FUNCTORS WHOSE DOMAIN IS A CATEGORY OF MORPHISMS

BY

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Summary

Connected sequences of functors whose domain is the category of morphisms of an arbitrary abelian category $\mathcal{A}$ and whose range category $\mathcal{B}$ is also abelian are compared with the composition functors of Eckmann and Hilton acting between the same categories Sequences of functors of both types are obtained from any half-exact functor $\mathcal{A} \to \mathcal{B}$ if $\mathcal{A}$ has enough injectives and projectives.

1. Introduction

It was observed by Eckmann and Hilton that the homotopy sequence of a triple could be generalized to the following situation: let the pair of maps (f, g)

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{1.1}$$

be any two composable, base-point preserving maps of topological spaces. There is then a long exact sequence (l.e.s.) of homotopy groups

$$\dots \to \pi_n(f) \to \pi_n(gf) \to \pi_n(g) \xrightarrow{\omega_n} \pi_{n-1}(f) \to \dots . \tag{1.2}$$

ω_n can be thought of as either a natural transformation between two appropriate functors which behaves properly under maps of pairs, or as a functor from the category whose objects are the composable pairs (f, g) to the category of morphisms of abelian groups. The latter is preferable. In [3] they generalized this notion of functor as follows: let $\mathcal{C}$ be any category, $\mathcal{C}^2$ the category of morphisms of $\mathcal{C}$, $\mathcal{C}^3$ the category whose objects are the composable pairs (f, g) of maps of $\mathcal{C}$, and $\mathcal{B}$ an arbitrary abelian category. Let $\pi: \mathcal{C}^2 \to \mathcal{B}$ be a graded functor, that is, $\pi = \{\pi_n: \mathcal{C}^2 \to \mathcal{B}, n \in Z\}$. Let $\omega: \mathcal{C}^3 \to \mathcal{B}^2$ be a graded functor such that to each $(f, g) \in \mathcal{C}^3$ there exists a l.e.s. of the form (1.2) in $\mathcal{B}$. The pair (π, ω) was called a composition functor in [3], but is called an Eckmann–Hilton functor (E–H functor) from $\mathcal{C}^2$ to $\mathcal{B}$ here. For any E–H functor (π, ω) it can easily be shown that $\pi_n(e)=0$ for all n and every equivalence e of $\mathcal{C}$.

A connected sequence of functors is defined in [2] to be a graded functor $\sigma: \mathcal{A} \to \mathcal{B}$, for abelian categories $\mathcal{A}$ and $\mathcal{B}$, together with a graded natural transformation ∂ with the following properties: to each short exact sequence (s.e.s.) of $\mathcal{A}$ there is the usual l.e.s. of $\mathcal{B}$, and ∂ behaves properly under maps of s.e.s.'s. The role of ∂ can be best expressed by defining ∂ to be a functor from the category of s.e.s.'s of $\mathcal{A}$ to $\mathcal{B}^2$. Each σ_n must be a half-exact functor. Grothendieck [7] gives the name homological functors to connected sequences of functors, with the conventions used in [2], so the pair (σ, ∂) is called a homological functor (h-functor) from $\mathcal{A}$ to $\mathcal{B}$.

These two notions of functor are distinctly different because they require totally different kinds of objects to manufacture connecting homomorphisms and l.e.s.'s. Even if (τ, ω) and (σ, ∂) are respectively an E–H functor and an h-functor from the abelian category $\mathcal{A}^2$ to $\mathcal{B}$, ω acts on pairs (f, g) of $\mathcal{A}^3$, while ∂ acts on s.e.s.'s of $\mathcal{A}^2$.

That is, given $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ $(f, g) \in \mathcal{A}^3$, then there is a l.e.s.

$$\ldots \to \tau_n(f) \to \tau_n(gf) \to \tau_n(g) \xrightarrow{\omega_n(f,g)} \tau_{n-1}(f) \to \ldots . \tag{1.3}$$

Given the s.e.s. $\ddot{0} \to a \to b \to c \to \ddot{0}$ of $\mathcal{A}^2$, or

$$\begin{tikzcd} 0 \arrow[r] & \cdot \arrow[r] \arrow[d, "a"] & \cdot \arrow[r] \arrow[d, "b"] & \cdot \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & 0 \end{tikzcd}$$

then there is a l.e.s.

$$\ldots \to \sigma_n(a) \to \sigma_n(b) \to \sigma_n(c) \xrightarrow{\partial_n} \sigma_{n-1}(a) \to \ldots . \tag{1.4}$$

The only general requirement on the domain category for an E–H functor is that it be a category of morphisms; the only requirement for an h-functor is that the domain be an abelian category. One of the interests of this paper is to study how these concepts relate when the domain is $\mathcal{A}^2$. That is, if $(\pi, \omega): \mathcal{A}^2 \to \mathcal{B}$ is an E–H functor, can a ∂' be defined so that (π, ∂') is an h-functor? If there is a ∂' such that (π, ∂') is an h-functor, then π must be half-exact. Conversely it is proven that if π is half-exact, then a ∂' can be explicitly given in terms of π and ω such that (π, ∂') is an h-functor. Conversely, given an h-functor (σ, ∂) from $\mathcal{A}^2$ to $\mathcal{B}$, a functor $\omega': \mathcal{A}^3 \to \mathcal{B}^2$ can be explicitly constructed from σ and ∂ such that (σ, ω') is an E–H functor iff $\sigma_n(1)=0$ for every n and every identity map of $\mathcal{A}$. Moreover,

∂' (respectively ω') is uniquely determined by the property that ∂' (respectively ∂) on the trivial s.e.s. (2.10) is equal to $\omega(0A, A0)$ (respectively $\omega'(0A, A0)$) for each A. These results are proven in section 3.

A concept of excision in preadditive categories is introduced in section 2. The special relationship of this property to E–H functors is explored. Section 4 contains several examples and counterexamples. In 4.A it is shown how one can totally avoid using the category of relations in obtaining the ker-coker sequence.

In section 5 the injectives of a category $\mathcal{A}^2$ are proven to be precisely the maps isomorphic to maps of the type

$$\langle 1, \mathrm{o} \rangle : I \oplus J \to I$$

where I and J are both injectives of $\mathcal{A}$. $\mathcal{A}$ has enough injectives iff $\mathcal{A}^2$ has enough injectives.

Let $\varrho_0 : \mathcal{A} \to \mathcal{B}$ be any half-exact functor, and assume that $\mathcal{A}$ has enough injectives and projectives. All the satellites $S^n \varrho_0$ can be found and together they give an h-functor (ϱ, ε). Let $\alpha : \mathcal{A} \to \mathcal{A}^2$ be the embedding functor given by $\alpha(A) = 0A$, the unique map from 0 to A. A half-exact functor $\varrho'_0 : \mathcal{A}^2 \to \mathcal{B}$ is then defined together with a natural epimorphism of functors $\varrho_0 \to \varrho'_0 \alpha$ which extends uniquely to a natural transformation of satellite h-functors $(\varrho, \varepsilon) \to (\varrho' \alpha, \varepsilon' \alpha')$. It follows directly from the definition that

(a) $\varrho'_0(p) = 0$ for every projective p of $\mathcal{A}^2$,

(b) $\varrho'_0(1_A) = 0$ for every A of $\mathcal{A}$,

(c) the induced map $\varrho_n \to \varrho'_n \alpha$ is a natural equivalence for $n < 0$. Furthermore, ϱ'_0 is the unique half-exact functor satisfying these conditions. Also $\varrho'_0(\mathrm{i}) = 0$ for each injective i of $\mathcal{A}^2$; there exists an ω' such that (ϱ', ω') is an E–H functor; and to each f one can produce a canonical $C \in \mathcal{A}$ such that $\varrho'_n(f) \cong \varrho_n(C)$ for $n < 0$.

If in addition $\varrho_0(P) = 0$ for every projective P of $\mathcal{A}$, then the map of satellite h-functors induced by α is a natural equivalence. In this case one can find a canonical D such that $\varrho'_n(f) \cong \varrho_{n-1}(D)$ for $n > 0$. This is contained in section 5. A weaker form of this result was obtained in [10] for the special case where ϱ_0 was the zero'th projective homotopy group. The main technical difference is that the emphasis has been shifted from derived functors to satellite functors.

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2. Notation and terminology

The terminology used for category theory is that of MacLane [12]. The zero object of any preadditive category is denoted by 0, and $\ddot{o}: 0 \to 0$ is the unique map of hom(0, 0). Unless a functor is specifically said to be contravariant, it will be covariant. $\mathcal{A}$ and $\mathcal{B}$ will always represent two arbitrary but fixed abelian categories. The zero element of hom(A, B) is denoted by $A0B$, and also (ambiguously) as o; $A00$ and $00B$ are shortened to $A0$ and $0B$. The identity map $1_A: A \to A$ is often written ambiguously as 1. The expression short (long) exact sequence is abbreviated throughout to s.e.s. (l.e.s.). Denote the s.e.s. of $\mathcal{A}$

$$0 \longrightarrow A \xrightarrow{d} B \xrightarrow{e} C \longrightarrow 0 \tag{2.1}$$

by $d \| e$. The category of all s.e.s.'s of $\mathcal{A}$, Ses$(\mathcal{A})$, is an additive category but not abelian.

The category "2" has two objects, X and Y, and morphisms 1_X, 1_Y, and $p: X \to Y$. The category "3" has objects U, V, and W, and morphisms 1_U, 1_V, 1_W, $r: U \to V$, $q: V \to W$, $s = qr: U \to W$. If $\mathcal{S}$ is a small category, then denote by $\mathcal{C}^{\mathcal{S}}$ the category whose objects are all the (covariant) functors $\mathcal{S} \to \mathcal{C}$, and whose morphisms are the natural transformations of the functors. It has been proven that if $\mathcal{C}$ is abelian, then so is $\mathcal{C}^{\mathcal{S}}$ [7, Prop. 1.6.1]. Notice that $\mathbf{2^2 = 3}$.

Therefore $\mathcal{A}^2$ is an abelian category. Its objects are the morphisms $f, g, h, \ldots$ of $\mathcal{A}$, its morphisms are pairs $\begin{pmatrix} f \\ g \end{pmatrix} : a \to b$ of morphisms such that $bf = ga$. $\begin{pmatrix} f \\ g \end{pmatrix}$ is a monic (epic) of $\mathcal{A}^2$ iff both f and g are monics (epics) of $\mathcal{A}$. A s.e.s. of $\mathcal{A}^2$ is always of the form (2.2), where $d \| e$, $d' \| e'$, $db = d'a$, and $ce = e'b$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "d"] \arrow[d, "a"] & B \arrow[r, "e"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "d'"] & B' \arrow[r, "e'"] & C' \arrow[r] & 0 \end{tikzcd} \tag{2.2}$$

(2.2) can also be written as $\begin{pmatrix} d \\ d' \end{pmatrix} \| \begin{pmatrix} e \\ e' \end{pmatrix}$ or

$$\ddot{o} \to a \to b \to c \to \ddot{o}. \tag{2.3}$$

An object of $\mathcal{C}^3$ consists of a pair of morphisms (f, g) such that gf is defined.

A graded functor $\tau: \mathcal{C} \to \mathcal{D}$ means that there is an ordered set of functors $\tau_n: \mathcal{C} \to \mathcal{D}$ for all integers n, $-\infty < n < \infty$. If τ is a graded functor, then $\tau(C) = 0$ means that $\tau_n(C) = 0$ for every n. All the graded functors that are encountered here are assumed to be *additive*.

