EXTREMAL ELEMENTS IN CERTAIN CLASSES OF CONFORMAL MAPPINGS OF AN ANNULUS

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1. For $0<r<1$, let R_r be the annulus $\{z; r<|z|<1\}$. Denoting by E the unit disk and by C its circumference, $\mathfrak{F}_r$ will be the class of schlicht analytic functions f with $f(R_r)\subset E-\{0\}=E_0$ and $f(C)=C$. For $B\in R_r$ we put

$$b_r'(B)=\inf_{f\in\mathfrak{F}_r}|f(B)|, \quad b_r''(B)=\sup_{f\in\mathfrak{F}_r}|f(B)| \tag{1.1}$$

and denote by $\mathfrak{F}_r(B;b)\subset\mathfrak{F}_r$ the subclass of functions satisfying $|f(B)|=b$ where $b_r'(B)\leqslant b\leqslant b_r''(B)$. We shall study the following problem: *Which functions minimize or maximize* $|f'(B)|$ *in the class* $\mathfrak{F}_r$ *and in the classes* $\mathfrak{F}_r(B;b)$?

In order to avoid the consideration of uninteresting rotations about the origin we shall mainly deal with the equivalent problems:

$P_{\min}(B)$; *Which functions, positive at* $B>0$, *minimize* $|f'(B)|$ *within* $\mathfrak{F}_r$;

$P_{\max}(B)$; *Which functions, positive at* $B>0$, *maximize* $|f'(B)|$ *within* $\mathfrak{F}_r$;

$P_{\min}(B;b)$; *Which functions, positive at* $B>0$, *minimize* $|f'(B)|$ *within* $\mathfrak{F}_r(B;b)$;

$P_{\max}(B;b)$; *Which functions, positive at* $B>0$, *maximize* $|f'(B)|$ *within* $\mathfrak{F}_r(B;b)$.

P. L. Duren [1] posed already the problems $P_{\min}(B)$ and $P_{\max}(B)$; using the variational Lemma of Schiffer [4], Duren [1] showed in sections 3–5: (i) $P_{\min}(B)$ *has the unique solution* Ψ_r; (ii) $P_{\max}(B)$ *has the unique solution* Φ_r *provided* B *is in a precise sense sufficiently far away from the inner boundary component* $|z|=r$ *of* R_r. [The functions Φ_r, Ψ_r will be defined in the following section 2.]. In the case when B is no more sufficiently far away from the inner boundary component, Duren [1] achieved a partial solution of $P_{\max}(B)$.

In the following we shall determine all solutions of each of the problems $P_{\min}(B)$, $P_{\max}(B)$, $P_{\min}(B; b)$, $P_{\max}(B; b)$; certain quadratic differerentials will play here a key role. It turns out that the extremal functions in the problems $P_{\min}$ are always unique while the extremal functions of the problems $P_{\max}$ usually form a one-parameter family. Since the problems $P_{\max}$ are more complicated they will be treated below in much more detail.

We shall use the notations common in the theory of extremal length:

$$L(\gamma, \varrho) = \int_\gamma \varrho(z)\,|dz|$$

for the length of the curve γ in the metric ϱ which is a non-negative Borel measurable function $\varrho(z)$ of two real variables;

$$L(\Gamma, \varrho) = \inf_{\gamma \in \Gamma} L(\gamma, \varrho)$$

for a family Γ of curves γ;

$$A(G, \varrho) = \iint_G \varrho^2\,dx\,dy$$

for the ϱ-area of a domain G;

$$\lambda(\Gamma, \varrho) = \frac{L^2(\Gamma, \varrho)}{A(G, \varrho)}$$

for the ϱ-length of the family Γ of curves γ lying in G provided the metric ϱ is *admissible* for Γ, i.e. the above quotient is defined;

$$\lambda(\Gamma) = \sup_{\varrho} \lambda(\Gamma, \varrho)$$

(the supremum being taken over all admissible metrics ϱ) for the extremal length of Γ; the metric ϱ^* is extremal if $\lambda(\Gamma) = \lambda(\Gamma, \varrho^*)$. If $\lambda(\Gamma) \neq 0, \infty$ and if there is an extremal metric ϱ^* then any other extremal metric is almost everywhere a constant multiple of ϱ^*.

2. For the determination of the quantities $b_r'(B), b_r''(B)$ in (1.1) we use (here and also later) the following notation.

For $0 < b < 1$, E_b' is the unit disk slit along the two segments $(-1, 0]$, $[b, 1)$. For $-1 < a < 1$, E_a is the unit disk in which the closed segment S_a between a and 0 is deleted, with degenerate case $E_0 = E - \{0\}$. $a(r) > 0$ is such that R_r and $E_{a(r)}$ are conformally equivalent. Φ_r is the function mapping R_r conformally onto $E_{-a(r)}$ with

$\Phi_r(1)=1$; Ψ_r given by $\Psi_r(z)=-\Phi_r(-z)$ for $z\in R_r$ maps then R_r conformally onto $E_{a(r)}$; both Φ_r and Ψ_r leave the points ± 1 fixed.

THEOREM 1. *Let $B>0$ be a point of R_r. In the notation* (1.1) *we have then*

(i) $b'_r(B)=\Phi_r(B)$, $b''_r(B)=\Psi_r(B)$;

(ii) *if $f\in\mathfrak{F}_r$ and $f(B)>0$, $f(B)=b'_r(B)$ implies $f=\Phi_r$ and $f(B)=b''_r(B)$ implies $f=\Psi_r$.*

To prove the theorem we will use several lemmata.

We consider for $0<b<1$ the family Γ'_b of curves in E'_b joining the upper half $C^+=\{z; z\in C, \Im z>0\}$ and the lower half $C^-=\{z; z\in C, \Im z<0\}$ of C. To determine the extremal length $\lambda(\Gamma'_b)$ we introduce the quadratic differential

$$\sigma_b=[z(z-b)(z-b^{-1})]^{-1}\,dz^2. \tag{2.1}$$

Since the function q_b in E'_b given by

$$q_b(z)=\text{ branch of }\int_0^z \sigma_b^{\frac{1}{2}}, \tag{2.2}$$

integration within E'_b, maps E'_b conformally onto a rectangle with horizontal sides $q_b(C^+)$, $q_b(C^-)$, the metric

$$\varrho_b(z)=\left|\frac{\sigma_b}{dz^2}\right|^{\frac{1}{2}} \tag{2.3}$$

is extremal for Γ'_b, and a curve $\gamma\in\Gamma'_b$ is of shortest ϱ_b-length if and only if γ satisfies the differential equation $\sigma_b<0$. Denoting this shortest ϱ_b-length within Γ'_b (which equals twice the ϱ_b-length of either of the segments $[-1,0]$, $[b,1]$ after having extended ϱ_b continuously to the full unit disk) by L'_b and the ϱ_b-length of the segment $[0,b]$ by H_b we obtain

$$\lambda(\Gamma'_b)=\frac{L'^2_b}{A(E'_b,\varrho_b)}=\frac{L'_b}{H_b}. \tag{2.4}$$

Similarly, the extremal length $\lambda(\Gamma_b)$ of the family Γ_b of curves joining the two segments $[-1,0]$ and $[b,1]$ in E'_b is given by

$$\lambda(\Gamma_b)=\frac{H_b}{L'_b}=\lambda^{-1}(\Gamma'_b). \tag{2.5}$$

LEMMA 1. *For $0<b<1$, $\lambda(\Gamma'_b)$ decreases strictly and $\lambda(\Gamma_b)$ increases strictly in b.*

Proof. For $0<b_1<b<1$, $\Gamma'_{b_1}\subset\Gamma'_b$ and thus

$$\lambda(\Gamma'_b)=\lambda(\Gamma'_b,\varrho_b)=\lambda(\Gamma'_{b_1},\varrho_b)\leqslant\lambda(\Gamma'_{b_1}). \tag{2.6}$$

The last inequality in (2.6) is strict unless ϱ_b is extremal for Γ'_{b_1}. ϱ_b, however, is not almost everywhere a constant multiple of ϱ_{b_1} which is extremal for Γ'_{b_1} since ϱ_b is bounded near $z=b_1$ while $\varrho_{b_1}\to\infty$ as $z\to b_1$ in E'_{b_1}, by (2.1) and (2.3). Hence the last inequality in (2.6) is indeed strict. The statement about $\lambda(\Gamma_b)$ then follows from (2.5).

For $-1<a\leqslant 0$ we denote by Γ'_{ab} the family of curves which have both endpoints on C which lie up to the endpoints in $E_a-\{b\}$, and which separate the boundary segment S_a of E_a from the point $b\in(0,1)$. For $0\leqslant a\leqslant b\in(0,1)$ we denote by Γ_{ab} the family of closed curves in E_a which do not pass through b and which separate $S_a\cup\{b\}$ from C.

LEMMA 2. *For $a\in(-1,0]$, $\lambda(\Gamma'_{ab})=\lambda(\Gamma'_b)$ and ϱ_b is extremal for Γ'_{ab}; for $a\in[0,b]$, $\lambda(\Gamma_{ab})=4\lambda(\Gamma_b)$ and ϱ_b is extremal for Γ_{ab}.*

To prove the first statement we observe that $a\in(-1,0]$ implies $\Gamma'_b\subset\Gamma'_{ab}\subset\Gamma'_{0b}$ whence

$$\lambda(\Gamma'_{0b})\leqslant\lambda(\Gamma'_{ab})\leqslant\lambda(\Gamma'_b). \tag{2.7}$$

On the other hand, the ϱ_b-length of $\gamma\in\Gamma'_{0b}$ is at least twice the ϱ_b-distance $\frac{1}{2}L'_b$ between C and the segment $(0,b)$ of the real axis since each γ meets that segment.

Hence
$$\lambda(\Gamma'_{0b})\geqslant\lambda(\Gamma'_{0b},\varrho_b)=\frac{L'^2_b}{A(E_a,\varrho_b)}=\frac{L'^2_b}{A(E'_b,\varrho_b)}=\lambda(\Gamma'_b); \tag{2.8}$$

(2.7) and (2.8) together give the first equality of Lemma 2 and the extremality of ϱ_b for Γ'_{ab}. To obtain the second statement of Lemma 2 we observe first that in E_b

$$q_b(z)=\int_0^z(\text{branch of } \sigma_b^{\frac{1}{2}}),\ \text{integration within } E_b,$$

has for periods only integral multiples of $2H_b$ whence $\exp[i\pi H_b^{-1}q_b(z)]$ is a function in E_b which, moreover, maps E_b conformally onto an annulus. Thus $\pi H_b^{-1}\varrho_b$ is extremal metric for Γ_{bb}, the family of curves separating the two boundary components of E_b, and

$$\lambda(\Gamma_{bb})=\frac{(2\pi)^2}{2\pi\frac{1}{2}\pi L'_bH_b^{-1}}=\frac{4H_b}{L'_b}=4\lambda(\Gamma_b). \tag{2.9}$$

With $\pi H_b^{-1}\varrho_b$ also ϱ_b is extremal for Γ_{bb}, and a similar reasoning as above in proving the first part of Lemma 2 yields: with $a\in[0,b]$, ϱ_b is extremal metric for any Γ_{ab} and $\lambda(\Gamma_{ab})=\lambda(\Gamma_{bb})$, so with (2.9) also the second assertion of Lemma 2 follows.

3. LEMMA 3. *Let $\mathfrak{G} \subset E_0$ be a doubly connected domain containing the point $b > 0$ and having C as one boundary component; let $\mathbf{\Gamma}'_b$ be the family of curves which have both endpoints on C, which lie up to the endpoints in $\mathfrak{G}$ and which separate the complementary component $\mathbf{K} \subset E$ of $\mathfrak{G}$ from b; let $\mathbf{\Gamma}_b$ be the family of closed curves in $\mathfrak{G}$ which separate C and $\mathbf{K} \cup \{b\}$. Then*

(i) $\lambda(\mathbf{\Gamma}'_b) \geqslant \lambda(\Gamma'_{0b})$ *with equality if and only if $\mathfrak{G}$ is an E_a for some $a \in (-1, 0)$;*

(ii) $\lambda(\mathbf{\Gamma}_b) \geqslant \lambda(\Gamma_{0b})$ *with equality if and only if $\mathfrak{G}$ is an E_a for some $a \in [0, b)$.*

Proof. I. From $\mathbf{\Gamma}'_b \subset \Gamma'_{0b}$ and $\mathbf{\Gamma}_b \subset \Gamma_{0b}$ we obtain at once the two inequalities stated in Lemma 3.

