_m$ degenerate in clumps of at most $[(n+2)/2]$. (Recall that $n+1$ is the dimension of the underlying space.) More precisely, for k large enough we can find fixed disjoint spheres $\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_t$ such that each of the spheres $S_{l+1}(k), \dots, S_m(k)$ lies inside one of $\tilde{S}_1, \dots, \tilde{S}_t$ and no $\tilde{S}_j$ contains more than $[(n+2)/2]$ of the $S_1, S_2, \dots, S_m$.

In this case we say $\Omega_k \overset{\text{simp}}{\to} \Omega$ where Ω is the Schottky domain given by $S_1, S_2, \dots, S_l$. Notice that as far as hyperbolic geometry goes there is no strong sense in which Ω_k and Ω can be thought of as close.

Examples. (a) In every dimension degenerating one side is always simple.

(b) In $\mathbf{H}^3$, consider Ω_k as pictured in Figure 4.5 with fixed S_1, S_2 and S_3, and

$$S_4(k) = \left((5,0), \frac{1}{k}\right), \quad S_5(k) = \left(\left(5 + \frac{2}{k}, 0\right), \frac{1}{k}\right).$$

Here $(n+2)/2 = 2$, and Ω is given by S_1, S_2, S_3. In this case $\Omega_k \overset{\text{simp}}{\to} \Omega$, since clearly S_4 and S_5 may be "capped off" by the dotted circle $\tilde{S}$ shown.

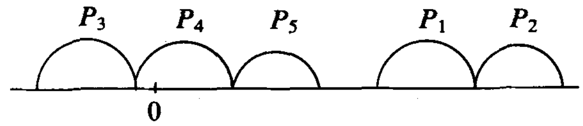


Fig. 4.6

COROLLARY 4.5. *For Schottky domains, if* $\Omega_k \overset{\text{simp}}{\to} \Omega$, *then* $\lambda_0(\Omega_k) \to \lambda_0(\Omega)$.

Proof. We need only show that $\Omega_k \overset{\text{deg}}{\to} \Omega$ and apply Theorem 4.4. Indeed for large k it follows from the definition of simple degeneration that there exist disjoint spheres $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_t$ containing all of the degenerating spheres $S_{l+1}, \ldots, S_m$. Let $\tilde{\Omega}$ denote the domain with bounding spheres

$$S_1, S_2, \ldots, S_l, \tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_t$$

We now check that the conditions in Definition 4.4 with this $\tilde{\Omega}$ are met. First $\Omega \setminus \Omega_k$, inasmuch as being the disjoint union of domains, each bounded by one hyperplane, is free. Clearly the second condition, $\Omega_k \to \Omega$, is satisfied. The sides of $\tilde{\Omega}$ contain $S_1, S_2, \ldots, S_l$ which are the sides of Ω. Also $\Omega_k \setminus \tilde{\Omega}$ is a disjoint union of domains bounded by $\tilde{S}_r$ and at most $[(n+2)/2]$ other sides, hence by Theorem 3.7 each of these is free, from which it follows that $\Omega_k \setminus \tilde{\Omega}$ is free. These remarks show $\Omega_k \overset{\text{deg}}{\to} \Omega$.

We end the discussion of continuity by remarking that the clumping condition in the definition of a simple degeneration cannot be dropped. For example consider the domains Ω_k in H^2 described in Figure 4.6.

Suppose that P_1, P_2 are fixed while P_3, P_4, P_5 are degenerated by scaling $z \to \lambda z$, $\lambda \to 0$. Since $z \to \lambda z$ is a hyperbolic isometry, if we ignore P_1, P_2, it is clear that λ_0(exterior (P_3, P_4, P_5)) is constant, say equal to λ_0. Since the exterior (P_3, P_4) has a null vector, it follows by the excision property (Theorem 2.10) that $\lambda_0 < 1/4$. It follows by monotonicity that $\lambda_0(\Omega_k) \leq \lambda_0$. On the other hand $\Omega_k \to \Omega$ where Ω, defined by P_1 and P_2, is free. Therefore $\lambda_0(\Omega) = 1/4$. So $\overline{\lim}\, \lambda_0(\Omega_k) < \lambda_0(\Omega)$.

5. Applications to Kleinian groups

We will now apply the theory developed in Sections 2–4 to problems concerning the size of the limit set of a Kleinian group, and related problems. The Patterson-Sullivan theorem described in the introduction is central in this respect. Let Γ be a discrete subgroup of G^*, the full group of isometries of X^{n+1}. We denote by $\lambda_0(\Gamma)<\lambda_1(\Gamma)\leq\ldots$ the discrete eigenvalues of Δ on the Hilbert space of Γ automorphic functions. It is understood that these λ_j's correspond to the discrete eigenvalues below the continuous spectrum, while if $\lambda_j(\Gamma)$ does not exist then it is taken to be $(n/2)^2$, that is the bottom of the continuous spectrum. We first relate the Neumann spectrum which was the subject matter of Sections 2–4 to the $\lambda_j(\Gamma)$'s.

PROPOSITION 5.1. *If Ω is a fundamental domain for Γ then*

$$\lambda_j(\Omega)\leq\lambda_j(\Gamma). \tag{5.1}$$

Proof. This follows immediately from the minimax characterization of the eigenvalues [Courant-Hilbert [7], p. 407] and the fact that $\lambda_j(\Omega)$ corresponds to free boundary conditions (i.e. no boundary conditions) while for the $\lambda_j(\Gamma)$ we have a more restricted class of admissible functions, namely those with periodic boundary conditions.

Thus the general analysis of Sections 2–4 will provide us with *lower* bounds for any X^{n+1}/Γ eigenvalue problem. There are a number of interesting cases where one has equality in (5.1). For example let $\Omega\subset H^{n+1}$ be a convex domain bounded by hyperplanes P_j. Let R_j be the reflection in P_j. If $R_1, R_2, \ldots$ generate a discrete subgroup in G^*, we call Ω a reflection domain and the corresponding Γ a reflection group. It is clear that for a reflection group Γ with domain Ω, $\lambda_j(\Gamma)=\lambda_j(\Omega)$ since the eigenfunctions being Γ automorphic satisfy Neumann boundary conditions, and conversely an eigenfunction which has normal derivative zero on a hyperplane is invariant by reflection in that hyperplane. For reference we state this as:

PROPOSITION 5.2. *If Ω is a reflection domain with reflection group Γ then*

$$\lambda_j(\Omega)=\lambda_j(\Gamma).$$

It follows from Propositions 5.1 and 5.2 and the Patterson-Sullivan theorem that:

