The Corona theorem for Denjoy domains

by

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§ 1. Introduction

Denote by $H^\infty(\mathscr{D})$ the space of bounded analytic functions on a plane domain $\mathscr{D}$ and give functions in $H^\infty(\mathscr{D})$ the supremum norm

$$\|f\| = \sup_{z\in\mathscr{D}} |f(z)|.$$

A Denjoy domain is a connected open subset Ω of the extended complex plane $\mathbf{C}^*$ such that the complement $E=\mathbf{C}^*\setminus\Omega$ is a subset of the real axis $\mathbf{R}$.

THEOREM. *If Ω is any Denjoy domain and if $f_1, \dots, f_N \in H^\infty(\Omega)$ satisfy*

$$0<\eta\leq \max_j |f_j(z)| \leq 1 \tag{1.1}$$

for all $z\in\Omega$, then there exist $g_1,\dots,g_N\in H^\infty(\Omega)$ such that

$$\sum f_j(z)\, g_j(z) = 1, \quad z\in\Omega. \tag{1.2}$$

Such a theorem is called a corona theorem (had the theorem been false for Ω the unit disc, there would have been a set of maximal ideals suggestive of the sun's corona), and the g_j are called corona solutions. It follows from the methods in Gamelin [6] that the theorem is equivalent to itself plus the further conclusion

$$\|g_j\| \leq C(N,\eta),$$

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where $C(N,\eta)$ does not depend on Ω, and the proof below gives such bounds on the solutions. Thus, by normal families, the reader can make the simplifying but unused assumption that E is a finite union of intervals.

For Ω a Denjoy domain, $H^\infty(\Omega)$ consists only of constants if and only if E has Lebesgue measure zero, $|E|=0$, (see [1]). We will not need that result, but we will repeatedly use the idea of its proof: if

$$f(z)=\frac{1}{\pi i}\int_E \frac{dt}{t-z},\quad z\in\Omega$$

then $e^f\in H^\infty(\Omega)$. Our proof depends expressly on the symmetry of Denjoy domains and implicitly on the fact that for linear sets there are simple relations between length, harmonic measure relative to the upper half plane, and analytic capacity [8].

The first corona theorem, for Ω simply connected, is due to Lennart Carleson [3]. Several authors have extended his theorem to other types of domains; see [15] for a historical discussion. The deepest extension is also due to Carleson [4], who proved the theorem when Ω is a Denjoy domain for which E is uniformly thick:

$$|E\cap(x-t,\ x+t)|\geqslant ct$$

for all $x\in E$ and all $t>0$. (See [15] for another proof of that result and [14] for a generalization to non-Denjoy domains.) Similarily, here the construction will take place inside a set $\Omega_1(\varepsilon)$ where E is thick in the sense that

$$\frac{1}{\pi}\int_E \frac{|y|}{y^2+(x-t)^2}\,dt>\varepsilon.$$

We also use Carleson's first theorem and the construction from its proof. In fact, our proof is quite close to his original argument. The differences are that instead of estimating norms by duality, we solve the corresponding $\bar\partial$ problem constructively, as originated in [12] and [13] and as used in [4], and that our contour can be taken in $\Omega_1(\varepsilon)$, because Ω is a Denjoy domain.

In section 2 we solve an interpolation problem needed for the theorem and in section 3 we prove the theorem. Section 4 has further remarks and complementary results.

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§ 2. Some interpolating sequences

Write U for the upper half plane. Fix a Denjoy domain $\Omega=\mathbf{C}^*\setminus E$, let $\varepsilon>0$, and define

$$\Omega_1(\varepsilon)=\left\{x+iy:\frac{1}{\pi}\int_E\frac{|y|\,dt}{y^2+(x-t)^2}>\varepsilon\right\}$$

and

$$\Omega_1^+(\varepsilon)=U\cap\Omega_1(\varepsilon).$$

The harmonic function

$$\omega(z,E)=\frac{1}{\pi}\int_E\frac{y\,dt}{y^2+(x-t)^2},\quad z=x+iy\in U,$$

is by definition the harmonic measure of E in U, and $\Omega_1^+(\varepsilon)=\{z:\omega(z,E)>\varepsilon\}$.