II. Assume now

$$\lambda(\mathbf{\Gamma}'_b) = \lambda(\Gamma'_{0b}). \tag{3.1}$$

Since $\lambda(\mathbf{\Gamma}'_b) \geqslant \lambda(\mathbf{\Gamma}'_b, \varrho_b) \geqslant \lambda(\Gamma'_{0b}, \varrho_b) = \lambda(\Gamma'_{0b})$, (3.1) implies first of all that ϱ_b is extremal for $\mathbf{\Gamma}'_b$. $\mathfrak{G}$ is conformally equivalent to some E_a, $-1 < a \leqslant 0$; let the function g with $g(b) > 0$ and $g(C) = C$ map $\mathfrak{G}$ conformally onto that E_a. The conformal invariance of extremal length gives with Lemma 2 the relation

$$\lambda(\Gamma'_{0b}) = \lambda(\mathbf{\Gamma}'_b) = \lambda(\Gamma'_{a,g(b)}) = \lambda(\Gamma'_{0,g(b)}),$$

and from Lemma 1 we obtain

$$g(b) = b. \tag{3.2}$$

Putting $h(z) = [z(z-b)(z-b^{-1})]^{-1}$ and applying the transformation $z = g(w)$, the quadratic differential σ_b in E_a has in $\mathfrak{G}$ the expression

$$h[g(w)]\, g'^2(w)\, dw^2, \quad w \in \mathfrak{G};$$

the metric $\boldsymbol{\rho}(w) = |h[g(w)]|^{\frac{1}{2}} \cdot g'(w)|$ is thus also extremal for $\mathbf{\Gamma}'_b$, and from the already observed extremality of $\varrho_b(w) = |h(w)|^{\frac{1}{2}}$ together with the continuity of both $\boldsymbol{\rho}$ and ϱ_b in $\mathfrak{G} - \{b\}$ we obtain with some positive constant $\mathfrak{c}$ the relation

$$|h[g(w)]|^{\frac{1}{2}} \cdot |g'(w)| = \mathfrak{c} \cdot |h(w)|^{\frac{1}{2}}, \quad w \in \mathfrak{G} \cup C. \tag{3.3}$$

Using the fact that $h[g(w)]\, g'^2(w)$ as well as $h(w)$ is regular analytic in $[\mathfrak{G} - \{b\}] \cup C$ we conclude from (3.3) for some complex number $c \neq 0$ the relation

$$h[g(w)]\, g'^2(w) = c\, h(w), \quad w \in \mathfrak{G} \cup C. \tag{3.4}$$

From $h[g(w)]\,g'^2(w)\,dw^2>0$ and $h(w)\,dw^2>0$ on each line element of C, from the invariance of C under g and from (3.4) we conclude $c>0$, and from

$$\int_C |h(w)|^{\frac{1}{2}}\,|dw| = \int_C |\sigma^{\frac{1}{2}}| = \int_C |h[g(w)]|^{\frac{1}{2}}\,|g'(w)|\,|dw| = |c|\cdot\int_C |h(w)|^{\frac{1}{2}}|dw|$$

we conclude $|c|=1$, thus $c=1$, and the conformal map g satisfies in $\mathfrak{G}-\{b\}$ the differential equation

$$g'^2(w)=\frac{g(w)[g(w)-b]\,[b^{-1}-g(w)]}{w(w-b)\,(b^{-1}-w)}. \tag{3.5}$$

Letting $w\to b$ in (3.5) and observing (3.2) we obtain $g'^2(b)=g'(b)$ which leads to

$$g'(b)=1 \tag{3.6}$$

since $g'(w)\neq 0$ in $\mathfrak{G}$.

The most elementary way to determine g from (3.5) near b under the conditions (3.2), (3.6) is to represent $g(w)-b$ by the Taylor series $w-b+\sum_{n=2}^{\infty} c_n(w-b)^n$ and to compare coefficients. A simple computation gives $c_n=0$ for $n\geqslant 2$. Hence $g(w)=w$ near b, and g is the identity in $\mathfrak{G}$.

This proves: if $\lambda(\boldsymbol{\Gamma}'_b)=\lambda(\Gamma'_{0b})$ then $\mathfrak{G}=E_a$ for some $a\in(-1,0]$.

III. Reasoning similarly as above in II. one obtains the condition for equality in Lemma 3 (ii) as stated.

4. *Proof of Theorem 1.* Let $B>0$ be a point of R_r and let $f\in\mathfrak{F}_r$ be such that $f(B)>0$. We denote by $\Gamma'_{R_r;B}$ the family of curves having both endpoints on C, lying up to the endpoints in R_r, and separating B from $C_r=\{z;|z|=r\}$; we denote by $\Gamma_{R_r;B}$ the family of closed curves in R_r separating C and $C_r\cup\{B\}$. Putting $f(R_r)=\mathfrak{G}$ and using the notation of Lemma 3 we have by conformal invariance of extremal length and by the lemmata 2 and 3

$$\lambda(\Gamma'_{\Phi_r(B)})=\lambda(\Gamma'_{-a(r),\,\Phi_r(B)})=\lambda(\Gamma'_{R_r;B})=\lambda(\boldsymbol{\Gamma}'_{f(B)})\geqslant\lambda(\Gamma'_{0,f(B)})=\lambda(\Gamma'_{f(B)}), \tag{4.1}$$

and similarly

$$4\,\lambda(\Gamma_{\Psi_r(B)})=\lambda(\Gamma_{a(r),\,\Psi_r(B)})=\lambda(\Gamma_{R_r;B})=\lambda(\boldsymbol{\Gamma}_{f(B)})\geqslant\lambda(\Gamma_{0,f(B)})=4\,\lambda(\Gamma_{f(B)}). \tag{4.2}$$

From (4.1) and (4.2), the relation

$$\Phi_r(B)\leqslant f(B)\leqslant\Psi_r(B)$$

follows (Lemma 1) which with (1.1) is the first statement of Theorem 1. The equality $f(B)=\Phi_r(B)$ gives the equality $\lambda(\Gamma'_{f(B)})=\lambda(\Gamma'_{0,f(B)})$ in (4.1) which, again by Lemma 3, implies that $g=\Phi_r\circ f^{-1}$ is the identity or that $f=\Phi_r$; similarly the equality $f(B)=\Psi_r(B)$ implies $f=\Psi_r$. This proves Theorem 1.

Remark. If $b'_r(B)<b<b''_r(B)$, there are obviously functions $f\in\mathfrak{F}_r$ with $f(B)=b$; one such function is obtained applying to Φ_r the linear transformation which leaves C and the real axis invariant and carries $b'_r(B)$ to b; that function, as we shall see later (Theorem 9), is actually the unique solution of the problem $P_{\min}(B;b)$.

As a trivial consequence of Theorem 1 we note the

COROLLARY. *The problems $P_{\min}(B;b'_r(B))$ and $P_{\max}(B;b'_r(B))$ have both the unique solution Φ_r; the problems $P_{\min}(B;b''_r(B))$ and $P_{\max}(B;b''_r(B))$ have both the unique solution Ψ_r.*

5. We turn now to the problem $P_{\max}(B;b)$ for $b'_r(B)<b<b''_r(B)$. We put $A=\{(a,b);-1<a<b,0<b<1\}$, and associate with each point $(a,b)\in A$ the quadratic differential

$$\sigma_{ab}=\frac{(b^{-1}-b)^2}{b^{-1}+b-a^{-1}-a}\,\frac{(z-a)(z-a^{-1})}{z(z-b)^2(z-b^{-1})^2}\,dz^2 \text{ for } a\neq 0,$$

$$=(b^{-1}-b)^2(z-b)^{-2}(z-b^{-1})^{-2}\,dz^2 \text{ for } a=0. \tag{5.1}$$

We call R_{ab} the rational function σ_{ab}/dz^2 and observe

$$R_{ab}(z)\,dz^2=R_{ab}(z^{-1})\,[d(z^{-1})]^2. \tag{5.2}$$

With

$$Z_{ab}=\frac{2}{b^{-1}-b}-\frac{1}{b}\,\frac{b^{-1}-b}{b^{-1}+b-a^{-1}-a} \text{ for } a\neq 0,$$

$$=\lim_{a\to 0} Z_{ab}=\frac{2}{b^{-1}-b} \text{ for } a=0, \tag{5.3}$$

R_{ab} has at b the principal part $(z-b)^{-2}+Z_{ab}(z-b)^{-1}$.

Any maximal curve (maximal as a point set) on which $\sigma_{ab}<0$ is called a *trajectory* of σ_{ab}. Because a and b are real and because of (5.2), all trajectories of σ_{ab} are symmetric to the real axis and to C; C itself is also a trajectory. For $a\neq 0$, the open segment $K^{(1)}_{ab}$ between a and 0 is a trajectory of σ_{ab}, there is also one other non-closed trajectory $K^{(2)}_{ab}$ of σ_{ab} in E which has the limit point a at both ends; all but the just mentioned trajectories $K^{(1)}_{ab}$, $K^{(2)}_{ab}$ in E are closed curves separating b and C. For $a=0$, the trajectories of σ_{0b} in E consist of the non-Euclidean circles about b. Each point of E, except the point b, and except the points a, 0 for $a\neq 0$, is contained in exactly one trajectory of σ_{ab}.

For $a \neq 0$, we but $K_{ab} = K_{ab}^{(1)} \cup K_{ab}^{(2)} \cup \{a\} \cup \{0\}$; K_{0b} denotes the non-Euclidean circle about b through 0; for convenience we put $K_{0b}^{(1)} = \emptyset$, $K_{0b}^{(2)} = K_{0b} - \{0\}$. In each case, K_{ab} is the closure of the union of the trajectories of σ_{ab} with limit point a.

$E - K_{ab}$ has two components, a simply connected domain $G'_{ab} \ni b$ and a doubly connected domain G''_{ab} having C on the boundary. The reduced modulus(1) of G'_{ab} at b is called M'_{ab}, the modulus(2) of G''_{ab} is called M''_{ab}.

We denote by $R_{ab}^{\frac{1}{2}}(z)$ the branch in the interior of $E - K_{ab}^{(1)}$ which is positive on the segment $(b, 1)$ of the real axis, and put correspondingly $\sigma_{ab}^{\frac{1}{2}} = R_{ab}^{\frac{1}{2}}(z)\,dz$. The periods of $\int \sigma_{ab}^{\frac{1}{2}}$ in G'_{ab} are integral multiples of $2\pi i$, the periods of $\int \sigma_{ab}^{\frac{1}{2}}$ in G''_{ab} are integral multiples of ih''_{ab} where h''_{ab} is the $|R_{ab}|^{\frac{1}{2}}$-length of C. This gives immediately an analytic expression for the function q^* mapping G'_{ab} conformally onto a disk about the origin and for the function q^{**} mapping G''_{ab} onto a concentric annulus about the origin. With $z_{ab} = K_{ab} \cap (b, 1)$ we have

$$q^*(z) = \exp \int_{z_{ab}}^{z} \sigma_{ab}^{\frac{1}{2}}, \text{ integration within } G'_{ab}, \tag{5.4}$$

and $q^*(b) = 0$ in particular, and

$$q^{**}(z) = \exp \left[\frac{2\pi}{h''_{ab}} \int_{z_{ab}}^{z} \sigma_{ab}^{\frac{1}{2}} \right], \text{ integration within } G''_{ab}. \tag{5.5}$$

Denoting further by k''_{ab} the $|R_{ab}|^{\frac{1}{2}}$-length of the segment of the negative real axis within G''_{ab}, and by k'_{ab} the reduced $|R_{ab}|^{\frac{1}{2}}$-length of the segment of the positive real axis on the left of b within G'_{ab}, i.e.

$$k'_{ab} = \lim_{\varepsilon \to 0+} \left\{ \int_{\max(a,0)}^{b-\varepsilon} |R_{ab}(x)|^{\frac{1}{2}}\,dx + \log \varepsilon \right\}, \tag{5.6}$$

$$k''_{ab} = \int_{-1}^{\min(a,0)} |R_{ab}(x)|^{\frac{1}{2}}\,dx, \tag{5.7}$$

we have
$$M'_{ab} = \frac{k'_{ab}}{2\pi}, \quad M''_{ab} = \frac{k''_{ab}}{h''_{ab}}. \tag{5.8}$$

6. We compare the decomposition of E by K_{ab} with the decomposition by other continua. $K_b \not\ni b$ will be a continuum in E which contains 0 and separates $b \in (0, 1)$ and

(1) The *reduced modulus* of a simply connected domain G' at $b \in G'$ is $(2\pi)^{-1} \log t$, when φ maps G' conformally onto a disk of radius t about the origin with $\varphi(b) = 0$, $|\varphi'(b)| = 1$.

(2) The *modulus* of a doubly connected domain G'' is $(2\pi)^{-1} \log t''/t'$, when φ maps G'' conformally onto an annulus $t' < |w| < t''$.