PROPOSITION 5.3. *If Γ is a reflection group with fundamental domain Ω, then for any Γ^* with the same fundamental domain Ω we have*

$$\lambda_j(\Gamma)\leq\lambda_j(\Gamma^*)$$

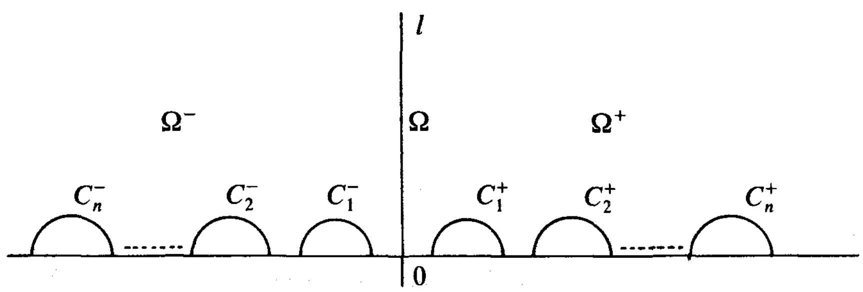


Fig. 5.1

and in particular if $\delta(\Gamma) \geqslant n/2$ *then*

$$\delta(\Gamma) \geqslant \delta(\Gamma^*).$$

Here $\delta(\Gamma)$ is the exponent of convergence of the Poincaré series as introduced in Section 1. A further application of part (ii) of the Patterson-Sullivan theorem then says that for geometrically finite groups, of all groups whose fundamental domain is equal to that of a reflection group (whose limit set has dimension $\geqslant n/2$), the reflection group has maximal Hausdorff dimension for its limit set.

Notice that a Schottky domain, is a reflection domain. In many cases one has equality in (5.1) even for discrete groups Γ which are orientation preserving. For example consider the case of a symmetric Schottky group. Its fundamental domain, shown in Figure 5.1 is obtained by removing $2n$ semicircles (noncrossing) from H^2. They are situated so as to be symmetric about the y axis l. The Schottky group Γ, in question, may be constructed by the use of the hyperbolic transformations $T_j = R_0 R_j$ where R_j is the reflection in C_j^+ and R_0 in l. T_j has isometric circle C_j^+ while T_j^{-1} has C_j^- as its isometric circle. The free group generated by $T_1, \ldots, T_n$ is the Schottky group, and it has Ω as a fundamental domain.

The spectral problem for $L^2(H^2/\Gamma)$ splits into Dirichlet and Neumann problems for Ω^+. The reflection R_0 about l stabilizes Γ, i.e., $R_0\Gamma = \Gamma R_0$. Thus we may decompose $L^2(H^2/\Gamma)$ into even and odd functions, and these spaces are invariant by Δ. An odd function satisfies $f(z) = f(T_j z)$ for $z \in C_j^-$ and also $f(z) = -f(T_j z)$ for such z since f is odd. One sees from this that the odd functions are zero on the boundary $\partial(\Omega^+)$. Thus the odd spectrum corresponds to the Dirichlet spectrum for Ω^+. In a similar way one sees that the even functions correspond to the Neumann spectrum on Ω^+. This explains what we mean by the $L^2(H^2/\Gamma)$ problem splitting into Dirichlet and Neumann problems for Ω^+.

The usual integration by parts, say with respect to y alone first, as in (2.7) with

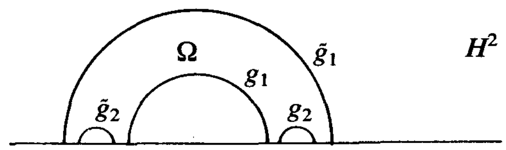


Fig. 5.2

$\varphi \equiv 1$, shows that any domain with Dirichlet boundary conditions can have no L^2 eigenfunctions in $[0, 1/4)$ (we are in dimension two here). It follows that the Dirichlet problem contributes no discrete spectrum to the $L^2(H^2/\Gamma)$ problem and therefore

$$\lambda_j(\Gamma) = \lambda_j(\Omega^+) \tag{5.2}$$

for a symmetric Schottky domain.

It is also easy to construct examples for which say $\lambda_0(\Omega) \neq \lambda_0(\Gamma)$. For example consider Figure 5.2. Let Γ be the group generated by the hyperbolic transformations taking $g_1 \to \tilde{g}_1$ and $g_2 \to \tilde{g}_2$. Γ has fundamental domain Ω. By Theorem 5.6 (to be proved later in this section) we can ensure Ω is free if the separation between $g_1, \tilde{g}_1, g_2, \tilde{g}_2$ is big enough, this clearly can be done. Since $\lambda_0(\Gamma) \geq \lambda_0(\Omega)$ we see that $\lambda_0(\Gamma) = 1/4$. However we could just as well choose the fundamental domain for Γ to be Ω' as pictured in Figure 5.3. Since a single cusp domain in H^2 has a null vector and hence the excision property, Theorem 2.10 shows that $\lambda_0(\Omega') < 1/4$. Thus Ω' is a fundamental domain for Γ but $\lambda_0(\Gamma) \neq \lambda_0(\Omega')$.

By a Schottky group we mean a discrete group which has a fundamental domain which is a Schottky domain. The main Theorem 3.9 when translated via the Patterson-Sullivan theorem, together with Proposition 5.1 leads to the following fundamental result.

THEOREM 5.4. *For $n \geq 3$ there is $c_n < n$ such that any Schottky group Γ acting on H^{n+1} satisfies*

$$\delta(\Gamma) \leq c_n.$$

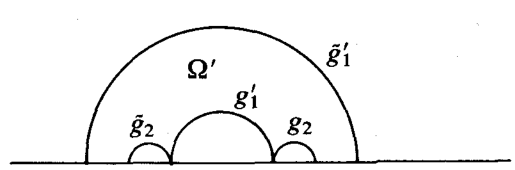


Fig. 5.3

Furthermore if Γ is geometrically finite then the Hausdorff dimension $d(\Lambda_\Gamma)$ (see the introduction) also satisfies

$$d(\Lambda_\Gamma) \leqslant c_n.$$

The following table, computed from (3.14), gives values for c_n.

Table 5.1

n	d_n	c_n
3	0.0263	2.9912
4	0.0412	3.9897
5	0.0446	4.9911
6	0.0416	5.9930
7	0.0357	6.9949
8	0.0288	7.9964

Proof. We may assume $\delta(\Gamma) \geqslant n/2$ for otherwise the claim is obvious. Once $\delta(\Gamma) \geqslant n/2$ we may apply Patterson-Sullivan to conclude that $\delta(n-\delta)=\lambda_0(\Gamma)$. However $\lambda_0(\Gamma) \geqslant \lambda_0(\Omega)$ by Proposition 5.1 where Ω is a fundamental domain for Γ which is Schottky. Applying Theorem 3.9 we have $\lambda_0(\Gamma) \geqslant \lambda_0(\Omega) \geqslant d_n$ which translates to the upper bound on $\delta(\Gamma)$. In the case Γ is geometrically finite, part (ii) of the Patterson-Sullivan theorem yields $d(\Lambda_\Gamma)=\delta(\Gamma)$.

Remark. Given a finite set of spheres in $\mathbf{R}^k$ which are mutually exterior to one another, let Γ be the group generated by inversions in these spheres. Let Λ_Γ be its singular set (i.e. its limit set) in $\mathbf{R}^k$. Γ is naturally a discrete group of isometries of H^{k+1} and is a Schottky group. Theorem 5.4 then shows that when $k \geqslant 3$, $d(\Lambda_\Gamma)$ is uniformly bounded away from k. This is a partial answer to a question raised by Beardon [5] as to whether the Hausdorff dimension of a singular set of an inversion group of $\mathbf{R}^k$ can be made arbitrarily close to k. The case $k=2$ remains unsolved. Also see Akaza [4] where this problem is raised. For the case $k=2$ see the numerical computations at the end of Section 6.