A *homological functor* (h-functor) is a pair $(\sigma, \partial): \mathcal{A} \to \mathcal{B}$ of graded functors $\sigma: \mathcal{A} \to \mathcal{B}$ and ∂: $\mathrm{Ses}(\mathcal{A}) \to \mathcal{B}^2$ such that to each s.e.s. (2.1) of $\mathcal{A}$ there is a l.e.s. in $\mathcal{B}$

$$\ldots \to \sigma_n(A) \xrightarrow{\sigma_n(d)} \sigma_n(B) \xrightarrow{\sigma_n(e)} \sigma_n(C) \xrightarrow{\partial_n(d \| e)} \sigma_{n-1}(A) \ldots. \tag{2.4}$$

This is the usual connected sequence of functors [12, p. 386].

An *Eckmann–Hilton composition functor* (E–H functor) is a pair $(\tau, \omega): \mathcal{C}^2 \to \mathcal{B}$ of graded functors $\tau: \mathcal{C}^2 \to \mathcal{B}$ and $\omega: \mathcal{C}^3 \to \mathcal{B}^2$, where $\mathcal{C}$ is any category, such that to each object (f, g) of $\mathcal{C}^3$ there is a l.e.s. in $\mathcal{B}$

$$\ldots \to \tau_n(f) \xrightarrow{\tau_n\binom{1}{g}} \tau_n(gf) \xrightarrow{\tau_n\binom{f}{1}} \tau_n(g) \xrightarrow{\omega_n(f,g)} \tau_{n-1}(f) \to \ldots. \tag{2.5}$$

It has been shown [3, Prop. 2.2] that for any E–H functor (τ, ω) as above, and any equivalence e of $\mathcal{C}$, one has $\tau(e) = 0$. Assume that $\mathcal{C}$ contains a zero object. The object $(0Y, Y0)$ of $\mathcal{C}^3$ gives $\ddot{o} = 0 \to Y \to 0$ under composition, and since $\ddot{o}$ is an equivalence one obtains the l.e.s. with each $\tau_j(\ddot{o}) = 0$

$$\ldots \to \tau_n(\ddot{o}) \to \tau_n(Y0) \to \tau_{n-1}(0Y) \to \tau_{n-1}(\ddot{o}) \to \ldots.$$

It follows that $w_n(Y) = \omega_n(0Y, Y0)$ is an isomorphism for all n. We shall call this process "flipping". Clearly there is a graded functor $w: \mathcal{C} \to \mathcal{B}^2$, and w will be used to represent the isomorphism $w_m(X)$ for all X and m. It was also shown [3, Thm. 2.8] that for any object (f, g) of $\mathcal{C}^3$ of the form (1.1) one has

$$\omega_n(f, g) = \tau_{n-1}\begin{pmatrix}0X\\1\end{pmatrix} w_n(Y)\, \tau_n\begin{pmatrix}1\\Z0\end{pmatrix}. \tag{2.6}$$

The map $\binom{f}{g}: f \to g$ can be factored into $\binom{f}{1}: f \to 1$ and $\binom{1}{g}: 1 \to g$. Therefore $\tau_n\binom{f}{g}$ is the zero map of $\hom(\tau_n(f), \tau_n(g))$ for all n, because it factors through $\tau_n(1) = 0$.

Any half-exact functor between two abelian categories is additive. Therefore if (τ, ∂) is an h-functor, the graded functor τ is additive (i.e. each τ_n is additive). If (τ, ω) is an E–H functor and if τ is not additive, then there can be no ∂ such that (τ, ∂) is an h-functor. In (4.C) an E–H functor which is not half-exact is introduced.

In the following discussion $\mathcal{D}$ will always denote a preadditive category, $\mathcal{E}$ a subclass of $\{(f, g) \mid gf = 0\} \subseteq \mathcal{D}^3$, and $\mathcal{E}'$ the subclass $\{(f, g) \mid f \| g\}$. Let $f: X \to Y$, $g: Y \to Z$ give an arbitrary element $(f, g) \in \mathcal{E}$. If $\varphi : \mathcal{D}^2 \to \mathcal{B}$ is any functor, then φ is said to have *the $\mathcal{E}$ left-excision property* or simply *$\mathcal{E}$ left-excision* (respectively, the *$\mathcal{E}$ right-excision property*) if for each $(f, g) \in \mathcal{E}$

$$\varphi\begin{pmatrix} f \\ 0Z \end{pmatrix} : \varphi(X0) \to \varphi(g) \quad \left(\varphi\begin{pmatrix} X0 \\ g \end{pmatrix} : \varphi(f) \to \varphi(0Z)\right)$$

is an isomorphism. In section 3 $\mathcal{D}$ is abelian, $\mathcal{E} = \mathcal{E}'$, and φ is simply said to have left- or right-excision.

If a and b are maps of $\mathcal{D}$, then $\hom(a, b)$ in $\mathcal{D}^2$ is an abelian group under componentwise additon of morphisms.

LEMMA 2.7. *Let $(\tau, \omega) : \mathcal{D}^2 \to \mathcal{B}$ be an E–H functor, and assume that τ is additive. Then τ has $\mathcal{E}$ left-excision iff τ has $\mathcal{E}$ right-excision. If τ is also a half-exact functor, then τ has the $\mathcal{E}'$ left- and right-excision properties.*

Proof. Take the objects $(0X, f)$ and $(0Y, g)$ of $\mathcal{D}^3$, and apply (τ, ω) to form the diagram (2.8). The rows are exact, the first two squares commute, and the third square anticommutes.

$$\begin{tikzcd}[column sep=large]
\cdots \arrow[r] & \tau_n(0X) \arrow[r, "\tau_n\begin{pmatrix} \ddot{0} \\ f \end{pmatrix}"] \arrow[d, "\tau_{n+1}\begin{pmatrix} f \\ 0Z \end{pmatrix} w^{-1}"] & \tau_n(0Y) \arrow[r, "\tau_n\begin{pmatrix} 0X \\ 1 \end{pmatrix}"] \arrow[d, "1"] & \tau_n(f) \arrow[r, "w\tau_n\begin{pmatrix} 1 \\ Y0 \end{pmatrix}"] \arrow[d, "\tau_n\begin{pmatrix} X0 \\ g \end{pmatrix}"] & \tau_{n-1}(0X) \arrow[r] \arrow[d, "\tau_n\begin{pmatrix} f \\ 0Z \end{pmatrix} w^{-1}"] & \cdots \\
\cdots \arrow[r] & \tau_{n+1}(g) \arrow[r, "w\tau_{n+1}\begin{pmatrix} 1 \\ Z0 \end{pmatrix}"'] & \tau_n(0Y) \arrow[r, "\tau_n\begin{pmatrix} \ddot{0} \\ g \end{pmatrix}"'] & \tau_n(0Z) \arrow[r, "\tau_n\begin{pmatrix} 0Y \\ 1 \end{pmatrix}"'] & \tau_n(g) \arrow[r] & \cdots
\end{tikzcd} \tag{2.8}$$

To check the anticommutativity notice that $\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ \mathrm{o} \end{pmatrix} + \begin{pmatrix} \mathrm{o} \\ g \end{pmatrix}$ in $\mathcal{D}^2$. Therefore $\tau_n\begin{pmatrix} f \\ g \end{pmatrix} = \tau_n\begin{pmatrix} f \\ \mathrm{o} \end{pmatrix} + \tau_n\begin{pmatrix} \mathrm{o} \\ g \end{pmatrix} = \mathrm{o}$ because τ_n is additive for each n. But the maps of the third square are

$$\tau_n\begin{pmatrix} 0Y \\ 1 \end{pmatrix} \tau_n\begin{pmatrix} X0 \\ g \end{pmatrix} = \tau_n\begin{pmatrix} \mathrm{o} \\ g \end{pmatrix} \text{ and } \tau_n\begin{pmatrix} f \\ 0Z \end{pmatrix} w^{-1} w \tau_n\begin{pmatrix} 1 \\ Y0 \end{pmatrix} = \tau_n\begin{pmatrix} f \\ \mathrm{o} \end{pmatrix},$$

Thus $\tau_n\begin{pmatrix} \mathrm{o} \\ g \end{pmatrix} = -\tau_n\begin{pmatrix} f \\ \mathrm{o} \end{pmatrix}$, that is, the square is anticommutative.

If $(f, g) \in \mathcal{E}$ and τ has $\mathcal{E}$ right-excision, then every map $\tau_n\begin{pmatrix} X0 \\ g \end{pmatrix}$ is an isomorphism. By an argument analogous to that of the 5-lemma, $\tau_n\begin{pmatrix} f \\ 0Z \end{pmatrix} w^{-1}$ is an isomorphism for each n. Since w^{-1} is an isomorphism, $\tau_n\begin{pmatrix} f \\ 0Z \end{pmatrix}$ is also an isomorphism. That is, τ has $\mathcal{E}$ left-excision. The argument in the other direction is clear.

Let $f\|g$ and consider the s.e.s. (2.9) of $\mathcal{D}^2$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & Z & \xrightarrow{1} & Z & \longrightarrow & 0 \end{array} \tag{2.9}$$

If τ is a half-exact functor, then from (2.9) there is the exact sequence of $\mathcal{B}$

$$\tau_n(X0)\to\tau_n(g)\to\tau_n(1)$$

for each n. $\tau_n(1)=0$ because (τ, ω) is an E–H functor. Therefore $\tau_n\begin{pmatrix} f \\ 0Z \end{pmatrix}$ is an epimorphism. Dually $\tau_n\begin{pmatrix} X0 \\ g \end{pmatrix}$ is monic. If one inserts this information in (2.8) then by [12, Lemma I.3.3] $\tau_n\begin{pmatrix} f \\ 0Z \end{pmatrix}$ is monic (the anticommutativity of the third square of (2.8) poses no problem) and $\tau_n\begin{pmatrix} X0 \\ g \end{pmatrix}$ is epic. That is, τ has both the $\mathcal{E}'$ left- and right-excision properties.

For every A of an abelian category $\mathcal{A}$, there is a s.e.s. (2.10) of $\mathcal{A}^2$ which will be denoted by ses$|A|$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{1} & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{1} & A & \longrightarrow & 0 & \longrightarrow & 0 \end{array} = \text{ses } |A| \tag{2.10}$$

If (τ, ∂) is an h-functor $\mathcal{A}^2\to\mathcal{B}$, then there is a map $\partial_n(\text{ses}|A|):\tau_n(A0)\to\tau_{n-1}(0A)$ of $\mathcal{B}$. If $(\tau, \omega):\mathcal{A}^2\to\mathcal{B}$ is an E–H functor, then there is also a map $\omega_n(0A, A0):\tau_n(A0)\to\tau_{n-1}(0A)$ of $\mathcal{B}$. This will be used in the next section.

3. Transmutation of functors

If (τ, ω) is an E–H functor $\mathcal{A}^2\to\mathcal{B}$, then a complete answer is given to the problem of whether or not one can prescribe a ∂ such that (τ, ∂) is an h-functor. The converse is also completely solved. (τ, ω) and (τ, ∂) are really quite different kinds of functors because they act on distinctly different diagrams of objects to get l.e.s.'s in their range category. This is seen in (1.3) and (1.4).

