# ANALYTIC RAMIFICATIONS AND FLAT COUPLES OF LOCAL RINGS

BY

TOMAS LARFELDT and CHRISTER LECH

*University of Stockholm, Sweden*

## Introduction

In a paper of 1935 Akizuki constructed an analytically ramified (Noetherian) local domain of Krull dimension one ([1], Section 3). We shall present another, similar construction. It effects a transformation

$$(R_0, R_1) \to \{R, \mathfrak{p}\}$$

where on the left stands an arbitrary equidimensional flat couple of local rings and on the right a local ring together with a prime ideal (of coheight one) whose analytic ramification reflects the structure of the couple to the left. More precisely, the completion $\hat{R}$ of $R$ contains just one prime ideal $\mathfrak{p}^*$ contracting to $\mathfrak{p}$, and the couple $(R_{\mathfrak{p}}, \hat{R}_{\mathfrak{p}^*})$ mirrors the structure of $(R_0, R_1)$ inasmuch as there exists a commutative diagram

$$\begin{array}{ccc} R_0 & \longrightarrow & R_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & \hat{R}_{\mathfrak{p}^*} \end{array}$$

with unramified flat ring injections as horizontal maps. (See below for definitions.)

Two conclusions can be drawn from this construction (cf. further [11]). One is simply that there are plenty of analytic ramifications. The prime information in this respect is obtained already by taking for $R_0$ a field $K$ and for $R_1$ a ring $A$ of the form $K[Z_1, ..., Z_n]/I$ with $I$ primary for $(Z_1, ..., Z_n)$. Then $\mathfrak{p}$ must be equal to (0) so that $R$ becomes a one-dimensional local domain with the property that $\hat{R}_{\mathfrak{p}^*}$, the formal fiber of its zero ideal, is an unramified flat extension of $A$. Actually $\hat{R}_{\mathfrak{p}^*} \simeq A \otimes_K K((x))$ (where $K((x)) = K[[x]][x^{-1}]$), as is easily derived from the explicit formulas $\hat{R} = \hat{R}_1[[x]]$, $\mathfrak{p}^* = \mathfrak{m}_1 \hat{R}$ (cf. below).

The other conclusion depends on the fact that $\mathfrak{p}^*$ has a rather special position in $\hat{R}$, which entails that the Hilbert and Poincaré invariants of $\hat{R}_{\mathfrak{p}^*}$ are essentially the same as those of $\hat{R}$, the differences being referable entirely to the change in Krull dimension. As a consequence the mapping $(R_0, R_1) \to (R_{\mathfrak{p}}, R)$, of couples of local rings, preserves the Hilbert functions and Poincaré series of the respective components except for a dimension shift in the second. In the light of known results this implies an equivalence between certain general, possibly valid inequalities for couples of local rings (namely the inequalities (A) and (B) below, where the symbol $F$ represents both Hilbert and Poincaré invariants).

The crude outcome of our construction is expressed by the theorem stated below. A few explanations, mainly of a terminological nature, precede.

All rings are commutative, equipped with identity elements, and, unless otherwise stated or else apparent, Noetherian.

A couple $(R_0, R_1)$ of local rings, with maximal ideals $(\mathfrak{m}_0, \mathfrak{m}_1)$, is called *flat* if there is given a ring injection $R_0 \to R_1$ that makes $R_1$ into a flat $R_0$-module and maps $\mathfrak{m}_0$ into $\mathfrak{m}_1$. Such a couple $(R_0, R_1)$, or injection $R_0 \to R_1$, is called *equidimensional* if the Krull dimensions of $R_0$ and $R_1$ are equal or, equivalently, if $\mathfrak{m}_0 R_1$ is primary for $\mathfrak{m}_1$. It is called *unramified* if $\mathfrak{m}_0 R_1$ is actually equal to $\mathfrak{m}_1$. The components of an unramified flat couple exhibit great similarities. In particular they have identical Hilbert functions and Poincaré series.