Another application via the Patterson-Sullivan theorem can be based on Corollary 4.5. It leads to

PROPOSITION 5.5. *If $\Gamma_k \overset{\text{simp}}{\to} \Gamma$ is a simple degeneration of reflection groups then as $k \to \infty$*

$$\delta(\Gamma_k) \vee \left(\frac{n}{2}\right) \to \delta(\Gamma) \vee \left(\frac{n}{2}\right)$$

where

$$f \vee g = \max\{f, g\}.$$

Notice that we cannot drop the $\vee$ in the above, since δ itself need not be continuous under a simple degeneration. An example of this is the case of the Hecke groups Γ_μ (see Section 6) where we have $\delta(\Gamma_\mu)>1/2$ for all finite μ, but $\delta(\Gamma_\infty)=0$.

We now turn to applications in the other direction, that is we use the $\delta(\Gamma)$, λ_0 connection to study further the notion of free domains and null vectors. The following two theorems are concerned only with Schottky domains and their reflection groups. The technique used to estimate δ in Theorem 5.6 is due to Beardon [6].

THEOREM 5.6. *Let Ω be a Schottky domain in H^{n+1} with m sides and denote by $\tau(\Omega)$ the separation measure defined in* (3.14). *If*

$$-\log(\tau(\Omega)) > \frac{\log m}{n}, \tag{5.3}$$

then Ω is free.

Proof. Since Ω is a fundamental domain for the reflection group described above, a Γ-invariant smooth function has to satisfy Neumann boundary conditions on $\partial\Omega$. Consequently $\lambda_0(\Gamma)=\lambda_0(\Omega)$ and we need only show that $\lambda_0(\Gamma)=(n/2)^2$ (see (1.7) in the introduction for the definition of $\lambda_0(\Gamma)$). Now the number $\lambda_0(\Gamma)$ is related to the exponent of convergence δ of the series

$$\sum_{\gamma\in\Gamma} \exp(-s(\gamma z, z)); \tag{5.4}$$

here $(\gamma z, z)$ denotes the hyperbolic distance from γz to z. In fact

$$\lambda_0(\Gamma) = \begin{cases} \delta(n-\delta) & \text{if } \delta > n/2, \\ (n/2)^2 & \text{if } \delta \leq n/2. \end{cases}$$

Proofs of this relation appear in [12] and [20]. To establish the theorem it therefore suffices to show that the hypotheses imply

$$\sum_{\gamma\in\Gamma} \exp\left(-\frac{n}{2}(\gamma z, z)\right) < \infty.$$

By assumption $\tau(\Omega)<m^{-1/n}$; as a consequence for some p in $B\cap\bar{\Omega}$

$$\tau(p) < m^{-1/n}. \tag{5.5}$$

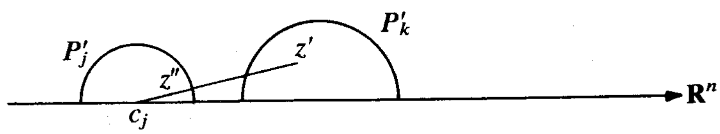


Fig. 5.4

Mapping p into ∞, Ω goes into the domain Ω' pictured in Figure 3.2. We now analyze the action of the group Γ on any point z in Ω'. A word of length k in Γ is of the form:

$$\gamma = R_{j_k} R_{j_{k-1}} \dots R_{j_2} R_{j_1}, \quad R_j \in (R_1, \dots, R_m). \tag{5.6}$$

We also require that this word is of genuine length k, i.e. that there are no repetitions of the kind $R_l R_l$ in the word. As k ranges over the integers, these words yield all the elements of Γ.

For such a word we now estimate $y(\gamma z)$. Notice that the action of each R_j in the word is to take a point from the exterior of P_j' to its interior. Not only that, but also the point in question must have come from the interior of some point P_k', $k \neq j$, except in the case of the initial R_j. The action $z''=R_j z'$, $z=(x, y)$, is illustrated in Figure 5.4. Let $|z'|$ denote the Euclidean distance in $\mathbf{R}^{n+1}$ from z' to the center c_j of P_j'. It follows from the definition of d_j and r_j that

$$|z'| \geq d_j \quad \text{and} \quad y'' = \frac{r_j^2}{|z'|^2} y'.$$

Consequently

$$y'' \leq \left(\frac{r_j}{d_j}\right)^2 y' \leq [\tau(p)]^2 y'.$$

Thus for a word of length k such as γ in (5.6) we have

$$y(\gamma z_0) \leq [\tau(p)]^{2k} y_0;$$

and hence

$$\sum_{\gamma \in \Gamma} y(\gamma z_0)^s \leq y_0^s \sum_{k=0}^{\infty} m^k [\tau(p)]^{2ks}. \tag{5.7}$$

For $s=n/2$ this series converges if $m[\tau(p)]^n < 1$, that is if (5.3) holds.

It is easy to see that

$$(\gamma z_0, z_0) \geq \log (y_0/y(\gamma z_0)).$$

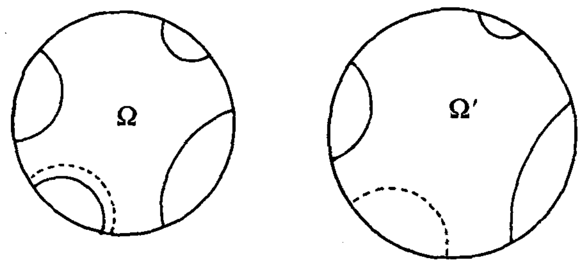


Fig. 5.5

It follows that $\exp(-s(\gamma z_0, z_0)) \leqslant [y(\gamma z_0)/y_0]^s$ and hence by (5.7) that the series (5.4) converges for $s=n/2$. This completes the proof of Theorem 5.6.

The other application is concerned with null vectors. If Ω is free then clearly $\delta(\Gamma)$ can be anything from 0 to $n/2$ inclusive, while $\lambda_0(\Omega)=(n/2)^2$. For the extreme case, $\delta=n/2$, where Ω is barely free, we now show that though there is no L^2 spectrum, there is a null vector. It follows that null vectors, as introduced in Section 2 occur in abundance.

THEOREM 5.7. *Let Ω be a free Schottky domain without cusps, then Ω has a null vector iff $\delta(\Gamma)=n/2$.*

Remarks. (i) The proof of Theorem 5.7 may easily be adapted to show that if Γ is a "convex co-compact" group, with $\delta(\Gamma)=n/2$ then X^{n+1}/Γ has a null vector. (For the definition of a convex co-compact group of isometries in X^{n+1} and related material see Sullivan [20].)

(ii) The assumption that Ω has no cusps, possibly could be dropped.