If (τ, ω) is given, then the process of production of the functor ∂: $\mathrm{Ses}(\mathcal{A}^2) \to \mathcal{B}^2$ proceeds as follows: a rule is given which converts any s.e.s. (2.3) into a s.e.s.

$$\ddot{o} \to a \to b' \to c' \to \ddot{o}, \tag{3.1}$$

where b' and c' are epic. A second rule converts (3.1) into a s.e.s.

$$\ddot{o} \to a' \to b'' \to c' \to \ddot{o} \tag{3.2}$$

where a' is monic and b'' is an isomorphism. A commutative diagram (3.3) is then obtained.

$$\begin{tikzcd}
\ddot{o} \arrow[r] & a \arrow[r] \arrow[d, "1"] & b \arrow[r] \arrow[d] & c \arrow[r] \arrow[d] & \ddot{o} \\
\ddot{o} \arrow[r] & a \arrow[r] & b' \arrow[r] & c' \arrow[r] & \ddot{o} \\
\ddot{o} \arrow[r] & a' \arrow[r] \arrow[u] & b'' \arrow[r] \arrow[u] & c' \arrow[r] \arrow[u, "1"'] & \ddot{o}
\end{tikzcd} \tag{3.3}$$

If τ is half-exact, then an isomorphism can be given $\tau_n(c') \to \tau_{n-1}(a')$. Using the maps induced by (3.3), then ∂ is given by

$$\tau_n(c) \to \tau_n(c') \to \tau_{n-1}(a') \to \tau_{n-1}(a).$$

THEOREM 3.4. *Let (τ, ω) be an E–H functor $\mathcal{A}^2 \to \mathcal{B}$. There is a unique graded functor ∂: $\mathrm{Ses}(\mathcal{A}^2) \to \mathcal{B}^2$ such that $\partial(\mathrm{ses}\,|A\,|) = \omega(0A, A0)$ for each A of $\mathcal{A}$, and such that (τ, ∂) is an h-functor iff τ is a half-exact functor.*

Proof. By $\{d, a\}$: $A \to B \oplus A'$ we mean the map given by $d: A \to B$, $a: A \to A'$, and the universal property of direct sums. $\langle c, e' \rangle$ is obtained dually. Take any s.e.s. (2.2) of $\mathcal{A}^2$. By the rules which are implicitly contained within their diagrams, the s.e.s.'s (3.5) and (3.6) of $\mathcal{A}^2$ are formed from (2.2).

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f = \{d, \mathrm{o}\}"] \arrow[d, "a"] & B \oplus B' \arrow[r, "g = e \oplus 1"] \arrow[d, "b' = \langle b, 1 \rangle"] & C \oplus B' \arrow[r] \arrow[d, "c' = \langle c, e' \rangle"] & 0 \\
0 \arrow[r] & A' \arrow[r, "d'"'] & B' \arrow[r, "e'"'] & C' \arrow[r] & 0
\end{tikzcd} \tag{3.5}$$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "a' = \{d, a\}"] & B \oplus B' \arrow[r, "g"] \arrow[d, "b''"] & C \oplus B' \arrow[r] \arrow[d, "c'"] & 0 \\
0 \arrow[r] & B \oplus A' \arrow[r, "f' = 1 \oplus d'"'] & B \oplus B' \arrow[r, "g' = \langle e', \mathrm{o} \rangle"'] & C' \arrow[r] & 0
\end{tikzcd} \tag{3.6}$$

$b'' = \{\langle 1, \mathrm{o} \rangle, \langle b, 1 \rangle\}$ has a two-sided inverse $b^* = \{\langle 1, \mathrm{o} \rangle, \langle -b, 1 \rangle\}$, so it is an isomorphism.

Also, b' and c' are epic, a' is monic. These s.e.s.'s correspond to (3.1) and (3.2). Define the maps

$$\begin{aligned} p&=\langle \mathrm{o}, 1\rangle : B\oplus A'\to A' \qquad & r&=\{1, \mathrm{o}\}: B\to B\oplus B' \\ q&=\langle \mathrm{o}, 1\rangle : B\oplus B'\to B' \qquad & s&=\{1, \mathrm{o}\}: C\to C\oplus B'. \end{aligned} \tag{3.7}$$

There are then the maps of s.e.s's $\left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}r\\1\end{pmatrix}, \begin{pmatrix}s\\1\end{pmatrix}\right]$: (2.2) → (3.5) *and* $\left[\begin{pmatrix}1\\p\end{pmatrix}, \begin{pmatrix}1\\q\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right]$: (3.6) → (3.5) which give the commutative diagram (3.3).

Clearly $\qquad pa'=a,\ qb''=b',\ b'r=b,\ \text{and}\ c's=c.$

Take the canonical factorization of the map gb^*f' of (3.6) into an epic h and a monic k. That is, $gb^*f'=kh$. To see that $a'\|h$, notice first that

$$kha'=gb^*f'a'=gb^*b''f=gf=\mathrm{o};$$

and if $hx=\mathrm{o}$ for some map x, then $gb^*f'x=\mathrm{o}$. But since $f\|g$, there is a unique map x' such that $b^*f'x=fx'$. Therefore

$$f'x=b''fx'=f'a'x',\ \text{and since } f' \text{ is monic, } x=a'x'.$$

That is, each left annihilator of h factors through the monomorphism a', so $a'\|h$. Dually $k\|c'$. Let D denote the image of h.

Let us assume now that τ is a half-exact functor. By Lemma 2.7 one has for each s.e.s. $f\|g$ of $\mathcal{A}$ the pair $(f, g)\in\mathcal{A}^3$ for which τ has left- and right-excision. There are then well-defined isomorphisms

$$\nu_n=\tau_{n-1}\begin{pmatrix}\mathrm{o}\\h\end{pmatrix}^{-1} w_n(D)\,\tau_n\begin{pmatrix}k\\\mathrm{o}\end{pmatrix}^{-1} : \tau_n(c')\to\tau_{n-1}(a'). \tag{3.8}$$

If one should choose some other canonical factorization of gb^*f', such as (k', h', D'), then it can be immediately verified that this gives the same ν. Thus ν depends only on the map gb^*f which in turn depends only on the s.e.s. (2.3). It follows easily that ν is a well-defined function $\mathrm{Ses}(\mathcal{A}^2)\to\mathcal{B}^2$. The proof that ν is a functor is now easy. Moreover, since τ and ω are both additive by hypothesis, then w is additive, and ν is additive too. Also, ν depends directly on ω.

$\partial:\mathrm{Ses}(\mathcal{A}^2)\to\mathcal{B}^2$ is given for any s.e.s. (2.3) by

$$\partial_n=\tau_{n-1}\begin{pmatrix}1\\p\end{pmatrix}\nu_n\,\tau_n\begin{pmatrix}s\\1\end{pmatrix} : \tau_n(c)\to\tau_{n-1}(a). \tag{3.9}$$

The proof that ∂ is an additive functor is straight-forward.

By using the pair (τ, ∂) a long sequence of objects and morphisms of $\mathcal{B}$ is obtained from each s.e.s. (2.3) of $\mathcal{A}^2$. (τ, ∂) is an h-functor iff every such sequence is exact. The exactness is first proved for all s.e.s.'s of the form (3.1).

It is important to notice that throughout this proof only the fact that τ has left- and right-excision is really needed. One can, in fact, derive that τ is half-exact from the following arguments and the assumption that τ has excision.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a'"] & B\oplus B' \arrow[r, "g"] \arrow[d, "b''"] & C\oplus B' \arrow[d, "c'"] \\
B\oplus A' \arrow[r, "f'"] \arrow[d, "p"] & B\oplus B' \arrow[r, "g'"] \arrow[d, "q"] & C' \arrow[d, "1"] \\
A' \arrow[r, "d'"] & B' \arrow[r, "e'"] & C'
\end{tikzcd} \tag{3.10}$$

If one examines the long sequence that is obtained from the s.e.s. $\ddot{o}\to p\to q\to 1\to\ddot{o}$ of $\mathcal{A}^2$, it is clear that the corresponding ∂ must be zero because $\tau(1)=0$. It is only necessary to prove that $\tau_n(p)\to\tau_n(q)$ is an isomorphism for all n to have exactness of the long sequence. Define $j=\{1, o\}: B\to B\oplus A'$. Then $j\|p$ and $f'j\|q$. These give the maps $B0\to p\to q$ and the excision isomorphisms $\tau_n\begin{pmatrix} j \\ o\end{pmatrix}$ and $\tau_n\begin{pmatrix} f'j \\ o\end{pmatrix}$ Therefore $\tau_n\begin{pmatrix} f' \\ d'\end{pmatrix}=\tau_n\begin{pmatrix} f'j \\ o\end{pmatrix}\tau_n\begin{pmatrix} j \\ o\end{pmatrix}^{-1}: \tau_n(p)\to\tau_n(q)$ is an isomorphism.

Apply the E–H functor (τ, ω) to the pairs of morphisms of the columns of (3.10). The columns of (3.11) represent a part of what is obtained, where the maps correspond to those of (2.5). The horizontal maps are either induced by the pairs of the commutative diagram (3.10), or are given by the connecting homomorphisms o, ν_n, and $\tau_{n-1}\begin{pmatrix} 1 \\ p\end{pmatrix}\nu_n$. With the possible exception of the third row, all the rows and columns are exact, and (3.11) is commutative.

$$\begin{tikzcd}
\tau_{n+1}(p) \arrow[r] \arrow[d] & \tau_{n+1}(q) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \tau_n(p) \arrow[r] \arrow[d] & \tau_n(q) \arrow[d] \\
\tau_n(a') \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \tau_n(c') \arrow[r, "\nu"] \arrow[d, "1"] & \tau_{n-1}(a') \arrow[r] \arrow[d] & 0 \arrow[d] \\
\tau_n(a) \arrow[r] \arrow[d] & \tau_n(b') \arrow[r] \arrow[d, "\cong"] & \tau_n(c') \arrow[r] \arrow[d] & \tau_{n-1}(a) \arrow[r] \arrow[d] & \tau_{n-1}(b') \arrow[d] \\
\tau_n(p) \arrow[r, "\cong"] \arrow[d] & \tau_n(q) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \tau_{n-1}(p) \arrow[r] \arrow[d] & \tau_{n-1}(q) \arrow[d] \\
\tau_{n-1}(a') \arrow[r] & 0 \arrow[r] & \tau_{n-1}(c') \arrow[r] & \tau_{n-2}(a') \arrow[r] & 0
\end{tikzcd} \tag{3.11}$$

If one uses the isomorphisms available, then one obtains two distinct maps $\tau_n(p)\to\tau_n(c')$. The map through $\tau_{n-1}(a')$ is

$$\tau_n\begin{pmatrix}k\\ \text{o}\end{pmatrix} w^{-1}\tau_{n-1}\begin{pmatrix}\text{o}\\ h\end{pmatrix}\tau_{n-1}\begin{pmatrix}\text{o}\\ 1\end{pmatrix} w\tau_n\begin{pmatrix}1\\ \text{o}\end{pmatrix}=\tau_n\begin{pmatrix}kh\\ \text{o}\end{pmatrix}.$$

Since $\tau_n\begin{pmatrix}b''\\ 1\end{pmatrix}^{-1}=\tau_n\begin{pmatrix}b^*\\ \text{o}\end{pmatrix}$, the map through $\tau_n(q)$ and $\tau_n(b')$ is

$$\tau_n\begin{pmatrix}g\\ e'\end{pmatrix}\tau_n\begin{pmatrix}b^*\\ 1\end{pmatrix}\tau_n\begin{pmatrix}f'\\ d'\end{pmatrix}=\tau_n\begin{pmatrix}gb^*f'\\ e'\,d'\end{pmatrix}=\tau_n\begin{pmatrix}kh\\ \text{o}\end{pmatrix}.$$

Therefore these two maps are the same. It follows that

$$\tau_n(b')\to\tau_n(c')\to\tau_{n-1}(a')=\tau_n(b')\to\tau_n(q)\leftarrow\tau_n(p)\to\tau_{n-1}(a').$$

From this observation the exactness of the third row of (3.11) can be established by simple diagram chasing. For example:

$$\begin{aligned}\tau_n(b')\to\tau_n(c')\to\tau_{n-1}(a)&=\tau_n(b')\to\tau_n(c')\to\tau_{n-1}(a')\to\tau_{n-1}(a)\\ &=\tau_n(b')\to\tau_n(p)\to\tau_{n-1}(a')\to\tau_{n-1}(a)=\text{o}\end{aligned}$$

by the exactness of the first column of (3.11).

