Isometry groups of simply connected manifolds of nonpositive curvature II

by

PATRICK EBERLEIN([1])

University of North Carolina
Chapel Hill, N.C., U.S.A.

Introduction

Let H denote a complete simply connected Riemannian manifold of nonpositive sectional curvature, and let $I(H)$ denote the group of isometries of H. In this paper we consider density properties of subgroups $D \subseteq I(H)$ that satisfy the *duality condition* (defined below). These density properties also yield characterizations of Riemannian symmetric spaces of noncompact type and results about lattices in H that strengthen several of the results of [11] and [15]. If H is a symmetric space of noncompact type and if D is a subgroup of $I_0(H)$, then the duality condition for D is implied by the Selberg property (S) for D [20, pp. 4–6] or [10]. A partial converse is obtained in [10]. It is an interesting question whether the two conditions are equivalent in this context.

Our density results are very similar to those of [5]. In Proposition 4.2 we obtain a differential geometric version of the Borel density theorem (cf. Corollary 4.2 of [5]): Let H admit no Euclidean de Rham factor, and let $G \subseteq I(H)$ be a subgroup whose normalizer D in $I(H)$ satisfies the duality condition. Then either (1) G is discrete or (2) there exist manifolds H_1, H_2 such that (a) H is isometric to the Riemannian product $H_1 \times H_2$, (b) H_1 is a symmetric space of noncompact type, (c) $(\bar{G})_0 = I_0(H_1)$ and (d) there exists a discrete subgroup $B \subseteq I(H_2)$, whose normalizer in $I(H_2)$ satisfies the duality condition, such that $I_0(H_1) \times B$ is a subgroup of $\bar{G}$ of finite index in $\bar{G}$. Using the result just quoted or the main theorem of section 3 we then obtain the following decomposition of a manifold H whose isometry group $I(H)$ satisfies the duality condition (Proposition 4.1): Let $I(H)$ satisfy the duality condition. Then there exist manifolds H_0, H_1 and H_2, two of which may have dimension zero, such that (1) H is isometric to

---

([1]) Supported in part by NSF Grant MCS-7901730.

the Riemannian product $H_0 \times H_1 \times H_2$, (2) H_0 is a Euclidean space, (3) H_1 is a symmetric space of noncompact type and (4) $I(H_2)$ is discrete but satisfies the duality condition.

We now define both the duality condition and the property (S) in a way that will make apparent the close relationship of the two conditions. If G is a topological group then a subgroup Γ of G is said to satisfy property (S) in G if for every neighborhood U of the identity in G and every element of $g \in G$ there exists an integer $n>0$ such that $g^n \in U \cdot \Gamma \cdot U$ [5]. Following [12] one may define the property (S) equivalently as follows. Let $\Gamma \subseteq G$ be a subgroup and let the left coset space G/Γ be given the quotient topology. Left translations act by homeomorphisms on G/Γ. A point $x \in G/\Gamma$ is a nonwandering point for a left translation T_g if for every open set U containing x there exists a sequence of integers n_k diverging to $+\infty$ such that $\left[(T_g)^{n_k}(U)\right] \cap U$ is nonempty for every k. The group Γ determines a nonwandering set $\Omega(G, \Gamma)=\{g \in G$: every point $x \in G/\Gamma$ is nonwandering relative to the left translation $T_g\}$. Then Γ satisfies property (S) if and only if $\Omega(G, \Gamma)=G$.