If $R$ is a local ring, and $\mathfrak{p}$ a prime ideal of $R$, then $\hat{R}$ will denote the completion of $R$ and $R_{\mathfrak{p}}$ the localization of $R$ at $\mathfrak{p}$. The prime ideal $\mathfrak{p}$ is said to be *analytically ramified* if the extension $\mathfrak{p}\hat{R}$ is not an intersection of prime ideals, in particular if there exists a minimal prime ideal $\mathfrak{p}^*$ of $\mathfrak{p}\hat{R}$ such that the equidimensional flat couple $(R_{\mathfrak{p}}, \hat{R}_{\mathfrak{p}^*})$ is ramified. That couple can be thought of as describing "the ramification of $\mathfrak{p}$ at $\mathfrak{p}^*$".

THEOREM. *Let $(R_0, R_1)$ be an equidimensional flat couple of local rings with maximal ideals $(\mathfrak{m}_0, \mathfrak{m}_1)$. Then there exist a local ring $R$ and flat ring injections $R_0 \to R$ and $R_1 \to \hat{R}$ such that*

(i) *the diagram*

$$\begin{array}{ccc} R_0 & \longrightarrow & R \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & \hat{R} \end{array}$$

*is commutative;*

(ii) $\mathfrak{p} = \mathfrak{m}_0 R$ *is a prime ideal of* $R$;

(iii) *the map* $R_1 \to \hat{R}$ *has the form* $R_1 \xrightarrow{\text{nat.}} \hat{R}_1[[x]]$ *where* $x$ *is a variable.*

The proof of this theorem will be our main occupation. But we shall first develop its consequences so as to complete our previous description.

The transformation $(R_0, R_1) \to \{R, \mathfrak{p}\}$ is obtained, in a non-unique manner, directly from the theorem by keeping the symbols as they stand. (For our present purpose it is unessential to enforce uniqueness.)

Put $\mathfrak{p}^* = \mathfrak{m}_1 \hat{R} = \mathfrak{m}_1 \hat{R}_1[[x]]$. Then $\mathfrak{p}^*$ is a prime ideal of coheight one, i.e. Krull dim $(\hat{R}/\mathfrak{p}^*) = 1$. Moreover, the inclusions $\mathfrak{p}\hat{R} \subseteq \mathfrak{p}^* \subseteq \operatorname{rad}(\mathfrak{p}\hat{R})$ show that $\mathfrak{p}^*$ is the only minimal prime ideal of $\mathfrak{p}\hat{R}$, and that therefore $\mathfrak{p}$ is also of coheight one. The stated square-diagram relation between the couples $(R_0, R_1)$ and $(R_\mathfrak{p}, \hat{R}_{\mathfrak{p}^*})$ is evident.

Consider the following numerical invariants of a local ring: the values of the Hilbert function (for the maximal ideal) and the so-called deviations (connected with the Poincaré series). Let us agree to arrange these invariants in some infinite array and to denote the result for a given local ring $S$ by $F(S)$. Define $F^{(1)}(S) = F(S[Z]_{(\mathfrak{n}, Z)})$ where $\mathfrak{n}$ is the maximal ideal of $S$ and $Z$ a variable. To be sure, $F(S)$ and $F^{(1)}(S)$ mutually determine each other in a simple and well-known manner, independent of $S$. As in effect we have already mentioned, $F$ takes equal values on the two components of an unramified flat couple. Using this and the special relation between our $\hat{R}$ and $\hat{R}_{\mathfrak{p}^*}$, we get the formulas

$$F(R_\mathfrak{p}) = F(R_0), \quad F(R) = F^{(1)}(R_1),$$

which express the preservation of invariants under the mapping $(R_0, R_1) \to (R_\mathfrak{p}, R)$.

Now on the basis of partial results one may conjecture that whenever $R$ is a local ring and $\mathfrak{p}$ a prime ideal of coheight one in $R$, then

$$F^{(1)}(R_\mathfrak{p}) \leqslant F(R) \tag{A}$$

in the sense of inequality for each entry (the "total-order" sense). In view of the above formulas regarding the mapping $(R_0, R_1) \to (R_\mathfrak{p}, R)$, this would imply that for any equi-dimensional flat couple $(R_0, R_1)$ of local rings

$$F^{(1)}(R_0) \leqslant F^{(1)}(R_1). \tag{B}$$

Since the reverse implication is already known, we conclude that the general assertions expressed by (A) and (B) are in fact equivalent.—As references for the subject here touched upon, see [9], [3] (especially Theorem 2), [7], [12] in what concerns Hilbert functions and [6] (Chapter 3), [2] in what concerns deviations. Let us also mention [10] which contains an independent, comparatively direct proof of Theorem 2 of [3], unsusceptible, however, to the improvement in [12].