C. $E-K_b$ has several components; $G'(K_b)$ is the simply connected component containing b, $G''(K_b)$ is the doubly connected component having C on the boundary. $M'(K_b, b)$ is the reduced modulus of $G'(K_b)$ at b, $M''(K_b)$ is the modulus of $G''(K_b)$. We shall use(1) ([2], Theorem 1)

THEOREM 2. *Let $b \in (0,1)$ be fixed; let $0 \in K_b$ be a continuum in E separating b and C. Then for any $a \in (-1, b)$ we have*

$$(2\pi)^2 M'(K_b, b) + h''^2_{ab} M''(K_b) \leqslant (2\pi)^2 M'_{ab} + h''^2_{ab} M''_{ab}\,; \tag{6.1}$$

equality in (6.1) *is possible for at most one value $a_0 \in (-1, b)$, and equality for a_0 implies $K_b = K_{a_0, b}$.*

THEOREM 3. *Let a, b be fixed with $-1<a<b$, $0<b<1$; let $\mathbf{K}_{ab} \ni 0$ be a proper subcontinuum of K_{ab}; let g map $G_{ab}=E-\mathbf{K}_{ab}$ conformally into E_0 with $g(b)=b$ and $g(C)=C$. Then* (i) $|g'(b)| \leqslant 1$; (ii) $|g'(b)|=1$ *implies that both G'_{ab} and G''_{ab} are invariant under g.*

Proof. I. $K=E-g(G_{ab})$ is the bounded complementary component of $g(G_{ab})$ with respect to the complex plane, thus $0 \in K$ and K is compact and connected. Further, $g(K_{ab}-\mathbf{K}_{ab})$ is connected, and so is its closure which has at least one point in common with K. Thus $\mathbf{K}=K \cup [\text{closure of } g(K_{ab}-\mathbf{K}_{ab})]$ is conected, it is also compact being the union of two compact sets. $\mathbf{K}$ further separates $g(G'_{ab}) \ni b$ and $g(G''_{ab})$ whence $\mathbf{K}$ separates b and C, we may write thus K_b for $\mathbf{K}$, and we have $g(G'_{ab})=G'(K_b)$, $g(G''_{ab})=G''(K_b)$. The conformal invariance of the modulus M''_{ab} of G''_{ab} gives $M''_{ab}=M''(K_b)$, and from the statement (6.1) in Theorem 2 now follows the inequality

$$M'(K_b, b) - M'_{ab} \leqslant 0. \tag{6.2}$$

The left side of (6.2), however, equals $(2\pi)^{-1} \log |g'(b)|$, which proves part (i) of Theorem 3.

II. Suppose now $|g'(b)|=1$. In this case we have in (6.1) equality, and by the second statement of Theorem 2 thus necessarily $K_b=K_{ab}$. Thus $g(G'_{ab})$ is the component of $E-K_{ab}$ which contains b whence $g(G'_{ab})=G'_{ab}$, similarly $g(G''_{ab})=G''_{ab}$, proving Theorem 3 part (ii).

7. We call $\mathfrak{G}(\mathbf{K}_{ab})$ the family of functions which map $G_{ab}=E-\mathbf{K}_{ab}$ conformally into E_0 with $g(b)=b$, $g(C)=C$, $|g'(b)|=1$, and we want to determine the elements of

(1) Closely related results are given in Jenkins [3].

$\mathfrak{G}(\mathbf{K}_{ab})$. We first introduce general coordinates related to (5.4), (5.5) in $G'_{ab}-\{b\}$ and in G''_{ab} as follows.

$$\zeta^*(z)=\log q^*(z) \text{ modulo } 2\pi i, \quad z\in G'_{ab}-\{b\}, \tag{7.1}$$

$$\zeta^{**}(z)=\frac{h''_{ab}}{2\pi}\log q^{**}(z) \text{ modulo } ih''_{ab}, \quad z\in G''_{ab}. \tag{7.2}$$

The conformal self-maps of G'_{ab} with fixed point b are precisely the maps

$$\zeta^*\to\zeta^*+i\vartheta', \quad \vartheta' \text{ real}, \tag{7.3}$$

and the conformal self-maps of G''_{ab} with C fixed are precisely the maps

$$\zeta^{**}\to\zeta^{**}+i\vartheta'', \quad \vartheta'' \text{ real}. \tag{7.4}$$

The map (7.3) is called *ϑ'-shift of G'_{ab}* (along the trajectories of σ_{ab}), the map (7.4) is called *ϑ''-shift of G''_{ab}* (along the trajectories of σ_{ab}).

We observe further that $\zeta^*(z)$ and $\zeta^{**}(z)$ may be defined continuously also on the boundary of G'_{ab} and G''_{ab} respectively provided that the two shores of the horizontal boundary segment S_a (of G'_{ab} for $a<0$, G''_{ab} for $a>0$) are distinguished. The shift maps (7.3), (7.4) are then continuous also on the boundary. If near $K^{(2)}_{ab}$ we restrict $\Im\zeta^*(z)$ within $(-\pi, \pi)$ and $\Im\zeta^{**}(z)$ within $(-\frac{1}{2}h''_{ab}, \frac{1}{2}h''_{ab})$ then $\zeta^*(z)=\zeta^{**}(z)$ on $K^{(2)}_{ab}$, and that common value is a conformal map of $K^{(2)}_{ab}$. This is immediate from the fact that in the unit disk slit along the segment $(-1, b]$

$$\zeta(z)=\int_{z_{ab}}^{z}\sigma_{ab}^{\frac{1}{2}}, \tag{7.5}$$

integration within the slit disk, is a (single-valued) function.

With a given $\mathbf{K}_{ab}$ we distinguish two cases.

Case I. $\mathbf{K}_{ab}$ is a subset of the real axis. We will then use for $\mathbf{K}_{ab}$ also the notation $\mathbf{K}_{ab}(0,0)$.

Case II. $\mathbf{K}_{ab}$ is not a subset of the real axis. Then the curved part $\mathbf{K}^{(2)}_{ab}=\mathbf{K}_{ab}\cap K^{(2)}_{ab}$ has positive $|R_{ab}|^{\frac{1}{2}}$-length less than $\min(2\pi, h''_{ab})$. $\mathbf{K}^{(2)}_{ab}$ may or may not be connected (i.e. may or may not stretch out from a in both directions along part of $K^{(2)}_{ab}$); we call ϑ_+ or ϑ_- respectively the $|R_{ab}|^{\frac{1}{2}}$-length of the component of $\mathbf{K}^{(2)}_{ab}$ starting out from a into the upper or lower half plane. Either of ϑ_+, ϑ_- may vanish, but

$$0<\vartheta_++\vartheta_-=\int_{\mathbf{K}^{(2)}_{ab}}|\sigma_{ab}|^{\frac{1}{2}}<\min(2\pi, h''_{ab}). \tag{7.6}$$

In this case we use for $\mathbf{K}_{ab}$ also the notation $\mathbf{K}_{ab}(\vartheta_+, \vartheta_-)$.

8. Lemma 4. *Let* $\mathbf{K}_{ab} = \mathbf{K}_{ab}(\vartheta_+, \vartheta_-)$ *be given; let* g_ϑ, $-\vartheta_- \leqslant \vartheta \leqslant \vartheta_+$, *be the function in* $G_{ab} = E - \mathbf{K}_{ab}$ *with the following properties:*

(i) $g(b) = b$, (ii) *the restriction of* g_ϑ *to both* G'_{ab} *and* G''_{ab} *is a* ϑ*-shift along the trajectories of* σ_{ab}, (iii) *with* ζ *of* (7.5),

$$\zeta[g_\vartheta(z)] = \zeta(z) + i\vartheta \quad \text{for} \quad z \in K^{(2)}_{ab} - \mathbf{K}^{(2)}_{ab}. \tag{8.1}$$

Then g_ϑ *is a conformal map of* G_{ab} *into* E_0 *leaving* C *invariant with* $g'_\vartheta(b) = e^{i\vartheta}$.

Proof. g_ϑ is apparently conformal in both G'_{ab} and G''_{ab} leaving these two domains invariant and satisfying $g'(b) = e^{i\vartheta}$. We conclude from (8.1) and (7.6) the relation $g_\vartheta(K^{(2)}_{ab} - \mathbf{K}^{(2)}_{ab}) \subset K^{(2)}_{ab}$: the shift is too short to carry a point of $K^{(2)}_{ab} - \mathbf{K}^{(2)}_{ab}$ away from $K^{(2)}_{ab}$ to the real axis. Further the equality (8.1) is valid also near $K^{(2)}_{ab} - \mathbf{K}^{(2)}_{ab}$ choosing near this set in G'_{ab} for ζ^* and in G''_{ab} for ζ^{**} the principal value ζ of (7.5), thus g_ϑ is also conformal on $K^{(2)}_{ab} - \mathbf{K}^{(2)}_{ab}$ mapping this set onto a piece of $K^{(2)}_{ab}$. This proves the lemma.

We will call the conformal map g_ϑ the *ϑ-shift* of G_{ab}. g_ϑ shifts also the continuum $\mathbf{K}_{ab}$ in an obvious sense by ϑ.

9. To complement Theorem 3 we prove

Theorem 4. *Let* $\mathbf{K}_{ab} = \mathbf{K}_{ab}(\vartheta_+, \vartheta_-)$ *be given; let* $\mathfrak{G}(\mathbf{K}_{ab})$ *be the family of conformal maps* g *of* $G_{ab} = E - \mathbf{K}_{ab}$ *into* E_0 *with* $g(b) = b$, $g(C) = C$, $|g'(b)| = 1$. *Then* $\mathfrak{G}(\mathbf{K}_{ab})$ *consists of the* ϑ*-shifts of* G_{ab} *with* $-\vartheta_- \leqslant \vartheta \leqslant \vartheta_+$*; if, in particular,* $\vartheta_- = 0 = \vartheta_+$, *then* $\mathfrak{G}(\mathbf{K}_{ab})$ *consists just of the identity.*

Remark. Theorem 4 is of Schwarz' Lemma type; if $\mathbf{K}_{ab}$ is short $[\mathbf{K}_{ab} = \mathbf{K}_{ab}(0,0)]$, the fact that a function has derivative of modulus 1 at a particular point implies that the function has its derivative even identically equal to 1. We refrain, however, here from pursuing this aspect of Theorem 4.

Proof of Theorem 4. The inclusion $\mathfrak{G}(\mathbf{K}_{ab}) \supset \{g_\vartheta;\ \vartheta_- \leqslant \vartheta \leqslant \vartheta_+,\ g_\vartheta \text{ is } \vartheta\text{-shift of } G_{ab}\}$ follows from Lemma 4. Let now $g \in \mathfrak{G}(\mathbf{K}_{ab})$. Theorem 3 (ii) gives $g(G'_{ab}) = G'_{ab}$, $g(G''_{ab}) = G''_{ab}$ which implies: the restriction of g to G'_{ab} is a ϑ'-shift and the restriction of g to G''_{ab} is a ϑ''-shift, ϑ' and ϑ'' both real. Let us consider the case $-1 < a < 0$: $[a, 0]$ is contained in the boundary of G'_{ab}, $h''_{ab} < 2\pi$. We may assume

$$\vartheta'' = \vartheta' \in [-\vartheta_-,\ -\vartheta_- + 2\pi).$$

If the open interval $(-\tfrac{1}{2}h''_{ab} + \vartheta_-, \tfrac{1}{2}h''_{ab} - \vartheta_+)$ is shifted by $\vartheta' \in (\vartheta_+, 2\pi - \vartheta_-)$ it will overlap with the open interval $(\tfrac{1}{2}h''_{ab}, 2\pi - \tfrac{1}{2}h''_{ab})$, thus such a ϑ'-shift will move at least one point z' of $(K_{ab} - \mathbf{K}_{ab}) \cap K^{(2)}_{ab}$ [on which set $\Im\zeta^*(z) \in (-\tfrac{1}{2}h''_{ab} + \vartheta_-, \tfrac{1}{2}h''_{ab} - \vartheta_+)$

mod 2π] to a point of the half open segment $(a, 0]$ on the real axis [on the two shores of which segment $\mathfrak{J}\zeta^*(z) \in (\frac{1}{2}h''_{ab}, 2\pi - \frac{1}{2}h''_{ab}) \bmod 2\pi$]. However, z' is a point of G_{ab} which is a common boundary point of G'_{ab} and G''_{ab} while for $\vartheta' \in (\vartheta_+, 2\pi - \vartheta_-)$ the ϑ'-shift moves z' to a point which is not a boundary point of G''_{ab}. Therefore the restriction of $g \in \mathfrak{G}(\mathbf{K}_{ab})$ to G'_{ab} cannot be a ϑ'-shift for $\vartheta' \in (\vartheta_+, 2\pi - \vartheta_-)$ since $g(G''_{ab}) = G''_{ab}$. Hence $\mathfrak{G}(\mathbf{K}_{ab}) \subset \{g_\vartheta;\ -\vartheta_- \leqslant \vartheta \leqslant \vartheta_+\}$, and for $\vartheta_- = 0 = \vartheta_+$, $\mathfrak{G}(\mathbf{K}_{ab})$ consists only of the identity g_0. This proves Theorem 4 in the case $-1 < a < 0$. A similar reasoning applies in the case $0 < a < b$, while for $a = 0$ the only functions leaving G'_{0b} and b or G''_{0b} and C invariant are the non-Euclidean rotations about b, in which case Theorem 4 is elementary.