Now let H be a complete simply connected Riemannian manifold of nonpositive sectional curvature, and let $D \subseteq I(H)$ be a subgroup. Using the notation of [16] we say that D satisfies the duality condition if for every geodesic γ of H there exists a sequence $\{\varphi_n\} \subseteq \mathbf{D}$ such that $\varphi_n p \to \gamma(\infty)$ and $\varphi_n^{-1}(p) \to \gamma(-\infty)$ as $n \to \infty$ for any point $p \in H$. Following [1] or [2] we restate this condition in equivalent form. For each group $D \subseteq I(H)$, we define a nonwandering set $\Omega(D) \subseteq T_1 H$ given by $\Omega(D)=\{v \in T_1 H$: for any open set O containing v there exist sequences $\{\varphi_n\} \subseteq D$ and $\{t_n\} \subseteq \mathbf{R}$ such that $t_n \to +\infty$ and $[(\varphi_n)_* T_{t_n}(O)] \cap O$ is nonempty for every $n\}$. Here $T_1 H$ denotes the unit tangent bundle of H and $\{T_t\}$ denotes the geodesic flow in $T_1 H$. One may then show by the argument of Proposition 3.7 of [13] that D satisfies the duality condition if and only if $\Omega(D)=T_1 H$.

If Γ is a subgroup of a topological group G such that the quotient space G/Γ admits a finite G-invariant measure that is positive on open sets then Γ satisfies the property (S). Similarly if $\Gamma \subseteq I(H)$ is a subgroup such that the quotient space H/Γ is a smooth manifold of finite volume (either compact or noncompact), then Γ satisfies the duality condition. In fact if $D \subseteq I(H)$ is a subgroup such that H/D is a smooth manifold, then D satisfies the duality condition if and only if every vector in $T_1(H/D)$ is nonwandering relative to the geodesic flow.

The main result of the paper lies in section 3 but we omit a statement here since the corollaries are of greater interest. Two corollaries have been stated already and we now present others, beginning with some results on lattices. A group $\Gamma \subseteq I(H)$ is a lattice if the quotient space H/Γ is a smooth manifold of finite Riemannian volume, and Γ is uniform or nonuniform according to whether H/Γ is compact or noncompact. A lattice Γ is reducible if H/Γ admits a finite Riemannian cover that is reducible (can be expressed as the Riemannian product of two manifolds of positive dimension). A lattice Γ is irreducible if it is not reducible.

The existence of irreducible lattices in reducible Riemannian symmetric spaces of noncompact type is well known. See for example [4] and [28, p. 64]. Surprisingly, irreducible lattices do not occur in reducible spaces H that are not Riemannian symmetric. More precisely, we establish the following (Proposition 4.5), which is a corollary of the more comprehensive Proposition 4.4: Let H be a reducible space with no Euclidean de Rham factor and let Γ be a lattice in H. If Γ is irreducible then H is a symmetric space of noncompact type.

Irreducible uniform lattices Γ^* in a reducible symmetric space H^* of noncompact type also satisfy a certain rigidity property. Let Γ be a lattice in an arbitrary space H such that Γ is isomorphic as a group to Γ^*. Then the quotient manifolds H/Γ and H^*/Γ^* are isometric up to a normalization of the H^*-metric provided that H and H^* satisfy certain conditions on their de Rham factorizations. See Proposition 4.6 for a precise statement. These conditions on the de Rham factorization of H and H^* may in fact be unnecessary. If so, one would then have a more general version of Mostow's rigidity theorem for irreducible uniform lattices in a reducible symmetric space of noncompact type.

Finally one may apply the main result of section 3 to answer some questions posed in [11] that concern the characterization of symmetric spaces of noncompact type. In particular (Proposition 4.8), we show that if H is an irreducible manifold whose isometry group $I(H)$ satisfies the duality condition, then either $I(H)$ is discrete or H is a symmetric space of noncompact type.

We describe the organization of the paper. Section 1 contains preliminary material. In section 2, we prove a "sandwich" lemma that generalizes the "flat strip" lemma of [16] or [29] from two complete geodesics whose Hausdorff distance is bounded to two complete totally geodesic submanifolds of arbitrary dimension whose Hausdorff distance is bounded. This lemma allows us to show that under certain conditions a complete totally geodesic submanifold B of H is a leaf of a parallel foliation N (Proposition 2.2). Consequently, if B^* is a leaf of the orthogonal foliation $N^\perp$, then H is isometric to the Riemannian product of B and B^* by the de Rham decomposition theorem.