### Description of a ring R that will satisfy the requirements of the theorem

With assumptions and notations as in the theorem, put $R_2 = \hat{R}_1[[x]]$ ($x$ a variable), $\mathfrak{m}_2 = (\mathfrak{m}_1, x)R_2$, $K_i = R_i/\mathfrak{m}_i$ ($i = 0, 1, 2$). We shall assume that $K_1$ ($= K_2$) is algebraic over $K_0$ as we can always reduce our proof to this case by an extension of $R_0$ within $R_1$; see Lemma A below.

Let $u = \{u_i \mid i \in I\}$ be a set of generators for the ring extension $R_1/R_0$. We can assume that the cardinality of the index set $I$ is not larger than the transcendence degree of the extension $K_0[[x]]/K_0[x]$, for that degree is not smaller than the cardinality of the entire set $R_1$; see Lemma B. Choose a map $\alpha: I \to R_0[[x]]$ which, composed with the natural map $R_0[[x]] \to K_0[[x]]$, gives an injection of $I$ into a transcendence basis for $K_0[[x]]/K_0[x]$. Put $u^* = \{u_i + \alpha(i) \mid i \in I\}$. Finally define

| | |
|---|---|
| $A = R_0[u^*, x, x^{-1}]$ | within $R_2[x^{-1}]$, |
| $B = A \cap R_2$, | |
| $C = B_{B \cap \mathfrak{m}_2}$ | within $R_2$, |
| $R =$ the $\mathfrak{m}_0 C$-adic completion of $C$ | within $R_2$. |

We wish $R$ to have the completion $R_2$ in the topology defined by its one maximal ideal $R \cap \mathfrak{m}_2$. We can then form the diagram of the theorem in a natural way. To establish the asserted properties, we get the list below of items to check. Within parenthesis we have added pertinent features of the construction.

| | |
|---|---|
| $R$ dense in $R_2$ | (choice of $u$), |
| $R$ and $R_2$ topologically concordant | (presence of $x^{-1}$ in $A$), |
| $\mathfrak{m}_0 R$ a prime ideal | (independence imposed on $u^*$), |
| $R$ Noetherian | (passage from $C$ to $R$), |
| $R$ flat as an $R_0$-module. | |

Let us verify the first point at once. Observe that

$$\text{the map } R_0[u^*, x] \xrightarrow{\text{nat.}} R_1[[x]]/(x^n) \text{ is surjective } \quad (n = 1, 2, 3, \ldots), \tag{*}$$

for clearly $R_0[u^*, x]$ has the same image in $R_1[[x]]/(x^n)$ as $R_0[u, x] = R_1[x]$. Attaching the map $R_1[[x]]/(x^n) \to R_2/\mathfrak{m}_2^n$, which is visibly surjective, we conclude that already the subring $R_0[u^*, x]$ of $R$ is dense in $R_2$.

### Main lines of the proof

For proof purposes we shall apply our construction not only to the given couple $(R_0, R_1)$ but also to the couple $(\tilde{R}_0, \tilde{R}_1) = (R_0/\mathfrak{m}_0, R_1/\mathfrak{m}_0 R_1)$. Distinguishing objects belonging to the latter application by the superscript ~ and using for $\tilde{u}^*$ the set naturally induced by $u^*$, we get natural ring homomorphisms $A \to \tilde{A}$, $B \to \tilde{B}$, $C \to \tilde{C}$, $R \to \tilde{R}$. These homomorphisms all contain $\mathfrak{m}_0$ in their kernels.