Let $\{z_n\}$ be a sequence of points in some plane domain $\mathcal{D}$. Then $\{z_n\}$ is called an *interpolating sequence* for $H^\infty(\mathcal{D})$ if, whenever $|w_n|\leq 1$, there exists $f\in H^\infty(\mathcal{D})$ such that

$$f(z_n)=w_n,\quad n=1,2,\ldots. \tag{2.1}$$

The bound

$$M(\{z_n\},\mathcal{D})=\sup_{|w_j|\leq 1}\inf\{\|f\|:f\in H^\infty(\mathcal{D})\text{ and (2.1) holds}\}$$

then is finite. Carleson's interpolation theorem [2], [9 p. 287] asserts that $\{z_n\}\subset U$ is an interpolating sequence for $H^\infty(U)$ if and only if

$$\delta(\{z_n\})=\inf_n\prod_{k,k\neq n}\left|\frac{z_n-z_k}{z_n-\bar{z}_k}\right|>0.$$

We need a similar result for Denjoy domains.

LEMMA 2.1. *Let $\varepsilon>0$ and let $\{z_n\}$ be a sequence in $\Omega_1^+(\varepsilon)$. There are positive constants $\gamma=\gamma(\varepsilon)$ and $M=M(\varepsilon)$ such that if*

$$\delta(\{z_n\})>1-\gamma \tag{2.2}$$

then (z_n) is an interpolating sequence for $H^\infty(\Omega)$ and

$$M(\{z_n\},\Omega)\leq M. \tag{2.3}$$

The constants $\gamma(\varepsilon)$ and $M(\varepsilon)$ do not depend on Ω. But if $|E|=0$, so that $H^\infty(\Omega)$ is trivial, then $\Omega_1^+(\varepsilon)=\emptyset$ and the lemma is vacuous. It is also true that if $\{z_n\}\subset\Omega_1^+(\varepsilon)$ and if $\delta(\{z_n\})>0$, then $\{z_n\}$ is an interpolating sequence for $H^\infty(\Omega)$, and a proof will be given in section 4. However, we need only Lemma 2.1 for the corona theorem.

Proof. By normal families it is sufficient to prove the lemma with (2.3) when $\{z_n\}=\{z_1,\dots,z_{n_0}\}$ is a finite sequence. We may also reorder the points so that

$$y_1 \geqslant y_2 \geqslant \dots \geqslant y_{n_0},$$

where $z_n=x_n+iy_n$. Fix

$$I_n=\{t\in\mathbf{R}: |t-x_n|<3\varepsilon^{-1}y_n\}$$

so that

$$\omega(z_n, I_n)=\frac{1}{\pi}\int_{-3\varepsilon^{-1}y_n}^{3\varepsilon^{-1}y_n}\frac{y_n\,dt}{t^2+y_n^2}>1-\varepsilon/3. \tag{2.4}$$

For $\beta=\beta(\varepsilon)<\varepsilon/3$ to be determined later, also fix

$$J_n=\{t\in\mathbf{R}: |t-x_n|<\beta^{-1}y_n\}$$

so that

$$\omega(z_n, J_n)>1-\beta. \tag{2.5}$$

The inequality

$$\sum_{k,\,k\neq n}\frac{4y_k y_n}{|z_n-\bar z_k|^2}\leqslant -\log\prod_{k,\,k\neq n}\left|\frac{z_n-z_k}{z_n-\bar z_k}\right|^2$$

which can be found on page 288 of [9], shows that

$$\sum_{k,\,k\neq n}\frac{y_k y_n}{|z_n-\bar z_k|^2}\leqslant\gamma$$

if (2.2) holds and $\gamma<1/2$. For $y_k\leqslant y_n$, elementary estimates on the Poisson kernel yield

$$\omega(z_n, J_k)\leqslant C(\beta)\frac{y_k y_n}{|z_n-\bar z_k|^2}.$$

Consequently there is $\gamma(\varepsilon)$ such that (2.2) implies

$$\omega\left(z_n, \bigcup_{k>n} J_k\right) < \beta. \tag{2.6}$$

Now set

$$E_n = \{E \cap I_n\} \setminus \bigcup_{k>n} J_k.$$

Then $E_n \subset E$ and $E_n \cap E_k = \emptyset$, $k \neq n$. By (2.5) and (2.6),

$$\omega\left(z_n, \bigcup_{k\neq n} E_k\right) \leq (1-\omega(z_n, J_n)) + \omega\left(z_n, \bigcup_{k>n} E_k\right) \leq 2\beta. \tag{2.7}$$