We shall now prove that the long sequence is exact for every s.e.s. of $\mathcal{A}^2$; that is, (τ,∂) is an h-functor, where ∂ is defined by (3.9).

$$\begin{array}{ccccc} A & \xrightarrow{d} & B & \xrightarrow{e} & C\\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s}\\ A & \xrightarrow{f} & B\oplus B' & \xrightarrow{g} & C\oplus B'\\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle c'}\\ A' & \xrightarrow{d'} & B' & \xrightarrow{e'} & C' \end{array}\tag{3.12}$$

Take the commutative diagram (3.12) and proceed exactly as above to obtain the commutative diagram (3.13). With the possible exception of the third row, all the rows and columns are exact. The first and fourth rows are exact from the above arguments; $\tau_n(r)\to\tau_n(s)$ is an isomorphism by a dual argument to the one above. Exactness of the third row follows as soon as we verify that the two maps $\tau_{n+1}(c')\to\tau_n(b)$ of (3.13) are the same. This will complete the proof of the theorem.

$$\begin{tikzcd}
\tau_{n+2}(b') \arrow[r] \arrow[d] & \tau_{n+2}(c') \arrow[r] \arrow[d] & \tau_{n+1}(a) \arrow[r] \arrow[d] & \tau_{n+1}(b') \arrow[r] \arrow[d] & \tau_{n+1}(c') \arrow[d] \\
\tau_{n+1}(r) \arrow[r] \arrow[d] & \tau_{n+1}(s) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \tau_n(r) \arrow[r, "\cong"] \arrow[d] & \tau_n(s) \arrow[d] \\
\tau_{n+1}(b) \arrow[r] \arrow[d] & \tau_{n+1}(c) \arrow[r, "\partial_{n+1}"] \arrow[d] & \tau_n(a) \arrow[r] \arrow[d, "1"] & \tau_n(b) \arrow[r] \arrow[d] & \tau_n(c) \arrow[d] \\
\tau_{n+1}(b') \arrow[r] \arrow[d] & \tau_{n+1}(c') \arrow[r] \arrow[d] & \tau_n(a) \arrow[r] \arrow[d] & \tau_n(b') \arrow[r] \arrow[d] & \tau_n(c') \arrow[d] \\
\tau_n(r) \arrow[r] & \tau_n(s) \arrow[r] & 0 \arrow[r] & \tau_{n-1}(r) \arrow[r] & \tau_{n-1}(s)
\end{tikzcd} \tag{3.13}$$

Define $u=\langle 0, 1\rangle : B\oplus B'\to B'$, $v=\langle o, 1\rangle : C\oplus B'\to B'$. Then $r\|u$ and $s\|v$ where r and s are defined in (3.7). Let $y=\{o, 1\}: B'\to B\oplus B'$. Then $b'y=uy=1$. It is clear that $1_{0B'}=\begin{pmatrix} o \\ uy\end{pmatrix}=\begin{pmatrix} B0 \\ u\end{pmatrix}\begin{pmatrix} 0B \\ y\end{pmatrix}: 0B'\to r\to 0B'$, so $\tau_n\begin{pmatrix} 0B \\ y\end{pmatrix}$ is the inverse of the excision isomorphism $\tau_n\begin{pmatrix} B0 \\ u\end{pmatrix}$. The factorization

$$\begin{pmatrix} 1 \\ b'\end{pmatrix}\begin{pmatrix} 0B \\ y\end{pmatrix}=\begin{pmatrix} 0B \\ 1\end{pmatrix}: OB'\to b \text{ gives } \tau_n\begin{pmatrix} 1 \\ b'\end{pmatrix}\tau_n\begin{pmatrix} 0B \\ y\end{pmatrix}=\tau_n\begin{pmatrix} 0B \\ 1\end{pmatrix},$$

or,

$$\tau_n\begin{pmatrix} 1 \\ b'\end{pmatrix}=\tau_n\begin{pmatrix} 0B \\ 1\end{pmatrix}\tau_n\begin{pmatrix} B0 \\ u\end{pmatrix}.$$

This does not follow from any commutative diagram of $\mathcal{A}^2$; it is true because τ has right-excision.

Apply τ_n to $\begin{pmatrix} B0 \\ u\end{pmatrix}=\begin{pmatrix} C0 \\ v\end{pmatrix}\begin{pmatrix} e \\ g\end{pmatrix}$ and use the above to obtain that the map $\tau_n(s)\to\tau_n(b)$ is

$$\tau_n\begin{pmatrix} 1 \\ b'\end{pmatrix}\tau_n\begin{pmatrix} e \\ g\end{pmatrix}^{-1}=\tau_n\begin{pmatrix} 0B \\ 1\end{pmatrix}\tau_n\begin{pmatrix} B0 \\ u\end{pmatrix}\tau_n\begin{pmatrix} B0 \\ u\end{pmatrix}^{-1}\tau_n\begin{pmatrix} C0 \\ v\end{pmatrix}=\tau_n\begin{pmatrix} o \\ v\end{pmatrix}.$$

Since $\tau_{n+1}(c')\to\tau_n(a)=\tau_{n+1}(c')\to\tau_n(a')\to\tau_n(a)$ it suffices to prove that for any n

$$\tau_n(a')\xleftarrow{\cong}\tau_{n+1}(c')\to\tau_n(s)\to\tau_n(b)=\tau_n(a')\to\tau_n(a)\to\tau_n(b).$$

The second map is $\tau_n\begin{pmatrix} d \\ d'p\end{pmatrix}$. The first map is

$$\tau_n\begin{pmatrix} o \\ v\end{pmatrix}\tau_n\begin{pmatrix} o \\ 1\end{pmatrix}w\tau_{n+1}\begin{pmatrix} 1 \\ o\end{pmatrix}\tau_{n+1}\begin{pmatrix} k \\ o\end{pmatrix}w^{-1}\tau_n\begin{pmatrix} o \\ h\end{pmatrix}=\tau_n\begin{pmatrix} o \\ vkh\end{pmatrix}.$$

Also, $vkh = vgb^*f' = \langle o, 1\rangle \{\langle 1, o\rangle, \langle -b, 1\rangle\}(1\oplus d') = \langle -b, 1\rangle(1\oplus d') = \langle -b, d'\rangle$. Let $t = \langle 1, o\rangle : B\oplus A' \to B$. Consider the map

$$\begin{pmatrix} d \\ bt \end{pmatrix} = \begin{pmatrix} 1 \\ b \end{pmatrix}\begin{pmatrix} d \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ t \end{pmatrix} : a' \to d \to 1 \to b.$$

Since this factors through 1, $\tau\begin{pmatrix} d \\ bt \end{pmatrix} = o$. Moreover,

$$\begin{pmatrix} d \\ d'p \end{pmatrix} - \begin{pmatrix} d \\ bt \end{pmatrix} = \begin{pmatrix} d-d \\ d'p - bt \end{pmatrix} = \begin{pmatrix} o \\ \langle -b, d'\rangle \end{pmatrix} = \begin{pmatrix} o \\ vkh \end{pmatrix}.$$

If one applies τ_n to this, then from the above remarks we have

$$\tau_n\begin{pmatrix} d \\ d'p \end{pmatrix} = \tau_n\begin{pmatrix} o \\ vkh \end{pmatrix}.$$

Therefore the two maps $\tau_{n+1}(c') \to \tau_n(b)$ are the same.

Therefore (τ, ∂) is an h-functor. By (3.8) we have $\nu_n(\text{ses}\,|A|) = w_n(A\oplus A)\tau_n\begin{pmatrix} b^* \\ \ddot{o} \end{pmatrix}^{-1}$ for $b'' = b^{*-1} = \{\langle 1, o\rangle, \langle 1, 1\rangle\}$ and $b^* = \{\langle 1, o\rangle, \langle -1, 1\rangle\} : A\oplus A \to A\oplus A$. Set $p = \langle o, 1\rangle : A\oplus A \to A$ and $s = \{1, o\} : A \to A\oplus A$; p induces $\tau_{n-1}\begin{pmatrix} \ddot{o} \\ p \end{pmatrix}$ $w_n(A\oplus A) = w_n(A)\tau_n\begin{pmatrix} p \\ \ddot{o} \end{pmatrix}$. From (3.9) it follows that since $pb^{*-1}s = 1 : A \to A$

$$\begin{aligned} \partial_n(\text{ses}\,|A|) &= \tau_{n-1}\begin{pmatrix} \ddot{o} \\ p \end{pmatrix} w_n(A\oplus A)\,\tau_n\begin{pmatrix} b^* \\ \ddot{o} \end{pmatrix}^{-1} \tau_n\begin{pmatrix} s \\ \ddot{o} \end{pmatrix} \\ &= w_n(A)\,\tau_n\begin{pmatrix} pb^{*-1}s \\ \ddot{o} \end{pmatrix} \\ &= w_n(A) \\ &= \omega_n(0A), A0). \end{aligned}$$

Let (τ, ∂') be any h-functor such that for any A $\partial'(\text{ses}\,|A|) = \omega_n(0A, A0)$. Following the notation of (3.6) a special s.e.s. which is denoted by $\underline{\bar{S}}$ is produced.

$$\underline{\bar{S}} = \begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & C\oplus B' & \xrightarrow{1} & C\oplus B' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle c'} & & \\ 0 & \longrightarrow & D & \xrightarrow{k} & C\oplus B' & \xrightarrow{c'} & C' & \longrightarrow & 0 \end{array}$$

From the map $\left[\begin{pmatrix} \ddot{o} \\ 1 \end{pmatrix}, \begin{pmatrix} k \\ k \end{pmatrix}, \begin{pmatrix} k \\ o \end{pmatrix}\right]$: ses $|D| \to \underline{\bar{S}}$ one obtains a commutative diagram

$$\begin{array}{ccc} \tau_n(D0) & \xrightarrow{\omega_n} & \tau_{n-1}(0D) \\ {\scriptstyle \tau_n\binom{k}{0}}\downarrow \cong & & \downarrow 1 \\ \tau_n(c') & \xrightarrow{\partial_n(\underline{\bar{S}})} & \tau_{n-1}(0D) \end{array}$$

by applying ∂_n. Therefore $\partial_n(\underline{\bar{S}}) = \omega_n(0D, D0)\,\tau_n\binom{k}{0}^{-1} = \partial'_n(\underline{\bar{S}})$. Now apply ∂_n to the map

$$\left[\binom{0}{h}, \binom{g}{gb_*}, \binom{1}{1}\right] : (3.6) \to \underline{\bar{S}}$$

and obtain the commutative diagram

$$\begin{array}{ccc} \tau_n(c') & \xrightarrow{\partial_n(3.6)} & \tau_{n-1}(a') \\ \downarrow 1 & & \downarrow \tau_{n-1}\binom{0}{h} \\ \tau_n(c') & \xrightarrow{\partial_n(\underline{\bar{S}})} & \tau_{n-1}(0D). \end{array}$$

It follows that $\partial_n(3.6) = \tau_{n-1}\binom{0}{h}^{-1}\partial_n(\underline{\bar{S}}) = \partial'_n(3.6)$. Now using the given maps $(3.6) \to (3.5)$ and $(2.2) \to (3.5)$ of diagram (3.3) one obtains successively that $\partial_n(3.5) = \partial'_n(3.5)$ and $\partial_n(2.2) = \partial'_n(2.2)$. But since (2.2) was an arbitrary s.e.s. of $\mathcal{A}^2$ it follows that $\partial_n = \partial'_n$. Since n was arbitrary ∂ is unique. Moreover, any ∂ which satisfies the hypotheses of (3.4) must have the form (3.9). This is the analogue of (2.6). The theorem now follows. We shall next prove a converse to Theorem 3.4.