Proof of Theorem 5.7. First we prove that if Ω has a null vector then $\delta(\Gamma)=n/2$. Since Ω is free it follows by the Patterson-Sullivan result that $\delta(\Gamma)\leqslant n/2$. We show that $\delta(\Gamma)<n/2$ leads to a contradiction. If Ω is a Schottky domain without cusps, then it looks like one of the domains in Figure 5.5.

Now consider deforming Ω a little to Ω', by increasing the radius of one of the bounding hyperplanes. Since the region $\Omega\setminus\Omega'$ is bounded by concentric hemispheres, it follows by Proposition 2.11 that it is strictly free. Therefore by Theorem 2.10, Ω' is not free, and this is so no matter how small the perturbation $\Omega\to\Omega'$. If Γ' is the reflection group generated by Ω', then $\delta(\Gamma')>n/2$ by the Patterson-Sullivan result. However we claim that $\delta(\Gamma)$ is continuous under the deformation in question, and hence for Γ' (or Ω') sufficiently close to Γ, $\delta(\Gamma')<n/2$ which is a contradiction. To see

that $\delta(\Gamma)$ is continuous under such a deformation we first observe that the deformation is quasiconformal, i.e. there exists a map $\varphi: X^{n+1} \to X^{n+1}$ which is quasiconformal and $\varphi\Gamma\varphi^{-1}=\Gamma'$. One can show this by arguments similar to the ones we used above to show that Ω and Ω' are K-quasi-isometric. Furthermore as $\Gamma' \to \Gamma$ the K in the quasi conformal mapping can be chosen to tend to 1. Now since φ conjugates Γ to Γ', the extension of φ to the boundary (see Ahlfors [1]) will take Λ_Γ to $\Lambda_{\Gamma'}$, where Λ_Γ, $\Lambda_{\Gamma'}$ are the limit sets of Γ and Γ' respectively. The K-quasiconformal map φ on the boundary will satisfy a Hölder estimate (see Ahlfors [1]), and it follows that the Hausdorff dimension of $\Lambda_{\Gamma'}$ and Λ_Γ will be close if K is near one. In view of the Patterson-Sullivan theorem $\delta(\Gamma') \to \delta(\Gamma)$. This completes the proof one way. For a similar discussion see Patterson [9].

To prove the converse we need to make use of the Patterson measure and associated eigenfunction (Sullivan [20]). The measure μ is positive and finite and is supported on the limit set Λ_Γ. For $w \in B^{n+1}$, $\beta \in B$, we let $\langle w,\beta\rangle$ be the signed distance from 0 to the horosphere through w tangent at β.

The relevance of μ to us, is that the function

$$\varphi(w) = \int_\Lambda \exp\left\{\frac{n}{2}\langle w,\beta\rangle\right\} d\mu(\beta)$$

satisfies

(i) $\Delta\varphi=(n/2)^2\varphi$

(ii) $\varphi(\gamma w)=\varphi(w)$, $\forall \gamma \in \Gamma$.

All this follows from $\delta(\Gamma)=n/2$, for proofs see (Sullivan [20]).

This positive Γ invariant eigenfunction is clearly a natural candidate for a null vector. Indeed all we need to show is that it is in the G-norm completion of functions in $W^1(\Omega)$. To see that this is the case we need some elementary estimates for the function $\psi(w,\beta)=\exp(\frac{1}{2}n\langle w,\beta\rangle)$. Let $r=(0,w)$ denote the hyperbolic distance from 0 to w.

LEMMA 5.8. (i) $|\psi(w,\beta)| \ll \exp\{-\frac{1}{2}nr\}$ *and*

(ii) $|\nabla\psi(w,\beta)| \ll \exp\{-\frac{1}{2}nr\}$,

if the angle between $\overrightarrow{0w}$ *and* $\overrightarrow{0\beta}$ *is bounded from below (i.e. the implied constants in* (i) *and* (ii) *depend only on the lower bound for this angle).*

Proof. It is more convenient to work in the upper half space model H^{n+1}. On mapping β into ∞, the condition that the angle be bounded away from zero goes into the

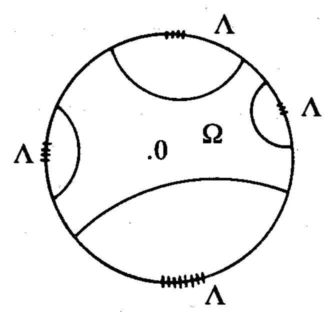


Fig. 5.6

condition that $w=(x,y)$ have $|x|$ bounded. We assume that 0 mapped into $j=(0,1)$. In this case $\langle w,\infty\rangle=\log y$.

$$\psi(w,\infty)=y^{n/2}$$
$$|\nabla\psi|=\left(\frac{n}{2}\right)y^{n/2}. \tag{5.8}$$

On the other hand

$$\cosh(j,w)=\cosh r=1+\frac{x^2+(y-1)^2}{2y}$$

and hence as $y\to 0$ with $|x|$ bounded we have

$$y\leqslant\frac{c}{\cosh r}.$$

The estimates (i) and (ii) follow immediately from this and (5.8).

Since Ω (pictured in Figure 5.6) is geometrically finite, Schottky with no cusps, it is clear that the limit set Λ is a bounded distance (in the spherical metric on S^n) from $\partial\Omega\cap S^n$. From this observation, Lemma 5.8, and the representation

$$\varphi(w)=\int_\Lambda \exp\left\{\frac{n}{2}\langle w,\beta\rangle\right\}d\mu(\beta)$$

it follows that for $w=(r,\theta)$

$$|\varphi|\ll\exp\left\{-\frac{n}{2}r\right\}$$
$$|\nabla\varphi|\ll\exp\left\{-\frac{n}{2}r\right\} \tag{5.9}$$

if θ is any direction in a small enough neighborhood, N_1, of those directions from 0 which see the boundary of X^{n+1} along geodesics in Ω.

Let R_k be a sequence of reals with $R_k \to \infty$ with k. Let χ_k, defined on $[0, \infty)$, be smooth and satisfy

$$\chi_k(r) = \begin{cases} 1 & \text{for } r \leq R_k \\ e^{-ar} e^{aR_k} & \text{for } r > R_k + 1 \end{cases}$$

for some $a > 0$. Also χ_k can be chosen so that

$$\left| \frac{\partial}{\partial r} \chi_k(r) \right| \leq C e^{aR_k - ar} \quad \text{for } r \geq R_k.$$

Define $\varphi_k(w) = \chi_k(r_w)\, \varphi(w)$ on Ω. In view of the estimates (5.9) we have

$$\int_\Omega |\varphi_k|^2 \, dV \ll \int_{\theta \in N_1} \int_0^\infty e^{-2ar} e^{-nr} e^{nr} \, dr \, d\theta < \infty.$$