Since $z_n \in \Omega_1^+(\varepsilon)$, (2.4) and (2.6) also give

$$\omega(z_n, E_n) \geq \omega(z_n, E) - (1-\omega(z_n, I_n)) - \omega\left(z_n, \bigcup_{k>n} E_k\right) \geq \varepsilon/3. \tag{2.8}$$

For a general function $v \in L^\infty(\mathbf{R})$ we denote by

$$v(z) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{y v(t)\,dt}{y^2+(x-t)^2}$$

the harmonic extension of v to U and by

$$\tilde{v}(z) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{(x-t)\,v(t)\,dt}{y^2+(x-t)^2}$$

its harmonic conjugate in U. It follows from (2.8) that there exists $u_n(t)$ supported on E_n such that

$$u_n(z_n) = 0 \tag{2.9}$$

$$\tilde{u}_n(z_n) = \pi \tag{2.10}$$

$$\int_{E_n} u_n(t)\,dt = 0 \tag{2.11}$$

and

$$\|u_n\|_\infty \leq C(\varepsilon). \tag{2.12}$$

Indeed, by a change of scale we may assume $x_n=0$, $y_n=1$. Then failure of (2.9)–(2.12) would mean

$$\lim_{j\to\infty} \inf_{a,b} \int_{L_j} \left| a+\frac{b}{1+t^2}-\frac{t}{1+t^2} \right| dt = 0$$

for some sequence $\{L_j\}$ of subsets of $[-3/\varepsilon, 3/\varepsilon]$ with $\inf|L_j|>0$, and that is impossible.

We also have

$$\sum_{k>n} |\tilde{u}_k(z_n)| \leq C_1(\varepsilon)\beta \tag{2.13}$$

where C_1 depends only on ε. To prove (2.13) again take $x_n=0$, $y_n=1$ and recall that $y_k \leq 1$ if $k>n$. Then by (2.11) and (2.12),

$$\sum_{k>n} |\tilde{u}_k(z_n)| \leq \frac{C(\varepsilon)}{\pi} \sum_{k>n} \int_{E_k} \left| \frac{t}{1+t^2} - \frac{x_k}{1+x_k^2} \right| dt.$$

But since $E_k \subset I_k$, we have

$$\left| \frac{t}{1+t^2} - \frac{x_k}{1+x_k^2} \right| \leq \frac{3\varepsilon^{-1}}{1+t^2}, \quad t \in E_k,$$

so that by (2.7) and the disjointness of the E_k,

$$\sum_{k>n} |\tilde{u}_k(z_n)| \leq 6\varepsilon^{-1} C(\varepsilon)\beta = C_1(\varepsilon)\beta.$$

For the interpolation we first assume $|w_n|=1$. Set

$$v = \sum_{n=1}^{n_0} c_n u_n, \quad -1 \leq c_n \leq 1,$$

and choose the c_n inductively so that

$$\exp\left(i \sum_{k\leq n} c_k \tilde{u}_k(z_n) \right) = w_n, \quad 1 \leq n \leq n_0.$$

This is possible because of (2.10). Set

$$F(z) = \exp(v(z) + i\tilde{v}(z)), \quad z \in U.$$

Since the E_n are disjoint, $e^{-C(\varepsilon)} \leq |F(z)| \leq e^{C(\varepsilon)}$, $z \in U$, and since v is supported on E, F reflects to be analytic on Ω and $|F(z)| \leq e^{C(\varepsilon)}$, $z \in \Omega$. Furthermore,

$$|F(z_n)-w_n| = \left|1-\exp v(z_n)\exp\left(i\sum_{k>n} c_k \tilde{u}_k(z_n)\right)\right|$$
$$\leq |1-\exp v(z_n)| + e^{v(z_n)}\left|1-\exp\left(i\sum_{k>n} c_k \tilde{u}_k(z_n)\right)\right|.$$