In section 3 we use the decomposition result in Proposition 2.2 of section 2 to prove the main theorem. If $G \subseteq I(H)$ is the group that appears in the statement of the theorem, then we let B denote the set of points in H that are fixed by K, a maximal compact subgroup of G. The set B is a complete totally geodesic submanifold of H to which one can apply Proposition 2.2 and obtain the main theorem.

In section 4, we present applications of the main result of section 3, several of which have been described above. We conclude with some appendices. At various points, especially in section 3, we have omitted or deferred proofs of some lemmas to clarify the exposition. Omitted proofs may be found in the appendices.

We take this opportunity to correct some inaccuracies in [11], notably in the statements of Proposition 2.5, Proposition 4.10, Theorem 4.5 and in the proofs of Proposition 4.12 and Theorem 4.13. Instead of proving or assuming as the case may be that a Lie group G of isometries of H is noncompact and semisimple we should be proving or assuming that the group G is semisimple with no compact factors. The necessary modifications are provided by Lemmas 3.1 and 3.2 of this paper.

Finally, we wish to acknowledge the influence of the ideas of M. Goto and M. Goto [17] and of E. Heintze [19]. In both cases the totally geodesic submanifold B of H that consists of the points in H fixed by K, a maximal compact subgroup of a semisimple isometry group G, plays a major role in some kind of decomposition of H. This manifold B also plays a major role in the proof of our main result in section 3. Heintze in fact sketched the idea for a proof of the main theorem in section 3 in the case that $G=I_0(H)$ and $I(H)$ contains a uniform lattice Γ.

Section 1. Preliminaries

All Riemannian manifolds in this paper will be assumed to be complete, connected and C^∞ and to have nonpositive sectional curvature. M will denote a nonsimply connected manifold and H a simply connected manifold, sometimes referred to as a Hadamard manifold. All geodesics in both H and M will be assumed to have unit speed. $T_1 H$, $T_1 M$ will denote the unit tangent bundles of H, M. $I(H)$ will denote the isometry group of H and $I_0(H)$ the connected component of $I(H)$ that contains the identity.

For manifolds of nonpositive sectional curvature we shall assume the notation, definitions and basic facts found in [16] and in shorter form in [11, pp. 76–78] or section 1 of [15].

Duality condition

This is a condition on subgroups D of $I(H)$ and is closely related to the Selberg property (S) for subgroups of a topological group G. See the introduction for a definition and discussion of both properties. We remark again that if $G=I_0(H)$, where H is a Riemannian symmetric space of noncompact type, then any subgroup $D\subseteq I_0(H)$ that satisfies property (S) must also satisfy the duality condition [10], [20]. A partial converse is given in [10]. Whether the two properties are equivalent in this context is interesting and unknown.

Now let $H=H_1\times H_2\times\ldots\times H_k, k\geq 2$ be a Riemannian product of Hadamard manifolds. One may show without difficulty that the duality condition is preserved by the projection homomorphisms p_i onto $I(H_i)$, $1\leq i\leq k$. To explain more precisely we consider a group $D\subseteq I(H)$ that leaves invariant the foliations of H corresponding to the factors H_i, $1\leq i\leq k$. Each element φ of D can be written $\varphi=\varphi_1\times\varphi_2\times\ldots\times\varphi_k$, where $\varphi_i\in I(H_i)$ for $1\leq i\leq k$, and we obtain projection homomorphisms $p_i: D\to I(H_i)$ given by $p_i(\varphi)=\varphi_i$. It is not difficult to show that if D satisfies the duality condition in H, then $D_i=p_i(D)$ satisfies the duality condition in H_i for $1\leq i\leq k$. In general, the groups D_i will not be discrete even if D is discrete.