Taking one further lemma for granted (Lemma C), we shall show that it suffices to prove the following four statements.

$$R \cap \mathfrak{m}_2 \text{ is a finitely generated ideal in } R; \tag{1; R}$$

$$R \cap \mathfrak{m}_2^n = (R \cap \mathfrak{m}_2)^n \quad (n = 1, 2, 3, \ldots); \tag{2; R}$$

$$R \cap \mathfrak{m}_1 R_2[x^{-1}] = \mathfrak{m}_0 R; \tag{3; R}$$

$$\text{the homomorphism } R/\mathfrak{m}_0 R \xrightarrow{\text{nat.}} \tilde{R} \text{ is bijective.} \tag{4; R}$$

We have seen that $R$ is dense in $R_2$. By (2; $R$) the two rings are topologically concordant. Thus $R$ is a possibly non-Noetherian local ring with the completion $R_2$. Applying the analogous result for $\tilde{R}$, we see that this possibly non-Noetherian local ring has the following properties: its maximal ideal is finitely generated according to (1; $\tilde{R}$), its completion is Noetherian and one-dimensional, and the single minimal prime ideal of its completion contracts to the zero ideal of the ring itself ((3; $\tilde{R}$)) (so that, in particular, $\tilde{R}$ is a domain). These properties allow us to conclude by Lemma C that $\tilde{R}$ is Noetherian. Then, in view of (4; $R$), the graded ring associated with the $\mathfrak{m}_0 R$-adic filtration of $R$ is Noetherian. Hence $R$ itself is Noetherian, as it is complete in the $\mathfrak{m}_0 R$-adic topology. It follows that the couple $(R, R_2) = (R, \hat{R})$ is flat. Considering that also $(R_0, R_2)$ is flat, we infer the same for $(R_0, R)$. Finally we deduce from (3; $R$) that $\mathfrak{m}_0 R$ is a prime ideal, the only remaining point.

It is a routine matter to see that the statements (1; $R$)–(4; $R$) can be derived from the analogous statements (1; $B$)–(4; $B$). We shall content ourselves with proving the latter together with Lemmas A–C.

## Completion of the proof

Let us start by proving (1; $B$) and (2; $B$). Any element $f$ of $B$ can be expanded in a power series $f = a_0 + a_1 x + a_2 x + \ldots \in R_1[[x]]$. Clearly $f \in \mathfrak{m}_2$ if and only if $a_0 \in \mathfrak{m}_1$. By (*), applied with $n = 1$, we can find finitely many elements $f_1, \ldots, f_r$ in $R_0[u^*, x] \subseteq B$ whose coefficients of index 0 generate $\mathfrak{m}_1$ in $R_1$. Then, by (*) again, for an arbitrary element $f \in B \cap \mathfrak{m}_2$, there exist elements $g_1, \ldots, g_r \in B$ such that the difference $f - (g_1 f_1 + \ldots + g_r f_r)$ has a vanishing constant term, hence is divisible by $x$ in $B$. This shows that $B \cap \mathfrak{m}_2 = (f_1, \ldots, f_r, x)$, which proves (1; $B$). Similarly we get (2; $B$) by observing that the element $f$ above belongs to $\mathfrak{m}_2^n$ $(n = 2, 3, 4, \ldots)$ if and only if its coefficients satisfy the conditions $a_i \in \mathfrak{m}_1^{n-i}$ $(i = 0, 1, \ldots, n-1)$.

Next, consider (3; $A$) and (4; $A$). The natural image of $R$ in $K_1[[x]][x^{-1}] \simeq R_2[x^{-1}]/\mathfrak{m}_1 R_2[x^{-1}]$ evidently has the form of a free polynomial extension of $K_0[x, x^{-1}]$

in $|I|$ variables corresponding to the elements of $u^*$. Hence we get (3; $A$) and, a fortiori, the injectivity part of (4; $A$). The surjectivity part of (4; $A$) is obvious.

To pass from (3; $A$) to (3; $B$) and from (4; $A$) to (4; $B$) we note that $A/B \simeq (A+R_2)/R_2$ has a natural representation on the form

$$A/B \simeq \coprod_{n=1}^{\infty} R_1 x^{-n},$$

cf. (*). Thus $A/B$ is $R_1$-free, hence $R_0$-flat. It follows that the exactness of the sequence $0 \to B \to A$ is preserved under tensoring with any $R_0$-module. Tensoring with $K_0$, we first derive that $B \cap \mathfrak{m}_0 A = \mathfrak{m}_0 B$, which, in combination with (3; $A$), gives (3; $B$). Secondly we conclude that the commutative diagram below has exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0 \otimes_{R_0} B & \longrightarrow & K_0 \otimes_{R_0} A & \longrightarrow & K_0 \otimes_{R_0} (A/B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{B} & \longrightarrow & \tilde{A} & \longrightarrow & \tilde{A}/\tilde{B} & \longrightarrow & 0. \end{array}$$

The right hand column represents a bijection, as $A/B$ is $R_1$-free and $K_0 \otimes_{R_0} R_1 \simeq \tilde{R}_1$, likewise the middle column by (4; $A$), hence also the left hand column by the Five Lemma. Thus (4; $B$) holds true, and the proof is complete.