By (2.9), (2.12) and (2.7),

$$|1-\exp v(z_n)| \leq |1-e^{2C(\varepsilon)\beta}| < 1/5$$

if β is small, and by (2.13),

$$\left|1-\exp\left(i\sum_{k>n} c_k \tilde{u}_k(z_n)\right)\right| < |1-e^{C_1(\varepsilon)\beta}| < 1/5$$

if β is small. Hence we can fix $\beta=\beta(\varepsilon)$ so that

$$|F(z_n)-w_n| \leq 1/2, \quad n=1,2,\ldots \tag{2.14}$$

can be solved with $F \in H^\infty(\Omega)$ and

$$\|F\| \leq e^{C(\varepsilon)} \tag{2.15}$$

whenever $|w_n|=1$.

It is well known that (2.14) and (2.15) imply interpolation. If $|w_n| \leq 1$, pick $|\alpha_n|=1/2$, such that $|\alpha_n-w_n| \leq 1/2$ and take $F_1 \in H^\infty(\Omega)$, $\|F_1\| \leq e^{C(\varepsilon)}/2$, such that

$$|\alpha_n - F_1(z_n)| < 1/4.$$

Then $|w_n-F_1(z_n)|<3/4$. Repeating this with w_n replaced by $4(w_n-F_1(z_n))/3$ and iterating, we obtain $F_j \in H^\infty(\Omega)$ with $\|F_j\| \leq (3/4)^{j-1} e^{C(\varepsilon)}/2$ and

$$\sum_{j=1}^{\infty} F_j(z_n) = w_n.$$

Thus (2.3) holds for $M=2e^{C(\varepsilon)}$.

LEMMA 2.2. *Suppose $S=\{z_n\}$ is a sequence in $\Omega_1^+(\varepsilon)$ such that*

$$\delta(\{z_n\}) \geq \delta > 0.$$

Then there are functions $h_n \in H^\infty(\Omega)$ *such that*

$$h_n(z_n) = 1 \tag{2.16}$$

$$\|h_n\| \leqslant M^2(\varepsilon) \tag{2.17}$$

and

$$\sum_n |h_n(z)| \leqslant K(\varepsilon, \delta), \quad z \in \Omega. \tag{2.18}$$

Proof. By a result due to Hoffman and Mills [10] (or [9] p. 407), S may be split into a disjoint union of subsequences S_m, $1 \leqslant m \leqslant 2^p$, so that

$$\delta(S_m) \geqslant (\delta(S))^{2^{-p}}, \quad 1 \leqslant m \leqslant 2^p.$$

Thus we can take $p = p(\varepsilon, \delta)$ such that

$$\delta(S_m) \geqslant 1 - \gamma(\varepsilon), \quad 1 \leqslant m \leqslant 2^p.$$

(With a different value of 2^p this can also be done by grouping the z_n into generations [5], [9] p. 416.) Then by Lemma 2.1 each S_m is an interpolating sequence for $H^\infty(\Omega)$. By a result of Varopoulos [16], [9] p. 298, there are $h_k \in H^\infty(\Omega)$ such that

$$h_k(z_n) = \delta_{n,k}, \quad z_n, z_k \in S_m$$

and

$$\sum_{z_k \in S_m} |h_k(z)| \leqslant M^2(\varepsilon), \quad z \in \Omega.$$

Indeed, suppose $S_m = \{z_1, \ldots, z_{n_0}\}$ is finite, let $\omega = e^{2\pi i/n_0}$ and take $f_j \in H^\infty(\Omega)$, $\|f_j\| \leqslant M = M(\varepsilon)$, $f_j(z_k) = \omega^{jk}$, $z_k \in S_m$. Then

$$h_k = \left(\frac{1}{n_0} \sum_{j=1}^{n_0} \omega^{-jk} f_j \right)^2$$

has $h_k(z_n) = \delta_{n,k}$ and

$$\sum_{k=1}^{n_0} |h_k(z)| = n_0^{-2} \sum_{k=1}^{n_0} \sum_{j,l} \omega^{-jk} \omega^{jl} f_j(z) \bar{f}_l(z)$$

$$= n_0^{-2} \sum_{j=1}^{n_0} n_0 |f_j(z)|^2 \leqslant M^2.$$

Therefore for $S = \bigcup S_m$ we have (2.16), (2.17), and (2.18) with $K(\varepsilon, \delta) = M^2 \cdot 2^p$.