If $H=H_1\times H_2\times\ldots\times H_k, k\geq 2$, is a nontrivial Riemannian product and if $D\subseteq I(H)$ is any subgroup, then there exists a subgroup $D^*\subseteq D$ of finite index in D whose elements leave invariant the foliations of H corresponding to the factors H_i, $1\leq i\leq k$. To see this we observe that if $\varphi\in I(H)$ is an arbitrary element then φ permutes the foliations of H corresponding to the factors in the de Rham decomposition of H [24, p. 192]. In particular any group $D\subseteq I(H)$ admits a finite index subgroup D^* whose elements leave invariant all of the de Rham foliations. If $H=H_1\times\ldots\times H_k$ is any Riemannian product decomposition of a Hadamard manifold H, then each factor H_i is isometric to the Riemannian product of some finite collection of de Rham factors of H and the foliation of H that corresponds to H_i is the orthogonal direct sum of some finite collection of the de Rham foliations of H. In particular, the subgroup D^* leaves invariant each foliation of H corresponding to a factor H_i, $1\leq i\leq k$. Using this argument, we note that if $D=I(H)$ then $D^{**}=I(H_1)\times\ldots\times I(H_k)$ has finite index in D.

For reasons partly explained in the preceding two paragraphs, it is often necessary to shift consideration from a given group $D\subseteq I(H)$ to a suitable subgroup D^* of finite index in D. In such a situation the following result is useful

PROPOSITION. *Let H be a Hadamard manifold and let $D\subseteq I(H)$ be a subgroup that satisfies the duality condition. If D^* is a subgroup of D with finite index in D, then D^* satisfies the duality condition.*

Proof. See Appendix I.

Symmetric spaces of noncompact type

We conclude section 1 with a quick summary of the basic facts about Riemannian symmetric spaces that we need in this paper. The facts described below are due to E. Cartan [6], [7], [8], [9]. They may also be found in [21, pp. 121–125; 156–159; 173–174; 205; 214–219] and [30, pp. 232–247].

A (real) Lie algebra $\mathfrak{g}$ is *semisimple* if the bilinear Killing form $B:(X, Y)\to \mathrm{Tr}(\mathrm{ad}\, X\ \mathrm{ad}\, Y)$ is nondegenerate on $\mathfrak{g}$. A semisimple Lie algebra $\mathfrak{g}$ is *compact* if B is negative definite on $\mathfrak{g}$. A noncompact semisimple Lie algebra $\mathfrak{g}$ admits a Cartan decomposition; that is, there exists a subalgebra $\mathfrak{k}$ and a vector subspace $\mathfrak{P}$ such that $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{P}, [\mathfrak{k}, \mathfrak{P}]\subseteq \mathfrak{P}, [\mathfrak{P}, \mathfrak{P}]\subseteq\mathfrak{k}$, B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{P}$.

A Riemannian manifold N is a *Riemannian symmetric space* if for each point $q\in N$ the locally defined geodesic symmetry $S_q: \exp_q(X)\to\exp_q(-X), X\in T_q(N)$, extends to a global isometry of N. Let G be a connected semisimple Lie group with noncompact Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{P}$ be a Cartan decomposition. If K is a Lie subgroup of G with Lie algebra $\mathfrak{k}$, then the left coset space G/K is a Hadamard manifold and a Riemannian symmetric space of the noncompact type relative to any G-invariant Riemannian metric. The tangent space to G/K at eK may be identified with the vector subspace $\mathfrak{P}$ and the geodesics starting at eK have the form $t\mapsto\exp(tX)K$ for $X\in\mathfrak{P}$. Moreover, $\tau(G)=I_0(G/K)$, where for $g\in G$, $\tau(g)$ is the left translation by g.