## The lemmas

LEMMA A. *Any equidimensional flat couple of local rings can be obtained as the composition of two such couples of which the first is unramified and the second residually algebraic.*

*Proof.* Let $(R_0, R_1)$ be a given equidimensional flat couple of local rings with maximal ideals $(\mathfrak{m}_0, \mathfrak{m}_1)$. Choose a system $\{z\}$ of representatives in $R_1$ for a transcendence basis of $R_1/\mathfrak{m}_1$ over $R_0/\mathfrak{m}_0$, and define $R_0'$ as the localization of $R_0[z]$ at $R_0[z] \cap \mathfrak{m}_1$. We shall show that the decomposition of $(R_0, R_1)$ into $(R_0, R_0')$ and $(R_0', R_1)$ has the desired properties.

Let us first prove that $\{z\}$ is algebraically independent over $R_0$, i.e., if $\{Z\}$ denotes a system of variables corresponding to $\{z\}$, that the natural map $R_0[Z] \to R_1$ is injective. It suffices to prove injectivity for the induced map of associated graded rings under the $(\mathfrak{m}_0)$-adic filtration. Due to flatness, this map is obtained by tensoring the original one with $\coprod \mathfrak{m}_0^n/\mathfrak{m}_0^{n+1}$ over $R_0$. But tensoring with a free $(R_0/\mathfrak{m}_0)$-module reduces, when one looks to injectivity alone, to tensoring with just $R_0/\mathfrak{m}_0$. Thus we arrive at the map that forms the first part of the composition

$$R_0[Z]/\mathfrak{m}_0 R_0[Z] \to R_1/\mathfrak{m}_0 R_1 \to R_1/\mathfrak{m}_1.$$

As the composed map is injective by our choice of $\{z\}$, our assertion follows, i.e. $\{z\}$ is algebraically independent over $R_0$.

The injectivity of the composed map also shows that $R_0[z] \cap \mathfrak{m}_1 = \mathfrak{m}_0 R_0[z]$. Hence $R_0'$ has the structure $R_0[Z]_{(\mathfrak{m}_0)}$. To see that this ring is Noetherian, consider all the subrings obtained by exchanging $\{Z\}$ for some finite subset. These subrings are Noetherian local rings with the same Krull dimension as $R_0$. Now, for every prime ideal in the full ring, we can find a subring at which the restriction of the prime ideal has maximal height. Clearly that restriction must generate the prime ideal. Thus $R_0[Z]_{(\mathfrak{m}_0)}$ has all its prime ideals finitely generated, hence is Noetherian.

It is evident that $(R_0, R_0')$ is flat and unramified, and that the residue field extension associated with $(R_0', R_1)$ is algebraic. All that remains to prove, is that $(R_0', R_1)$ is flat. Remembering that the maximal ideal of $R_0'$ is $\mathfrak{m}_0 R_0'$, we derive this last point from the proportionality between $\text{length}_{R_0'}(R_0'/\mathfrak{m}_0^n R_0')$ and $\text{length}_{R_1}(R_1/\mathfrak{m}_0^n R_1)$ as $n$ varies, which is implied by the flatness of $(R_0, R_0')$ and $(R_0, R_1)$ (cf. [4], Chapter III, § 5).

LEMMA B. *With the notation of the main text, the cardinality of $R_1$ (qua set) does not exceed the transcendence degree of $K_0[[x]]/K_0[x]$. (Recall that $R_1$ is a local ring whose residue class field $K_1$ is algebraic over $K_0$.)*