THEOREM 3.14. *Let (τ, ∂) be an h-functor $\mathcal{A}^2 \to \mathcal{B}$. There is a unique graded functor $\omega : \mathcal{A}^3 \to \mathcal{B}^2$ such that (τ, ω) is an E–H functor and $\omega(0A, A0) = \partial(\mathrm{ses}\,|\,A\,|)$ iff for every identity map 1_A of $\mathcal{A}$ $\tau(1_A) = 0$.*

Proof. If ω is given such that (τ, ω) is an E–H functor, then $\tau(e) = 0$ for every equivalence e.

If $\tau(1) = 0$ for each identity map of $\mathcal{A}$, then apply (τ, ∂) to $\mathrm{ses}\,|\,A\,|$.

In the resulting l.e.s. of $\mathcal{B}$ there are the isomorphisms

$$\xi_n(A) = \partial_n(\mathrm{ses}\,|\,A\,|) : \tau_n(A0) \to \tau_{n-1}(0A).$$

If (x, y) is an object of $\mathcal{A}^3$, $x : X \to Y$, $y : Y \to Z$, then define the functor $\omega : \mathcal{A}^3 \to \mathcal{B}^2$ to be

$$\omega_n(x, y) = \tau_{n-1}\binom{0X}{1}\,\xi_n(Y)\,\tau_n\binom{1}{Z0}.$$

It is obvious that $\omega_n(0A, A0) = \partial_n(\text{ses}|A|)$. ξ plays the same role as the flipping functor w. We shall verify that (τ, ω) is an E–H functor. Let $f: A \to B$ be any map of $\mathcal{A}$. Form the s.e.s. (3.15) of $\mathcal{A}^2$.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "1"] \arrow[d, "f"] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r, "1"] & B \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd} \tag{3.15}$$

Consider the map $\left[\begin{pmatrix}\ddot{0}\\ f\end{pmatrix}, \begin{pmatrix}1\\ f\end{pmatrix}, \begin{pmatrix}1\\ \ddot{0}\end{pmatrix}\right]$: ses$|A| \to (3.15)$. Apply the h-functor (τ, ∂) to this commutative diagram of s.e.s.'s to obtain the commutative diagram of l.e.s.'s (3.16)

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \tau_{n+1}(A0) \arrow[r, "\xi_{n+1}"] \arrow[d, "1"] & \tau_n(0A) \arrow[r] \arrow[d, "\tau_n\begin{pmatrix}\ddot{0}\\ f\end{pmatrix}"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \tau_{n+1}(f) \arrow[r, "\tau_{n+1}\begin{pmatrix}1\\ B0\end{pmatrix}"] & \tau_{n+1}(A0) \arrow[r, "\partial_{n+1}"] & \tau_n(0B) \arrow[r] & \tau_n(f) \arrow[r] & \cdots
\end{tikzcd} \tag{3.16}$$

Since $\tau_n\begin{pmatrix}\ddot{0}\\ 1\end{pmatrix} = 1 : \tau_n(0A) \to \tau_n(0A)$, then $\xi_{n+1}\tau_{n+1}\begin{pmatrix}1\\ B0\end{pmatrix} = \omega_{n+1}(0A, f)$. From (3.16) one then obtains the l.e.s. (3.17) of $\mathcal{B}$ which corresponds to (2.5) for the object $(0A, f)$ of $\mathcal{A}^3$.

$$\cdots \to \tau_{n+1}(f) \xrightarrow{\omega_{n+1}} \tau_n(0A) \xrightarrow{\tau_n\begin{pmatrix}\ddot{0}\\ f\end{pmatrix}} \tau_n(0B) \xrightarrow{\tau_n\begin{pmatrix}0A\\ 1\end{pmatrix}} \tau_n(f) \to \cdots. \tag{3.17}$$

Since f was arbitrary, it follows from [3, Thm. 2.13] that (τ, ω) is an E–H functor. It is a consequence of [3, Thm. 2.8] that ω is uniquely determined by the condition that $\omega(0A, A0) = \partial(\text{ses}|A|)$.

4. Some examples

A. Let (2.2) be any s.e.s. of $\mathcal{A}^2$. There is always a l.e.s. of $\mathcal{A}$

$$0 \to \ker a \to \ker b \to \ker c \xrightarrow{\partial} \operatorname{cok} a \to \operatorname{cok} b \to \operatorname{cok} c \to 0.$$

That is, there is an h-functor $(\tau, \partial): \mathcal{A}^2 \to \mathcal{A}$ given by $\tau_1 = \ker$, $\tau_0 = \operatorname{cok}$, $\partial_1 =$ the usual zig-zag homomorphism which is defined in the category of relations of $\mathcal{A}$, and all the other functors are zero.

Since $\tau(1) = 0$, by Theorem 3.14 there is an ω such that (τ, ω) is an E–H functor. This can be proved quite independently if one chooses

$$w_n(A) = 1_A : \ker A0 \to \operatorname{cok} 0A.$$

This is, for example, a problem in [12, p. 50]. It is a more interesting result to note that τ is half-exact so that by Theorem 3.4, if we are given the E–H functor (τ, ω), there then exists a ∂ such that (τ, ∂) is the usual h-functor. In this manner one can find the zig-zag homomorphism in any abelian category *without having to go to the category of relations.*

B. Let $\mathcal{M}$ be the category of left R-modules over some ring R (with unit), and let $\mathcal{A}b$ be the category of abelian groups. Consider the h-functor of [8, Theorem 13.16 dual] which is given by

$$\tau_j = \operatorname{Ext}^{-j}(A, -) \text{ for } j<0; \quad \tau_j = \underline{\pi}_j(A, -) \text{ for } j \geqslant 0,$$

where A is a fixed R-module. In [10] P. J. Hilton and the author proved that this functor could be generalized to produce an E–H functor $(\tau', \omega') : \mathcal{M}^2 \to \mathcal{A}b$, where $\tau'(0A) \cong \tau(A)$. It was also proven that τ' had right-excision, so by Lemma 2.7 and Theorem 3.4 there exists a ∂' such that (τ', ∂') is an h-functor $\mathcal{M}^2 \to \mathcal{A}b$.

$\underline{\pi}_j(A, -)$ is called the jth projective homotopy group functor. There are three other functors of this kind

$$\text{(i)}: \underline{\pi}_j(-, B), \quad \text{(ii)}: \bar{\pi}_j(A, -), \quad \text{(iii)}: \bar{\pi}_j(-, B).$$

Observe that (i) and (iii) are contravariant functors. We call (ii) and (iii) the injective homotopy groups. There is a "dual" h-functor to the one above involving (iii) and $\operatorname{Ext}^n(-, B)$. By [8, 13.15] (i) and (ii) are E–H functors; they are not however generally h-functors (see below). Kleisli [11] and Eckmann and Kleisli [4] have also studied these functors.

C. The functor $\tau_1 = \bar{\pi}_1(Z, -) : \mathcal{A}b^2 \to \mathcal{A}b$ is defined to be the abelian group given by $\hom(z, f)/\hom_0(z, f)$ where $z : Z \to Q$ is the usual monic of $\mathcal{A}b$, f is arbitrary, and $\hom_0(z, f)$ is the subgroup of maps which can be factored through any map of the type $j : I \to J$ of $\mathcal{A}b$, where I, J, and $\ker j$ are all injectives. It has been shown that this is part of an E–H functor. We shall show that τ_1 does not have right-excision, so by (3.4) and (2.7) it cannot be an h-functor. In fact $\bar{\pi}_0(Z, -)$: $\mathcal{A}b \to \mathcal{A}b$ is not even half-exact.

Let $0 \to Z \xrightarrow{g} E \xrightarrow{h} Q \to 0$ be a non-split s.e.s. of $\mathcal{A}$ (this exists because $\operatorname{Ext}^1(Q, Z) \neq 0$). Then by direct calculation

$$\tau_1(Z0) \cong Z \quad \text{and} \quad \tau_1(h) \cong E.$$

If τ_1 had right-excision these would be the same.

Similarly, if we apply $\bar{\pi}_0(Z, -)$ to this s.e.s. we obtain the sequence $Z \to E \to 0$ which is clearly not exact at E.

5. The extension of functors

Let $\mathcal{A}$ be a category with enough projectives and injectives, and let $\varrho_0: \mathcal{A} \to \mathcal{B}$ be a half-exact functor. The satellites of ϱ_0 exist and form an h-functor (ϱ, ∂). One would like to extend the definition of ϱ to act on morphisms of $\mathcal{A}$ also. It is proven here that a functor $\varrho_0': \mathcal{A}^2 \to \mathcal{B}$ can be defined which is half-exact and has a satellite h-functor (ϱ', ε'). Moreover, $\varrho'(1) = 0$ so that by (3.14) there is an ω' such that (ϱ', ω') is an E–H functor. Let $\alpha: \mathcal{A} \to \mathcal{A}^2$ denote the embedding functor given by $\alpha(f) = \begin{pmatrix} \ddot{o} \\ f \end{pmatrix}$. If ϱ_0 vanishes on all the projectives of $\mathcal{A}$, then there is a natural equivalence $\varrho_0 \to \varrho_0' \alpha$ which extends to a natural equivalence of the h-functors.

Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$, $\gamma: \mathcal{C}' \to \mathcal{C}$ the inclusion functor, and $\mu: \mathcal{C}' \to \mathcal{J}$ any functor. If there is a functor $\lambda: \mathcal{C} \to \mathcal{J}$ such that $\lambda\gamma$ and μ are naturally equivalent, $\lambda\gamma \cong \mu$, then λ *is said to extend* μ. If μ is a graded functor, then λ *extends* μ iff λ is graded and $\lambda_n \gamma \cong \mu_n$. Let γ': Ses$(\mathcal{C}') \to$ Ses$(\mathcal{C})$ be the functor induced by γ, then the h-functor (λ, ∂) *extends* the h-functor (μ, ε) iff $\lambda\gamma \cong \mu$ and $\partial\gamma' \cong \varepsilon$.

Example. Let 1 be the category with precisely one map $1_U = \ddot{u}$. Let $\mathcal{S}$ be a small category with an initial object E. There is a unique functor $\theta: \mathcal{S} \to 1$ which sends each map of $\mathcal{S}$ to $\ddot{u}$. Let $\eta: 1 \to \mathcal{S}$ send $\ddot{u}$ to 1_E. These induce the embedding functor $\theta': \mathcal{A} \to \mathcal{A}^{\mathcal{S}}$ and its adjoint functor $\eta': \mathcal{A}^{\mathcal{S}} \to \mathcal{A}$. Given any sequence of $\mathcal{A}^{\mathcal{S}}$

$$o \to \varrho' \to \varrho \to \varrho'' \to o$$

then by [12, IX.3.1] it is a s.e.s. iff for each Y of $\mathcal{S}$

$$0 \to \varrho'(Y) \to \varrho(Y) \to \varrho''(Y) \to 0$$

is a s.e.s. of $\mathcal{A}$. The functor η' induces η'': Ses$(\mathcal{A}^{\mathcal{S}}) \to$ Ses$(\mathcal{A})$ by choosing $Y = E$. For any h-functor $(\sigma, \partial): \mathcal{A} \to \mathcal{B}$ one then has the h-functor $(\sigma\eta', \partial\eta''): \mathcal{A}^{\mathcal{S}} \to \mathcal{B}$ which extends (σ, ∂).