If G, K are as above then there is an essentially unique G-invariant Riemannian metric on G/K. Identifying the tangent space at eK with $\mathfrak{P}$ implies that the G-invariant metrics on G/K are in one-one correspondence with the ad($\mathfrak{k}$)-invariant inner products Q on $\mathfrak{P}$, those inner products Q such that $Q(\mathrm{ad}\, Z(Y), X)=-Q(Y, \mathrm{ad}\, Z(X))$ for $Z\in\mathfrak{k}$ and $X, Y\in\mathfrak{P}$. If $\mathfrak{g}$ is simple then any ad($\mathfrak{k}$) invariant inner product on $\mathfrak{P}$ is of the form λB for some $\lambda>0$, where B is the Killing form of $\mathfrak{g}$. In general, $\mathfrak{g}$ is a direct sum $\oplus_{i=1}^n\mathfrak{g}_i$, where $\{\mathfrak{g}_i\}_{i=1}^n$ are the simple ideals of $\mathfrak{g}$. Any Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{P}$ is of the form $\mathfrak{k}=\oplus_{i=1}^n\mathfrak{k}_i$ and $\mathfrak{P}=\oplus_{i=1}^n\mathfrak{P}_i$, where $\mathfrak{g}_i=\mathfrak{k}_i\oplus\mathfrak{P}_i$ is a Cartan decomposition for $\mathfrak{g}_i$. Any ad($\mathfrak{k}$)-invariant inner product Q on $\mathfrak{P}$ is of the form $Q=\Sigma_{i=1}^n\lambda_i B_i$, where $\lambda_i>0$ and $B_i=B$ on $\mathfrak{P}_i$, zero on $\mathfrak{P}_j$ for $j\neq i$.

We conclude the section with a special case that arises later. If G is a noncompact, connected semisimple Lie group with finite center and if K is a maximal compact subgroup of G, then the coset space G/K is a Riemannian symmetric space of the noncompact type. One shows that the Lie algebra $\mathfrak{k}$ of K is a maximal compact subalgebra of $\mathfrak{g}$ and hence part of a Cartan decomposition for $\mathfrak{g}$. In this case, it is known that any two maximal compact subgroups of G are conjugate in G, and this conjugacy induces a conjugacy of the corresponding Lie algebras in $\mathfrak{g}$.

Section 2. The Sandwich lemma

Our model for this result is the following. Let γ_1, γ_2 be two maximal geodesics of H such that $\gamma_1(\infty)=\gamma_2(\infty)$ and $\gamma_1(-\infty)=\gamma_2(-\infty)$. Then it is known by Lemma 5.1 of [16] or [29] that γ_1 and γ_2 bound a *flat strip;* that is, there exists a number $c>0$ and an isometric, totally geodesic imbedding $F: \mathbf{R}\times[0,c]\to H$ such that $F(t,0)=\gamma_1(t)$ and $F(t,c)=\gamma_2(t)$ for suitable parametrizations of γ_1, γ_2 and all $t\in\mathbf{R}$. The number $c=d(\gamma_1 t, \gamma_2)=d(\gamma_2 t, \gamma_1)$ for every $t\in\mathbf{R}$. The result of this section is a generalization of the result just quoted. We present two applications of the "Sandwich lemma" that will be useful in the proof of the main theorem.

If B is any complete, totally geodesic submanifold of H, then one may naturally include $B(\infty)$ as a subset of $H(\infty)$. Given a point $x\in B(\infty)$ let x be written as $\gamma(\infty)$ for some geodesic γ of B. Since B is totally geodesic the curve γ is also a geodesic of H and determines a point $x^*=\gamma(\infty)$ in $H(\infty)$. The map $x\mapsto x^*$ imbeds $B(\infty)$ as a subset of $H(\infty)$.

We are now ready to state the lemma.

LEMMA 2.1. *Let B_1, B_2 be distinct complete, totally geodesic submanifolds of H such that $B_1(\infty)=B_2(\infty)\subseteq H(\infty)$. Then there exists a number $c>0$ and an isometric, totally geodesic imbedding $F: B_1\times[0,c]\to H$ such that $F(b,0)=b$ for all $b\in B_1$ and $F(B_1\times\{c\})=B_2$.*

The proof will involve some intermediate results. We recall that if B is any totally geodesic submanifold of H, then by Lemma 3.2 of [3] there is a unique perpendicular geodesic from a point not on B to B.