Similarly,

$$\int_\Omega |\nabla \varphi_k|^2 \, dV < \infty$$

so that φ_k lies in $W^1(\Omega)$. We calculate

$$E(\varphi_k) = D(\varphi_k) - \left(\frac{n}{2}\right)^2 H(\varphi_k).$$

We can write E as

$$\begin{aligned} E(\varphi_k) &= \int_{\Omega \cap \{w, r \leq R_k\}} |\nabla \varphi|^2 \, dV - \left(\frac{n}{2}\right)^2 \int_{\Omega \cap \{w, r \leq R_k\}} |\varphi|^2 \, dV \\ &\quad + \int_{\Omega \cap \{w, r > R_k\}} |\nabla \varphi_k|^2 \, dV - \left(\frac{n}{2}\right)^2 \int_{\Omega \cap \{w, r > R_k\}} |\varphi_k|^2 \, dV \\ &= \mathrm{I} - \mathrm{II} + \mathrm{III} - \mathrm{IV} \end{aligned}$$

Recalling that $\partial_n \varphi = 0$ along the sides of Ω, we integrate I by parts to get

$$\begin{aligned} \mathrm{I} &= \int_{\partial(\Omega \cap \{w, r \leq R_k\})} (\nabla \varphi \cdot n)\, \varphi \, dS + \int_{\Omega \cap \{w, r \leq R_k\}} (\Delta \varphi)\, \varphi \, dV \\ &= \mathrm{V} + \left(\frac{n}{2}\right)^2 \int_{\Omega \cap \{w, r \leq R_k\}} |\varphi|^2 \, dV, \end{aligned}$$

where

$$\mathrm{V}=\int_{\Omega\cap\{r=R_k\}}\frac{\partial\varphi}{\partial r}\varphi\,dS.$$

It follows that

$$E(\varphi_k)=\mathrm{V}+\mathrm{III}-\mathrm{IV}.$$

However each of these last terms is bounded independent of k. In applying Lemma 5.8 we get

$$\mathrm{V}\ll\int_{S^n}(e^{-\frac{1}{2}nr})^2 e^{nr}\,dS=O(1)$$

and

$$\begin{aligned}\mathrm{IV}&=\left(\frac{n}{2}\right)^2\int_{\Omega\cap\{w,r>R_k\}}|\chi_n(r)\,\varphi(w)|^2\,dw\\ &\ll\int_{S^n}\int_{R_k}^{\infty}e^{-2ar+2aR_k}e^{-nr}e^{nr}\,dr\,d\theta\\ &\ll e^{2aR_k}e^{-2aR_k}=O(1);\end{aligned}$$

and similarly for III.

Now from (2.13) and Remark 3, since $E\geq 0$ the G form can be chosen as

$$G=E+K\quad\text{where}\quad K(u)=\int_S u^2\,dV. \tag{5.10}$$

$S\subset\Omega$ any compact subset. Since eventually $\varphi_k=\varphi$ on such an S, it follows that $K(\varphi_k)=O(1)$.

We have therefore shown that

$$G(\varphi_k)\leq C\quad\text{for some } C.$$

It follows that some subsequence φ_{k_j} has a weak limit in $\mathbf{H}_G$, call it h. Since the G form majorizes the L^2 norm over any compact subset S^1 of Ω, the φ_{k_j} will also converge weakly to h in $L^2(S^1)$. However, $\varphi_k=\varphi$ in S for k sufficiently large and it follows from this that $h=\varphi$ locally and hence globally.

Finally for j sufficiently large and any smooth ψ vanishing near B, integration by parts yields

$$G(\psi,\varphi_{k_j})=\int_\Omega \nabla\psi\nabla\varphi_{k_j}\,dV-\left(\frac{n}{2}\right)^2\int_\Omega \psi\varphi_{k_j}\,dV+\int_S \psi\varphi_{k_j}\,dV$$
$$=\int_S \psi\varphi_{k_j}\,dV.$$

Taking the limit as $j\to\infty$, we get

$$G(\psi,\varphi)=\int_S \psi\varphi\,dV.$$

From (5.10) we see that $E(\psi,\varphi)=0$ and since such ψ are dense in $\mathbf{H}_G$, we conclude that $E(\varphi)=0$. It now follows by Lemma 2.6 that φ is indeed a null vector of Δ'. This completes the proof of Theorem 5.7.

6. Applications to particular examples and numerical calculations

In this section we apply our theory to examples of various kinds. We also study carefully the case of Hecke groups and give some numerical computations (done by machine) of the Hausdorff dimension of the limit sets of many Schottky groups. We begin this section with an example of a group Γ which is geometrically nonfinite, but which nevertheless has a discrete spectrum in the interval $[0,(n/2)^2)$. Moreover $\lambda_0(\Gamma)<(n/2)^2$ and the corresponding base eigenfunction is square integrable on the fundamental domain. This answers a question raised by Sullivan [21]. However it should be noted that the Γ in our example is not finitely generated.

Our example is a Schottky group whose fundamental domain Ω in H^3 is exterior to a set of hemispheres $S_1, S_2, \dots$ whose intersection with $B=\mathbf{R}^2$ are the circles $C_1, C_2, \dots$ shown in Figure 6.1. The circles C_1, C_2 and C_3 are mutually tangent, while $C_4, C_5, \dots$ march out to infinity in such a way that the dotted vertical straight lines L_j separate them as shown. A typical subdomain in H^3, say like Ω_4, which is bounded by the hyperplanes corresponding to L_4, L_5 and C_4, is free. This follows from Theorem 3.7. By Corollary 2.13 the region to the right of L_4 is free since it is the union of free regions. It follows that Ω is the union of a region which has the finite geometric property and a free region. Thus by Corollary 2.5 the Laplacian on Ω has a discrete spectrum in $[0,(n/2)^2)$.

In the region exterior to S_1, S_2, S_3 the augmented Laplacian Δ' has a null vector; in fact this region is isometric to the domain discussed just before Proposition 3.5. It therefore follows by the excision property (remove S_4 and use Theorem 2.10) and

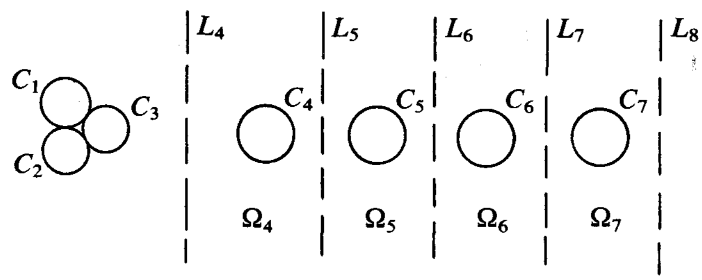


Fig. 6.1

monotonicity (remove the rest of the S_j's and use Proposition 2.12) that Ω is not free, i.e. that $\lambda_0(\Omega)<(n/2)^2$. Since the spectrum of the Laplacian is discrete below $(n/2)^2$, the bottom of the spectrum has an L^2 eigenfunction.

Now if Γ is generated by the reflections in the sides of Ω, then, as we have previously noted, the $L^2(H^3/\Gamma)$ spectral problem and the Neumann spectral problem coincide. Hence the group Γ corresponding to the domain in the above example has the asserted properties. If we want an example consisting only of orientation preserving isometries, we simply take the index two subgroup Γ_e of Γ of words of even length.

