ON THE REPRESENTATION OF A NUMBER AS THE SUM OF TWO SQUARES AND A PRIME

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1. Introduction

In their celebrated paper "Some problems of partitio numerorum: III" [6] Hardy and Littlewood state an asymptotic formula, suggested by a purely formal application of their circle method, for the number of representations of a number n as the sum of two squares and a prime number. The truth of this formula would imply that every sufficiently large number is the sum of two squares and a prime. No proof, even on the extended Riemann hypothesis (which we hereafter refer to as Hypothesis R), has hitherto been found. However, in another paper [7] they suggest that on Hypothesis R it should be possible to prove that *almost all* numbers can be so represented. This proof was effected by Miss Stanley [11], as were proofs (also on Hypothesis R) of asymptotic formulae for the number of representations of a number as sums of *greater* numbers of squares and primes. The dependence of her results on the unproved hypothesis was gradually removed by later writers, in particular by Chowla [2], Walfisz [13], Estermann [4] and Halberstam [5].

It is the purpose of this paper to shew that the original formula of Hardy and Littlewood is true on Hypothesis R. Our method depends on the fact that, as is easily seen, the number of representations of n in the required form is equal to the sum

$$\sum_{p<n} r(n-p),$$

where $r(\nu)$ denotes the number of representations of ν as the sum of two integral squares. On noting that $r(\nu)$ may be expressed as a sum over the divisors of ν, we see that our problem is related in character to the problem of determining the asymptotic behaviour of the sum

$$\sum_{0<p+a\leq x} d(p+a),$$

where a is a fixed non-zero integer and $d(\nu)$ denotes the number of divisors of ν. The latter problem is due to Titchmarsh [12] and was solved by him on Hypothesis R. Let $\pi(m; b, k)$ denote the number of primes not exceeding m which belong to the arithmetical progression $b \pmod{k}$. Then the two problems are similar in that each sum can be expressed as a combination of terms of the type $\pi(m; b, k)$, where k belongs to a certain range that depends on a parameter m. In each case the strength of Hypothesis R is sufficient to estimate $\pi(m; b, k)$ over nearly all the required range of k, while elementary methods will suffice to estimate the contributions to the sums due to the exceptional values of k. In our problem, however, this elementary estimation presents a more fundamental difficulty and requires a different method, since it is necessary to take into account the changes of sign due to the presence of the quadratic character in the expression for $r(\nu)$. In fact the major part of this paper is devoted to this estimation. We shall require repeatedly, here, the ideas of Brun's modification of the Eratosthenian sieve. An important feature of our method is the application we make of asymptotic formulae, and not merely upper bounds, for sums depending on an invariant sieve.

It is natural to ask whether it would not now be possible to prove this result independently of Hypothesis R. It seems unlikely, however, that such a proof can be achieved at present. The main difficulty with this problem, as with the much harder Goldbach problem (concerning numbers as sums of *two* primes), is that the number of representations of a large number is too small for the circle method in its present form to be effective, whether or not Hypothesis R be assumed. Moreover, our result must probably depend in some essential manner on properties of either exponential sums or primes in arithmetical progression that can only be proved at this time on the full strength of Hypothesis R.

We may consider the *conjugate* sum

$$\sum_{0 < p+a \leq x} r(p+a)$$

in a similar way. The details are a little simpler, since the term a in the summand is independent of the limit of summation, x. The asymptotic formula, which is stated without proof in the final section, shews that there are infinitely many primes of the form $u^2 + v^2 + a$, where u and v are integers.

I am very much indebted to Mr Ingham for reading a preliminary draft of this paper.

Notation and terminology

A_i is a positive absolute constant; the equation $f = O(|g|)$ denotes an inequality of the type $|f| < A_i |g|$, true for all values of the variables consistent with stated conditions. $A_i(a)$ is a constant depending at most on the parameter a.

n is a positive integer exceeding e^e; p is a prime number; d, k, l, m, q, r, s, t, μ and ν are positive integers; u and v are integers, except in Section 4; y is a real number not less than 1; z is a complex variable.

$[l, m]$ and (l, m) denote respectively the least common multiple and the highest common factor of l and m. $\omega(\nu)$ is the number of different prime factors of ν; $\Omega(\nu)$ is the total number of prime factors of ν (counted according to their multiplicity).

2. Decomposition of sum

Let $\nu(n)$ denote the number of representations of the integer n in the form

$$n = p + u^2 + v^2.$$

Then
$$\nu(n) = \sum_{n = p + u^2 + v^2} 1 = \sum_{p \leq n} \sum_{u^2 + v^2 = n - p} 1 = \sum_{p < n} r(n - p) + O(1). \tag{1}$$

Using the fact that
$$r(\nu) = 4 \sum_{l \mid \nu} \chi(l),$$

where $\chi(l)$ is the non-principal character (mod 4), we have

$$\sum_{p<n} r(n-p) = 4 \sum_{\substack{lm = n - p \\ p < n}} \chi(l)$$
$$= 4 \left(\sum_{l \leq n^{\frac{1}{2}} \log^{\;3} n} + \sum_{n^{\frac{1}{2}} \log^{-3} n < l < n^{\frac{1}{2}} \log^3 n} + \sum_{l \geq n^{\frac{1}{2}} \log^3 n} \right) = 4 (\Sigma_A + \Sigma_B + \Sigma_C), \text{ say.} \tag{2}$$

We have thus reduced our problem to that of the estimation of three different sums. Each sum is considered separately, Σ_A and Σ_C in Section 3 and Σ_B in Section 4. It will appear that Σ_A gives rise to the dominant term of the final asymptotic formula, Σ_B and Σ_C being of a lower order of magnitude.

3. Estimation of Σ_A and Σ_C

In order to estimate Σ_A and Σ_C we shall assume the extended Riemann hypothesis, which we state explicitly as follows:

Every zero of every Dirichlet's function

$$L(z) = \sum_{m=1}^{\infty} \frac{\chi_q(m)}{m^z},$$

where $\chi_q(m)$ *is a character* (mod q), *has a real part which does not exceed* $\frac{1}{2}$ *for all* q *and all* χ_q.

It should be noted that Lemma 1 depends on this hypothesis; but that the remainder of the paper does not, except indirectly through this lemma.

Preliminary lemmata

Lemma 1 is due to Titchmarsh [12].

LEMMA 1. *If* $(a, k) = 1$, *then on the extended Riemann hypothesis*

$$\sum_{\substack{p<y \\ p\equiv a \pmod k}} 1 = \frac{1}{\varphi(k)} \operatorname{Li} y + O(y^{\frac{1}{2}} \log 2y).$$

LEMMA 2. *For any* m, *we have*

$$\sum_{\substack{l\le y \\ (l,m)=1}} \frac{\chi(l)}{l} = \frac{\pi}{4} \prod_{p|m} \left(1 - \frac{\chi(p)}{p}\right) + O\left(\frac{1}{y} d(m; y)\right) + O\{\sigma_{-1}(m; y)\},$$

where

$$d(m; y) = \sum_{\substack{d|m \\ d\le y}} 1 \quad \text{and} \quad \sigma_{-1}(m; y) = \sum_{\substack{d|m \\ d>y}} \frac{1}{d}.$$

We have

$$\sum_{\substack{l\le y \\ (l,m)=1}} \frac{\chi(l)}{l} = \sum_{l\le y} \frac{\chi(l)}{l} \sum_{\substack{d|l \\ d|m}} \mu(d) = \sum_{\substack{d|m \\ d\le y}} \mu(d) \sum_{\substack{l=dt \\ l\le y}} \frac{\chi(l)}{l}$$

$$= \sum_{\substack{d|m \\ d\le y}} \frac{\mu(d)\chi(d)}{d} \sum_{t\le \frac{y}{d}} \frac{\chi(t)}{t}$$

$$= \sum_{\substack{d|m \\ d\le y}} \frac{\mu(d)\chi(d)}{d} \left\{\frac{\pi}{4} + O\left(\frac{d}{y}\right)\right\}$$

$$= \frac{\pi}{4} \sum_{d|m} \frac{\mu(d)\chi(d)}{d} + O\left(\frac{1}{y} \sum_{\substack{d|m \\ d\le y}} 1\right) + O\left(\sum_{\substack{d|m \\ d>y}} \frac{1}{d}\right)$$

$$= \frac{\pi}{4} \prod_{p|m} \left(1 - \frac{\chi(p)}{p}\right) + O\left(\frac{1}{y} d(m; y)\right) + O\{\sigma_{-1}(m; y)\}.$$