If $\mathcal{S} = 2$ then $\sigma\eta'(1_A) = \sigma(A)$ which need not be 0. Therefore there is no ω such that $(\sigma\eta', \omega)$ is an E–H functor. Extensions which vanish on every identity map will be produced.

A class of objects $\mathcal{P} \subset \mathcal{A}$ is a class of *projectives* iff hom$(P, -)$ is exact for every $P \in \mathcal{P}$. The class of *injectives* $\mathcal{I}$ is defined dually. It is assumed that $\mathcal{A}$ has enough projectives and enough injectives. This will be taken to mean that for each object $A \in \mathcal{A}$ there are two selected objects, $IA \in \mathcal{I}$ and $PA \in \mathcal{P}$, such that there are the two s.e.s.'s

$$\Gamma(0A) = 0 \to \Omega A \to PA \xrightarrow{p} A \to 0, \tag{5.1 a}$$

and

$$\Gamma(A0) = 0 \to A \xrightarrow{i} IA \xrightarrow{\hat{\imath}} \Sigma A \to 0. \tag{5.1 b}$$

If $A=0$, then both IA and PA are set equal to zero. If A is injective (projective), then IA (PA) is chosen to be A. For any map $f\colon A\to B$ define $\Gamma(f)$ as follows: let C be the pushout of f and $i\colon A\to IA$, let D be the pullback of f and $p\colon PB\to B$, and observe that both squares of (5.2) are bicartesian.

$$\begin{tikzcd} D \arrow[r, "n"] \arrow[d, "e"] & A \arrow[r, "i"] \arrow[d, "f"] & IA \arrow[d, "g"] \\ PB \arrow[r, "p"] & B \arrow[r, "j"] & C \end{tikzcd} \tag{5.2}$$

By [9, Theorem 3.7] it follows that the composite of the two squares is bicartesian and there is the s.e.s.

$$\Gamma(f)=\left[0\to D\xrightarrow{\{-in,\,e\}} IA\oplus PB\xrightarrow{\langle g,\,jp\rangle} C\to 0\right].$$

LEMMA 5.3. *If $\mathcal{A}$ has enough injectives, then so does $\mathcal{A}^2$.*

Proof. Let $k\colon J\to K$ be an injective of $\mathcal{A}^2$. J cannot be 0, because the monomorphism $\begin{pmatrix} o \\ 1 \end{pmatrix}\colon 0K\to 1_K$ does not split. Clearly both J and K must be injectives of A. The s.e.s. (5.4) must split.

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "\{1,2k\}"] \arrow[d, "k"'] & J\oplus K \arrow[r] \arrow[d, "\langle -k,1\rangle"] & K \arrow[r] \arrow[d, dash] & 0 \\ 0 \arrow[r] & K \arrow[r, "1"] & K \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd} \tag{5.4}$$

In particular there must be a map $\langle m,n\rangle\colon J\oplus K\to J$ such that $k\langle m,n\rangle=\langle -k,1\rangle$. Therefore $kn=1$, so k is epic and splits and there is an injective I such that $s\colon J\cong I\oplus K$ and $ks^{-1}=\langle o,1\rangle$. Since k was arbitrary *every injective* of $\mathcal{A}^2$ is isomorphic to a map of the form $q=\langle o,1\rangle\colon I\oplus K\to K$. Conversely any map of this form is an injective. To see this let $\begin{pmatrix} a \\ c \end{pmatrix}\colon b\to d$ be any monic of $\mathcal{A}^2$, and $\begin{pmatrix} e \\ f \end{pmatrix}\colon b\to q=\langle o,1\rangle$ be a morphism.

$$\begin{tikzcd} I & I\oplus K \arrow[l, "\langle 1,o\rangle"'] \arrow[d, "\langle o,1\rangle"] & A \arrow[l, "e"'] \arrow[r, "a"] \arrow[d, "b"] & C \arrow[d, "d"] \\ & K & B \arrow[l, "f"'] \arrow[r, "c"] & D \end{tikzcd}$$

Since I and K are injective there are maps $g\colon D\to K$, and $h\colon C\to I$ such that $gc=f$ and $ha=\langle 1,o\rangle e$. The map $\begin{pmatrix} \{h,gd\} \\ g \end{pmatrix}\colon d\to\langle o,1\rangle$ extends $\begin{pmatrix} e \\ f \end{pmatrix}$.

If $f: A \to B$ is any map, let $i: A \to IA$ and $j: B \to IB$ be monics into injectives. The map $\begin{pmatrix} \{i, jf\} \\ j \end{pmatrix}: f \to \langle o, 1 \rangle$ is the required monic into an injective. That is, $\mathcal{A}^2$ has enough injectives. This dualizes immediately to give the corresponding result for projectives.

In [8, Chapter 13] all the maps of injectives $i: I \to J$ with injective kernels (not necessarily epic) were chosen as the class of injectives. It follows from the theory of Eilenberg and Moore [5] that the class of s.e.s.'s on which the hom$(-, i)$ are exact must be reduced—which was indeed the case.

In the subsequent discussion $\mathcal{A}$ will denote an abelian category with enough injectives and projectives. By generalizing the results of Cartan and Eilenberg [1, Chapter III], or by a remark of Mitchell [13, p. 203], any half-exact functor $\varrho_0: \mathcal{A} \to \mathcal{B}$ gives rise to an h-functor (ϱ, ε); $\mathcal{A} \to \mathcal{B}$ where ϱ_n is the nth left (right) satellite $S^{-n}\varrho_0$ for $n > 0$ $(n < 0)$. (ϱ, ε) is called the satellite h-functor of ϱ_0. Since ϱ_n is half-exact, it is additive; ε_n is additive by the construction procedure. Let α': Ses$(\mathcal{A}) \to$ Ses$(\mathcal{A}^2)$ be the functor induced by $\alpha: \mathcal{A} \to \mathcal{A}^2$. We now state the main theorem of this section:

THEOREM 5.5. *Let $\varrho_0: \mathcal{A} \to \mathcal{B}$ be a half-exact functor with satellite h-functor (ϱ, ε). One can define*

(i) *a half-exact functor $\varrho'_0: \mathcal{A}^2 \to \mathcal{B}$ with satellite h-functor (ϱ', ε'), together with*

(ii) *a natural epimorphism of functors $\varrho_0 \to \varrho'_0\alpha$ which extends uniquely to a natural transformation $(\varrho, \varepsilon) \to (\varrho'\alpha, \varepsilon'\alpha')$*

such that

(iii) *$\varrho'_0(1_A) = 0$ for every A of $\mathcal{A}$,*

(iv) *$\varrho'_0(p) = 0$ for every projective p of $\mathcal{A}^2$, and*

(v) *the induced natural transformation $\varrho_n \to \varrho'_n\alpha$ is an equivalence for $n < 0$.*

Any functor ϱ'_0 which satisfies the above conditions also satisfies

(vi) *there exists a unique $\omega': \mathcal{A}^3 \to \mathcal{B}^2$ such that $\omega'(0A, A0) = \varepsilon'(\text{ses}\,|A|)$ for each A in $\mathcal{A}$, and such that (ϱ', ω') is an E–H functor,*

(vii) *$\varrho'_0(i) = 0$ for every injective i of $\mathcal{A}^2$,*

(viii) *using the notation of (5.2)*

$$\varrho'_n(f) \cong \varrho_n(C) \quad \textit{for} \quad n < 0.$$

Moreover, it can also be shown that

(ix) *up to natural equivalence, ϱ'_0 is uniquely determined by* (i)$\to$(v),

(x) *if $\varrho_0(P) = 0$ for every projective P of $\mathcal{A}$ then the natural transformation of* (ii) *is a natural equivalence. That is, (ϱ', ε') extends (ϱ, ε). Also $\varrho'_n(f) \cong \varrho_{n-1}(D)$ for $n > 0$.*

Proof. Define $\varrho'_0(f) = \mathrm{im}\ \varepsilon_0(\Gamma(f)) : \varrho_0(C) \to \varrho_{-1}(D)$ where f is as in (5.2). To show that this definition is independent of the choice of injective, let $z : A \to J$ be a monic into some other injective J. Let H be the pushout of i and z, and $h : H \to IH$ a monic into an injective. The maps $IA \to H \to IH$ and $J \to H \to IH$ are both monic and split. That is, there is an injective K and an isomorphism $IH \cong IA \oplus K$ such that the composition $IA \to IH \to IA \oplus K$ is $\{1, \mathrm{o}\}$. Using the monic $\{i, \mathrm{o}\} : A \to IA \oplus K$ instead of i in (5.2) one obtains a new s.e.s. $\Gamma^{\sim}(f) = \Gamma(f) \oplus (\mathrm{o} \| 1_K)$. It is clear that $\varepsilon(\mathrm{o} \| 1_X) = \mathrm{o}$ for every X. It follows from the additivity of ε that $\varepsilon_0\ (\Gamma^{\sim}(f)) = \varepsilon_0(\Gamma(f)) \oplus \mathrm{o}$; that is im $\varepsilon_0(\Gamma^{\sim}(f)) = \varrho'_0(f)$.

Now let C'' be the pushout of f and z, and let K'' be an injective such that $IH \cong J \oplus K''$. It follows that there is an isomorphism $x : IA \oplus K \to J \oplus K''$. Build a diagram (5.2) using the monic $\{z, \mathrm{o}\} : A \to J \oplus K''$ (P is kept fixed throughout). There is a map induced from the diagram built from $\{i, \mathrm{o}\}$ to this one, and there is a unique map $y : C \oplus K \to C'' \oplus K''$ induced by the universal property of pushouts. Moreover, this induces a map of s.e.s.'s.

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] \arrow[d, "1"] & P \oplus IA \oplus K \arrow[r] \arrow[d, "\cong"] & C \oplus K \arrow[r] \arrow[d, "y"] & 0 \\
0 \arrow[r] & D \arrow[r] & P \oplus J \oplus K'' \arrow[r] & C'' \oplus K'' \arrow[r] & 0
\end{tikzcd}$$

from which it follows that y is an isomorphism. Apply (ϱ, ε).

$$\begin{tikzcd}
\varrho_0(C) \oplus \varrho_0(K) \arrow[r, "\varepsilon_0"] \arrow[d, "\cong"] & \varrho_{-1}(D) \arrow[d, "1"] \\
\varrho_0(C'') \oplus \varrho_0(K'') \arrow[r, "\varepsilon_0''"] & \varrho_{-1}(D)
\end{tikzcd}$$

It follows that $\mathrm{im}\, \varepsilon_0$ is isomorphic to $\mathrm{im}\, \varepsilon_0''$, and thus $\varrho'_0(f)$ is independent of the choice of injective (dually for projectives). That is, $\varrho'_0(f)$ is well-defined.