§ 3. Proof of theorem

For a Denjoy domain $\Omega=\mathbf{C}^*\setminus E$ and for $\varepsilon>0$ define

$$\Omega_2(\varepsilon)=\Omega\setminus\bar{\Omega}_1(\varepsilon).$$

Note that $\Omega_2(\varepsilon)$ is symmetric about the axis $\mathbf{R}$ and that $\mathbf{R}\cap\Omega\subset\Omega_2(\varepsilon)$.

Assume $f_1,\dots,f_N\in H^\infty(\Omega)$ satisfy (1.1). By the maximum principle applied to $\omega(z,E)$, each component of $\Omega_1(\varepsilon)$ is simply connected, so that by Carleson's theorem, there exist functions $G_{1,1},\dots,G_{N,1}\in H^\infty(\Omega_1(\varepsilon))$ such that $\Sigma f_j G_{j,1}=1$ on $\Omega_1(\varepsilon)$ and $\|G_{j,1}\|\le C(N,\eta)$. We need solutions in the region $\Omega_2(\varepsilon)$. Note that $\Omega_2(\varepsilon)$ may have infinitely many connected components, and each of these components may be multiply connected. Now fix

$$\varepsilon=\eta/\sqrt{16N}.$$

LEMMA 3.1. *There exist* $G_{1,2},\dots,G_{N,2}\in H^\infty(\Omega_2(\varepsilon))$ *such that for all* $z\in\Omega_2(\varepsilon)$

$$|G_{j,2}(z)|\le 8\eta^{-2}$$

and

$$\sum_j f_j(z)\,G_{j,2}(z)=1.$$

Proof. Write

$$f_j^+(z)=\tfrac{1}{2}(f_j(z)+\overline{f_j(\bar z)}),$$
$$f_j^-(z)=\frac{1}{2i}(f_j(z)-\overline{f_j(\bar z)}).$$

Then $f_j^\pm\in H^\infty(\Omega)$ and $\|f_j^\pm\|\le 1$ by (1.1). Also, $\operatorname{Im}(f_j^\pm)=0$ on $\mathbf{R}\cap\Omega$, so that by the Poisson integral formula

$$|\operatorname{Im} f_j^\pm(z)|\le\varepsilon,\quad z\in\Omega_2(\varepsilon).$$

Since $f_j=f_j^+ + if_j^-$, we have $|f_j^+|^2+|f_j^-|^2\ge\frac{1}{2}|f_j|^2$, and with (1.1) the inequality $\operatorname{Re}(z^2)\ge |z|^2-2(\operatorname{Im} z)^2$ yields

$$\begin{aligned}\left|\sum_j ((f_j^+(z))^2+(f_j^-(z))^2)\right| &\ge \sum_j \operatorname{Re}\{(f_j^+(z))^2+(f_j^-(z))^2\}\\ &\ge \eta^2/2-4N\varepsilon^2\\ &=\eta^2/4,\quad z\in\Omega_2(\varepsilon).\end{aligned}$$

Set

$$G_{j,2}=(f_j^+-if_j^-)\left\{\sum_k (f_k^+)^2+(f_k^-)^2\right\}^{-1}$$

Then $|G_{j,2}(z)|\leq 8\eta^{-2}$ and $\Sigma f_j(z)\,G_{j,2}(z)=1$ for $z\in\Omega_2(\varepsilon)$.

If $|E|=0$ then $\Omega_2(\varepsilon)=\Omega$ and our proof stops here.