LEMMA 3. *For any* m, *we have*

$$\sum_{\substack{l\le y \\ (l,m)=1}} \frac{\chi(l)}{\varphi(l)} = \frac{\pi}{4} C E(m) + O\left(\frac{\log 2y}{y} d(m)\right),$$

where

$$C = \prod_{p>2} \left(1 + \frac{\chi(p)}{p(p-1)}\right)$$

and
$$E(m)=\prod_{\substack{p\mid m\\ p\equiv 1\,(\mathrm{mod}\,4)}}\frac{(p-1)^2}{p^2-p+1}\prod_{\substack{p\mid m\\ p\equiv 3\,(\mathrm{mod}\,4)}}\frac{p^2-1}{p^2-p-1}.$$

Also, there exist positive absolute constants A_1 and A_2 such that

$$\frac{A_1}{\log\log 10\,m}<E(m)<A_2\log\log 10\,m.$$

Let l' denote a general square-free number. Then

$$\frac{l}{\varphi(l)}=\prod_{p\mid l}\left(1-\frac{1}{p}\right)^{-1}=\prod_{p\mid l}\left(1+\frac{1}{p-1}\right)=\sum_{l'\mid l}\frac{1}{\varphi(l')}.$$

So
$$\sum_{\substack{l\le y\\ (l,m)=1}}\frac{\chi(l)}{\varphi(l)}=\sum_{\substack{l\le y\\ (l,m)=1}}\frac{\chi(l)}{l}\sum_{l'r=l}\frac{1}{\varphi(l')}=\sum_{\substack{l'\le y\\ (l'm)=1}}\frac{\chi(l')}{l'\varphi(l')}\sum_{\substack{r\le y/l'\\ (r,m)=1}}\frac{\chi(r)}{r}.$$

Hence, since
$$\frac{1}{y}d(m)\ge\begin{cases}\dfrac{1}{y}d(m;y)\\ \sigma_{-1}(m;y),\end{cases}$$

we have, by Lemma 2,

$$\begin{aligned}\sum_{\substack{l\le y\\ (l,m)=1}}\frac{\chi(l)}{\varphi(l)}&=\frac{\pi}{4}\prod_{p\mid m}\left(1-\frac{\chi(p)}{p}\right)\sum_{\substack{l'\le y\\ (l',m)=1}}\frac{\chi(l')}{l'\varphi(l')}+O\left(\frac{d(m)}{y}\sum_{l'\le y}\frac{1}{\varphi(l')}\right)\\ &=\frac{\pi}{4}\prod_{p\mid m}\left(1-\frac{\chi(p)}{p}\right)\sum_{(l',m)=1}\frac{\chi(l')}{l'\varphi(l')}+O\left\{\prod_{p\mid m}\left(1+\frac{1}{p}\right)\sum_{l>y}\frac{1}{l\varphi(l)}\right\}+O\left\{\frac{d(m)}{y}\sum_{l\le y}\frac{1}{\varphi(l)}\right\}.\end{aligned}\tag{3}$$

Now, since[1] $l/\varphi(l)=O\{\sigma_{-1}(l)\}$ and

$$\sum_{l\le x}\sigma_{-1}(l)=O(x),\tag{4}$$

we have, by partial summation,

$$\sum_{l>y}\frac{1}{l\varphi(l)}=O\left(\frac{1}{y}\right)\quad\text{and}\quad\sum_{l\le y}\frac{1}{\varphi(l)}=O(\log 2y).\tag{5}$$

Also
$$\prod_{p\mid m}\left(1+\frac{1}{p}\right)=O(2^{\omega(m)})=O\{d(m)\}.\tag{6}$$

We deduce from (3), (5) and (6)

$$\sum_{\substack{l\le y\\ (l,m)=1}}\frac{\chi(l)}{\varphi(l)}=\frac{\pi}{4}\prod_{p\mid m}\left(1-\frac{\chi(p)}{p}\right)\sum_{(l',m)=1}\frac{\chi(l')}{l'\varphi(l')}+O\left(\frac{\log 2y}{y}d(m)\right).\tag{7}$$

Furthermore, by Euler's theorem on the factorisation of infinite series,

[1] For the proof, see, for example, HARDY and WRIGHT, *The Theory of Numbers*, Chapter XVIII.

$$\sum_{(l',m)=1} \frac{\chi(l')}{l'\varphi(l')} = \prod_{\substack{p\nmid m\\ p>2}} \left(1 + \frac{\chi(p)}{p(p-1)}\right) = C \prod_{\substack{p\mid m\\ p>2}} \frac{p(p-1)}{p^2-p+\chi(p)} \tag{8}$$

and

$$\prod_{p\mid m} \left(1 - \frac{\chi(p)}{p}\right) \prod_{\substack{p\mid m\\ p>2}} \frac{p(p-1)}{p^2-p+\chi(p)} = \prod_{\substack{p\mid m\\ p>2}} \frac{(p-1)(p-\chi(p))}{p^2-p+\chi(p)} = E(m). \tag{9}$$

The first part of the lemma follows from (7), (8) and (9).

The second part of the lemma is an easy deduction from the relation

$$\prod_{p\mid\nu} \left(1 - \frac{1}{p}\right)^{-1} = O(\log\log 10\,\nu).$$

Estimation of Σ_A

By (2), we have

$$\Sigma_A = \sum_{l\le n^{\frac{1}{2}}\log^{-3}n} \chi(l) \sum_{\substack{p\equiv n \,(\mathrm{mod}\, l)\\ p<n}} 1 = \sum_{\substack{l\le n^{\frac{1}{2}}\log^{-3}n\\ (l,n)=1}} \chi(l) \sum_{\substack{p\equiv n \,(\mathrm{mod}\, l)\\ p<n}} 1 + O\left(\sum_{\substack{l\le n^{\frac{1}{2}}\log^{-3}n\\ (l,n)>1}} \sum_{\substack{p\equiv n \,(\mathrm{mod}\, l)\\ p<n}} 1\right).$$

Since the arithmetical progression $n \,(\mathrm{mod}\, l)$ contains at most one prime if $(l, n) > 1$, we have

$$\Sigma_A = \sum_{\substack{l\le n^{\frac{1}{2}}\log^{-3}n\\ (l,n)=1}} \chi(l) \sum_{\substack{p\equiv n \,(\mathrm{mod}\, l)\\ p<n}} 1 + O\left(\frac{n^{\frac{1}{2}}}{\log^3 n}\right).$$

Hence, by Lemma 1,

$$\Sigma_A = \mathrm{Li}\, n \sum_{\substack{l\le n^{\frac{1}{2}}\log^{-3}n\\ (l,n)=1}} \frac{\chi(l)}{\varphi(l)} + O\left(\sum_{\substack{l\le n^{\frac{1}{2}}\log^{-3}n\\ (l,n)=1}} n^{\frac{1}{2}}\log n\right) + O\left(\frac{n^{\frac{1}{2}}}{\log^3 n}\right)$$

$$= \mathrm{Li}\, n \sum_{\substack{l\le n^{\frac{1}{2}}\log^{-3}n\\ (l,n)=1}} \frac{\chi(l)}{\varphi(l)} + O\left(\frac{n}{\log^2 n}\right),$$

and so, by Lemma 3,

$$\Sigma_A = \frac{\pi}{4} C E(n)\, \mathrm{Li}\, n + O\left(\mathrm{Li}\, n \frac{\log^4 n}{n^{\frac{1}{2}}} d(n)\right) + O\left(\frac{n}{\log^2 n}\right)$$

$$= \frac{\pi}{4} C E(n)\, \mathrm{Li}\, n + O\left(\frac{n}{\log^2 n}\right)$$

$$= \frac{\pi}{4} C E(n) \frac{n}{\log n} + O\left(\frac{n}{\log^2 n} \log\log n\right). \tag{10}$$

Estimation of Σ_C

In Σ_C, the condition $l \geq n^{\frac{1}{2}} \log^3 n$ implies $m < n^{\frac{1}{2}} \log^{-3} n$. Therefore

$$\Sigma_C = \sum_{m < n^{\frac{1}{2}} \log^{-3} n} \sum_{\substack{lm = n - p \\ mn^{\frac{1}{2}} \log^3 n \leq lm < n}} \chi(l). \tag{11}$$

We denote the inner sum in (11) by Σ_m. The summand in Σ_m is 1 if $l \equiv 1 \pmod 4$, is -1 if $l \equiv 3 \pmod 4$, and is 0 otherwise. Therefore

$$\Sigma_m = \sum_{\substack{p \equiv n + 3m \pmod{4m} \\ p \leq n - mn^{\frac{1}{2}} \log^3 n}} 1 - \sum_{\substack{p \equiv n + m \pmod{4m} \\ p \leq n - mn^{\frac{1}{2}} \log^3 n}} 1.$$