A map $\begin{pmatrix} a \\ b \end{pmatrix} : f \to f'$ of $\mathcal{A}^2$ induces a morphism $\varrho'_0(f) \to \varrho'_0(f')$. It will be shown that this is also well-defined. Place a prime on each object and map of (5.2) and denote it by (5.2)$'$. Choose maps $a' : IA \to IA'$, $b' : PB \to PB'$ such that $a'i = i'a$ and $bp = p'b'$. The maps a, b, a', and b' induce unique maps $d : D \to D'$ and $c : C \to C'$ which give a commutative diagram corresponding to a map (5.4)$\to$(5.4)$'$. This in turn induces a map $\Gamma(f) \to \Gamma(f')$. If (ϱ, ε) is applied to this, the commutative diagram (5.6) is obtained.

$$\begin{tikzcd}
\varrho_0(C) \arrow[r, "t"] \arrow[d, "\varrho_0(c)"] & \varrho_0'(f) \arrow[r, "s"] \arrow[d, dotted] & \varrho_{-1}(D) \arrow[r] \arrow[d, "\varrho_{-1}(d)"] & \varrho_{-1}(IA \oplus PB) \arrow[d] \\
\varrho_0(C') \arrow[r, "t'"] & \varrho_0'(f') \arrow[r, "s'"] & \varrho_{-1}(D') \arrow[r] & \varrho_{-1}(IA' \oplus PB')
\end{tikzcd} \tag{5.6}$$

Let $st = \varepsilon_0(\Gamma(f))$ and $s't' = \varepsilon_0(\Gamma(f'))$ be the canonical factorizations into an epic followed by a monic. Since s and s' are the kernels of the maps on their right, there is a unique map $\varrho'_0(f) \to \varrho'_0(f')$ induced. Dually there is a unique map induced because t and t' are cokernels, and this must be *the same map*. If one varies IA, IA', or the choice of a', one does not change s, s', or $\varrho_{-1}(d)$ so the induced map does not change either. Dually if one changes PA, PA', and b' the induced map doesn't change. Therefore $\varrho'_0\begin{pmatrix} a \\ b \end{pmatrix}: \varrho'_0(f) \to \varrho'_0(f')$ is well-defined. The fact that $\varrho'_0(b)$ is well-defined could have been deduced, with a little work, from this result. (This argument, which is due to P. J. Hilton, appears also in [10].) The remainder of the proof that ϱ'_0 is a functor is now trivial.

Take any s.e.s. (2.2) of $\mathcal{A}^2$ and let $i: B \to I$ and $j: C \to J$ be monics into injectives. Form the s.e.s. (5.7), and note that each vertical map is monic.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "d"] \arrow[d, "\{a, id\}"] & B \arrow[r, "e"] \arrow[d, "\{b, \{i, je\}\}"] & C \arrow[r] \arrow[d, "\{c, j\}"] & 0 \\
0 \arrow[r] & A' \oplus I \arrow[r, "d' \oplus \{1, 0\}"] & B' \oplus (I \oplus J) \arrow[r, "e' \oplus \langle 0, 1 \rangle"] & C' \oplus J \arrow[r] & 0
\end{tikzcd} \tag{5.7}$$

The sequence of cokernels $0 \to X \to Y \to Z \to 0$ is exact. Each cokernel is a pushout, i.e. X is the pushout of a and id, and their pushout squares are bicartesian.

Dually take epics $p: P \to A'$ and $q: Q \to B'$ where P and Q are projectives. By a dual argument three new bicartesian squares are formed, and the pullbacks form a s.e.s. $0 \to U \to V \to W \to 0$.

Each map a, b, and c stands as the common edge of two bicartesian squares. The s.e.s.'s resulting from the composition of these pairs of squares are represented by the columns of the commutative diagram (5.8). The lines are also exact. The middle line splits.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \oplus P \arrow[r, "\{1, 0\}"] \arrow[d] & (I \oplus P) \oplus (J \oplus Q) \arrow[r, "\langle 0, 1 \rangle"] \arrow[d] & J \oplus Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{5.8}$$

Apply (ϱ, ε) to (5.8) to obtain the commutative diagram (5.9). The maps $\varrho_0 \to \varrho_0'$ are all epic. The top line splits at $\varrho_0((I \oplus P) \oplus (J \oplus Q))$. The exactness of the bottom line is easily verified by a diagram chasing argument, and this proves that ϱ'_0 is a half-exact functor. Let (ϱ', ε') denote the satellite functor of ϱ_0'.

$$\begin{tikzcd}
\varrho_0(I \oplus P) \arrow[r, "\{1, o\}"] \arrow[d] & \varrho_0((I \oplus P) \oplus (J \oplus Q)) \arrow[r, "\langle o, 1 \rangle"] \arrow[d] & \varrho_0(J \oplus Q) \arrow[r, "o"] \arrow[d] & \varrho_{-1}(I \oplus P) \arrow[d] \\
\varrho_0(X) \arrow[r] \arrow[d] & \varrho_0(Y) \arrow[r] \arrow[d] & \varrho_0(Z) \arrow[r, "\varepsilon_0"] \arrow[d] & \varrho_{-1}(X) \\
\varrho_0'(a) \arrow[r] & \varrho_0'(b) \arrow[r] & \varrho_0'(c) &
\end{tikzcd} \tag{5.9}$$

If $\Gamma(0A)$ is taken as in (5.1a), then there is a natural epimorphism given by the factorization of $\varepsilon_0(\Gamma(0A))$ into an epic followed by a monic, which maps $\varrho_0(A)$ onto $\varrho'_0(0A)$. That is, there is a natural transformation $\varrho_0 \to \varrho'_0 \alpha$. By the techniques of Cartan and Eilenberg [1, Proposition III.5.2] (these certainly can be immediately carried over to the abelian categories under discussion here), this natural transformation extends uniquely to a natural transformation of h-functors $(\varrho, \varepsilon) \to (\varrho' \alpha, \varepsilon' \alpha')$, where $\alpha' : \mathrm{Ses}(\mathcal{A}) \to \mathrm{Ses}(\mathcal{A}^2)$ extends α. The only hypothesis that must be checked is that $\varrho'_n \alpha(P) = 0$ for all projectives P and all $n > 0$. But $0P$ is a projective of $\mathcal{A}^2$ by (5.3), so $\varrho'_n(0P) = 0$. This gives (ii).

Let 1_A be any identity map of $\mathcal{A}$. Denote IA and PA by I and P. From the properties of pullback and pushout it is clear that in the situation of (5.2) one could take $C = I$, $D = P$, $g = 1$, $e = 1$, and

$$\Gamma(1_A) = 0 \to P \xrightarrow{\{1, -ip\}} P \oplus I \xrightarrow{\langle ip, 1 \rangle} I \to 0.$$

This is a split s.e.s. so $\varrho'_0(1_A) = \mathrm{im}\, \varepsilon_0(\Gamma(1_A)) = 0$. This is condition (iii).

Form the s.e.s. $\ddot{o} \to 1_A \to 1_I \to 1_{\Sigma A} \to \ddot{o}$ of $\mathcal{A}^2$. 1_I is an injective of $\mathcal{A}^2$ by (5.3), so $\varrho'_n(1_I) = 0$ for all $n < 0$. Apply the h-functor (ϱ', ε') to the above s.e.s. Note that since $\varrho'_0(1_A) = 0$ for every A it follows that $\varrho'_{-1}(1_A) = 0$ for all A also. Similarly one can show that $\varrho'_n(1_X) = 0$ for all n and all X. It follows from Theorem 3.14 that there is a unique functor $\omega' : \mathcal{A}^3 \to \mathcal{B}^2$ such that (ϱ', ω') is an E–H functor, as in condition (vi). By Theorem 3.4 and Lemma 2.7 one obtains that ϱ' has left- and right-excision.

Let P be any projective of $\mathcal{A}$. $\Gamma(0P)$ is the s.e.s. $o \| 1_P$ of $\mathcal{A}^2$, so $\varrho'_0(0P) = 0$. Let $\{o, 1\}$: $Q \to P \oplus Q$ be any projective of $\mathcal{A}^2$. By excision $\varrho'_0(\{o, 1\}) \cong \varrho'_0(0P) = 0$.

There is a dual proof for injectives $\langle o, 1 \rangle : I \oplus J \to J$. However, if one wished to avoid using the definition of ϱ'_0, one could simply observe that if one assumes statement (v)

$$\varrho'_0(\langle o, 1 \rangle) \cong \varrho'_0(I0) \cong \varrho'_{-1}(0I) \cong \varrho_{-1}(I) = 0.$$

This does not dualize for projectives. This lack of symmetry, which also appears in (x), is due to our choice of the embedding functor $\alpha: \mathcal{A} \to \mathcal{A}^2$. It also follows that $\varrho'_n(0I)=0$ for all $n<0$ and all injectives I, even though $0I$ isn't itself injective. Therefore (iv) and (vii) are satisfied.

Let $R=P(IA)$ be a projective and $r: R \to IA$ an epimorphism. One obtains the commutative diagram (5.10) where the second line is a part of the l.e.s. used to obtain $\varrho'_0(A0)$ from (5.1b). The maps s, t, and u are all cokernels of the maps preceding them in the diagram and can be thought of as the maps used to define the respective ϱ'_0 objects. ε'_0 is epic because $\varrho'_{-1}(0IA)=0$.

$$\begin{tikzcd}
\varrho_0(R) \arrow[r, "1"] \arrow[d, "\varrho_0(r)"] & \varrho_0(R) \arrow[d] & & \\
\varrho_0(IA) \arrow[r, "\varrho_0(\hat{\imath})"] \arrow[d, "s"] & \varrho_0(\Sigma A) \arrow[r, "u"] \arrow[d, "t"] & \varrho'_0(A0) \arrow[r, "\cong"] & \varrho_{-1}(A) \arrow[d, "y"] \\
\varrho'_0(0IA) \arrow[r, "\varrho'_0\binom{0}{\hat{\imath}}"] & \varrho'_0(0\Sigma A) \arrow[rr, "\varepsilon'_0"] & & \varrho'_{-1}(0A)
\end{tikzcd} \tag{5.10}$$

The map $\varrho_0(\Sigma A) \to \varrho_{-1}(A)$ is $\varepsilon_0(i \| \hat{\imath})$, which is a cokernel of $\varrho_0(\hat{\imath})$, so the map $y: \varrho_{-1}(A) \to \varrho'_{-1}(0A)$ is uniquely determined. By chasing the diagram one finds that y is an isomorphism. But this is precisely the procedure of [1] in the proof of the universality of the right satellites. If one iterates this procedure one has that $\varrho_n \to \varrho'_n \alpha$ is an isomorphism for all $n<0$. This proves statement (v).

Take any map f as in (5.2). Consider the maps $\langle 1, 0\rangle: B \oplus I \to B$ and $\{f, i\}: A \to B \oplus I$ and notice that their composition is f. By excision and the results above

$$\varrho'_n(\{f, i\}) \cong \varrho'_n(0C) \cong \varrho_n(C) \quad \text{for} \quad n<0,$$

$$\varrho'_n(\langle 1, 0\rangle) \cong \varrho'_n(I0) = 0 \qquad \text{for} \quad n \leqslant 0.$$

Apply the E–H functor (ϱ', ω') of (vi) to the maps $(\{f, i\}, \langle 1, 0\rangle)$ and use the isomorphisms above to obtain from the usual l.e.s. of $\mathcal{B}$ the isomorphisms of condition (viii)

$$\varrho_n(C) \cong \varrho'_n(f) \quad \text{for} \quad n<0.$$

This result is independent of the I or monic $A \to I$ used to obtain a C. Dually one obtains an isomorphism

$$\varrho'_n(f) \cong \varrho'_{n-1}(0D) \quad \text{for} \quad n>0$$

but $\varrho'_{n-1}(0D)$ is not generally isomorphic to $\varrho_{n-1}(D)$ [see example 5.15].