Theorem 4 has the

COROLLARY. *Let $-1 < a_1 < a_2 < b$, $0 < b < 1$; let $\mathbf{K}_{a_1,b}$ and $\mathbf{K}_{a_2,b}$ be proper subcontinua of $K_{a_1,b}$ and $K_{a_2,b}$ respectively such that these two subcontinua both contain the point 0. Then there is no conformal map of $G_{a_1,b} = E - \mathbf{K}_{a_1,b}$ onto $G_{a_2,b} = E - \mathbf{K}_{a_2,b}$ which leaves b and C fixed, unless $G_{a_1,b} = G_{a_2,b}$.*

Proof. Assuming, on the contrary, for $G_{a_1,b} \neq G_{a_2,b}$ the existence of a conformal map g_{12} of $G_{a_1,b}$ onto $G_{a_2,b}$ with $g_{12}(b) = b$, $g_{12}(C) = C$, and putting $g_{12}^{-1} = g_{21}$, we obtain from Theorem 4 the contradictory statements $|g'_{12}(b)| < 1$ and $|g'_{21}(b)|^{-1} = |g'_{21}(b)| < 1$ since clearly $g_{12} \notin \mathfrak{G}(\mathbf{K}_{a_1,b})$ and $g_{21} \notin \mathfrak{G}(\mathbf{K}_{a_2,b})$.

The problem $P_{\max}(B; b)$ was solved in the corollary of Theorem 1 for $b = b'_r(B)$ and $b = b''_r(B)$. In the remaining cases, $P_{\max}(B; b)$ is solved by

THEOREM 5. *Let $B > 0$ be a point of $R_r = \{z; 0 < r < |z| < 1\}$ and let $b'_r(B) < b < b''_r(B)$. Then*

(i) *there is a unique $a_b \in (-1, b)$ such that $R_r - \{B\}$ can be mapped conformally onto some $E - (\mathbf{K}_{a_b,b} \cup \{b\})$ by a function of $\mathfrak{F}_r$, moreover, $|a_b| < a(r)$;*

(ii) *the $|R_{a_b,b}|^{\frac{1}{2}}$-length $2\vartheta_{B;b}$ of the curved part $\mathbf{K}^{(2)}_{a_b,b}$ of $\mathbf{K}_{a_b,b}$ above is positive;*

(iii) *if $f_b \in \mathfrak{F}_r$ maps $R_r - \{B\}$ conformally onto $G_b - \{b\}$ where $G_b = E - \mathbf{K}_b$ and $\mathbf{K}_b = \mathbf{K}_{a_b,b}(\vartheta_{B;b}, \vartheta_{B;b})$ are symmetric to the real axis, $f \in \mathfrak{F}_r(B; b)$ implies $|f'(B)| \leqslant f'_b(B)$;*

(iv) *if $f \in \mathfrak{F}_r(B; b)$ and $f(B) = b > 0$, $|f'(B)| = 1$ if and only if $f = g_\vartheta \circ f_b$ with $|\vartheta| \leqslant \vartheta_{B;b}$ where g_ϑ is the ϑ-shift of G_b.*

Proof. If a_b is such that $R_r - \{B\}$ can be mapped conformally onto $E - (\mathbf{K}_{a_b,a} \cup \{b\})$ by a function $f \in \mathfrak{F}_r$ then from the strict inequality $b'_r(B) < b < b''_r(B)$ together with $f(C) = C$ we infer $f(R_r) \neq \Phi_r(R_r)$ as well as $f(R_r) \neq \Psi_r(R_r)$. $\mathbf{K}_{a_b,a}$ then neither contains the segment

$[-a(r), 0]$ nor the segment $[0, a(r)]$ which gives $|a_b| < a(r)$ as well as the positivity of the $|R_{a_b,b}|^{\frac{1}{2}}$-length of $\mathbf{K}^{(2)}_{a_b,b}$; $a_b < b$ since $b \in E - \mathbf{K}_{a_b,b}$. There is at most one such a_b, by the corollary of Theorem 4; that there is actually one such a_b will be shown later (Theorem 6) by a direct construction [which also will give f_b of (iii) above]. This proves (i) and (ii). (iii) is immediate from Theorem 3 (i), observing that the function f_b (which is symmetric to the real axis) has positive derivative at B. (iv) is immediate from Theorem 4.

10. We turn now to the construction of the function f_b in Theorem 5 (iii).

With two positive parameters s, h we consider the pentagon P containing the half-strip $\{z; \Re z < 0, 0 < \Im z < \pi\}$ the sides of which are the following segments parallel to the axes:

$$\{z; -\infty < \Re z \leqslant 0, \Im z = \pi\}, \quad \{z; -\infty < \Re z \leqslant h, \Im z = 0\},$$
$$\{z; \Re z = h, 0 \leqslant \Im z \leqslant s\}, \qquad \{z; 0 \leqslant \Re z \leqslant h, \Im z = s\},$$

and the closed segment between is, $i\pi$ (degenerate to the point $i\pi$ for $s = \pi$ when P is degenerate to a half strip); the sides of P form the angle 0 at ∞, the angles $\pi/2$ at h and $h + is$; for $s < \pi$ the sides form the angles $3\pi/2$ at is and $\pi/2$ at $i\pi$, for $s > \pi$ the sides form the angles $\pi/2$ at is and $3\pi/2$ at $i\pi$, for $s = \pi$ at $is = i\pi$ the angle π is formed.

The function mapping the upper half plane conformally onto P with $\infty, -1, +1$ corresponding in this order to the boundary points $\infty, h, h + is$ of P is

$$p(w) = i\pi + \int_{\max(x,y)}^{w} \sqrt{\frac{t-x}{t-y}} \frac{dt}{\sqrt{t^2-1}}, \quad \text{integrand} > 0 \text{ for } t < -1, \tag{10.1}$$

where $x > 1$, $y > 1$ correspond to the remaining vertices of P with angles $3\pi/2, \pi/2$ in that order $(s \neq \pi)$, and $x = y > 1$ corresponds to the vertex $i\pi$ of P $(s = \pi)$. P is determined by the pair (s, h), $0 < s, h < \infty$, as well as by the pair (x, y), $1 < x, y < \infty$. The correspondence between (x, y) and (s, h) is one-one, and we have the expressions

$$s = \int_{-1}^{1} \sqrt{\frac{x-t}{y-t}} \frac{dt}{\sqrt{1-t^2}} \tag{10.2}$$

and

$$h = \int_{1}^{\min(x,y)} \sqrt{\frac{x-t}{y-t}} \frac{dt}{\sqrt{t^2-1}} \tag{10.3}$$

or alternatively

$$h = \int_{\max(x,y)}^{\infty} \left\{\sqrt{\frac{x+t}{y+t}} - \sqrt{\frac{t-x}{t-y}}\right\} \frac{dt}{\sqrt{t^2-1}} + \int_{1}^{\max(x,y)} \sqrt{\frac{x+t}{y+t}} \frac{dt}{\sqrt{t^2-1}}. \tag{10.4}$$

In calculating partial derivatives of h, (10.3) is more convenient if $x<y$ and (10.4) is more convenient if $x>y$.

11. LEMMA 5. *The mapping* $\varphi:(x,y)\to(s,h)$ *is a sense preserving homeomorphism of* $(1,\infty)\times(1,\infty)$ *onto* $(0,\infty)\times(0,\infty)$.

Proof. We saw already that φ maps $(1,\infty)\times(1,\infty)$ onto $(0,\infty)\times(0,\infty)$ in a one-one fashion. From (10.2), (10.3) we infer the continuity of φ. By the Jordan curve theorem it then follows that any neighbourhood of a point (x_0, y_0) is mapped by φ onto a neighbourhood of the image point (s_0, h_0), thus the inverse φ^{-1} is also continuous. φ is sense preserving since it is sense preserving on the line $x=y$: that line is mapped onto the line $s=\pi$ while $x<y$ corresponds to $s<\pi$ and $x>y$ corresponds to $s>\pi$.

We will see, however, in addition that for $x\neq y$ the partial derivatives of s,h with respect to x,y are continuous with positive Jacobian $J(x,y)=s_x h_y - s_y h_x$; further applying the map $(s,h)\to(s-\pi)(1-\log|s-\pi|), h)=(s',h')$ near $s=\pi$, and applying the maps $(x,y)\to(x+y, y-x)=(\xi,\eta)$ and $(\xi,\eta)\to(\xi,\eta(1-\log|\eta|))=(x',y')$ near the line $x=y$, a direct calculation (which we omit) shows that the partial derivatives of (s',h') with respect to (x',y') are continuous near $x'>2$, $y'=0$ corresponding to $x=y>1$, and that the Jacobian of $(x',y')\to(s',h')$ is positive there, which proves the lemma again.

We put

$$A(t;x,y)=[(x-t)(y-t)(1-t^2)]^{-\frac{1}{2}} \text{ for } 0<t^2<1, \tag{11.1}$$

$$B(t;x,y)=[(x-t)(y-t)(t^2-1)]^{-\frac{1}{2}} \text{ for } t^2>1 \text{ with } t\notin[\min(x,y), \max(x,y)], \tag{11.2}$$

$$A(x,y)=\int_{-1}^{1} A(t;x,y)\,dt, \quad A^*(x,y)=\int_{-1}^{1}\frac{x-t}{y-t}A(t;x,y)\,dt. \tag{11.3}$$

Abbreviating
$$\int_{\max(x,y)}^{\infty}+\int_{-\infty}^{-1} \text{ by } \oint_{\max(x,y)}^{-1}$$
we put further for $x\neq y$

$$B(x,y)=\int_{1}^{\min(x,y)} B(t;x,y)\,dt=\oint_{\max(x,y)}^{-1} B(t;x,y)\,dt \tag{11.4}$$

and
$$B^*(x,y)=\int_{1}^{\min(x,y)}\frac{x-t}{y-t}B(t;x,y)\,dt \quad \text{for } x<y,$$

$$=\oint_{\max(x,y)}^{-1}\frac{x-t}{y-t}B(t{:}\ x,y)\,dt \quad \text{for } x>y. \tag{11.5}$$

Denoting partial derivatives by subscripts we obtain from (10.2)

$$s_x = \tfrac{1}{2} A(x, y), \quad s_y = -\tfrac{1}{2} A^*(x, y), \tag{11.6}$$

and from (10.3), (10.4) for $x \neq y$

$$h_x = \tfrac{1}{2} B(x, y), \quad h_y = -\tfrac{1}{2} B^*(x, y). \tag{11.7}$$

The Jacobian $J(x, y) = s_x h_y - s_y h_x$ of the map $(x, y) \to (s, h)$ for $x \neq y$ is given by

$$4\, J(x, y) = -A(x, y)\, B^*(x, y) + A^*(x, y)\, B(x, y). \tag{11.8}$$

Observing that for some $t_A \in (-1, 1)$ and for some t_B, where $t_B \in (1, x)$ for $x < y$ and $t_B \in (x, +\infty] \cup [-\infty, -1)$ for $x > y$, we have

$$A^*(x, y) = \frac{x - t_A}{y - t_A} A(x, y), \quad B^*(x, y) = \frac{x - t_B}{y - t_B} B(x, y),$$

and we obtain from (11.8)

$$4\, J(x, y) = A(x, y)\, B(x, y) \left(-\frac{x - t_B}{y - t_B} + \frac{x - t_A}{y - t_A} \right) > 0 \quad \text{for} \quad x \neq y. \tag{11.9}$$