Also the conditions $(n+m, 4m) = 1$ and $(n + 3m, 4m) = 1$ are equivalent. Hence, if $(n+m, 4m) = 1$, we have, by Lemma 1,

$$\Sigma_m = O(n^{\frac{1}{2}} \log n); \tag{12}$$

whereas, if $(n+m, 4m) > 1$, then trivially

$$\Sigma_m = O(1). \tag{13}$$

We thus have, by (11), (12) and (13),

$$\Sigma_C = O\left(\sum_{m < n^{\frac{1}{2}} \log^{-3} n} n^{\frac{1}{2}} \log n\right) = O\left(\frac{n}{\log^2 n}\right). \tag{14}$$

4. Estimation of Σ_B

Our estimation of Σ_B depends basically on the sieve method. The virtue of the sieve method from our point of view is that we are able to prove by its means results that embody information about the distribution of primes belonging to sequences of low density. The conventional result of this type, usually proved by either Brun's method [1] or Selberg's method [10], is an upper or lower bound for the number of primes in a sequence of a given class, and for the proof it is found convenient to choose a sieve that depends on the particular sequence in question. Lemma 4 is in these respects rather different. It is required for a part of our investigation (the estimation of Σ_E), where it is necessary to work in terms of *asymptotic equalities* that relate throughout to the *same sieve*. We use here a weak variant of the Brun sieve, in order to minimize the complications caused by our special requirements. Lemma 4 is thus imperfect in that it does not imply the best possible upper bounds; it is, however, quite adequate for our purposes, and any improvement would have a negligible effect on the error term in our final result. Lemma 5 belongs to the more conventional category of results.

Lemmata on sieve method

We commence by defining a sieve and then develop briefly its analytical formulation.

Let
$$x = n^{\frac{1}{(\log\log n)^2}}; \qquad P = \prod_{p \le x} p;$$

and d_1 a typical divisor of P. Also for any positive integer represented by an arbitrary letter t, let
$$t^{(1)} = \prod_{p|t;\ p \le x} p^{\alpha}; \qquad t^{(2)} = \prod_{p|t;\ p > x} p^{\alpha},$$
where
$$t = \prod p^{\alpha}.$$

We use the familiar relation
$$\sum_{d|\nu} \mu(d) = \begin{cases} 1, & \text{if } \nu = 1, \\ 0, & \text{if } \nu > 1. \end{cases} \tag{15}$$

We define the function $f(\nu) \equiv f_n(\nu)$ by the equation
$$f(\nu) = g(\nu) + h(\nu),$$
where
$$g(\nu) = \begin{cases} 1, & \text{if } \nu \text{ is a prime not exceeding } x, \\ 0, & \text{otherwise,} \end{cases}$$
and
$$h(\nu) = \begin{cases} 1, & \text{if } \nu \text{ is prime to } P, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $f(\nu)$ is a non-negative function which equals 1 when ν is a prime number. In virtue of (15),
$$h(\nu) = \sum_{d|\nu^{(1)}} \mu(d) = \sum_{d_1|\nu} \mu(d_1). \tag{16}$$

To use this formula, however, would involve considerable complications. Instead we shall approximate to $h(\nu)$ by a formula similar to (16) but much easier to apply to our problem.

For any r, let $s_r(\nu)$ be given by
$$s_r(\nu) = \sum_{\substack{d|\nu \\ \omega(d) \le r}} \mu(d).$$

Then we can shew that

$$s_r(\nu)\begin{cases} =1, & \text{if } \nu=1, \\ \geq 0, & \text{if } \nu>1 \text{ and } r \text{ even}, \\ \leq 0, & \text{if } \nu>1 \text{ and } r \text{ odd}. \end{cases} \tag{17}$$

This special case of Brun's basic formula admits of a particularly simple proof. The case $\nu=1$ is trivial. For the case $\nu>1$, we are content to give the proof for r even, as the proof for r odd is similar. Let then r be even. If $\omega(\nu)=t\geq 2r$, then

$$s_r(\nu)=\sum_{s=0}^{r}\binom{t}{s}(-1)^s>0,$$

since $\binom{t}{s}$ is an increasing function of s for $s\leq r$. If $\omega(\nu)=t<2r$, then, by (15),

$$\begin{aligned} s_r(\nu) &= -\sum_{\substack{d\mid\nu \\ \omega(d)>r}}\mu(d) \quad \text{(where the sum is possibly empty)} \\ &= -\sum_{s=r+1}^{t}\binom{t}{s}(-1)^s \\ &= \sum_{s=r+1}^{t}\binom{t}{s}(-1)^{s-1}\geq 0, \end{aligned}$$

since $\binom{t}{s}$ is a decreasing function of s for $s>r$.

We now have for any r

$$\sum_{d\mid\nu}\mu(d)=\sum_{\substack{d\mid\nu \\ \omega(d)\leq r}}\mu(d)+O\Bigg(\sum_{\substack{d\mid\nu \\ \omega(d)=r+1}}1\Bigg), \tag{18}$$

since, by (17), the left-hand side of the above equation lies betweeen $s_r(\nu)$ and $s_{r+1}(\nu)$. We deduce from (16) and (18)

$$h(\nu)=\sum_{\substack{d_1\mid\nu \\ \omega(d_1)\leq r}}\mu(d_1)+O\Bigg(\sum_{\substack{d_1\mid\nu \\ \omega(d_1)=r+1}}1\Bigg). \tag{19}$$

LEMMA 4. *Let $y\leq n$ and $k=O(n^\theta)$, where θ is a positive absolute constant less than* 1. *Then*

$$\sum_{\substack{\nu\leq y \\ \nu\equiv a\,(\mathrm{mod}\,k)}} f(\nu)=\begin{cases} \dfrac{1}{\varphi(k)}B(n)\,y+O\left(\dfrac{n}{k\log^5 n}\right), & \text{if } (a^{(1)},k)=1, \\ O\left(\dfrac{n}{k\log^5 n}\right), & \text{if } (a^{(1)},k)>1, \end{cases}$$

where $B(n)$ *depends only on* n *and satisfies*

$$B(n)=O\left(\frac{(\log\log n)^2}{\log n}\right).$$

In the above conditions $(a^{(1)}, k)$ *may be replaced by* $(a, k^{(1)})$.

Firstly, we have

$$\sum_{\substack{\nu\le y\\ \nu\equiv a\ (\mathrm{mod}\ k)}} g(\nu)=O\left(\sum_{\nu\le x}1\right)=O(x)=O\left(\frac{n}{k\log^5 n}\right). \tag{20}$$

Secondly, suppose $(a^{(1)}, k)=1$. Then, by (19),

$$\sum_{\substack{\nu\le y\\ \nu\equiv a\ (\mathrm{mod}\ k)}} h(\nu)=\sum_{\substack{\nu\le y\\ \nu\equiv a\ (\mathrm{mod}\ k)}}\left\{\sum_{\substack{d_1\mid\nu\\ \omega(d_1)\le r}}\mu(d_1)+O\left(\sum_{\substack{d_1\mid\nu\\ \omega(d_1)=r+1}}1\right)\right\}=\sum_{\omega(d_1)\le r}\mu(d_1)\sum_{\nu}1+O\left(\sum_{\omega(d_1)=r+1}\sum_{\nu}1\right), \tag{21}$$

where, in the inner summations on the right-hand side, ν satisfies the conditions

$$\nu\le y,\ \nu\equiv a\ (\mathrm{mod}\ k),\ \nu\equiv 0\ (\mathrm{mod}\ d_1).$$

Now the simultaneous congruence $\nu\equiv a\ (\mathrm{mod}\ k)$, $\nu\equiv 0\ (\mathrm{mod}\ d_1)$ has a solution, if and only if $(d_1, k)\mid a$, in which case the solutions form a residue class mod $([d_1, k])$. This condition is equivalent to $(d_1, k)\mid a^{(1)}$, which in turn is equivalent to $(d_1, k)=1$, since $(a^{(1)}, k)=1$. Hence the right-hand side of (21) becomes

$$\sum_{\substack{(d_1,k)=1\\ \omega(d_1)\le r}}\mu(d_1)\left(\frac{y}{k\,d_1}+O(1)\right)+O\left\{\sum_{\substack{(d_1,k)=1\\ \omega(d_1)=r+1}}\left(\frac{y}{k\,d_1}+O(1)\right)\right\}$$

$$=\frac{y}{k}\sum_{(d_1,k)=1}\frac{\mu(d_1)}{d_1}+O\left(\frac{y}{k}\sum_{\omega(d_1)>r}\frac{1}{d_1}\right)+O\left(\sum_{\omega(d_1)\le r+1}1\right)=\frac{y}{k}\Sigma_1+O\left(\frac{y}{k}\Sigma_2\right)+O(\Sigma_3),\ \text{say}. \tag{22}$$