SUBLEMMA 1. *For each $b\in B_1$ let $V(b)$ denote the unit vector tangent to the unique perpendicular geodesic from b to B_2. Then*

(i) *there exists a constant $c>0$ such that $d(b, B_2)=d(b', B_1)=c$ for every $b\in B_1$, $b'\in B_2$,*

(ii) *V is a C^∞ normal vector field on B_1,*

(iii) $B_2=\{\exp_b(cV(b)): b\in B_1\}$.

Proof. To establish (i) it suffices to show that given $b\in B_1$ we have $d(b, B_2)\leq d(b', B_2)$ for any point $b'\in B_1$. Let $b, b'\in B_1$ be given and assume that $b\neq b'$. Let γ be the unit speed geodesic of H such that $\gamma(0)=b$ and $\gamma(t_0)=b'$, where $t_0=d(b, b')$. Note that $\gamma(\mathbf{R})\subseteq B_1$ since B_1 is totally geodesic and contains both b and b'. Now let b^* be the foot of b on B_2, and let γ^* be the unit speed geodesic of H such that $\gamma^*(0)=b^*$ and $\gamma^*(\infty)=\gamma(\infty)$. It follows that $\gamma^*(\mathbf{R})\subseteq B_2$ since B_2 is totally geodesic and

$B_2(\infty)=B_1(\infty)$. By Theorem 4.1 of [3] the function $f(t)=d^2(\gamma t, \gamma^*)$ is a C^∞ convex function and f is bounded above for $t\geq 0$ since γ, γ^* are asymptotic. Hence, $f(t)$ is nonincreasing in t and we see that $d^2(b', B_2)=d^2(\gamma t_0, B_2)\leq d^2(\gamma t_0, \gamma^*)\leq d^2(\gamma 0, \gamma^*)= d^2(b, b^*)=d^2(b, B_2)$. Since b, b' are arbitrary points in B_1, it follows that $b\mapsto d(b, B_2)$ is constant on B_1. Similarly $b^*\mapsto d(b^*, B_1)$ is constant on B_2.

We prove (ii) and (iii). It follows from the previous paragraph that any perpendicular from B_1 to B_2 or from B_2 to B_1 is mutually perpendicular to both B_1 and B_2. Hence the unit vector field V is a normal vector field on B_1. Moreover $B_2=\{\exp_b(cV(b): b\in B_1\}$, where $c>0$ is the constant of (i). Finally, $cV=(\exp)^{-1}(B_2)$, where $\exp^\perp: (B_1)^\perp \to H$ is a diffeomorphism by Lemma 3.1 of [3]; $B_1^\perp$ denotes the normal bundle of B_1. It follows that cV and hence V is a C^∞ vector field on B_1.

SUBLEMMA 2. *Let $P_2: H\to B_2$ be the orthogonal projection. If $\gamma(t)$ is a unit speed geodesic in B_1, then $\gamma^*(t)=(P_2\circ\gamma)(t)=\exp_{\gamma t}(cV(\gamma t))$ is a unit speed geodesic in B_2, and γ, γ^* bound a flat strip in H.*

Proof. Let $P_1: H\to B_1$ be the orthogonal projection on B_1. The fact that any perpendicular from B_1 to B_2 or from B_2 to B_1 is a mutual perpendicular implies that $P_1\circ P_2$ and $P_2\circ P_1$ are the identity maps when restricted to B_1 and B_2. Hence $P_2: B_1\to B_2$ and $P_1: B_2\to B_1$ are diffeomorphisms. By Lemma 3.2 of [3] we know that $d(P_1 b, P_1 b')\leq d(b, b')$ and $d(P_2 a, P_2 a')\leq d(a, a')$ for all points b, b' in B_2 and a, a' in B_1. It follows that P_1 and P_2 are distance preserving maps when restricted to B_2 and B_1 and hence are isometries. Therefore $\gamma^*(t)=(P_2\circ\gamma)(t)$ is a unit speed geodesic of B_2 and also of H. By Lemma 5.1 of [16] the geodesics γ, γ^* bound a flat strip in H since $d(\gamma t, \gamma^* t)\equiv c$ for all t.