We have solutions in $\Omega_1(\varepsilon)$ and in $\Omega_2(\varepsilon)$, but we must solve a $\bar\partial$ problem to get solutions which agree on $\Omega\cap\partial\Omega_2(\varepsilon)$. First we perturb the level set $\{\omega(z,E)=\varepsilon\}=U\cap\partial\Omega_2(\varepsilon)$. Define a *Carleson contour* to be a countable union Γ of rectifiable arcs in U such that every interval $I\subset\mathbf{R}$,

$$\text{length}\,(\Gamma\cap(I\times(0,|I|]))\leq C(\Gamma)\,|I|.$$

Thus arc length on Γ is a Carleson measure with constant $C(\Gamma)$.

LEMMA 3.2. *Let* $0<\varepsilon\leq 1/4$. *There exists a constant* $A>1$, *independent of E and* ε, *and there exists a Carleson contour* Γ *such that*

$$C(\Gamma)\leq A, \tag{3.1}$$

$$\Gamma\subset\Omega_2(\varepsilon)\cap\Omega_1^+(\varepsilon/A), \tag{3.2}$$

and if $\tilde\Gamma$ *is the closure of* $\Gamma\cup\{z:\bar z\in\Gamma\}=\Gamma\cup\bar\Gamma$ *then*

$$\tilde\Gamma \textit{ separates } \Omega_2(\varepsilon/A) \textit{ from } \Omega_1(\varepsilon). \tag{3.3}$$

The proof is well known. One applies the reasoning of section 3 of Carleson's original paper [3] to $F\in H^\infty(U)$ with $\log|F(z)|=(-1/\varepsilon)\,\omega(z,E)$. Or see pages 342–347 of [9]. We omit the details.

Let $M^2(\varepsilon/A)$ be the constant in (2.17) (with ε replaced by ε/A) and fix

$$\alpha=(6M^2(\varepsilon/A))^{-1}.$$

Define $d(z)=|y|^{-1}\inf_{\zeta\in\tilde\Gamma}|z-\zeta|$, $z\in\Omega$, and set $\mathscr{D}=\{z\in\Omega_2(\varepsilon): d(z)\leq\alpha\}$. By (3.2) and Harnack's inequality applied to $\omega(z,E)$,

$$\mathscr{D}\subset\Omega_1(\varepsilon/2A).$$

Standard arguments plus (3.3) show there is $\psi\in C^\infty(\Omega)$, $\psi\equiv 1$ on $\Omega_2(\varepsilon/2A)$, $\psi\equiv 0$ on $\Omega_1(\varepsilon)$, gradient $\psi\equiv 0$ on $\Omega\setminus\mathscr{D}$, and $|y|\cdot|\text{grad}\,\psi(z)|\leq C\alpha^{-1}$. Let $G_{1,1},\ldots,G_{N,1}$ be corona solutions on $\Omega_1(\varepsilon/2A)$, let $G_{1,2}, \dots, G_{N,2}$ be the corona solutions of Lemma 3.1, and set

$$\varphi_j = G_{j,1}(1-\psi)+G_{j,2}\psi.$$

Then $\varphi_j \in C^\infty(\Omega)$ and $\Sigma_j f_j \varphi_j \equiv 1$ on Ω. The φ_j are not analytic, but by the construction of ψ and the bounds on $G_{j,1}$ and $G_{j,2}$,

$$\left|\frac{\partial \varphi_j(z)}{\partial \bar{z}}\right| \leqslant C(N,\eta)\,|y|^{-1}\chi_{\mathscr{D}}(z). \tag{3.4}$$

A well known argument due to Hörmander [11] (see also [9], p. 325) allows one to reduce to solving a $\bar\partial$ problem. Suppose Φ, Ψ are in $L^1(\mathrm{loc})$ on Ω. Then $\bar\partial\Phi=\Psi$ in the sense of distributions on Ω if for all $\Xi \in C^\infty$ with compact support on Ω,

$$\int_\Omega \Phi(z)\frac{\partial}{\partial \bar z}\Xi(z)\,dx\,dy = -\int_\Omega \Psi(z)\,\Xi(z)\,dx\,dy.$$

Weyl's lemma asserts that if $\Phi \in L^1(\mathrm{loc})$ on Ω and $\bar\partial\Phi\equiv 0$ there, Φ is almost everywhere equal to a function analytic on Ω. Suppose we can solve for each j, k the problem

$$\bar\partial a_{j,k} = \varphi_j\bar\partial\varphi_k, \quad a_{j,k}\in L^\infty(\Omega).$$