We now choose $r=[10\log\log n]+1.$

We have

$$\Sigma_1=\prod_{\substack{p\le x\\ (p,k)=1}}\left(1-\frac{1}{p}\right)=\prod_{p\le x}\left(1-\frac{1}{p}\right)\prod_{p\mid k}\left(1-\frac{1}{p}\right)^{-1}\prod_{\substack{p\mid k\\ p>x}}\left(1-\frac{1}{p}\right)$$

$$=\frac{k}{\varphi(k)}B(n)\prod_{\substack{p\mid k\\ p>x}}\left(1-\frac{1}{p}\right),\ \text{say}, \tag{23}$$

where, by Mertens' formula,

$$B(n)\sim\frac{e^{-\gamma}}{\log x}=O\left(\frac{(\log\log n)^2}{\log n}\right). \tag{24}$$

Now $$\log \prod_{\substack{p\mid k\\ p>x}} \left(1-\frac{1}{p}\right) = \sum_{\substack{p\mid k\\ p>x}} \log\left(1-\frac{1}{p}\right) = O\left(\sum_{\substack{p\mid k\\ p>x}} \frac{1}{p}\right) = O\left(\frac{\log n}{x}\right),$$

since k has at most $\log_2 k$ prime factors; and so

$$\prod_{\substack{p\mid k\\ p>x}} \left(1-\frac{1}{p}\right) = 1 + O\left(\frac{\log n}{x}\right) = 1 + O\left(\frac{1}{\log^5 n}\right). \tag{25}$$

We deduce from (23), (24) and (25)

$$\Sigma_1 = \frac{k}{\varphi(k)} B(n) + O\left(\frac{k}{\varphi(k)} \cdot \frac{(\log\log n)^2}{\log^6 n}\right) = \frac{k}{\varphi(k)} B(n) + O\left(\frac{1}{\log^5 n}\right). \tag{26}$$

We have

$$\Sigma_2 = \sum_{s>r} \sum_{\omega(d_1)=s} \frac{1}{d_1} = O\left\{\sum_{s\geq r} \frac{1}{s!}\left(\sum_{p<n} \frac{1}{p}\right)^s\right\} = O\left\{\sum_{s\geq r} \frac{1}{s!} (\log\log n + A_3)^s\right\}.$$

By Stirling's formula $$\frac{1}{s!} = O\left\{\frac{1}{\sqrt{s}} \left(\frac{e}{s}\right)^s\right\},$$

and so, if $s \geq r$,

$$\frac{1}{s!} = O\left\{\left(\frac{e}{10 \log\log n}\right)^s\right\} = O\left\{\left(\frac{1}{e(\log\log n + A_3)}\right)^s\right\}.$$

Hence $$\Sigma_2 = O\left\{\sum_{s\geq r} \left(\frac{1}{e}\right)^s\right\} = O\left\{\left(\frac{1}{e}\right)^r\right\} = O\left(\frac{1}{\log^{10} n}\right) = O\left(\frac{1}{\log^5 n}\right). \tag{27}$$

In Σ_3, we have $$d_1 \leq \left(n^{\frac{1}{(\log\log n)^2}}\right)^{\omega(d_1)} \leq n^{\frac{11}{\log\log n}}.$$

Hence $$\Sigma_3 = O\left(n^{\frac{11}{\log\log n}}\right) = O\left(\frac{n}{k \log^5 n}\right). \tag{28}$$

From (22), (26), (27) and (28), we deduce the first part of

$$\sum_{\substack{\nu\equiv a \pmod k\\ \nu\leq y}} h(\nu) = \begin{cases} \dfrac{1}{\varphi(k)} B(n)\, y + O\left(\dfrac{n}{k \log^5 n}\right), & \text{if } (a^{(1)}, k) = 1, \\ 0, & \text{if } (a^{(1)}, k) > 1. \end{cases} \tag{29}$$

The second part is trivial, since $(a^{(1)}, k) > 1$ implies $(\nu, P) > 1$.

The first part of the lemma follows from (20) and (29); the second part from $(a^{(1)}, k) = (a, k^{(1)})$.

LEMMA 5. *For any $r < \frac{1}{2} n$, the number of solutions of the equation*

$$n - p = p' r; \quad p < n;$$

in primes p, p' is $$O\left(\frac{n}{r} \cdot \frac{\log\log n}{\log^2 (n/r)}\right).$$

This lemma is analogous to a result of Erdös [3], and the proof almost identical.

If $(n, r) = 1$, an immediate application of Brun's method gives the number of solutions as

$$O\left\{\frac{n}{r}\prod_{\substack{p \nmid 2nr \\ p<(n/r)^{A_4}}}\left(1-\frac{2}{p}\right)\prod_{\substack{p \mid nr \\ p<(n/r)^{A_4}}}\left(1-\frac{1}{p}\right)\right\} \qquad (A_4<1)$$

$$= O\left\{\frac{n}{r}\prod_{p<(n/r)^{A_4}}\left(1-\frac{1}{p}\right)^2\prod_{\substack{p \mid nr \\ p<(n/r)^{A_4}}}\left(1-\frac{1}{p}\right)^{-1}\right\} = O\left\{\frac{n}{r}\cdot\frac{\log\log n}{\log^2 (n/r)}\right\}.$$

If $(n, r) > 1$, the lemma is trivial, since then the number of solutions is at most 1.

Further lemmata

We require a lemma concerning the distribution of numbers m, for which the value of $\Omega(m)$ is restricted by certain conditions. A method of Hardy and Ramanujan [8] is applicable to problems of this nature. We prefer, however, to use another method which, though non-elementary, has the advantage of requiring less computation. Also, for problems concerning $\Omega(m)$ our method appears to yield more accurate estimations, although for those concerning $\omega(m)$ the two methods are equivalent in power.

LEMMA 6. *If* $\frac{1}{2} \le a \le \frac{7}{4}$, *then*

$$\sum_{m \le y} a^{\Omega(m)} = O(y \log^{a-1} 2y).$$

We merely indicate the proof, as it follows very closely that of a theorem due to Ramanujan and Wilson [14]. For $R(z) > 1$,

$$\sum_{m=1}^{\infty}\frac{a^{\Omega(m)}}{m^z} = \prod_{p=2}^{\infty}\left(1-\frac{a}{p^z}\right)^{-1},$$

since the infinite product is absolutely convergent if $a < 2$. This product equals $[\zeta(z)]^a f(z)$, where $f(z)$ is given by a Dirichlet series that converges absolutely for $R(z) > 1-\theta$, where θ is a positive absolute constant. Hence

$$\sum_{m \le y} a^{\Omega(m)} = A_1(a)\, y \log^{a-1} y + O(y \log^{a-2} y) \qquad (\text{as } y \to \infty),$$

from which the lemma follows.

LEMMA 7. *If* $\frac{1}{2} \le \alpha < 1 < \beta \le \frac{3}{2}$ *and* $y > e^e$, *then*

(a) $$\sum_{\substack{y^{\frac{1}{2}}\log^{-6} y < m < y^{\frac{1}{2}}\log^3 y \\ \Omega(m) \le \alpha \log\log y}} \frac{1}{m} = O(\log^{\gamma_\alpha - 1} y \log\log y),$$

(b) $$\sum_{\substack{m\leq y \\ \Omega(m)\geq\beta\log\log y-1}} \frac{1}{m} = O(\log^{\gamma_\beta} y),$$

where $$\gamma_c = c - c\log c.$$

If $\frac{1}{2}\leq a\leq 1$, then by Lemma 6 and partial summation,

$$\sum_{y^{\frac{1}{2}}\log^{-5} y<m<y^{\frac{1}{2}}\log^3 y} \frac{a^{\Omega(m)}}{m} = O(\log^{a-1} y\log\log y).$$

Hence

$$\sum_{\substack{y^{\frac{1}{2}}\log^{-5} y<m<y^{\frac{1}{2}}\log^3 y \\ \Omega(m)\leq\alpha\log\log y}} \frac{1}{m} \leq a^{-\alpha\log\log y} \sum_{y^{\frac{1}{2}}\log^{-5} y<m<y^{\frac{1}{2}}\log^3 y} \frac{a^{\Omega(m)}}{m} = O(\log^{\gamma_{\alpha,a}-1} y\log\log y), \quad (30)$$

where $$\gamma_{c,a} = a - c\log a.$$

Similarly, if $1\leq a\leq\frac{3}{2}$,

$$\sum_{\substack{m\leq y \\ \Omega(m)\geq\beta\log\log y-1}} \frac{1}{m} \leq a^{1-\beta\log\log y}\sum_{m\leq y}\frac{a^{\Omega(m)}}{m} = O(\log^{\gamma_{\beta,a}} y). \quad (31)$$

Now for c fixed, $\gamma_{c,a}$ attains its minimum $c-c\log c$ when $a=c$. The lemma is thus true, since we may legitimately set $a=\alpha$ and $a=\beta$ in (30) and (31) respectively.