12. Before constructing the function f_b of Theorem 5 (iii) we will consider still further auxiliary maps. We start from the unit disk slit along the segment $[-a(r), 0]$ of the real axis with the point $b_r'(B)$ removed, which is $\Phi_r(R_r - \{B\}), B > 0$. We map the upper half of the unit disk conformally onto the upper half plane so that the points $+1$, -1, $b_r'(B)$ go into the points $-1, +1, \infty$ in that order. Let α and β, $1 < \alpha < \beta$, be the points corresponding to $-a(r)$ and 0 respectively. Choosing the parameter $\delta \in [\alpha, \beta]$ and mapping the upper half plane by

$$\zeta(w) = i\pi + \int_\beta^w \frac{t - \delta}{\sqrt{(t-\alpha)\,(t-\beta)}} \frac{dt}{\sqrt{t^2 - 1}}, \quad \text{integrand } > 0 \text{ for } t < -1,$$

we obtain a hexagon H_δ which is degenerate to a pentagon as considered in section 10 for $\delta = \alpha$ (with $x = \alpha, y = \beta$) and for $\delta = \beta$ (with $x = \beta$, $y = \alpha$), and which is otherwise such a pentagon slit along the segment $[\zeta(\delta), i\pi]$ of the imaginary axis. We denote the parameters x, y of this pentagon thus corresponding to H_δ by $x(\delta), y(\delta)$, and that pentagon itself by $P = P_{xy}$. Writing $s(P)$, $h(P)$ for the other pair of parameters s, h determining P and putting

$$s_H(H_\delta) = |\zeta(1) - \zeta(-1)| = \int_{-1}^{1} \frac{\delta - t}{\sqrt{(\alpha - t)\,(\beta - t)}} \frac{dt}{\sqrt{1 - t^2}}, \tag{12.1}$$

$$h_H(H_\delta) = |\zeta(\alpha) - \zeta(1)| = \int_1^\alpha \frac{\delta - t}{\sqrt{(\alpha-t)(\beta-t)}} \frac{dt}{\sqrt{t^2-1}}, \tag{12.2}$$

for the corresponding sides of H_δ, we see that $x(\delta), y(\delta)$ satisfy the equations

$$s_H(H_\delta) = s(P_{xy}), \quad h_H(H_\delta) = h(P_{xy}). \tag{12.3}$$

By (12.1), $s_H(H_\delta)$ is strictly increasing in $\delta \in [\alpha, \beta]$ from $s_H(H_\alpha) < \pi$ to $s_H(H_\beta) > \pi$, so the equation $s_H(H_\delta) = \pi$ determines a unique number $\delta_0 \in (\alpha, \beta)$, and $s_H(H_\delta) < \pi$ for $\delta \in [\alpha, \delta_0)$, $s_H(H_\delta) > \pi$ for $\delta \in (\delta_0, \beta]$. Since the partial derivatives (11.6), (11.7) of $s(P_{xy})$, $h(P_{xy})$ with respect to x, y are continuous for $x \neq y$ we obtain differentiating (12.3) with respect to δ

$$\begin{aligned} s'_H(H_\delta) &= s_x(P_{xy})\, x'(\delta) + s_y(P_{xy})\, y'(\delta) \\ h'_H(H_\delta) &= h_x(P_{xy})\, x'(\delta) + h_y(P_{xy})\, y'(\delta), \quad \delta \neq \delta_0, \end{aligned} \tag{12.4}$$

the dash denoting differentiation with respect to δ, right-sided at $\delta = \alpha$, left-sided at $\delta = \beta$.

Solving (12.4) for x', y' and observing $s'_H(H_\delta) \equiv A(\alpha, \beta), h'_H(H_\delta) \equiv B(\alpha, \beta)$ we obtain from (11.6), (11.7) and observing (11.8), (11.9) for $x(\delta)$, $y(\delta)$ with $\delta \in [\alpha, \beta] - \{\delta_0\}$ the system of differential equations

$$\begin{aligned} x' &= 2\frac{A^*(x,y)\, B(\alpha,\beta) - A(\alpha,\beta)\, B^*(x,y)}{A^*(x,y)\, B(x,y) - A(x,y)\, B^*(x,y)}, \\ y' &= 2\frac{A(x,y)\, B(\alpha,\beta) - A(\alpha,\beta)\, B(x,y)}{A^*(x,y)\, B(x,y) - A(x,y)\, B^*(x,y)} \end{aligned} \tag{12.5}$$

with the boundary conditions

$$x(\alpha) = \alpha, \quad y(\alpha) = \beta; \quad x(\beta) = \beta, \quad y(\beta) = \alpha. \tag{12.6}$$

The denominator in (12.5), equal to $4\,J(x,y)$, is positive.

13. We put $Q = \{(x, y);\ x > 1,\ y > 1\}$, $Q_0 = \{(x, y);\ \alpha \leqslant x \leqslant \beta,\ \alpha \leqslant y \leqslant \beta\}$, $l = \{(x, y);\ x = y > 1\}$, $l_0 = l \cap Q_0$, and we shall determine some properties of the functions $x(\delta)$, $y(\delta)$ considering the differential equation (12.5) in $Q - l$.

With the notation

$$M(x,y) = \frac{B(x,y)}{A(x,y)}, \quad M^*(x,y) = \frac{B^*(x,y)}{A^*(x,y)} \tag{13.1}$$

and the abbreviations $A(x,y) = A$, $A^*(x,y) = A^*$, $A(\alpha,\beta) = a$, $B(x,y) = B$, $B^*(x,y) = B^*$, $B(\alpha,\beta) = b$, $M(x,y) = M$, $M^*(x,y) = M^*$, $M(\alpha,\beta) = m$ [the notation $a = A(\alpha,\beta)$, $b = B(\alpha,\beta)$

will be used only in this section where a confusion with the different previous meaning of the symbols a and b as in σ_{ab} is not to be feared], the differential equation (12.5) gets the form

$$x' = 2\,\frac{a}{A}\,\frac{m-M^*}{M-M^*}, \quad y' = 2\,\frac{a}{A^*}\,\frac{m-M}{M-M^*}; \quad (x,y)\in Q-l. \tag{13.2}$$

The expressions A, B, M are symmetric in x and y; $M(x,y)$ is the extremal distance between the segments $[-1,1]$, $[\min(x,y), \max(x,y)]$ in the upper half plane, thus $M(x,y)\geqslant m$ for $(x,y)\in Q_0-l_0$ with equality only at (α,β) and (β,α). We have $M-M^* = A^{-1}A^{*-1}4J>0$ for $(x,y)\in Q-l$, and we notice $M\to\infty$ and $M^*\to\infty$ as (x,y) tends to a point of l. While (13.2) is not defined on l, the direction dy/dx prescribed by (13.2) in $Q-l$ has limiting value $+1$ on l as is seen upon a little consideration.

Lemma 6. *For $\delta\in[\alpha,\beta]$, the function $y(\delta)$ is strictly decreasing from β to α with $y'(\delta)<0$ in $(\alpha,\delta_0)\cup(\delta_0,\beta)$.*

Proof. (12.6) gives already $y(\alpha)=\beta$, $y(\beta)=\alpha$, so we have merely to show that $y(\delta)$ is strictly decreasing with negative derivative in $(\alpha,\delta_0)\cup(\delta_0,\beta)$.

From (12.5) or (13.2) we have immediately

$$x'(\alpha)=2=x'(\beta), \quad y'(\alpha)=0=y'(\beta). \tag{13.3}$$

Since M, M^* have on $Q-l$ continuous first order partial derivatives with respect to x and y we obtain from (12.5) using (13.3)

$$y''(\alpha)=\tfrac{1}{2}\,\frac{1}{J(\alpha,\beta)}\,\frac{\partial}{\partial x}\,[A(x,y)\cdot b\,-\,a\cdot B(x,y)]_{\substack{x=\alpha\\ y=\beta}}\cdot 2. \tag{13.4}$$

At $(x,y)=(\alpha,\beta)$ we have from (11.1) through (11.4)

$$\frac{\partial}{\partial x}\,A(x,y) = -\tfrac{1}{2}\int_{-1}^{1}\frac{1}{\alpha-t}\,A(t;\alpha,\beta)\,dt = -\tfrac{1}{2}\,\frac{1}{\alpha-t_0}\cdot a$$

and

$$\frac{\partial}{\partial x}\,B(x,y) = -\tfrac{1}{2}\oint_{\beta}^{-1}\frac{1}{\alpha-t}\,B(t;\alpha,\beta)\,dt = -\tfrac{1}{2}\,\frac{1}{\alpha-t_1}\cdot b$$

with suitable $t_0\in(-1,1)$ and suitable $t_1\in(\beta,\infty]\cup[-\infty,-1)$. Therefore

$$\frac{\partial}{\partial x}\,[A(x,y)\cdot b\,-\,a\cdot B(x,y)]_{\substack{x=\alpha\\ y=\beta}} = -\tfrac{1}{2}\left(\frac{1}{\alpha-t_0}-\frac{1}{\alpha-t_1}\right)ab<0$$

whence from (13.4)

$$y''(\alpha) < 0. \tag{13.5}$$

In a similar way we obtain

$$y''(\beta) > 0. \tag{13.6}$$

(13.3), (13.5), (13.6) show that $y'(\delta) < 0$ on the rigth of α near α and on the left of β near β, and Lemma 6 is proved once it is shown that $y'(\delta)$ never vanishes in $(\alpha, \delta_0) \cup (\delta_0, \beta)$ since $(x(\delta), y(\delta)) \in l$ only for $\delta = \delta_0$. Assuming now that $y'(\delta)$ vanishes somewhere in (α, δ_0) we denote by δ_1 the smallest zero of y' in (α, δ_0) so that $y'(\delta) < 0$ for $\delta \in (\alpha, \delta_1)$. Since $M^* < M$, (13.2) gives $x'(\delta_1) > 0$; thus on the left of δ_1 and near δ_1, $x(\delta)$ is increasing and $y(\delta)$ is decreasing whence the extremal distance $M[x(\delta), y(\delta)]$ is increasing at δ_1, and from (13.2) the contradiction $y'(\delta) > 0$ near δ_1 on the left of δ_1 would follow. Therefore indeed $y'(\delta) < 0$ for $\delta \in (\alpha, \delta_0)$, and similarly $y'(\delta) < 0$ for $\delta \in (\delta_0, \beta)$ is obtained. This proves the lemma.

LEMMA 7. *There is a (unique) number $\delta_{1x} \in (\alpha, \delta_0)$ such that sign $x'(\delta) =$ sign $(\delta_{1x} - \delta)$ for $\delta \in [\alpha, \delta_0)$; there is a (unique) number $\delta_{2x} \in (\delta_0, \beta)$ such that sign $x'(\delta) =$ sign $(\delta - \delta_{2x})$ for $\delta \in (\delta_0, \beta]$.*

Proof. We saw already $M_\delta^* = M^*[x(\delta), y(\delta)] \to \infty$ for $\delta \to \delta_0$. From (13.2), (13.3) we conclude that $x'(\delta)$ vanishes at least once in (α, δ_0) and once in (δ_0, β). On the other hand, a simple calculation gives

$$\operatorname{sign} \frac{\partial M^*}{\partial y} = \operatorname{sign}(x - y), \quad (x, y) \in Q - l \tag{13.7}$$

whence, by (13.2), $x'(\delta_x) = 0$ for $\delta_x \in (\alpha, \delta_0) \cup (\delta_0, \beta)$ implies that x' is strictly decreasing or strictly increasing at δ_x as $y(\delta_x) - x(\delta_x)$ is positive or negative, i.e. as $\delta_x \in (\alpha, \delta_0)$ or $\delta_x \in (\delta_0, \beta)$. The lemma follows.

LEMMA 8. *The curve $\gamma_\delta = \{(x, y); x = x(\delta), y = y(\delta), \alpha \leqslant \delta \leqslant \beta\}$ lies up to the initial point $\gamma_\alpha = (\alpha, \beta)$ and the terminal point $\gamma_\beta = (\beta, \alpha)$ in the interior of Q_0; its slope dy/dx is a continuous function of δ [assuming the value $\pm\infty$ exactly twice continuously at $\delta = \delta_{1x}$ and $\delta = \delta_{2x}$]; γ_δ passes at δ_0 through the line l with slope $+1$.*

Intuitively, Lemma 8 says that the curve γ_δ has the shape of a question-mark intersecting its tangent l at γ_{δ_0} from left and above to right and below. It might seem surprising that $x(\delta)$ is not monotonic in $[\alpha, \beta]$, and is at δ_0 decreasing just as fast as $y(\delta)$.

Proof. The piece $\{(x, y); x = x(\delta), y = y(\delta), \alpha \leqslant \delta \leqslant \delta_{1x}\}$ of γ_δ on which $x(\delta)$ is strictly increasing and $y(\delta)$ is strictly decreasing [Lemmata 6 and 7] lies up to the initial point (α, β) in the open triangle Δ_1 with vertices (α, β), (α, α), (β, β). Since γ_{δ_0} is (Lemma 6) a point of the open segment between (α, α), (β, β), the curve γ_δ cannot leave Δ_1 for $\delta_{1x} \leqslant \delta < \delta_0$ (Lemma 7). Similarly it is seen that the piece $\{(x, y); x = x(\delta), y = y(\delta), \delta_0 < \delta < \beta\}$ lies within the open triangle Δ_2 with vertices (β, β), (α, α), (β, α), thus the first statement of Lemma 8 holds. The continuity of the slope dy/dx is obvious for $\delta \neq \delta_0$, and for $\delta = \delta_0$ it follows from $\lim_{\delta \to \delta_0}(dy/dx) = 1$ which fact was remarked before Lemma 6. The third statement of Lemma 8 is now obvious.