We are now ready to prove Lemma 2.1. Let B_1, B_2 be distinct complete, totally geodesic submanifolds of H with $B_1(\infty)=B_2(\infty)$. Let V be the C^∞ unit normal vector field defined on B_1 as in Sublemma 1. Let $c>0$ be the value of the constant functions $b\mapsto d(b, B_2)$ and $b'\mapsto d(b', B_1)$, where $b\in B_1$ and $b'\in B_2$. Define a map $F: B_1\times[0, c]\to H$ by $F(b, t)=\exp_b(tV(b))$. Clearly F is C^∞ and F is one-one since each point of H has a unique closest point on B_1. Moreover $F(B_1\times\{c\})=B_2$ by the earlier sublemmas. Any geodesic in $B_1\times[0, c]$ of the form $t\mapsto(b, t_0+t)$ is clearly carried by F into a geodesic of H of the same speed that is mutually perpendicular to B_1 and B_2. To prove that F is an isometric totally geodesic imbedding it suffices to show that if $\gamma(s)=(\gamma_1(s), t_0+\alpha s)$ is a constant speed geodesic in $B_1\times[0, c]$ with $\|\gamma_1'(0)\|>0$, then $(F\circ\gamma)(s)$ is a geodesic in H with the same speed as γ. We may assume without loss of generality that γ has unit speed. If $\|\gamma_1'(0)\|=\beta>0$ then $\sigma^*(s)=\gamma_1(s/\beta)$ is a unit speed geodesic in B_1. If we define

$G:\mathbf{R}\times[0,c]\to H$ by $G(s,t)=\exp_{\sigma^*s}(tV(\sigma^*s))$, then G is a totally geodesic, isometric imbedding by Sublemma 2 and Lemma 5.1 of [16]. Now $(F\circ\gamma)(s)=F(\gamma_1 s, t_0+\alpha s)=\exp_{\gamma_1 s}([t_0+\alpha s]V(\gamma_1 s))=G(\beta s, t_0+\alpha s)$. Therefore $(F\circ\gamma)(s)$ is a geodesic of H since $s\mapsto(\beta s, t_0+\alpha s)$ is a geodesic of $\mathbf{R}\times[0,c]$. Note that $1=\|\gamma'(0)\|^2=\beta^2+\alpha^2$. Hence $\|(F\circ\gamma)'(0)\|^2=\beta^2\|G_*\,\partial/\partial s\|^2+\alpha^2\|G_*\,\partial/\partial t\|^2=\beta^2+\alpha^2=1$. This completes the proof of Lemma 2.1.

Riemannian product decompositions

We apply the lemma above to obtain two results that have some independent interest. Here they are applied to the main theorem of the next section and to the last two results of section 4.

PROPOSITION 2.2. *Let $B\subseteq H$ be a complete totally geodesic submanifold and let $X=B(\infty)\subseteq H(\infty)$. Let $D=\{\varphi\in I(H):\varphi(X)=X\}$. If $L(D)=H(\infty)$ then there exists a complete, totally geodesic submanifold B' of H such that H is isometric to the Riemannian product of B and B'.*

COROLLARY 2.3. *Let $\Gamma\subseteq I(H)$ be a subgroup such that Γ leaves invariant a complete totally geodesic submanifold B of H and $L(\Gamma)=B(\infty)$. Let D be the normalizer of Γ in $I(H)$. If $L(D)=H(\infty)$ then there exists a Riemannian product decomposition $H_1\times H_2$ of H such that* (i) *$H_1(\infty)=B(\infty)$ and B is one of the leaves of the decomposition,* (ii) *every element $\varphi\in D$ has the form $\varphi_1\times\varphi_2$, $\varphi_i\in I(H_i)$ for $i=1,2$,* (iii) *every element $\gamma\in\Gamma$ has the form $\gamma=\gamma_1\times\{1\}$ for some $\gamma_1\in I(H_1)$.*