LEMMA 8. *If* $(rs,m)=1$ *and* $r,s,m\leq n$, *then*

$$\sum_{\substack{l\geq y \\ (l,ms)=1}} \frac{\chi(l)}{\varphi(rsl)} = O\{\log\log n\cdot R_m(r;s;y)\} + O\left(\log\log n\frac{\sigma_{-1}(s)}{rs}\sigma_{-1}(m;y)\right) + O\left(\frac{(\log\log n)^2}{rsy}\right),$$

where $$R_m(r;s;y) = \frac{\log 2y}{y}\frac{d(s)}{rs}d(m;y).$$

The proof is similar to that of Lemma 3. We use the relation

$$\frac{1}{\varphi(rsl)} = \frac{1}{l\varphi(rs)}\prod_{\substack{p\mid l \\ p\nmid r}}\left(1-\frac{1}{p}\right)^{-1},$$

valid for $(l,s)=1$. This gives rise to the identity

$$\frac{1}{\varphi(rsl)} = \frac{1}{\varphi(rs)}\cdot\frac{1}{l}\sum_{\substack{q\mid l \\ (q,r)=1}}\frac{1}{\varphi(q)},$$

where q denotes a general square-free number. Therefore

$$\varphi(rs)\sum_{\substack{l\geq y \\ (l,ms)=1}}\frac{\chi(l)}{\varphi(rsl)} = \sum_{\substack{qt\geq y \\ (q,rsm)=1 \\ (t,sm)=1}}\frac{\chi(qt)}{qt\varphi(q)}$$

$$= \sum_{\substack{q \le y \\ (q, r s m)=1}} \frac{\chi(q)}{q \varphi(q)} \sum_{\substack{t \ge y/q \\ (t, s m)=1}} \frac{\chi(t)}{t} + \sum_{(t, s m)=1} \frac{\chi(t)}{t} \sum_{\substack{q > y \\ (q, r s m)=1}} \frac{\chi(q)}{q \varphi(q)};$$

and this, by Lemma 2, is equal to

$$O\left\{\sum_{q \le y} \frac{1}{q \varphi(q)} \cdot \frac{q}{y} d\left(m s; \frac{y}{q}\right)\right\} + O\left\{\sum_{q \le y} \frac{1}{q \varphi(q)} \sigma_{-1}\left(m s; \frac{y}{q}\right)\right\} + O\left\{\frac{\log \log n}{y}\right\}$$

$$= O\left\{\frac{\log 2 y}{y} d(s) d(m; y)\right\} + O\left\{\sum_{l \le y} \frac{1}{l \varphi(l)} \sigma_{-1}\left(m s; \frac{y}{l}\right)\right\} + O\left\{\frac{\log \log n}{y}\right\}. \qquad (32)$$

Now

$$\sum_{l \le y} \frac{1}{l \varphi(l)} \sigma_{-1}\left(m s; \frac{y}{l}\right) = O\left(\sum_{l=1}^{\infty} \frac{1}{l \varphi(l)} \sum_{\substack{d \mid m s \\ d > y/l}} \frac{1}{d}\right) = O\left(\sum_{\substack{d \mid m s \\ d \le y}} \frac{1}{d} \sum_{l > y/d} \frac{1}{l \varphi(l)}\right) + O\left(\sum_{\substack{d \mid m s \\ d > y}} \frac{1}{d} \sum_{l=1}^{\infty} \frac{1}{l \varphi(l)}\right)$$

$$= O\left(\frac{1}{y} d(m s; y)\right) + O\{\sigma_{-1}(m s; y)\}$$

$$= O\left(\frac{1}{y} d(s) d(m; y)\right) + O\{\sigma_{-1}(m s; y)\}. \qquad (33)$$

We transform $\sigma_{-1}(m s; y)$. If $\mu \mid m s$, then $\mu = \mu_1 \mu_2$, where $\mu_1 \mid m$ and $\mu_2 \mid s$. Hence

$$\sigma_{-1}(m s; y) \le \sum_{\mu_1 \mid m} \frac{1}{\mu_1} \sum_{\substack{\mu_2 \mid s \\ \mu_2 > y/\mu_1}} \frac{1}{\mu_2} = \sum_{\substack{\mu_1 \mid m \\ \mu_1 \le y}} \frac{1}{\mu_1} \sum_{\substack{\mu_2 \mid s \\ \mu_2 > y/\mu_1}} \frac{1}{\mu_2} + \sum_{\substack{\mu_1 \mid m \\ \mu_1 > y}} \frac{1}{\mu_1} \sum_{\mu_2 \mid s} \frac{1}{\mu_2}$$

$$= \sum_{\substack{\mu_1 \mid m \\ \mu_1 \le y}} \frac{1}{\mu_1} O\left(\frac{\mu_1}{y} d(s)\right) + \sigma_{-1}(m; y) \sigma_{-1}(s)$$

$$= O\left(\frac{d(s) d(m; y)}{y}\right) + O\{\sigma_{-1}(s) \sigma_{-1}(m; y)\}. \qquad (34)$$

The lemma follows from (32), (33), (34) and the relation $r s / \varphi(r s) = O(\log \log n)$.

LEMMA 9. *If*[1] $0 < u < n$; $u' \ge u$; *and* $m \le n$, *then we have*

(a) $$\sum_{d \le u} \cdot \sum_{u/d < l < (u \log^6 n)/d} R_m\left(d; l; \frac{u'}{d}\right) = O\{(\log \log n)^4\},$$

(b) $$\sum_{d \le u} \sum_{u/d < l < (u \log^6 n)/d} \frac{\sigma_{-1}(l)}{d l} \sigma_{-1}\left(m; \frac{u'}{d}\right) = O\{(\log \log n)^3\},$$

(c) $$\sum_{d \le u} \sum_{u/d < l < (u \log^6 n)/d} \frac{d}{u} \cdot \frac{1}{d l} = O\{\log \log n\}.$$

[1] Here, in accordance with the notation introduced in Section 1, u, u' are not necessarily integral.

The first sum is equal to

$$\sum_{d \le u} \frac{d}{u'} \log \frac{2u'}{d} d\left(m; \frac{u'}{d}\right) \sum_{u/d < l < (u \log^6 n)/d} \frac{d(l)}{dl}$$

$$= O\left\{\frac{1}{u'} \sum_{d \le u} \log \frac{2u'}{d} d\left(m; \frac{u'}{d}\right) \left[\log\left(\frac{u \log^6 n}{d}\right) \log\log n\right]\right\}$$

$$= O\left\{\frac{1}{u'} \sum_{d \le u'} \log \frac{2u'}{d} d\left(m; \frac{u'}{d}\right) \left[\log\left(\frac{u' \log^6 n}{d}\right) \log\log n\right]\right\}$$

$$= O\left\{\frac{\log\log n}{u'} \sum_{d \le u'} \log^2 \frac{2u'}{d} d\left(m; \frac{u'}{d}\right)\right\} + O\left\{\frac{(\log\log n)^2}{u'} \sum_{d \le u'} \log \frac{2u'}{d} d\left(m; \frac{u'}{d}\right)\right\}. \tag{35}$$

Now for $k = 1$ or 2, we have

$$\sum_{d \le u'} \log^k \frac{2u'}{d} d\left(m; \frac{u'}{d}\right) = \sum_{\substack{l \mid m \\ l \le u'}} \sum_{d \le \frac{u'}{l}} \log^k \frac{2u'}{d} = O\left(\sum_{\substack{l \mid m \\ l \le u'}} \frac{u'}{l} \log^k 2l\right) = O\left(u' \sum_{l \mid m} \frac{\log^k 2l}{l}\right). \tag{36}$$

Also, as we shall shew below,

$$\sum_{l \mid m} \frac{\log^k 2l}{l} = O\{(\log\log 10m)^{k+1}\}. \tag{37}$$

The first part of the lemma follows from (35), (36) and (37).