14. In order to construct the function f_b of Theorem 5 we use the following notation. $\Theta_\varkappa$, $\varkappa > 1$, is the function mapping the upper half of the unit disk conformally onto the upper half plane with the boundary correspondence $\Theta_\varkappa(+1) = -1$, $\Theta_\varkappa(-1) = +1$, $\Theta_\varkappa(0) = \varkappa$. With $1 < \varkappa_1 < \varkappa_2$ and $\varkappa_1 \leqslant \delta \leqslant \varkappa_2$, $\zeta_{\varkappa_1, \varkappa_2; \delta} = \zeta_{\varkappa_2, \varkappa_1; \delta}$ is the function defined in the upper half plane by

$$\zeta_{\varkappa_1, \varkappa_2; \delta}(z) = i\pi + \int_{\varkappa_2}^{z} \frac{t - \delta}{\sqrt{(t-\varkappa_1)(t-\varkappa_2)}} \frac{dt}{\sqrt{t^2 - 1}}, \quad \text{integrand} > 0 \text{ for } t < -1, \tag{14.1}$$

mapping the upper half plane onto a hexagon which is degenerate to a pentagon for $\delta = \varkappa_1$ and $\delta = \varkappa_2$. [Cf. section 12.]

Consider the annulus R_r with the point $B > 0$ of R_r distinguished. With $\Phi_r(R_r) = E_{-a(r)}$, $\Phi_r(B) = b_r'(B)$, the number $\beta = \beta(r, B) > 1$ is uniquely determined by the requirement $\Theta_\beta[b_r'(B)] = \infty$; let $\Theta_\beta[-a(r)] = \alpha = \alpha(r, B) > 1$. Let $\zeta_{\alpha, \beta; \delta}(1) = h + is$ where $h = h(r, B; \delta) > 0$, $s = s(r, B; \delta) > 0$. There is a unique pair (x, y) with $x = x(r, B; \delta) \in [\alpha, \beta]$ and $y = y(r, B; \delta) \in [\alpha, \beta]$ such that $\zeta_{x, y; x}(1) = h + is$ (Lemmata 5, 8), and by Lemma 6 the correspondence between $\delta \in [\alpha, \beta]$ and $y \in [\alpha, \beta]$ is homeomorphic for fixed r, B. We put further $\Theta_y^{-1}(\infty) = b$ with $b = b(r, B; \delta)$ and $\Theta_y^{-1}(x) = a$ with $a = a(r, B; \delta)$, and we shall see (Lemma 9) that a, b depend likewise homeomorphically upon δ for fixed r, B.

THEOREM 6. *Let there be given the annulus R_r and a point $B > 0$ of R_r. Let α and β, both depending on r and B, be as above, and let $b^* \in [b_r'(B), b_r''(B)]$ be given. Then there is a unique $\delta^* \in [\alpha, \beta]$ such that $b^* = b(r, B; \delta^*)$ with the above function $b(r, B; \delta)$; further with $x^* = x(r, B; \delta^*)$, $y^* = y(r, B; \delta^*)$, $a^* = a(r, B; \delta^*)$, depending each on r, B, δ as above, the function f_{b^*} mapping R_r conformally onto the domain $E - \mathbf{K}_{a^* b^*}$ symmetric to the real axis and maximizing $|f'(B)|$ within the class $\mathfrak{F}_r(B, b^*)$ is given in the upper half of R_r through*

$$f_{b^*} = \Theta_{y^*}^{-1} \circ \zeta_{x^*, y^*; x^*}^{-1} \circ \zeta_{\alpha, \beta; \delta^*} \circ \Theta_\beta \circ \Phi_r.$$

Proof. We prove first that for any $\delta \in [\alpha, \beta]$ with corresponding x, y, b, a the analytic function f_b given in the upper half of R_r by

$$f_b = \Theta_y^{-1} \circ \zeta_{x,y;x}^{-1} \circ \zeta_{\alpha,\beta;\delta} \circ \Theta_\beta \circ \Phi_r \tag{14.2}$$

maps R_r conformally onto $G = E - \mathbf{K}_{ab}$ where $\mathbf{K}_{ab} \ni 0$ is a continuum symmetric to the real axis on K_{ab} [cf. section 5 for the definition of K_{ab}]. The statement is obvious for $\delta = \alpha$ since then, by (12.6), $f_b = \Phi_r$ in (14.2), it is also obvious for $\delta = \beta$ since then the right side of (14.2) equals $\Theta_\alpha^{-1} \circ \zeta_{\beta,\alpha;\beta}^{-1} \circ \zeta_{\alpha,\beta;\beta} \circ \Theta_\beta \circ \Phi_r$ which by $\zeta_{\alpha,\beta;\beta} = \zeta_{\beta,\alpha;\beta}$ equals $\Theta_\alpha^{-1} \circ \Theta_\beta \circ \Phi_r = \Psi_r$ because $\Theta_\alpha^{-1} \circ \Theta_\beta$ is the map $z \to (z + a(r))/(1 + a(r) z)$. So let now $\delta \in (\alpha, \beta)$. Since f_b of (14.2) is real on the intersection of R_r with the real axis it is sufficient to show that the set $\mathbf{K}$ omitted in the upper half unit disk by $f_b(R_r^+)$, where R_r^+ is the upper half of R_r, lies on a trajectory of $\sigma_{ab} < 0$ and has a as a limit point.

$\mathbf{K}$ is the image by $\Theta_y^{-1} \circ \zeta_{x,y;x}$ of the segment $[\zeta_{\alpha,\beta;\delta}(\delta), i\pi) = \mathbf{k}_\delta$ by which the corresponding hexagon $H_\delta = \zeta_{\alpha,\beta;\delta} \circ \Theta_\beta \circ \Phi_r(R_r^+)$ and pentagon $\zeta_{x,y;x} \circ \Theta_\beta \circ \Phi_r(R_r^+)$ differ. $\mathbf{k}_\delta$ lies on a trajectory $\sigma < 0$ of the quadratic differential $\sigma = 1 \cdot d\zeta^2$, and the transformation $\Theta = \zeta_{x,y;x}^{-1}(\zeta)$ gives σ the expression

$$\sigma = \frac{\Theta - x}{\Theta - y} \frac{1}{\Theta^2 - 1} d\Theta^2. \tag{14.3}$$

With

$$y = \tfrac{1}{2}(b^{-1} + b) = y(b), \quad 0 < b < 1, \tag{14.4}$$

we have

$$\Theta_y(z) = \tfrac{1}{2}\left[\frac{b - z}{1 - bz} + \frac{1 - bz}{b - z}\right], \tag{14.5}$$

and the application of the transformation $z = \Theta_y^{-1}(\Theta)$ to (14.3) gives σ after a simple calculation the form σ_{ab} of (5.1) where $a = \Theta_y^{-1}(x)$, therefore the set $\mathbf{K}$ omitted in the upper half unit disk by $f_b(R_r^+)$ is indeed the piece of the trajectory $\sigma_{ab} < 0$ between a and $\Theta_y^{-1} \circ \zeta_{x,y;x}^{-1} \circ \zeta_{\alpha,\beta;\delta}(\delta)$. By the Theorems 3 and 4, the conformal map f_b given in R_r^+ by (14.2) is the unique function positive at $B > 0$ and symmetric to the real axis which maximizes $|f_b'(B)|$ within $\mathfrak{F}_r(B; b)$. To complete the proof of Theorem 6 it remains to show that for any $b^* \in [b_r'(B), b_r''(B)]$ there is a unique $\delta^* \in [\alpha, \beta]$ such that $b(r, B; \delta^*) = b^*$.

Let there be given $b^* \in [b_r'(B), b_r''(B)]$. (14.4) determines uniquely $y^* = y(b^*)$, and $y^* \in [\alpha, \beta]$ since $y[b_r'(B)] = \beta$ and $y[b_r''(B)] = \alpha$; $y^* \in [\alpha, \beta]$ determines (Lemma 6) uniquely $\delta^* \in [\alpha, \beta]$, and thus for fixed r and B the equation $b^* = b(r, B; \delta)$ is satisfied for exactly one $\delta^* \in [\alpha, \beta]$. This concludes the proof.

We remark that $a^* = a(r, B; \delta^*)$, which is uniquely determined also by b^*, is the unique a_{b^*} whose existence was asserted in Theorem 5 but not proved there. Denoting for fixed r, B, the functions $a(r, B; \delta)$ and $b(r, B; \delta)$ simply by $a(\delta)$ and $b(\delta)$ we prove

LEMMA 9. *The derivatives $a'(\delta)$, $b'(\delta)$ exist and are continuous for $\delta \in [\alpha, \beta] - \{\delta_0\}$, with $a'(\delta) > 0$ in $[\alpha, \delta_0) \cup (\delta_0, \beta]$ and $b'(\delta) > 0$ in $(\alpha, \delta_0) \cup (\delta_0, \beta)$.*

Proof. I. Since $y'(\delta)$ is continuous in $[\alpha, \delta_0) \cup (\delta_0, \beta]$ the assertion for $b(\delta)$ is immediate from Lemma 6 and (14.4).

II. The continuity of $a'(\delta)$ in $[\alpha, \delta_0) \cup (\delta_0, \beta]$ is again obvious. For $x \neq y$, the relation $a = \Theta_y^{-1}(x)$ has, by (14.5), the form

$$2x = \frac{b-a}{1-ab} + \frac{1-ab}{b-a}, \tag{14.6}$$

and using (14.4) we obtain from (14.6) the relation $a^2(x-y) - 2a(xy-1) + (x-y) = 0$; since $|a| < 1$ and since from $x(\delta_0) = y(\delta_0)$ follows $a(\delta_0) = 0$ we have explicitly

$$a = \frac{1}{x-y}\left(xy - 1 - \sqrt{(x^2-1)(y^2-1)}\right), \quad \delta \in [\alpha, \delta_0) \cup (\delta_0, \beta], \quad a(\delta_0) = 0. \tag{14.7}$$

We consider first the case $\alpha \leqslant \delta < \delta_0$. Since then $x < y$ we have from (14.7)

$$-a = \frac{xy-1}{y-x} - \sqrt{\left(\frac{xy-1}{y-x}\right)^2 - 1} \tag{14.8}$$

whence the statements

$$a'(\delta) > 0 \quad \text{and} \quad \frac{d}{d\delta}\frac{xy-1}{y-x} > 0$$

are equivalent for $\alpha \leqslant \delta < \delta_0$. So it is sufficient to prove

$$(y-x)\frac{d}{d\delta}(xy-1) - (xy-1)\frac{d}{d\delta}(y-x) = x'(y^2-1) - y'(x^2-1) > 0$$

or equivalently, using (12.5) and observing the positivity of the denominator there, the relation

$$A(\alpha, \beta) \cdot B^{**}(x, y) - B(\alpha, \beta)\, A^{**}(x, y) > 0 \tag{14.9}$$

where
$$B^{**}(x, y) = (x^2-1)\,[B(x, y) - B^*(x, y)] - (y^2 - x^2)\, B^*(x, y),$$
$$A^{**}(x, y) = (x^2-1)\,[A(x, y) - A^*(x, y)] - (y^2 - x^2)\, A^*(x, y).$$

From (11.3), (11.4), (11.5) we obtain

$$B(x, y) - B^*(x, y) = (y-x)\int_1^x \frac{B(t; x, y)}{y-t}\, dt,$$

$$A(x,y)-A^*(x,y)=(y-x)\int_{-1}^{1}\frac{A(t;x,y)}{y-t}\,dt$$

whence with $x^2-1-(y+x)(x-t)=t(y+x)-xy-1$

$$B^{**}(x,y)=(y-x)\int_{1}^{x}\frac{t(y+x)-xy-1}{y-t}\,B(t;x,y)\,dt,$$

$$A^{**}(x,y)=(y-x)\int_{-1}^{1}\frac{t(y+x)-xy-1}{y-t}\,A(t;x,y)\,dt.$$

Using again (11.3), (11.4) we obtain with a suitable value $t_0\in(-1,1)$ and a suitable value $t_1\in(1,x)$

$$A(\alpha,\beta)\,B^{**}(x,y)-B(\alpha,\beta)\,A^{**}(x,y)$$
$$=(y-x)\left[\frac{t_1(y+x)-xy-1}{y-t_1}\,A(\alpha,\beta)\,B(x,y)-\frac{t_0(y+x)-xy-1}{y-t_0}\,B(\alpha,\beta)\,A(x,y)\right], \tag{14.10}$$

and both factors of the second term in (14.10) are positive. So (14.9) is indeed correct whence $a'(\delta)>0$ for $\alpha\leqslant\delta<\delta_0$.

III. Reasoning similarly as in II. above one concludes also $a'(\delta)>0$ for $\delta_0<\delta\leqslant\beta$. This proves the lemma.