*Proof.* The cardinality of $R_1$ is majorized by that of $K_1[[x_1, ..., x_n]]$ for a suitable $n$, hence also by the cardinality of $K_0[[x]]$. ($K_0$ and $K_1$ have the same cardinality, or $K_0$ is finite and $K_1$ denumerable.) Thus it suffices to show for every field $K$ that the transcendence degree of $K[[x]]/K$ equals the cardinality of $K[[x]]$. As $K[[x]]$ is non-denumerable, its cardinality coincides with its transcendence degree over the prime field $P$ of $K$. Therefore it suffices to show for the two successive field extensions $P \hookrightarrow K \hookrightarrow K[[x]]$ that the latter does not have a smaller transcendence degree than the former. Let $J$ be an index set for a transcendence basis of $K/P$. We may assume that the cardinality $|J|$ is infinite. Then, with $\omega = \{0, 1, 2, ...\}$, we have $|J \times \omega| = |J|$ which means that we can find $|J|$ elements of $K[[x]]$ all of whose coefficients are algebraically independent over $P$. It is not difficult to see that such a system of elements of $K[[x]]$ is algebraically independent over $K$. Hence the result.

LEMMA C. *A local domain is Noetherian (and 1-dimensional) if* (i) *its maximal ideal is finitely generated,* (ii) *its completion is Noetherian and 1-dimensional, and* (iii) *the minimal prime ideals of the completion contract to the zero ideal of the ring itself.*

*Proof.* Call the domain $R$ and its maximal ideal $\mathfrak{m}$. Let $\mathfrak{a}$ be a non-zero ideal of $R$. As the ideal $\mathfrak{a}\hat{R}$ cannot be contained in any of the minimal prime ideals of $\hat{R}$, it must be primary for $\mathfrak{m}\hat{R}$. Let us choose $k$ so that $\mathfrak{m}^k\hat{R}\subseteq\mathfrak{a}\hat{R}$. Then $\mathfrak{m}^k\subseteq\mathfrak{a}+\mathfrak{m}^{k+1}$ as $R/\mathfrak{m}^{k+1}\simeq\hat{R}/\mathfrak{m}^{k+1}\hat{R}$ and, observing that $\mathfrak{m}^k$ is finitely generated, we infer by Nakayama's lemma that $\mathfrak{m}^k\subseteq\mathfrak{a}$. It follows that $\mathfrak{a}$ is finitely generated, which gives the result.

## Final remarks

In the main the result of the present paper was obtained already around 1970. It was communicated on a minor scale through the preprint [8] and through lectures on various occasions. A partial parallel is contained in [5], applications together with a proof of a simplified version in [11].

We thank L. Avramov and D. Eisenbud for valuable comments on the preprint. The former made us aware, among other things, that the statement of Lemma A is not self-evident.

## References

[1] Akizuki, Y., Einige Bemerkungen über primäre Integritätsbereiche mit Teilerkettensatz. *Proc. Phys.-Math. Soc. Japan*, 17 (1935), 327–336.
[2] Avramov, L., Homology of local flat extensions and complete intersection defects. *Math. Ann.*, 228 (1977), 27–37.
[3] Bennett, B. M., On the characteristic functions of a local ring. *Ann. of Math.*, 91 (1970), 25–87.
[4] Bourbaki, N., *Éléments de mathématique; Algèbre commutative*. Paris, 1961.
[5] Ferrand, D. & Raynaud, M., Fibres formelles d'un anneau local noethérien. *Ann. Sci. École Norm. Sup., 4e série*, 3 (1970), 295–311.
[6] Gulliksen, T. & Levin, G., Homology of local rings. *Queen's papers in pure and applied math.*, 20 (1969). Queen's university, Kingston, Ont.
[7] Hironaka, H., Certain numerical characters of singularities. *J. Math. Kyoto Univ.*, 10 (1970), 151–187.
[8] Larfeldt, T. & Lech, C., Flat couples, analytic ramifications and Hilbert functions. *Preprint series, Stockholm Univ.*, 8 (1973), 7 pp.
[9] Lech, C., Inequalities related to certain couples of local rings. *Acta Math.*, 112 (1964), 69–89.
[10] Ljungström, A., An inequality between the Hilbert functions of certain prime ideals one of which is immediately included in the other. *Report, Stockholm Univ.* (1970), 14 pp.
[11] Marinari, M. G., Examples of bad noetherian local rings. *Nagoya Math. J.*, 70 (1978), 105–110.
[12] Singh, B., Effect of a permissible blowing up on the local Hilbert functions. *Invent. Math.*, 26 (1974), 201–112.

*Received October 20, 1980*