We are content to sketch the proof of (37), as it depends essentially on a method due to Ingham [9]. Let

$$G(z) = \prod_{p \mid m} \left(1 - \frac{1}{p^z}\right)^{-1}.$$

Then clearly $$\sum_{l \mid m} \frac{\log^k 2l}{l} < 1 + A_5 \sum_{l \mid m} \frac{\log^k l}{l} \le 1 + A_5 (-1)^k G^{(k)}(1).$$

An easy calculation shews that

$$-G'(1) = \prod_{p \mid m} \left(1 - \frac{1}{p}\right)^{-1} \sum_{p \mid m} \left(1 - \frac{1}{p}\right)^{-1} \frac{\log p}{p}$$

and $$G''(1) = \prod_{p \mid m} \left(1 - \frac{1}{p}\right)^{-1} : \left\{\left[\sum_{p \mid m} \left(1 - \frac{1}{p}\right)^{-1} \frac{\log p}{p}\right]^2 + \sum_{p \mid m} \left(1 - \frac{1}{p}\right)^{-2} \frac{\log^2 p}{p}\right)\right\}.$$

But $$\sum_{p \mid m} \frac{\log^k p}{p} \le \sum_{p \le \log^k 10m} \frac{\log^k p}{p} + \frac{1}{\log^k 10m} \sum_{p \mid m} \log^k p$$

$$\le \sum_{p \le \log^k 10m} \frac{\log^k p}{p} + \frac{1}{\log^k 10m} \left(\sum_{p \mid m} \log p\right)^k$$

$$= O\{(\log\log 10m)^k\}.$$

Therefore $$(-1)^k G^{(k)}(1) = O\{(\log\log 10\, m)^{k+1}\},$$

and so (37) is established.

The second sum is equal to

$$\sum_{d\le u} \frac{1}{d}\sigma_{-1}\left(m; \frac{u'}{d}\right) \sum_{u/d<l<(u\log^6 n)/d} \frac{\sigma_{-1}(l)}{l} = O\left\{\log\log n \sum_{d\le u}\frac{1}{d}\sigma_{-1}\left(m;\frac{u}{d}\right)\right\},$$

by (4) and partial summation. It is therefore equal to

$$O\left(\log\log n \sum_{l|m}\frac{1}{l}\sum_{u/l<d\le u}\frac{1}{d}\right).$$

If $l\le u$, the inner sum is $O(\log 2l)$; if $l>u$, it is $O(\log 2u) = O(\log 2l)$. Hence the double sum is

$$O\left(\log\log n \sum_{l|m}\frac{\log 2l}{l}\right) = O\{(\log\log n)^3\},$$

by (37).

The third part of the lemma is almost immediate.

Inequality for Σ_B

We are now in a position to commence our assessment of Σ_B.

Let $$D(m) = \sum_{\substack{l|m \\ n^{\frac{1}{2}}\log^{-3} n<l<n^{\frac{1}{2}}\log^3 n}} 1 \quad\text{and}\quad F(m) = \sum_{\substack{l|m \\ n^{\frac{1}{2}}\log^{-3} n<l<n^{\frac{1}{2}}\log^3 n}} \chi(l).$$

Then $$\Sigma_B = \sum_{p<n} F(n-p) = \sum_{\substack{p<n \\ D(n-p)\neq 0}} F(n-p).$$

Therefore, by the Cauchy–Schwarz inequality,

$$\Sigma_B = O\left\{\left(\sum_{\substack{p<n \\ D(n-p)\neq 0}} 1\right)^{\frac{1}{2}}\left(\sum_{p<n} F^2(n-p)\right)^{\frac{1}{2}}\right\} = O\{(\Sigma_D)^{\frac{1}{2}}(\Sigma_E)^{\frac{1}{2}}\},\ \text{say.} \tag{38}$$

Estimation of Σ_D

Let α satisfy the inequality $1<\alpha\le\frac{3}{2}$. We have

$$\Sigma_D \le \sum_{\substack{p<n \\ \Omega(n-p)\le\alpha\log\log n}} D(n-p) + \sum_{\substack{p<n \\ \Omega(n-p)>\alpha\log\log n}} 1 = \Sigma_{D,1}+\Sigma_{D,2},\ \text{say.} \tag{39}$$

We estimate $\Sigma_{D,1}$ first. Since here, and also later, sums over complicated ranges of the variables will occur, we shall adopt an abbreviated notation for some of the more lengthy conditions of summation. When possible, a capital letter will be used to denote a condition satisfied by the corresponding small letter. We now define

(L), (M), (P) to be the conditions $n^{\frac{1}{2}}\log^{-3}n<l<n^{\frac{1}{2}}\log^{3}n$, $n^{\frac{1}{2}}\log^{-5}n<m<n^{\frac{1}{2}}\log^{3}n$, $lm=n-p$, respectively. If (P) holds, the conditions of summation in $\Sigma_{D,1}$ imply that at least one of $\Omega(l)$ and $\Omega(m)$ does not exceed $\frac{1}{2}\alpha\log\log n$. Therefore

$$\Sigma_{D,1}\leq\sum_{\substack{(L),(P)\\ \Omega(l)\leq\frac{1}{2}\alpha\log\log n}}1+\sum_{\substack{(L),(P)\\ \Omega(m)\leq\frac{1}{2}\alpha\log\log n}}1.$$

In the second sum the conditions of summation imply $m<n^{\frac{1}{2}}\log^{3}n$. Therefore

$$\Sigma_{D,1}\leq\sum_{\substack{(L),(P)\\ \Omega(l)\leq\frac{1}{2}\alpha\log\log n}}1+\sum_{\substack{(M),(P)\\ \Omega(m)\leq\frac{1}{2}\alpha\log\log n}}1+\sum_{\substack{l<n^{\frac{1}{2}}\log^{3}n\\ m\leq n^{\frac{1}{2}}\log^{5}n}}1=\Sigma_1+\Sigma_2+\Sigma_3,\text{ say.}\tag{40}$$

Clearly
$$\Sigma_1=O(\Sigma_2)$$
and
$$\Sigma_3=O\left(\frac{n}{\log^2 n}\right).\tag{41}$$

Now
$$\Sigma_2=\sum_{\substack{(M)\\ \Omega(m)\leq\frac{1}{2}\alpha\log\log n}}\sum_{\substack{p\equiv n\,(\mathrm{mod}\,m)\\ p<n}}1=O\Bigg(\sum_{\substack{(M)\\ \Omega(m)\leq\frac{1}{2}\alpha\log\log n}}\frac{n(\log\log n)^2}{\varphi(m)\log n}\Bigg),$$

be Lemma 4. Hence

$$\Sigma_2=O\Bigg(\frac{n(\log\log n)^3}{\log n}\sum_{\substack{(M)\\ \Omega(m)\leq\frac{1}{2}\alpha\log\log n}}\frac{1}{m}\Bigg)=O\left(\frac{n}{\log n}\log^{\gamma_{\frac{1}{2}\alpha}-1}n\,(\log\log n)^4\right),\tag{42}$$

by Lemma 7, since $\frac{1}{2}<\frac{1}{2}\alpha\leq\frac{3}{4}$. Because $\gamma_{\frac{1}{2}\alpha}>0$, we deduce from (40), (41) and (42)

$$\Sigma_{D,1}=O\left(\frac{n}{\log n}\log^{\gamma_{\frac{1}{2}\alpha}-1}n\,(\log\log n)^4\right).\tag{43}$$

Our estimation of $\Sigma_{D,2}$, with its dependence on Lemma 5, is due essentially to Erdös [3].

$$\Sigma_{D,2}\leq\sum_{\substack{m<n\\ \Omega(m)>10\log\log n}}1+\sum_{\substack{p<n\\ \alpha\log\log n<\Omega(n-p)\leq 10\log\log n}}1=\Sigma_4+\Sigma_5,\text{ say.}\tag{44}$$

We have
$$\Sigma_4<\frac{1}{\log^5 n}\sum_{m<n}(\sqrt{e})^{\Omega(m)}=O(n\log^{\sqrt{e}-6}n),$$

by Lemma 6, since $\frac{1}{2}<\sqrt{e}<\frac{7}{4}$; so
$$\Sigma_4=O\left(\frac{n}{\log^2 n}\right).\tag{45}$$