15. We investigate now which $f_b'(B)$ for $b\in[b_r'(B),\,b_r''(B)]$ is maximal. We place R_r into the w-plane; after having Θ_β continued analytically into the full slit disk $E_{-a(r)}$ we place $\Theta_\beta\circ\Phi_r(R_r)$, which is the extended plane slit along the segments $[-1,1]$, $[\alpha,\beta]$ of the real axis, into the Θ-plane, and $f_b(R_r)$ into the z-plane. In $f_b(R_r)$ we take the quadratic differential σ_{ab} determined in the last section; since a and b depend both on $\delta\in[\alpha,\beta]$ and since $f_b(R_r)$ lies in the z-plane we write now $\sigma_{\delta;z}$ for σ_{ab} in $f_b(R_r)$. For the analytic expression obtained from $\sigma_{\delta;z}$ by application of f_b^{-1} we write $\sigma_{\delta;w}$, and for the analytic expression obtained from $\sigma_{\delta;w}$ by application of $\Theta_\beta\circ\Phi_r$ (or from $\sigma_{\delta;z}$ by application of $\zeta^{-1}_{\alpha,\beta;\delta}\circ\zeta_{x,y;x}\circ\Theta_y$) we write $\sigma_{\delta;\Theta}$. $\sigma_{\delta;z}, \sigma_{\delta;\Theta}, \sigma_{\delta;w}$ are different expressions of the same quadratic differential.

We choose $\varrho_\Theta>0$ large and denote by $F_{\delta;\Theta}(\varrho_\Theta)$ the area of the disk $|\Theta|<\varrho_\Theta$ in the metric $|\sigma_{\delta;\Theta}/d\Theta^2|^{\frac12}$; similarly we choose $\varrho_z>0$, $\varrho_w>0$ both small and denote by $F_{\delta;z}(\varrho_z)$ and $F_{\delta;w}(\varrho_w)$ respectively the area of $E-\{z;\,|z-b|\leqslant\varrho_z\}$ in the metric $|\sigma_{\delta;z}/dz^2|^{\frac12}$ and of $R_r-\{w;\,|w-b_r'(B)|\leqslant\varrho_w\}$ in the metric $|\sigma_{\delta;w}/dw^2|^{\frac12}$.

The equality $$F_{\delta;z}(\varrho_z) = F_{\delta;\Theta}(\varrho_\Theta) = F_{\delta;w}(\varrho_w) \tag{15.1}$$

relates the three numbers $\varrho_z, \varrho_\Theta, \varrho_w$ in such a way that one of them determines the two others and that the conditions $\varrho_z \to 0$, $\varrho_\Theta \to \infty$, $\varrho_w \to 0$ are equivalent. From now on we will have these three numbers always related by (15.1). The relation

$$f_b'(B) = \lim_{\varrho_w \to 0} \frac{\varrho_z}{\varrho_w} \tag{15.2}$$

as well as the relations

$$\frac{d}{dw}\left[\frac{1}{\Theta_\beta \circ \Phi_r(w)}\right]_{w=B} = \lim_{\varrho_w \to 0} \frac{1}{\varrho_\Theta \cdot \varrho_w} = D_1, \tag{15.3}$$

$$\frac{d}{dz}\left[\frac{1}{\zeta_{\alpha,\beta;\delta}^{-1} \circ \zeta_{x,y;x} \circ \Theta_y(z)}\right]_{z=b} = \lim_{\varrho_z \to 0} \frac{1}{\varrho_\Theta \cdot \varrho_z} = D_2 \tag{15.4}$$

follow. Observing that the right side of (6.1) is the at b reduced area of the unit disk in the metric $|\varrho_{\delta;z}/dz^2|^{\frac{1}{2}}$ we have

$$\lim_{\varrho_z \to 0} [F_{\delta;z}(\varrho_z) + 2\pi \log \varrho_z] = (2\pi)^2 M'_{ab} + h''^2_{ab} M''_{ab}, \tag{15.5}$$

which gives with (15.1) the relation

$$F_{\delta;\Theta}(\varrho_\Theta) = (2\pi)^2 M'_{ab} + h''^2_{ab} M''_{ab} - 2\pi \log \varrho_z + o(1) \quad \text{for} \quad \varrho_\Theta \to \infty. \tag{15.6}$$

Subtracting $2\pi \log \varrho_\Theta$ in (15.6) and taking the limit we obtain with (15.4)

$$-2\pi \log D_2 = (2\pi)^2 M'_{ab} + h''^2_{ab} M''_{ab} - \lim_{\varrho_\Theta \to \infty} [F_{\delta;\Theta}(\varrho_\Theta) - 2\pi \log \varrho_\Theta] \tag{15.7}$$

which becomes with (15.2), (15.3) after addition of $2\pi \log D_1$

$$2\pi \log f_b'(B) = (2\pi)^2 M'_{ab} + h''_{ab} M''_{ab} + 2\pi \log D_1 - \lim_{\varrho_\Theta \to \infty} [F_{\delta;\Theta}(\varrho_\Theta) - 2\pi \log \varrho_\Theta]. \tag{15.8}$$

The function $$\zeta_{\alpha,\beta;\delta}(\Theta) = i\pi + \int_\beta^\Theta \frac{t-\delta}{\sqrt{(t-\alpha)(t-\beta)}} \frac{dt}{\sqrt{t^2-1}},$$

integrand >0 for $t < -1$, maps the upper half plane onto the hexagon H_δ; the integrand being a branch of $\sigma^{\frac{1}{2}}_{\delta;\Theta}$; we get thus in the notation of section 12

$$\lim_{\varrho_\Theta \to \infty} [F_{\delta;\Theta}(\varrho_\Theta) - 2\pi \log \varrho_\Theta]$$
$$= 2 s_H(H_\delta) \cdot h_H(H_\delta) + 2\pi \int_\beta^\infty \left\{\frac{t-\delta}{\sqrt{(t-\alpha)(t-\beta)}} \frac{1}{\sqrt{t^2-1}} - \frac{1}{t}\right\} dt - 2\pi \log \beta, \tag{15.9}$$

and we have finally from (15.8), (15.9)

$$2\pi \log f_b'(B) = (2\pi)^2 M'_{ab} + h''^2_{ab} M''_{ab} - 2 s_H(H_\delta) \cdot h_H(H_\delta)$$
$$- 2\pi \int_\beta^\infty \left\{ \frac{t-\delta}{\sqrt{(t-\alpha)(t-\beta)}} \frac{1}{\sqrt{t^2-1}} - \frac{1}{t} \right\} dt + 2\pi \log (D_1 \cdot \beta). \quad (15.10)$$

16. In order to find the maximal value of (15.10) in $b \in [b_r'(B), b_r''(B)]$ we will use several properties of the functions M'_{ab}, M''_{ab}.

LEMMA 10. *The map* $(a, b) \to (M'_{ab}, M''_{ab})$ *is continuous on*

$$A = \{(a, b);\ -1 < a < b, 0 < b < 1\}.$$

We omit the easy proof. (The lemma is amply contained in Theorem 8 of [2].)

LEMMA 11. *The partial derivatives* $(\partial/\partial a) M'_{ab}$, $(\partial/\partial a) M''_{ab}$ *are continuous on* A *except on the line* $a = 0$; *the partial derivatives* $(\partial/\partial b) M'_{ab}$, $(\partial/\partial b) M''_{ab}$ *are continuous on* A.

This is part of Lemma 7 of [2].

LEMMA 12. $(2\pi)^2\ (\partial/\partial a) M'_{ab} + h''^2_{ab} (\partial/\partial a) M''_{ab} = 0$ *on* A *except on the line* $a = 0$.

With the continuity of $(\partial/\partial a) M'_{ab}$, $(\partial/\partial a) M''_{ab}$ for $a \neq 0$ the lemma is immediate from Theorem 2 part (i) of [2].

LEMMA 13. $(2\pi)^2 (\partial/\partial b) M'_{ab} + h''^2_{ab} (\partial/\partial b) M''_{ab} = -2\pi Z_{ab}$ *on* A *where* Z_{ab} *is given in* (5.3).

This is Theorem 5 of [2].

LEMMA 14. *The equation* $Z_{ab} = 0$ *defines on* A *an implicit function* $a = j(b)$ *where* $0 < b < \sqrt{2} - 1$, *with* $\lim_{b \to 0} j(b) = 0$, $\lim_{b \to \sqrt{2}-1} j(b) = -1$; $j(b)$ *has a continuous always negative derivative on* $0 < b < \sqrt{2} - 1$.

This follows at once from Lemma 5 of [2]; it may be obtained also from (5.3). Denoting by $A^* = \{(a, b);\ (a, b) \in A, Z_{ab} = 0\}$ the graph of j we have

LEMMA 15. *On* $A - A^*$, Z_{ab} *has the sign of* $a - j(b)$ *if* $b \in (0, \sqrt{2} - 1)$, *and* $Z_{ab} > 0$ *if* $b \in [\sqrt{2} - 1, 1)$; *in particular,* $Z_{ab} > 0$ *if* $a \geqslant 0$ *on* A.

This follows from (5.3) using e.g. the explicit expression

$$j(b) = -(\sinh \beta - \sqrt{\sinh^2 \beta - 1})(\sinh^2 \beta - \sqrt{\sinh^4 \beta - 1})$$

where $\sinh \beta = \frac{1}{2}(b^{-1} - b)$ for $b \in (0, \sqrt{2} - 1)$, which expression is given in (8.1) of [2].

17. Let $\delta \in [\alpha, \beta] - \{\delta_0\}$. Since then $x \neq y$ and $a = \Theta_y^{-1}(x) \neq 0$ we obtain differentiating (15.10) with respect to δ (using Lemma 10 and the continuity of $a' = da/d\delta$, $b' = db/d\delta$) the relation

$$2\pi \frac{d}{d\delta} \log f_b'(B) = \left[(2\pi)^2 \frac{\partial}{\partial a} M_{ab}' + h_{ab}''^2 \frac{\partial}{\partial a} M_{ab}''\right] a'$$

$$+ \left[(2\pi)^2 \frac{\partial}{\partial b} M_{ab}' + h_{ab}''^2 \frac{\partial}{\partial b} M_{ab}''\right] b' + 2 h_{ab}'' M_{ab}'' \frac{d}{d\delta} h_{ab}''$$

$$- 2 \frac{d}{d\delta} [s_H(H_\delta) \cdot h_H(H_\delta)] + 2\pi \int_\beta^\infty \frac{1}{\sqrt{(t-\alpha)(t-\beta)}} \frac{dt}{\sqrt{t^2-1}}. \tag{17.1}$$

To simplify (17.1) we use the

Lemma 16. *For* $\delta \in [\alpha, \beta] - \{\delta_0\}$,

$$h_{ab}'' M_{ab}'' \frac{d}{d\delta} h_{ab}'' - \frac{d}{d\delta} [s_H(H_\delta) \cdot h_H(H_\delta)] + \pi \int_\beta^\infty \frac{1}{\sqrt{(t-\alpha)(t-\beta)}} \frac{dt}{\sqrt{t^2-1}} = 0. \tag{17.2}$$

Proof. We notice

$$h_{ab}'' = 2 s_H(H_\delta) \quad \text{and} \quad M_{ab}'' = \frac{h_H(H_\delta)}{2 s_H(H_\delta)}.$$

Thus the left side of (17.2) has also the form

$$h_H(H_\delta) \frac{d}{d\delta} s_H(H_\delta) - s_H(H_\delta) \cdot \frac{d}{d\delta} h_H(H_\delta) + \pi \int_\beta^\infty \frac{1}{\sqrt{(t-\alpha)(t-\beta)}} \frac{dt}{\sqrt{t^2-1}}; \tag{17.3}$$

from (12.1), (12.2), it is seen that (17.3) is actually independent of δ. Choosing in (17.3) for δ the value β we obtain therefore with (12.1) and (12.2) for the left side of (17.2) also the expression

$$\int_{-1}^1 \frac{1}{\sqrt{(t-\alpha)(t-\beta)}} \frac{dt}{\sqrt{1-t^2}} \cdot \int_1^\alpha (\beta - t) \frac{1}{\sqrt{(t-\alpha)(t-\beta)}} \frac{dt}{\sqrt{t^2-1}}$$

$$- \int_{-1}^1 (\beta - t) \frac{1}{\sqrt{(t-\alpha)(t-\beta)}} \frac{dt}{\sqrt{1-t^2}} \cdot \int_1^\alpha \frac{1}{\sqrt{(t-\alpha)(t-\beta)}} \frac{dt}{\sqrt{t^2-1}}$$

$$+ \pi \int_\beta^\infty \frac{1}{\sqrt{(t-\alpha)(t-\beta)}} \frac{dt}{\sqrt{t^2-1}}. \tag{17.4}$$