Our next example is a discrete Schottky group Γ for which the topological and radial limit sets have different Hausdorff dimension (see Sullivan [20] for the definition of the radial limit set). Examples of this phenomenon are known, see Patterson [16], however Theorem 5.4 allows for particularly simple constructions. Let $S_1, S_2, \ldots$ be a sequence of spheres in $\mathbf{R}^3$, which are mutually external to each other and for which

$$m\left(\mathbf{R}^3 \setminus \bigcup_i S_i\right) = 0$$

m being Lebesgue measure. Such a sequence of spheres obviously exists. Let Γ be the group generated by reflections in these spheres. It is easy to show that the topological limit set of Γ is all of $\mathbf{R}^3$. On the other hand, from Table 5.1 $\delta(\Gamma)\leq 2.992$. However it is known (see Sullivan [20] Theorem 24 or [24]) that if Λ_{rad} is the radial limit set then $d(\Lambda_{\mathrm{rad}})\leq\delta(\Gamma)$. Hence

$$d(\Lambda_{\mathrm{rad}}) \leq 2.992 < 3 = d(\Lambda_{\mathrm{top}}), \tag{6.1}$$

which proves our assertion. Notice that Γ is not geometrically finite; it cannot be if this phenomenon is to occur, since by Beardon, Maskit [24], Λ_{rad} and Λ_{top} differ by only a countable set in the geometrically finite case.

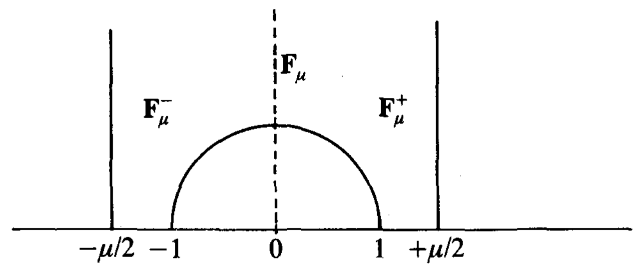


Fig. 6.2

A rather simple yet interesting family of groups are the so called Hecke groups. In dimension $n=1$ these groups are parametrized by a real number μ, where Γ_μ is generated by

$$T_\mu: z \to z+\mu$$
$$S: z \to -1/z.$$

We assume that $\mu \geq 2$, these being the only cases of interest in this paper. A fundamental domain $\mathbf{F}_\mu$ for Γ_μ is depicted in Figure 6.2.

It is clear that these Hecke groups are examples of symmetric Schottky groups which we discussed after Proposition 5.3. Therefore the discrete spectrum for $L^2(H/\Gamma_\mu)$ is the same as that of $\mathbf{F}_\mu^+$ with Neumann boundary conditions. Our theory is therefore applicable.

THEOREM 6.1. *For $\mu>2$ the Hecke groups Γ_μ have precisely one discrete eigenvalue $\lambda_0(\mu)$. As μ ranges from 2 to ∞, $\lambda_0(\mu)$ increases continuously and strictly monotonically from 0 to 1/4.*

Remark. The part of the theorem concerning the variation of $\lambda_0(\mu)$ from 0 to 1/4 when translated to $\delta(\Gamma_\mu)$ was proved by Beardon [6] by direct combinatorial methods. The monotonicity of δ was first proved by Elstrodt and independently by Patterson [9]. Our proof is quite different since it relies on the various continuity theorems that were proved in Section 4. Our proof of the uniqueness of $\lambda_0(\mu)$ is based on the method used in Sarnak [18], but in view of our theory of free domains is even simpler.

Proof of Theorem 6.1. We have pointed out that the discrete spectrum for $L^2(H^2/\Gamma_\mu)$ corresponds to the Neumann problem for $\mathbf{F}_\mu^+$. The continuity of the discrete spectrum for $2 \leq \mu < \infty$, follows from Theorem 4.2. Now $\mathbf{F}_2$ has finite volume so $\lambda_0(2)=0$. That $\lambda_0(\mu)$ strictly increases with μ follows from the monotonicity (Proposi-

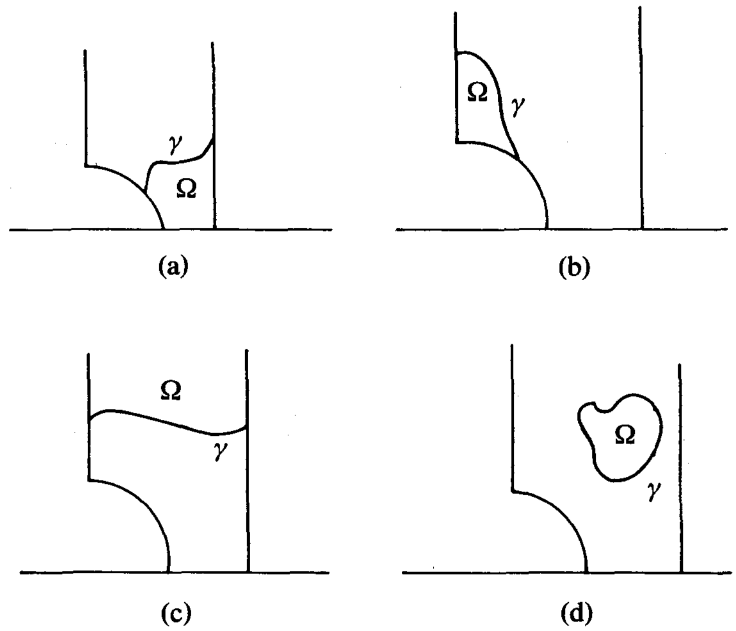


Fig. 6.3

tion 2.12) since $\mathbf{F}^+_{\mu_1} \setminus \mathbf{F}^+_{\mu_2}$ is free for $\mu_1 > \mu_2$. Also the continuity under a degeneration of a side (Corollary 4.6) shows that $\lambda_0(\mu) \to \lambda_0(\infty)$ as $\mu \to \infty$. Since $\mathbf{F}^+_\infty$ is free, being bounded by only two sides in H^2, it follows that $\lambda_0(\mu) \to 1/4$ as $\mu \to \infty$. We have therefore proven all but the fact that there are no other eigenvalues besides $\lambda_0(\mu)$. By Theorem 2.1, $[1/4, \infty)$ has no L^2 eigenvalues if $\mu > 2$, thus to prove the rest of Theorem 6.1, it suffices to show that $\lambda_1(\mu) \geq 1/4$.

Suppose $\lambda_1(\mu) < 1/4$, then by Lemma 2.3, there is an eigenfunction u_1 of Δ in $W^1(\mathbf{F}^+_\mu)$. Also u_1 is orthogonal to u_0, u_0 being the base eigenfunction corresponding to λ_0. As in Lemma 2.7 we know that $u_0 > 0$ on $\mathbf{F}^+_\mu$. It follows that u_1 takes on both positive and negative values. Let N_1 be its nodal set i.e. $\{z \in \mathbf{F}^+_\mu;\ u_1(z) = 0\}$, which is nonempty and which separates the set where $u_1 > 0$ from where $u_1 < 0$. As such we can clearly find a component γ of N_1 which looks like one of the arcs pictured in Figure 6.3.