Let R_n be the set of numbers m which are less than n and which are such that they have no prime factor exceeding $n^{\frac{1}{20\log\log n}}$. Then

$$\Sigma_5 \leq \sum_{\substack{m \in R_n \\ \Omega(m) \leq 10 \log\log n}} 1 + \sum_{\substack{n-p \notin R_n \\ \Omega(n-p) > \alpha \log\log n}} 1 = \Sigma_6 + \Sigma_7, \text{ say.} \tag{46}$$

Now if $\Omega(m) \leq 10 \log\log n$ and $m \in R_n$, then $m \leq n^{\frac{1}{2}}$. Therefore

$$\Sigma_6 = O(n^{\frac{1}{2}}). \tag{47}$$

If $n-p \notin R_n$ and $\Omega(n-p) > \alpha \log\log n$, then $n-p$ has at least one representation in the form $r p'$, where $p' > n^{1/(20 \log\log n)}$ and $\Omega(r) > \alpha \log\log n - 1$; in such a representation $r < n^{1-1/(20\log\log n)}$. Therefore

$$\Sigma_7 \leq \sum_{\substack{r < n^{1-1/20 \log\log n} \\ \Omega(r) > \alpha \log\log n - 1}} \sum_{\substack{p<n \\ n-p=rp'}} 1 = O\left(\sum_{\substack{r < n^{1-1/20 \log\log n} \\ \Omega(r) > \alpha \log\log n - 1}} \frac{n (\log\log n)^3}{r \log^2 n} \right),$$

by Lemma 5; so

$$\Sigma_7 = O\left(\frac{n(\log\log n)^3}{\log^2 n} \sum_{\substack{r \leq n \\ \Omega(r) > \alpha \log\log n - 1}} \frac{1}{r} \right) = O\left(\frac{n}{\log n} \log^{\gamma_\alpha - 1} n \, (\log\log n)^3 \right), \tag{48}$$

by Lemma 7. Since $\gamma_\alpha > 0$, (44), (45), (46), (47) and (48) imply

$$\Sigma_{D,2} = O\left(\frac{n}{\log n} \log^{\gamma_\alpha - 1} n \, (\log\log n)^3 \right). \tag{49}$$

If we choose α so that $\gamma_\alpha = \gamma_{\frac{1}{2}\alpha}$, then

$$\alpha - \alpha \log \alpha = \tfrac{1}{2}\alpha - \tfrac{1}{2}\alpha \log(\tfrac{1}{2}\alpha),$$

giving

$$\log \alpha = 1 - \log 2; \; \alpha = \tfrac{1}{2} e,$$

and the condition $1 < \alpha \leq \frac{3}{2}$ is satisfied. Also

$$\gamma_\alpha = \tfrac{1}{2} e \log 2.$$

We now deduce from (39), (43) and (49)

$$\Sigma_D = O\left(\frac{n}{\log n} \log^{-\gamma} n \, (\log\log n)^4 \right), \tag{50}$$

where

$$\gamma = 1 - \tfrac{1}{2} e \log 2 \qquad (>0).$$

Estimation of Σ_E

We have

$$\Sigma_E = \sum_{p<n} F^2(n-p) \leq \sum_{\nu<n} F^2(n-\nu) f(\nu) = \sum_{\substack{l_1' m_1 = l_2' m_2 = n-\nu \\ n^{\frac{1}{2}} \log^{-3} n < l_1', l_2' < n^{\frac{1}{2}} \log^3 n}} \chi(l_1') \chi(l_2') f(\nu).$$

Now, for given l_1', l_2', the values of $n-\nu$ in the above sum are equivalent to 0 $(\mathrm{mod}\,[l_1', l_2'])$. Furthermore, if $(l_1', l_2')=d$, then $l_1'=d\,l_1$, $l_2'=d\,l_2$ and $[l_1', l_2']=d\,l_1\,l_2$, where $(l_1, l_2)=1$. We now define (L_i), (H), (K), (K_1) by $(L_i)\equiv\{(n^{\frac{1}{2}}\log^{-3}n)/d<l_i<(n^{\frac{1}{2}}\log^3 n)/d\}$, $(H)\equiv\{(l_1, l_2)=1\}$, $(K)\equiv\{(d\,l_1\,l_2, n^{(1)})=1\}$, $(K_1)\equiv\{(d\,l_1, n^{(1)})=1\}$. Hence

$$\Sigma_E=\sum_{\substack{l_1 l_2 d\, m=n-\nu\\ (L_1),(L_2),(H)}}\chi^2(d)\,\chi(l_1)\,\chi(l_2)\,f(\nu)=\sum_{d\geq n^{1/8}}+\sum_{d<n^{1/8}}=\Sigma_1+\Sigma_2,\ \text{say.} \tag{51}$$

In Σ_1, we have $l_1\,l_2\,d<(n\log^6 n)/d\leq n^{7/8}\log^6 n$. Hence, using Lemma 4,

$$\begin{aligned}\Sigma_1&=\sum_{\substack{(L_1),(L_2),(H)\\ d\geq n^{1/8}}}\chi^2(d)\,\chi(l_1)\,\chi(l_2)\sum_{\substack{\nu\equiv n\,(\mathrm{mod}\,d\,l_1\,l_2)\\ \nu<n}}f(\nu)\\
&=B(n)\,n\sum_{\substack{(L_1),(L_2),(H),(K)\\ d\geq n^{1/8}}}\frac{\chi^2(d)\,\chi(l_1)\,\chi(l_2)}{\varphi(d\,l_1\,l_2)}+O\left(\frac{n}{\log^5 n}\sum_{\substack{(L_1),(L_2),(H)\\ d\geq n^{1/8}}}\frac{1}{d\,l_1\,l_2}\right)\\
&=B(n)\,n\sum_{\substack{(L_1),(L_2),(H),(K)\\ n^{1/8}\leq d\leq n^{\frac{1}{2}}\log^{-3}n}}\frac{\chi^2(d)\,\chi(l_1)\,\chi(l_2)}{\varphi(d\,l_1\,l_2)}+\\
&\qquad+B(n)\,n\sum_{\substack{(L_1),(L_2),(H),(K)\\ n^{\frac{1}{2}}\log^{-3}n<d<n^{\frac{1}{2}}\log^3 n}}\frac{\chi^2(d)\,\chi(l_1)\,\chi(l_2)}{\varphi(d\,l_1\,l_2)}+O\left(\frac{n}{\log^5 n}\sum_{d,\,l_1,\,l_2<n}\frac{1}{d\,l_1\,l_2}\right)\\
&=B(n)\,n\,\Sigma_3+B(n)\,n\,\Sigma_4+O\left(\frac{n}{\log^2 n}\right),\ \text{say.}\end{aligned}\tag{52}$$

Now
$$\Sigma_3=\sum_{\substack{(L_1),(K_1)\\ n^{1/8}\leq d\leq n^{\frac{1}{2}}\log^{-3}n}}\chi^2(d)\,\chi(l_1)\sum_{\substack{(L_2)\\ (l_2,\,l_1\,n^{(1)})=1}}\frac{\chi(l_2)}{\varphi(d\,l_1\,l_2)}.$$

Substituting for the inner sum by Lemma 8, and then deleting the conditions (K_1) and $d\geq n^{1/8}$ from the outer summation, we obtain

$$\begin{aligned}\Sigma_3=O\Bigg\{\sum_{\substack{(L_1)\\ d\leq n^{\frac{1}{2}}\log^{-3}n}}&\Bigg(\log\log n\left[R_{n^{(1)}}\left(d;l_1;\frac{n^{\frac{1}{2}}\log^{-3}n}{d}\right)+R_{n^{(1)}}\left(d;l_1;\frac{n^{\frac{1}{2}}\log^3 n}{d}\right)\right]+\\
&+\log\log n\,\frac{\sigma_{-1}(l_1)}{d\,l_1}\left[\sigma_{-1}\left(n^{(1)};\frac{n^{\frac{1}{2}}\log^{-3}n}{d}\right)+\sigma_{-1}\left(n^{(1)};\frac{n^{\frac{1}{2}}\log^3 n}{d}\right)\right]+\\
&+(\log\log n)^2\,\frac{d}{n^{\frac{1}{2}}\log^{-3}n}\cdot\frac{1}{d\,l_1}\Bigg)\Bigg\}.\end{aligned}\tag{53}$$

If we set successively $m=n^{(1)}$, $u=n^{\frac{1}{2}}\log^{-3}n$, $u'=u$ and $m=n^{(1)}$, $u=n^{\frac{1}{2}}\log^{-3}n$, $u'=n^{\frac{1}{2}}\log^3 n$, the conditions of Lemma 9 are satisfied. Therefore, by (53),

$$\Sigma_3=O\{(\log\log n)^5\}+O\{(\log\log n)^4\}+O\{(\log\log n)^3\}=O\{(\log\log n)^5\}.\tag{54}$$