Set $g_j=\varphi_j+\Sigma_{k=1}^N(a_{j,k}-a_{k,j})f_k$. Then $\Sigma f_j g_j\equiv 1$ on Ω and $\bar\partial g_j \equiv 0$ on Ω, i.e. $g_j \in H^\infty(\Omega)$. Because of inequality (3.4), the theorem will thus be an immediate consequence of

LEMMA 3.3. *Let $B(z)\in L^\infty(\Omega)$ and set $b(z)=y^{-1}B(z)$. If $b\equiv 0$ on $\Omega\setminus\mathscr{D}$, there is $F\in L^\infty(\Omega)$ such that*

$$\bar\partial F = b$$

in the sense of distributions on Ω, and

$$\|F\|_{L^\infty(\Omega)} \leqslant C(\varepsilon)\,\|B\|_{L^\infty}.$$

Proof. Write $b=b^+ + b^-$ where $b^-\equiv 0$ on U and $b^+\equiv 0$ on $\Omega\setminus U$. By a repetition we may assume $b=b^+$ and work only on the upper half-plane. Let $\{z_n\}$ be a collection of points on Γ satisfying

$$|z_m - z_n| \geqslant \alpha y_n, \quad m \neq n, \tag{3.5}$$

$$\inf_n \frac{|z-z_n|}{y_n} \leqslant 3\alpha, \quad z\in\mathscr{D}\cap U. \tag{3.6}$$

The existence of such a sequence follows by taking a maximal sequence satisfying (3.5). It is well known (see section 6 of [17] or [9], p. 341) that if a sequence $\{z_n\}$ lies on a Carleson contour Γ and (3.5) holds, then $\delta(\{z_n\})\geq\delta(\alpha, C(\Gamma))>0$. By (3.1) our points z_n thus satisfy

$$\delta(\{z_n\})\geq\delta(\varepsilon)>0.$$

Let h_n be the functions guaranteed by Lemma 2.2, write $\mathscr{D}\cap U$ as the disjoint union of sets $\mathscr{D}_n\subset\{z: |z-z_n|\leq 3\alpha y_n\}$, and write

$$F(\zeta)=\sum_n \frac{1}{\pi}\iint_{\mathscr{D}_n}\frac{h_n(\zeta)}{h_n(z)}\cdot\frac{b(z)}{\zeta-z}\,dx\,dy.$$

Then formally $\bar\partial F=b$, so we need only check the convergence of the sum. First notice that by (2.16), the definition of α, inequality (2.17), and Schwarz's lemma,

$$|h_n(z)|\geq 1/2,\quad z\in\mathscr{D}_n.$$

Notice also that if we write $F(\zeta)=\Sigma_n H_n(\zeta)$, then

$$\begin{aligned}|H_n(\zeta)|&\leq\frac{2}{\pi}|h_n(\zeta)|\int_{\mathscr{D}_n}\frac{|b(z)|}{|\zeta-z|}\,dx\,dy\\&\leq\frac{2}{\pi}|h_n(\zeta)|\,\|B\|_\infty\int_{\mathscr{D}_n}\frac{y^{-1}}{|\zeta-z|}\,dx\,dy\\&\leq 4\|B\|_\infty|h_n(\zeta)|.\end{aligned}$$

Consequently, (2.18) yields

$$|F(\zeta)|\leq\sum_n |H_n(\zeta)|\leq 4\|B\|_\infty K(\varepsilon/A, \delta(\varepsilon)).$$

§4. Remarks

Let S be a sequence in $\Omega_1(\varepsilon)$ for some Denjoy domain Ω and some $\varepsilon>0$. Write $S_+=S\cap U$, $S_-=S\setminus S_+$. The argument of section 2 yields:

THEOREM. *S is an interpolating sequence for $H^\infty(\Omega)$ if and only if*

$$\delta=\min(\delta(S_+), \delta(\bar S_-))>0 \tag{4.1}$$

where $\bar S_-$ is the reflection of S_- into U.