For each situation of the nodal curve let Ω be the domain indicated. Clearly u_1 is the base eigenfunction in Ω with Dirichlet boundary conditions on γ and hence $\lambda_1(\Gamma) = \lambda_0(\Omega)$. Since $u_1 = 0$ on γ, we may clearly, by Dirichlet monotonicity, claim that $\lambda_0(\Omega) \geq \lambda_0(\Omega')$ where for each of (a), (b), (c), and (d), Ω' is pictured in Figure 6.4. By

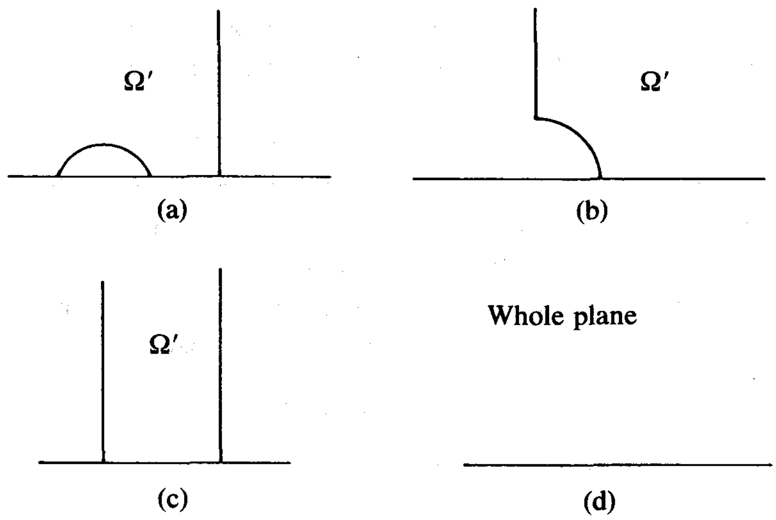


Fig. 6.4

theorem 3.7 each Ω' above is free. Thus $\lambda_0(\Omega')$ and hence $\lambda_0(\Omega)$ is $\geqslant 1/4$. This completes the proof of Theorem 6.1.

The proof just given of $\lambda_1(\mu)\geqslant 1/4$ holds just as well for $0<\mu\leqslant 2$; the case $\mu=1$ corresponds to the modular group, for which $\lambda_1\geqslant 1/4$ is a well known result.

In the case of 3 dimensions, i.e. $n=2$, we may consider analogues of the Hecke groups, viz. Γ_μ generated by

$$\begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & i\mu \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{in } SL_2(\mathbf{C}).$$

It is easy to see that the base eigenfunction u_0 corresponding to the lowest eigenvalue $\lambda_0(\mu)$ satisfies Neumann boundary conditions. In fact, denote the group generated by reflections through the surfaces $x_1=\mu/2$, $x_2=\mu/2$, $x_1=-\mu/2$, $x_2=-\mu/2$ and the hemisphere of radius 1 about $(0,0,0)$ by Γ_μ^*. Then averaging u_0 over Γ_μ^*/Γ_μ will give the base eigenfunction for Γ_μ^*. (Since $u_0>0$ there can be no cancellations in the averaging.)

It follows as in the proof of Theorem 4.2 that $\lambda_0(\mu)$ is continuous over the range $2\leqslant\mu<\infty$. Since the prism $\{|x_1|\leqslant\mu/2,\ |x_2|\leqslant\mu/2\}$ has a null vector, Theorem 2.10 shows that $\lambda_0(\mu)<1$ in this range. Proposition 2.12 asserts that $\lambda_0(\mu)$ is strictly increasing in μ. Finally we can use Theorem 4.4 to prove that $\lambda_0(\mu)\to 1$ as $\mu\to\infty$. To do so we set $\tilde{\Omega}=\Omega_{\mu_0}$ for a fixed μ_0.

As we saw in Section 3 when discussing 2-lattice Schottky domains, the value of $\lambda_0(2)$ is of special interest. We shall see shortly that a good approximation to this is 0.66.

In fact we present finally some data obtained with the help of a computer, on $\delta(\Gamma)$ for Schottky domains in H^3. A domain Ω, bounded by disjoint hemispheres $P_1, P_2, \ldots$, is a fundamental domain for the group Γ of inversions through the hemispheres $P_1, P_2, \ldots$. The P_i's are in turn determined by their intersections with B; that is the $C_i = P_i \cap B$. We shall compute $\delta(\Gamma)$ from the asymptotic distribution of orbital points. According to a result of Lax and Phillips [12], if $\delta(\Gamma) > 1$, then

$$N(w, s) \sim c(w) \exp(\delta(\Gamma) s) \tag{6.2}$$

as s becomes infinite, where

$$N(w, s) = \#[\gamma;\ \gamma \in \Gamma,\ (\gamma w, w) \leq s]. \tag{6.3}$$

It is clear that

$$\Delta(s) = \log[N(w, s)] - \log[N(w, s-1)] \tag{6.4}$$

approximates $\delta(\Gamma)$ for large s. As we shall see, this is even a good approximation for rather small s.

We have determined $N(j, s)$, $j = (0, 0, 1)$, for various Schottky domains characterized by different sets of circles chosen from the following:

	x_1	x_2	Radius
C_1	−0.577350	1.000000	1.000000
C_2	−0.577350	−1.000000	1.000000
C_3	1.154701	0.000000	1.000000
C_4	0.000000	0.000000	2.154701
C_5	0.000000	0.000000	0.154701
C_6	−1.672028	0.000000	0.482673
C_7	−1.073180	0.000000	0.116175
C_8	−0.217482	0.000000	0.062782
C_9	0.836014	1.448018	0.482673
C_{10}	0.836014	−1.448018	0.482673
C_{11}	0.108741	0.188345	0.062782
C_{12}	0.108741	−0.188345	0.062782
C_{13}	0.536590	0.929401	0.116175
C_{14}	0.536590	−0.929401	0.116175

These circles are pictured in Figure 6.5. Certain combinations are equivalent with respect to isometries. Along with a list of circles used to determine Ω we shall also

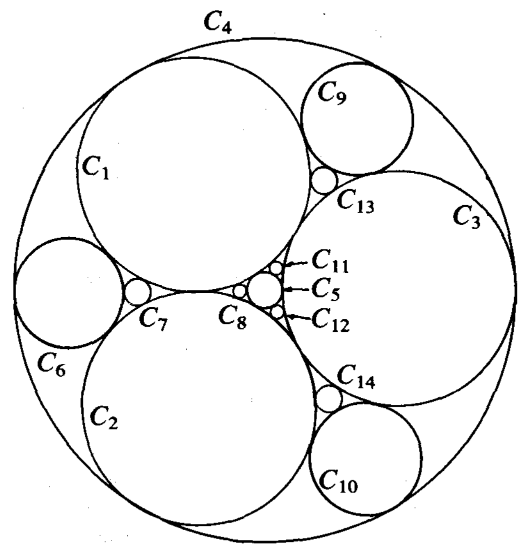
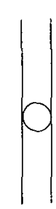


Fig. 6.5

picture a canonical form of the domain obtained by an inversion with respect to the point $(-0.577350,0)$.