Also $$\Sigma_4 = O\left(\sum_{\substack{n^{\frac12}\log^{-3}n<d<n^{\frac12}\log^3 n\\ l_1, l_2<\log^6 n}} \frac{1}{\varphi(d\,l_1\,l_2)}\right) = O\left(\log\log n \sum_{\substack{n^{\frac12}\log^{-3}n<d<n^{\frac12}\log^3 n\\ l_1, l_2<\log^6 n}} \frac{1}{d\,l_1\,l_2}\right)$$
$$= O\{(\log\log n)^4\}. \quad (55)$$

We deduce from (52), (54) and (55)

$$\Sigma_1 = O\left(\frac{n}{\log n}(\log\log n)^7\right). \quad (56)$$

To estimate Σ_2 we use the fact that, for given l_1, l_2,

$$\sum_{\substack{r\ t=l_1\\ s\ t=l_2}} \mu(t) = \begin{cases} 1, & \text{if } (l_1, l_2)=1,\\ 0, & \text{if } (l_1, l_2)>1.\end{cases}$$

We define (R), (S), (D), $(D\,T)$ by $(R) \equiv \{(n^{\frac12}\log^{-3} n)/d\,t < r < (n^{\frac12}\log^3 n)/d\,t\}$, $(S) \equiv \{(n^{\frac12}\log^{-3} n)/d\,t < s < (n^{\frac12}\log^3 n)/d\,t\}$, $(D) \equiv \{d < n^{1/4}\}$, $(D\,T) \equiv \{d < n^{1/4},\ \ t < n^{1/8}\}$. We thus have

$$\Sigma_2 = \sum_{\substack{r\,s\,t^2\,d\,m=n-\nu\\ (R),(S),(D)}} \mu(t)\,\chi^2(t)\,\chi^2(d)\,\chi(r)\,\chi(s)\,f(\nu) = \sum_{t<n^{1/8}} + \sum_{t\geq n^{1/8}} = \Sigma_5 + \Sigma_6, \text{ say.} \quad (57)$$

Now in Σ_5 the conditions of summation imply

$$r\,t^2\,d\,m < \frac{n}{s} < \frac{n}{n^{\frac12}\,d^{-1}\,t^{-1}\log^{-3} n} = n^{\frac12}\,d\,t\log^3 n < n^{\frac78}\log^3 n. \quad (58)$$

So

$$\Sigma_5 = \sum_{\substack{r\,t^2\,d\,m<n^{\frac78}\log^3 n\\ (R),(D\,T)}} \mu(t)\,\chi^2(t)\,\chi^2(d)\,\chi(r) \sum_{\substack{\nu=n-r\,s\,t^2\,d\,m\\ (S)}} \chi(s)\,f(\nu) =$$
$$= O\Big\{\sum_{\substack{r\,t^2\,d\,m<n^{\frac78}\log^3 n\\ (R),(D\,T)}} \Big|\sum_{\substack{\nu=n-r\,s\,t^2\,d\,m\\ y_1\leq\nu<y_2}} \chi(s)\,f(\nu)\Big|\Big\}, \quad (59)$$

where $y_2 = \max(n - n^{\frac12}\,r\,t\,m\log^{-3} n, 1)$ and $y_1 = \max([n - n^{\frac12}\,r\,t\,m\log^3 n]+1, 1)$. If we set $\lambda = r\,t^2\,d\,m$, we have that $\nu = n - \lambda\,s$. Also $\chi(s)$ is 1 if $s\equiv 1 \pmod 4$, is -1 if $s \equiv 3 \pmod 4$, and is 0 otherwise. Hence the inner sum in (59) is

$$\sum_{\substack{\nu\equiv n-\lambda \,(\mathrm{mod}\ 4\lambda)\\ y_1\leq\nu<y_2}} f(\nu) - \sum_{\substack{\nu\equiv n-3\lambda \,(\mathrm{mod}\ 4\lambda)\\ y_1\leq\nu<y_2}} f(\nu), \quad (60)$$

where ν is the variable of summation. The conditions $(n-\lambda, 4\,\lambda^{(1)}) = 1$ and $(n-3\,\lambda, 4\,\lambda^{(1)})$ are equivalent, because each is equivalent to $(n - \lambda^{(1)}\,\lambda^{(2)}, 2\,\lambda^{(1)}) = 1$. Also, by (58), $4\,\lambda = O(n^{\frac78}\log^3 n)$. Therefore, by Lemma 4, the right-hand side of (60) is equal to

$$\begin{cases} \dfrac{y_2-y_1}{\varphi(4\,\lambda)}B(n) - \dfrac{y_2-y_1}{\varphi(4\,\lambda)}B(n) + O\left(\dfrac{n}{\lambda\log^5 n}\right) = O\left(\dfrac{n}{\lambda\log^5 n}\right), & \text{if } (n-\lambda, 4\,\lambda^{(1)}) = 1,\\ O\left(\dfrac{n}{\lambda\log^5 n}\right), & \text{if } (n-\lambda, 4\,\lambda^{(1)}) > 1.\end{cases}$$

Hence, by this and (59),

$$\Sigma_5 = O\left(\sum_{r,\,d,\,m,\,t^2\le n} \frac{n}{r\,d\,m\,t^2\log^5 n}\right) = O\left(\frac{n}{\log^2 n}\right). \tag{61}$$

Also

$$\Sigma_6 = O\left(\sum_{\substack{r\,s\,t^2\,d\,m\le n\\ t\ge n^{1/8}}} 1\right) = O\left(\sum_{t\ge n^{1/8}}\ \sum_{\mu\le n/t^2} d_4(\mu)\right)$$

$$= O\left(n\log^3 n \sum_{t\ge n^{1/8}} \frac{1}{t^2}\right) = O\left(n^{7/8}\log^3 n\right) = O\left(\frac{n}{\log^2 n}\right). \tag{62}$$

We deduce from (57), (61) and (62)

$$\Sigma_2 = O\left(\frac{n}{\log^2 n}\right). \tag{63}$$

Therefore, finally, by (51), (56) and (63),

$$\Sigma_E = O\left(\frac{n}{\log n}(\log\log n)^7\right). \tag{64}$$

Estimation of Σ_B

We deduce from (38), (50) and (64)

$$\Sigma_B = O\left(\frac{n}{\log n}\log^{-\delta} n\,(\log\log n)^{\frac{11}{2}}\right), \tag{65}$$

where

$$\delta = \tfrac{1}{2}\left(1-\tfrac{1}{2}e\log 2\right) \qquad (>0).$$

5. The final sum

The first part of our final result is now immediate from (1), (2), (10), (14) and (65).

THEOREM 1. *On the extended Riemann hypothesis, the number of representations of the integer n in the form*

$$n = p + u^2 + v^2$$

is equal to

$$\frac{\pi n}{\log n}\prod_{p>2}\left(1+\frac{\chi(p)}{p(p-1)}\right)\prod_{\substack{p\mid n\\ p\equiv 1 \pmod 4}}\frac{(p-1)^2}{p^2-p+1}\prod_{\substack{p\mid n\\ p\equiv 3 \pmod 4}}\frac{p^2-1}{p^2-p-1} + O\left(\frac{n}{\log n}\log^{-\delta} n\,(\log\log n)^{\frac{11}{2}}\right).$$

Every sufficiently large number n is the sum of a prime and two integral squares.

The second part of the theorem follows at once from the first part, since by Lemma 3 the explicit term in the formula for the number of representations is greater than $A_6\, n/\log n \log\log n$.

As stated in the introduction, the conjugate theorem may be proved by a similar method.

THEOREM 2. *On the extended Riemann hypothesis, we have*

$$\sum_{0<p+a\leq x} r\,(p+a) = \frac{\pi x}{\log x} \prod_{p>2} \left(1+\frac{\chi(p)}{p\,(p-1)}\right) \prod_{\substack{p \mid a \\ p\equiv 1 \pmod 4}} \frac{(p-1)^2}{p^2-p+1} \prod_{\substack{p \mid a \\ p\equiv 3 \pmod 4}} \frac{p^2-1}{p^2-p-1} + $$
$$+\, O\left(\frac{x}{\log x} \log^{-\delta} x\, (\log\log x)^5\right),$$

where a is a fixed non-zero integer.

There exist infinitely many primes of the form u^2+v^2+a.

Finally, it may be of interest to note that the numerical value of the constant δ is given approximately by $\delta = 0.0289\ldots > \frac{1}{35}$.

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