Proof. Clearly (4.1) is necessary because S_+ and $\bar{S}_-$ must be interpolating sequences for $H^\infty(U)$.

First assume that $S=S_+$. Then the Hoffman-Mills lemma can be used to write $S=S_1\cup\ldots\cup S_N$ with $\delta(S_j)>1-\gamma$, $S_k\cap S_j=\emptyset$ and

$$\inf\left\{\left|\frac{z_k-z_j}{z_k-\bar{z}_j}\right| : z_k\in S_k,\ z_j\in S_j\right\}\geq\delta>0 \tag{4.2}$$

Fix $S_k\cup S_j$, $k\neq j$. If $\gamma=\gamma(\varepsilon,\delta)$ is sufficiently small, then by (4.2) and the proof of Lemma 2.1 we can find sets $E_n\subset E$ such that $\omega(z_n, E_n\}\geq\varepsilon/3$ and such that if $z_n\in S_k$ and $E_n\cap E_m\neq\emptyset$, $m\neq n$, then $z_m\in S_j$ and z_m is unique. Then $u_n(E)$ can be chosen supported on E_n such that if $E_n\cap E_m\neq\emptyset$,

$$(u_n+i\tilde{u}_n)(z_m)=0$$

and such that (2.9)–(2.13) hold with $\|u_n\|_\infty\leq C(\varepsilon,\delta)$. Hence each $S_k\cup S_j$ is an interpolating sequence for $H^\infty(\Omega)$. Now let $|w_n|\leq 1$ and let $\alpha_n^{N-1}=w_n$. If $F_{k,j}\in H^\infty(\Omega)$ satisfies

$$F_{k,j}(z_n)=\alpha_n,\quad z_n\in S_k,$$

$$F_{k,j}(z_n)=0,\quad z_n\in S_j,$$

then $F=\Sigma_k\Pi_{j\neq k}F_{k,j}\in H^\infty(\Omega)$ solves $F(z_n)=w_n$, $z_n\in S$.

In the general case we now know that S_+ and S_- are interpolating sequences for $H^\infty(\Omega)$ with constants $M\leq M(\varepsilon,\delta)$. Let

$$G(z)=\exp\left\{\frac{\log 2M}{\varepsilon}(\chi_E(z)+i\tilde{\chi}_E(z))\right\},\quad z\in U,$$

and extend G to Ω by reflection. Let $F\in H^\infty(\Omega)$ satisfy $F(z_n)=G(z_n)^{-1}$, $z_n\in S_+$, and $\|F\|\leq 1$. Then $f=FG$ satisfies

$$|f(z_n)|\leq\frac{1}{2M},\quad z_n\in S_-$$

$$f(z_n)=1,\quad z_n\in S_+,$$

and this means $S=S_+\cup S_-$ is an interpolating sequence.

The above theorem can be used to give stronger versions of Lemma 3.3. For example, suppose $B(z)\in L^\infty(\Omega)$, $b(z)=y^{-1}B(z)$, $b=0$ on $\Omega_2(\varepsilon)$, and $b(z)\,dx\,dy$ is a Carleson measure. Then there is $F\in L^\infty(\Omega)$ such that $\bar{\partial}F=b$. To see this, combine the last theorem with the argument of [12], [13] (see also [9] pp. 358–363).

Critical to our proof is the reflection argument of Lemma 3.1. It would be interesting to see some variant of $\Omega_2(\varepsilon)$ and Lemma 3.1 for general domains. In this connection we note that the results of [14], which generalize [4], give some progress on solutions of $\bar{\partial}$, on something like $\Omega_1(\varepsilon)$, for general domains. However, a proof of the corona theorem for all plane domains may require a better understanding of analytic capacity than we have today. Brian Cole has an example of a Riemann surface, covering a Denjoy domain, for which the corona theorem fails (see [7]). It depends on the fact that on a plane domain the boundary behavior of a bounded harmonic function is restricted only by Wiener series, while H^∞ functions must satisfy much more stringent conditions. On the other hand, a bounded harmonic function on Ω has the form $\log|f|$ where f is analytic on a covering surface of Ω. This permits one to build unsolvable corona problems on the covering surface.

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