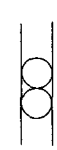
$\Omega_1: C_1, C_2, C_3$

s	$N(s,j)$	$\log[N(s,j)]$	$\Delta(s)$
5	123	4.8122	One circle
6	309	5.7333	0.921
7	831	6.7226	0.989
8	2 187	7.6903	0.968
9	6 033	8.7050	1.015
10	16 557	9.7146	1.009
11	44 997	10.7144	0.999
12	121 995	11.7117	0.997



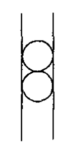
$\Omega_2: C_1, C_2, C_3, C_5$

s	$N(s,j)$	$\log[N(s,j)]$	$\Delta(s)$
5	151	5.0173	Two circles
6	529	6.2710	1.254
7	1 915	7.5575	1.287
8	6 832	8.8294	1.272
9	25 375	10.1415	1.312
10	94 135	11.4525	1.311
11	347 380	12.7582	1.306
12	1 278 563	14.0613	1.303



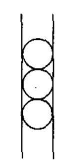
Ω_3: C_1, C_2, C_3, C_4

s	$N(s, j)$	$\log[N(s, j)]$	$\Delta(s)$
5	322	5.7746	Two circles
6	1 228	7.1131	1.339
7	4 708	8.4570	1.344
8	17 242	9.7551	1.298
9	63 796	11.0635	1.308
10	235 366	12.3689	1.305
11	868 942	13.6750	1.306
12	3 210 846	14.9821	1.307



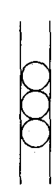
Ω_4: C_1, C_2, C_3, C_4, C_6

s	$N(s, j)$	$\log[N(s, j)]$	$\Delta(s)$
5	346	5.8464	Three circles
6	1 416	7.2556	1.409
7	5 781	8.6623	1.407
8	23 109	10.0480	1.386
9	94 051	11.4516	1.404
10	381 266	12.8513	1.400
11	1 553 242	14.2559	1.405
12	6 330 035	15.6608	1.405



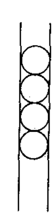
Ω_5: C_1, C_2, C_3, C_5, C_6

s	$N(s, j)$	$\log[N(s, j)]$	$\Delta(s)$
5	151	5.0173	Three circles
6	538	6.2879	1.271
7	2 013	7.6074	1.320
8	7 730	8.9529	1.346
9	30 973	10.3409	1.388
10	124 701	11.7337	1.393
11	504 242	13.1308	1.397
12	2 041 506	14.5292	1.398



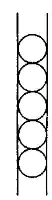
Ω_6: $C_1, C_2, C_3, C_4, C_5, C_6$

s	$N(s, j)$	$\log[N(s, j)]$	$\Delta(s)$
5	374	5.9243	Four circles
6	1 668	7.4194	1.495
7	7 186	8.8799	1.461
8	30 518	10.3261	1.446
9	131 287	11.7851	1.459
10	561 181	13.2378	1.453
11	2 410 514	14.6954	1.458
12	10 334 641	16.1510	1.456



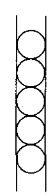
Ω_7: $C_1, C_2, C_3, C_4, C_5, C_6, C_7$

s	$N(s, j)$	$\log[N(s, j)]$	$\Delta(s)$
5	374	5.9243	Five circles
6	1 673	7.4224	1.498
7	7 255	8.8894	1.467
8	31 214	10.3486	1.459
9	136 428	11.8236	1.475
10	594 316	13.2952	1.472
11	2 606 290	14.7734	1.478



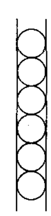
Ω_8: $C_1, C_2, C_3, C_4, C_5, C_6, C_8$

s	$N(s, j)$	$\log[N(s, j)]$	$\Delta(s)$
5	374	5.9243	Five circles
6	1 677	7.4248	1.501
7	7 284	8.8934	1.469
8	31 498	10.3577	1.464
9	138 120	11.8359	1.478
10	602 900	13.3095	1.474
11	2 651 303	14.7906	1.481



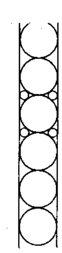
Ω_9: $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8$

s	$N(s, j)$	$\log[N(s, j)]$	$\Delta(s)$
5	374	5.9243	Six circles
6	1 682	7.4277	1.503
7	7 353	8.9029	1.475
8	32 194	10.3795	1.477
9	143 261	11.8724	1.493
10	636 035	13.3630	1.491
11	2 847 301	14.8619	1.499



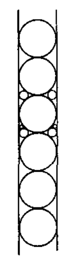
Ω_{10}: $C_1, C_2, C_3, C_4, C_5, C_6, C_8, C_9, C_{10}, C_{11}, C_{12}$

s	$N(s, j)$	$\log[N(s, j)]$	$\Delta(s)$
5	422	6.0450	Six large–four small
6	2 071	7.6358	1.591
7	9 626	9.1722	1.536
8	45 350	10.7221	1.550
9	215 720	12.2817	1.560
10	1 014 946	13.8304	1.549
11	4 817 451	15.3878	1.557



Ω_{11}: $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}, C_{13}, C_{14}$

s	$N(s, j)$	$\log[N(s, j)]$	$\Delta(s)$
5	422	6.0450	Six large–four small
6	2 086	7.6430	1.598
7	9 833	9.1935	1.551
8	47 438	10.7672	1.574
9	231 143	12.3508	1.584
10	1 114 627	13.9240	1.573
11	5 414 493	15.5046	1.581



A few remarks are in order. As we have already noted in Section 3 (just before Proposition 3.5), Δ' has a null vector over Ω_1. It therefore follows from Theorem 4.7 that $\delta(\Omega_1)=1$. Thus $\Delta(10)$ and $\Delta(11)$ are correct to within 0.003 while $\Delta(9)$ is correct to within 0.01. Since Ω_1 has the excision property by Theorem 2.10, the domains Ω_2 and Ω_3 are no longer free and $\delta(\Gamma)>1$ for these domains. Since they have only four sides, they provide examples for Proposition 3.5; these are essentially the same examples as those constructed by Akaza [2]. The limit set for the common canonical form Ω_2 and Ω_3 is the Appolonian grid, pictured on page 187 of Mandelbrot's book on Fractals [13]. The Hausdorff dimension of this grid can be shown to be the same as the Appolonian packing constant (Sullivan, private communication). For the definition of this constant D and the estimate $1.3002<D<1.3145$ see D. W. Boyd [8]. Using a computer to calculate D, Z. A. Melzak [14] arrived at the value 1.30695. In our computations $\Delta(9)$, $\Delta(10)$, $\Delta(11)$ and $\Delta(12)$ are in agreement with this number to within 0.005.

The domains Ω_j, $j=1,2,\ldots,9$, are partial linear lattices. We have also computed δ for the full linear lattice Ω. According to Theorem 3.12 $\delta(\Omega)=\delta(\Omega_0)$ for the domain Ω_0 described in (3.35). We found that $\delta(\Omega_0)=1.585\pm0.004$, which corresponds to $\lambda_0=0.658\pm0.002$.

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