In the situation of (5.9) one would therefore have that $\varrho_{-1}(X)$, $\varrho_{-1}(Y)$, and $\varrho_{-1}(Z)$ are respectively isomorphic to $\varrho'_0(a)$, $\varrho'_0(b)$, and $\varrho'_0(c)$. Also, the composition

$$\varepsilon_0(\varrho_0(J\oplus Q)\rightarrow\varrho_0(Z))=0$$

so ε_0 factors through $\varrho'_0(c)$. The map induced from $\varrho'_0(c)\rightarrow\varrho_{-1}(X)$ together with the isomorphism $\varrho_{-1}(X)\rightarrow\varrho'_{-1}(a)$ is actually ε'_0.

Following the notation of (5.2), $\varrho_{-1}(IA)=0$. If (ϱ, ε) is applied to $\Gamma(f)$, then by the definition of $\varrho'_0(f)$ there is an exact sequence

$$0\rightarrow\varrho'_0(f)\rightarrow\varrho_{-1}(D)\xrightarrow{\varrho_{-1}(e)}\varrho_{-1}(PB). \tag{5.11}$$

Let an arbitrary functor $\varrho\tilde{}_0: \mathcal{A}^2\rightarrow\mathcal{B}$ which also satisfies all the conditions (i)$\rightarrow$(v) be chosen. The satellite h-functor of $\varrho\tilde{}_0$ is denoted by $(\varrho\tilde{}, \varepsilon\tilde{})$. The kernel of the epimorphism p of (5.2) is denoted by ΩB. Since D is a pullback one obtains from (5.2) a s.e.s. (5.12) of $\mathcal{A}^2$.

$$\ddot{o}\rightarrow 1_{\Omega B}\rightarrow e\rightarrow f\rightarrow\ddot{o} \tag{5.12}$$

A l.e.s. is obtained from the action of $(\varrho\tilde{}, \varepsilon\tilde{})$ on (5.12). Since $\varrho\tilde{}_n(1_A)=0$ for every A, if $n=0$, it is also true for every n. It follows that the l.e.s. contains isomorphisms for each n

$$\varrho\tilde{}_n(e)\cong\varrho\tilde{}_n(f). \tag{5.13}$$

Let $(\varrho\tilde{}, \omega\tilde{})$ be the E–H functor which exists by (vi), and apply it to $(0D, e)$. Since $0\rightarrow PB$ is a projective of $\mathcal{A}^2$ then by (iv) one obtains from the l.e.s. the exact sequence

$$0=\tilde{\varrho}_0(0PB)\rightarrow\tilde{\varrho}_0(e)\rightarrow\tilde{\varrho}_{-1}(0D)\xrightarrow{\tilde{\varrho}_{-1}\begin{pmatrix}\ddot{o}\\ e\end{pmatrix}}\tilde{\varrho}_{-1}(0PB).$$

By (ii), (v), and (5.13) an isomorphic exact sequence is obtained

$$0\rightarrow\tilde{\varrho}_0(f)\rightarrow\varrho_{-1}(D)\xrightarrow{\varrho_{-1}(e)}\varrho_{-1}(PB).$$

But from (5.11) one has that $\varrho'_0(f)$ is also a kernel of $\varrho_{-1}(e)$. Therefore there exists a unique isomorphism $\varrho'_0(f)\rightarrow\varrho\tilde{}_0(f)$. Moreover, it is trivial to note that given any map $f\rightarrow f'$, the appropriate square of maps commutes. That is, there is a natural equivalence $\varrho'_0\rightarrow\varrho\tilde{}_0$, so ϱ'_0 is essentially *unique*. This proves (ix).

If $\varrho_0(P)=0$ for every projective P, then if ϱ_0 is applied to the s.e.s. (5.1 a) the sequence

$$0=\varrho_0(PA)\rightarrow\varrho_0(A)\xrightarrow{\text{epic}}\varrho'_0(0A)\xrightarrow{\text{monic}}\varrho_1(\Omega A)$$

is obtained. It follows that $\varrho_0(A)$ is isomorphic to $\varrho_0'(0A)$ for every A. This induces a natural equivalence $\varrho_0 \to \varrho'_0\alpha$ which extends uniquely to a natural equivalence $(\varrho, \varepsilon) \to (\varrho'\alpha, \varepsilon'\alpha')$. For categories in which $\varrho_0(P)=0$ for all P one could in fact define $\varrho'_0(0A)=\varrho_0(A)$ and $\varrho'_0\begin{pmatrix}\ddot{o}\\ f\end{pmatrix}=\varrho_0(f)$ for all A and all f of $\mathcal{A}$. Without loss of generality this will be assumed. To complete the proof of the theorem one notes that there are now isomorphisms

$$\varrho'_n(f) \cong \varrho'_{n-1}(0D) = \varrho_{n-1}(D) \quad \text{for} \quad n>0. \qquad \text{Q.E.D.}$$

Let $\mathcal{A}^{(n)}$ be the iterated morphism category $(\dots(\mathcal{A}^2)^2\dots)^2$, where $\mathcal{A}^{(0)}=\mathcal{A}$ and $\mathcal{A}^{(1)}=\mathcal{A}^2$ The functor $\alpha: \mathcal{A}^{(0)} \to \mathcal{A}^{(1)}$ can be iterated to give embeddings $\mathcal{A}^{(n)} \to \mathcal{A}^{(n+1)}$ for all n. If $\mathcal{A}$ has enough injectives (projectives) then so does $\mathcal{A}^{(n)}$. Define $\mathcal{A}^\infty = \bigcup_n \mathcal{A}^{(n)}$. This is an abelian category. If $\mathcal{A}$ has enough projectives, then so does $\mathcal{A}^\infty$. This is not true for injectives; $\mathcal{A}^\infty$ has none! The morphisms of $\mathcal{A}^\infty$ can all be identified with objects of $\mathcal{A}^\infty$. By interchanging ö and f in the definition of α one obtains a dual embedding.

Each occurrence of $\mathcal{A}$, ϱ_0, and ϱ'_0 in the statement of (5.5) can be replaced by $\mathcal{A}^{(1)}$, ϱ'_0, and $\varrho^{(2)}{}_0$ respectively to obtain a new theorem. No further proof is required. One can in fact make successive replacements with $\mathcal{A}^{(2)}$, $\varrho^{(2)}{}_0$, $\varrho^{(3)}{}_0$; $\mathcal{A}^{(3)}$, $\varrho^{(3)}{}_0$, $\varrho^{(4)}{}_0$; ...; $\mathcal{A}^{(n)}$, $\varrho^{(n)}{}_0$, $\varrho^{(n+1)}{}_0$ for every $n \geqslant 2$ to obtain new true theorems. There is a considerable connection between $\varrho^{(n)}$ and ϱ, because for $m<0$ $\varrho^{(n)}{}_m(X)$ is isomorphic to $\varrho_m(X')$ for some suitable X' of $\mathcal{A}$ (X is an object of $\mathcal{A}^{(n)}$).

COROLLARY 5.14. *Let (ϱ, ε) be given as in Theorem 5.5 and assume that $\varrho_0(P)=0$ for every projective P. An h-functor $(\varrho^\infty, \varepsilon^\infty): \mathcal{A}^\infty \to \mathcal{B}$ can then be defined which extends (ϱ, ε) and such that*

(i) *if $A \in \mathcal{A}^{(n)}$, $\varrho^\infty(A) = \varrho^{(n)}(A)$*
(ii) *there is an E–H functor $(\varrho^\infty, \omega^\infty)$.*

Proof. Each $A \in \mathcal{A}^\infty = \bigcup \mathcal{A}^{(n)}$ must belong to $\mathcal{A}^{(n)}$ for some smallest n, say N. By the convention adopted above

$$\varrho^{(N)}(A) = \varrho^{(N+1)}(A) = \dots,$$

where the distinction between A and its image $0A$ in $\mathcal{A}^{(N+1)}$ and its ultimate image in $\mathcal{A}^\infty$ is ignored. Set $\varrho^\infty(A) = \varrho^{(N)}(A)$. Clearly $\varrho^\infty{}_0$ vanishes on all projectives and identity maps of $\mathcal{A}^\infty$ (there are no injectives). If $f\|g$ is a s.e.s. of $\mathcal{A}^\infty$, it is also a s.e.s. of $\mathcal{A}^{(N)}$ for some N and one can define the action of $(\varrho^\infty, \varepsilon^\infty)$ to be that of $(\varrho^{(N)}, \varepsilon^{(N)})$. Similarily an E–H functor is obtained.

If one is given a ϱ_0 which is not zero on each P, then one could restate Corollary 5.14 using ϱ'_0 and the relationships between ϱ_0 and ϱ'_0 mentioned in (5.5).

Example 5.15. Let R be the ring of integers modulo 8, $\mathcal{A}$ the category of R-modules, $\mathcal{B}$ the category of abelian groups, and $\varrho_0 = \operatorname{Hom}_R(4R, -): \mathcal{A} \to \mathcal{B}$. There is a s.e.s. (5.16) of R-modules

$$0 \to 2R \xrightarrow{i} R \to 4R \to 0, \tag{5.16}$$

where i is the inclusion of the submodule. From the definition of ϱ'_0 one obtains

$$\varrho'_0(0 \to 2R) \cong Z/2Z \cong \varrho'_0(0 \to 4R).$$

Since $0 \to R$ is a projective $\varrho'_n(0R) = 0$ for $n \geqslant 0$. Apply α' to (5.16) to obtain a s.e.s. of $\mathcal{A}^2$ upon which (ϱ', ε') acts to give the exact sequence

$$0 = \varrho'_1(0R) \to \varrho'_1(04R) \to \varrho'_0(02R) \to \varrho'_0(0R) = 0.$$

Therefore $\varrho'_1(04R) \cong Z/2Z$. But $\varrho_1(4R) = 0$. Similarly $\varrho'_n(04R) \cong Z/2Z$ for all $n > 0$.

Thus the property of being left-exact is not preserved when we pass from ϱ_0 to ϱ'_0. On the other hand, if ϱ_0 is right-exact then $\varrho'_0 = 0$. Such cases can be handled by defining

$$\varrho''_0(f) = \operatorname{im} \varepsilon_1(\Gamma(f)).$$

By a similar argument to the one used in (5.5) there is a natural monomorphism $\varrho''_0 \alpha \to \varrho_0$ which is an equivalence if $\varrho_0(I) = 0$ for every injective I.

If ϱ_0 is contravariant, one can simply replace $\mathcal{B}$ by $\mathcal{B}^{\mathrm{op}}$ to find the corresponding theorems.

In [10, Theorem 5.3] the authors proved that $\operatorname{Ext}^n(A, \varphi) \cong \operatorname{Ext}^{n-1}(A, \Sigma\varphi)$, $n \geqslant 2$ for any suspension $\Sigma\varphi: \Sigma X \to \Sigma Y$ of $\varphi: X \to Y$. The same proof is not used here because injectives of $\mathcal{A}^2$ are restricted by (5.3) to be maps of the type $\langle 1, \mathrm{o} \rangle : I \oplus J \to I$, whereas any map $I \to J$ was allowed in [10]. This gives a slightly better result.

If one insists that Σf be the cokernel of a monomorphism $f \to i$, where i is an injective of $\mathcal{A}^2$, then by applying the h-functor (ϱ', ε') to the s.e.s.

$$\ddot{\mathrm{o}} \to f \to i \to \Sigma f \to \ddot{\mathrm{o}}$$

one obtains $\varrho'_n(\Sigma f) \cong \varrho'_{n-1}(f)$ for $n < 1$. This gives one "more" isomorphism on the left than was had before.

Although abelian categories have been almost exclusively used here, one can obtain corresponding results in additive categories with additional structure, such as triangulated categories.

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