The transformation

$$u = \int_\beta^\Theta \frac{1}{\sqrt{(t-\alpha)(t-\beta)}} \frac{dt}{\sqrt{t^2-1}}, \quad \text{integrand} > 0 \text{ for } t > \beta, \tag{17.5}$$

maps the upper half Θ-plane onto the rectangle $0 < \Re u < \omega_1$, $0 < \Im u < \omega_2$ with

$$\omega_1 = \int_1^\alpha \frac{1}{\sqrt{(\alpha - t)(\beta - t)}} \frac{dt}{\sqrt{t^2 - 1}} > 0, \quad \omega_2 = \int_{-1}^1 \frac{1}{\sqrt{(\alpha - t)(\beta - t)}} \frac{dt}{\sqrt{1 - t^2}} > 0,$$

where $\{2\omega_1, 2i\omega_2\}$ is a set of fundamental periods of the elliptic function $\Theta(u)$ inverse to $u(\Theta)$ of (17.5). Putting

$$u_\infty = \int_\beta^\infty \frac{1}{\sqrt{(t-\alpha)(t-\beta)}} \frac{dt}{\sqrt{t^2 - 1}} \in (0, \omega_1), \tag{17.6}$$

$\Theta(u)$ has simple poles at $u = \pm u_\infty$ modulo periods, with residue -1 at $+u_\infty$ and residue $+1$ at $-u_\infty$, Applying the transformation (17.5) to (17.4) we obtain (17.4) in the form

$$\omega_2 \cdot \int_{i\omega_2}^{i\omega_2 + \omega_1} [\beta - \Theta(u)]\, du - \frac{1}{i}\, \omega_1 \cdot \int_{\omega_1}^{\omega_1 + i\omega_2} [\beta - \Theta(u)]\, du + \pi u_\infty, \tag{17.7}$$

the path of integration in (17.7) being always a straight line segment. The elliptic function $\beta - \Theta(u)$ has in terms of Weierstrass' function $\zeta(u) = u^{-1} + O(u^3)$ near $u = 0$, corresponding to the period parallelogram with vertices $\pm\omega_1 \pm i\omega_2$, the expression

$$\beta - \Theta(u) = \zeta(u - u_\infty) - \zeta(u + u_\infty) + 2\zeta(u_\infty) \tag{17.8}$$

since both sides of (17.8) vanish at $u = 0$ and the difference of the two sides is an elliptic function without poles. (17.8) now gives

$$\int_{i\omega_2}^{i\omega_2 + \omega_1} [\beta - \Theta(u)]\, du = \omega_1 \cdot 2\zeta(u_\infty) + \int_{i\omega_2 - u_\infty}^{i\omega_2 + u_\infty} [\zeta(u) - \zeta(u + \omega_1)]\, du;$$

taking into account that $\zeta(u) - \zeta(i\omega_2)$ and $\zeta(u + \omega_1) - \zeta(i\omega_2 + \omega_1)$ are odd functions of $u - i\omega_2$ we obtain with the usual notation $\zeta(i\omega_2) - \zeta(i\omega_2 + \omega_1) = -\eta_1$

$$\int_{i\omega_2}^{i\omega_2 + \omega_1} [\beta - \Theta(u)]\, du = 2[\omega_1 \cdot \zeta(u_\infty) - \eta_1 u_\infty]. \tag{17.9}$$

Similarly we obtain

$$\frac{1}{i}\int_{\omega_1}^{\omega_1 + i\omega_2} [\beta - \Theta(u)]\, du = \omega_2 \cdot 2\zeta(u_\infty) + \frac{1}{i}\int_{\omega_1 - u_\infty}^{\omega_1 + i\omega_2 - u_\infty} \zeta(u)\, du - \frac{1}{i}\int_{\omega_1 + u_\infty}^{\omega_1 + i\omega_2 + u_\infty} \zeta(u)\, du$$

which becomes applying Cauchy's integral theorem to the rectangle with vertices $\omega_1 \pm u_\infty, \omega_1 + i\omega_2 \pm u_\infty$ in which $\zeta(u)$ is regular,

$$\frac{1}{i}\int_{\omega_1}^{\omega_1 + i\omega_2} [\beta - \Theta(u)]\, du = \omega_2 \cdot 2\zeta(u_\infty) + \frac{1}{i}\int_{\omega_1 - u_\infty}^{\omega_1 + u_\infty} [\zeta(u) - \zeta(u + i\omega_2)]\, du$$

from which, reasoning similarly as before (17.9), one obtains with the usual notation $\zeta(\omega_1)-\zeta(\omega_1+i\omega_2)=-i\eta_2$

$$\frac{1}{i}\int_{\omega_1}^{\omega_1+i\omega_2}[\beta-\Theta(u)]\,du=2[\omega_2\zeta(u_\infty)-\eta_2 u_\infty]. \tag{17.10}$$

Inserting (17.9), (17.10) into (17.7) and observing that (17.7) and the left side of (17.2) are identical we obtain

$$\begin{aligned}h''_{ab}M''_{ab}\frac{d}{d\delta}h''_{ab}-\frac{d}{d\delta}[s_H(H_\delta)\cdot h_H(H_\delta)]+\pi\int_\beta^\infty\frac{1}{\sqrt{(t-\alpha)(t-\beta)}}\frac{dt}{\sqrt{t^2-1}}\\ =2\omega_2[\omega_1\zeta(u_\infty)-\eta_1 u_\infty]-2\omega_1[\omega_2\zeta(u_\infty)-\eta_2u_\infty]+\pi u_\infty\\ =[2\omega_1\eta_2-2\omega_2\eta_1+\pi]\,u_\infty,\end{aligned} \tag{17.11}$$

and the last term in (17.11) vanishes by Legendre's relation about the additive periods of the function $\zeta(u)$. This proves Lemma 16.

18. Lemma 17. *Let* $\delta\in(\alpha,\beta)-\{\delta_0\}$; *let* $a=a(r,B;\delta)$ *and* $b=b(r,B;\delta)$ *be as in Theorem 6. Then*

$$\operatorname{sign}\left(\frac{d}{d\delta}\log f'_b(B)\right)=\operatorname{sign}(-Z_{ab}). \tag{18.1}$$

Proof. In (17.1), the factor of a' vanishes by Lemma 12; the factor of b' equals $-2\pi Z_{ab}$ by Lemma 13, and the remaining terms vanish by Lemma 16. Thus we have

$$\frac{d}{d\delta}\log f'_b(B)=-Z_{ab}\frac{db}{d\delta},\quad \delta\in[\alpha,\beta]-\{\delta_0\}, \tag{18.2}$$

and (18.1) follows from (18.2) by Lemma 9.

Theorem 7. *Let* $B>0$ *be a point of the annulus* R_r *and let* $\Phi_r(R_r)=E_{-a(r)}$, $\Phi_r(B)=b'_r(B)$. *Then the problem* $P_{\max}(B)$ *has the following solution:*

(i) *if* $Z_{-a(r),\,b'_r(B)}\geqslant 0$, Φ_r *is the only function of* $\mathfrak{F}_r$ *satisfying*

$$\Phi'_r(B)=\max_{f\in\mathfrak{F}_r}|f'(B)| \tag{18.3}$$

and any $\varphi\in\mathfrak{F}_r$ *with* $|\varphi'(B)|=\Phi'_r(B)$ *has the form* $\varphi(z)=e^{i\gamma}\Phi_r(z)$ *with* γ *real;*

(ii) *if* $Z_{-a(r),\,b'_r(B)}<0$, *there is in the previous notation a unique* $\delta^+\in(\alpha,\beta)$ *such that* $Z_{a^+b^+}=0$ *where* $a^+=(r,B;\delta^+)<0$, $b^+=b(r,B;\delta^+)$; $\delta^+<\delta_0$ *and* f_{b^+} *satisfies*

$$f'_{b^+}(B) = \max_{f \in \mathfrak{F}_r} |f'(B)|; \tag{18.4}$$

any $\varphi \in \mathfrak{F}_r$ *with* $|\varphi'(B)| = f'_{b^+}(B)$ *has the form*

$$\varphi(z) = e^{i\gamma} g_\vartheta[f_{b^+}(z)], \quad \gamma \text{ real}, \tag{18.5}$$

where g_ϑ *is the* ϑ*-shift of section 8 with* $|\vartheta| \leqslant \vartheta^+$ *and*

$$0 < \vartheta^+ = \int_\alpha^{\delta^+} \frac{\delta^+ - t}{\sqrt{(t-\alpha)(\beta-t)}} \frac{dt}{\sqrt{t^2-1}} = \frac{1}{2} h''_{a^+ b^+} < \pi;$$

further any φ *of* (18.5) *is in* $\mathfrak{F}_r$ *and has* $|\varphi'(B)| = f'_{b^+}(B)$;

(iii) *the cases* (i) *and* (ii) *above mean geometrically: case* (i) *corresponds to* $\beta - \alpha \leqslant b'_r(B)$ *and case* (ii) *corresponds to* $\beta - \alpha > b'_r(B)$.

Proof. Let $\delta_1 \in (\alpha, \beta)$ and put $a_1 = a(r, B; \delta_1)$, $b_1 = b(r, B; \delta_1)$. Since both a and b are strictly increasing (Lemma 9) in δ, $Z_{a_1 b_1} \geqslant 0$ implies $Z(\delta) = Z_{a(r,B;\delta), b(r,B;\delta)} > 0$ for $\delta_1 < \delta \leqslant \beta$: by Lemma 15, $Z_{ab} > 0$ above and $Z_{ab} < 0$ below A^*; by Lemma 14, A^* has negative slope. For $\delta_1 \neq \delta_0$, $f'_b(B)$ (which is positive) as function of δ is at δ_1 therefore increasing, decreasing or stationary with strict maximum as $Z_{a_1 b_1}$ is negative, positive or zero (Lemma 17). At δ_0, $f'_b(B)$ is decreasing in δ by Lemma 17 and the mean value theorem since $a(r, B; \delta_0) = 0$ and $Z_{0, b(r,B;\delta_0)} > 0$ by Lemma 15, and since $Z(\delta)$ is continuous in δ. Thus either $Z(\delta) \geqslant 0$ in $[\alpha, \beta]$ or else there is exactly one value $\delta^+ \in (\alpha, \delta_0)$ with $Z(\delta^+) = 0$ and $Z(\delta) < 0$ for $\alpha \leqslant \delta < \delta^+$ and $Z(\delta) > 0$ for $\delta^+ < \delta \leqslant \beta$. In the former case $[Z(\alpha) \geqslant 0]$, $f'_b(B)$ attains its maximal value in $[b'_r(B), b''_r(B)]$ only at $b(r, B; \alpha) = b'_r(B)$; in the latter case $[Z(\alpha) < 0]$, $f'_b(B)$ attains its maximal value only at $b^+ = b(r, B; \delta^+)$. Thus in case (i) of Theorem 7 we do have (18.3), and in case (ii) we have the unicity of δ^+ and the validity of (18.4); the remaining statements in the cases (i), (ii) follow readily from Theorems 4 and 5; statement (iii) is obtained upon simple computation.

19. The solution of the problems $P_{\min}(B)$ and $P_{\min}(B; b)$ is much simpler. Since any argumentation in solving these problems is similar to or even easier than some reasoning encountered above we shall give below the relevant statements refraining from proofs.

THEOREM 8. *Let* E_{cd} *be the unit disk slit along the segment* $[c, d]$ *with* $-1 < c < d < 1$, *and let* $d < b < 1$; *let* h *be a function mapping* E_{cd} *conformally into the unit disk* E *and satisfying* $h(b) = b$ *as well as* $h(C) = C$ *where* C *is the circumference of* E. *Then* $|h'(b)| \geqslant 1$ *with equality if and only if* h *is a non-Euclidean rotation about* b.

COROLLARY. *If in addition* $0 \in [c, d]$, *and if* h *satisfies also* $h(E_{cd}) \subset E_0 = E - \{0\}$, *then* $|h'(b)| = 1$ *implies that* h *is the identity.*

THEOREM 9. *Let* $B > 0$ *be a point of the annulus* R_r; *let* H_b, $b_r'(B) \leqslant b \leqslant b_r''(B)$, *be the function mapping* R_r *conformally onto* E_{cd} *with* $d < b$, $H_b(B) = b$, $H_b(1) = 1$; *let* $f \in \mathfrak{F}_r(B; b)$ *with* $f(B) > 0$. *Then* $|f'(B)| \geqslant H_b'(B)$ *with equality if and only if* $f = H_b$.

THEOREM 10. *With* $B > 0$ *as above, let* $f \in \mathfrak{F}_r$ *with* $f(B) > 0$. *Then* $|f'(B)| \geqslant \Psi_r'(B)$ *with equality if and only if* $f = \Psi_r$, *where* Ψ_r *is given in section 2.*

20. *Final remark.* It might be of interest to find out extremal properties of the functions $F_b^\eta(z) H_b^{1-\eta}(z)$ with $0 < \eta < 1$ and $F_b = g_\vartheta \circ f_b$ where g_ϑ is a ϑ-shift (section 8) and f_b is given by (14.2). This might lead to the determination of the range of $f'(B)$ for $f \in \mathfrak{F}_r$, $f(B) = b$.

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