We prove the corollary first, assuming the result of Proposition 2.2. Let $X=B(\infty)\subseteq H(\infty)$. Since D normalizes Γ it follows that $\varphi X=X$ for every $\varphi\in D$. By Proposition 2.2 there exists a Riemannian product decomposition $H_1\times H_2$ of H such that $H_1(\infty)=B(\infty)=X$. The proof of Proposition 2.2 will show that B is one of the leaves of the parallel foliation of H corresponding to H_1 and will show also that D leaves invariant the foliations of H corresponding to H_1 and H_2. Hence every element φ of D can be written $\varphi=\varphi_1\times\varphi_2$, where $\varphi_i\in I(H_i)$ for $i=1,2$.

We show that every element in Γ can be written $\gamma=\gamma_1\times\{1\}$ for some $\gamma_1\in I(H_1)$. For $i=1,2$ let D_i,Γ_i denote the subgroups of $I(H_i)$ that consist of the ith components of the elements φ of D,Γ in the expression $\varphi=\varphi_1\times\varphi_2$. The totally geodesic submanifold B can be identified with $H_1\times\{p_2\}$ for some $p_2\in H_2$ and p_2 is fixed by every element of Γ_2 since Γ leaves B invariant. The fixed points of Γ_2 in H_2 form a closed convex set that contains the orbit $D_2(p_2)$ since Γ is normal in D and hence Γ_i is normal in D_i for $i=1,2$. The fact that $L(D)=H(\infty)$ implies that $L(D_i)=H_i(\infty)$ for $i=1,2$ and therefore Γ_2 fixes every point of H_2, the smallest, closed convex set that contains $D_2(p_2)$. Therefore $\Gamma_2=\{1\}$ and the corollary is proved.

We now prove Proposition 2.2. Let $H^*=\{p\in H$: there exists a complete, totally geodesic submanifold $B^*\subseteq H$ with $p\in B^*$ and $B^*(\infty)=B(\infty)=X\}$. The submanifold B^* containing p is unique as one may see by inspection or by applying Lemma 2.1. The main step of the proof is to show that $H^*=H$. We then define a foliation N in H by letting $N(p)$ be the tangent space at p to B_p, the complete, totally geodesic submanifold with $p\in B_p$ and $B_p(\infty)=X$. Finally, we show that N is a parallel foliation in H. Hence the orthogonal distribution $N^\perp$ is also a parallel foliation and if B, B' are maximal integral manifolds of $N, N^\perp$ through a fixed point p of H, then H is isometric to the Riemannian Product $B\times B'$ by the de Rham decomposition theorem [24, p. 187].

Since $L(D)=H(\infty)$ it will follow that $H^*=H$ when we show that H^* is invariant under D, closed and convex in H. By hypothesis $H^*\supseteq B$ is nonempty and clearly H^* is invariant under D; if $p^*\in H^*$ then $p^*\in B^*$ where B^* is totally geodesic with $B^*(\infty)=X$ and hence $\varphi p\in\varphi B^*$ with $(\varphi B^*)(\infty)=\varphi B^*(\infty)=\varphi X=X$ for every $\varphi\in D$. We show that H^* is closed in H. Let $\{p_n\}\subseteq H^*$ be a sequence converging to a point $p\in H$. Let B_n be the unique complete, totally geodesic submanifold of H with $p_n\in B_n$ and $B_n(\infty)=X$. By the lemma above, the manifolds B_n have the same dimension $k\geq 1$ for every n. Let $\{\alpha_1^{(n)},\dots,\alpha_k^{(n)}\}$ be an orthonormal basis for $T_{p_n}(B_n)$ for every n. Passing to a subsequence let $\alpha_i^{(n)}\to\alpha_i\in T_p(H)$ as $n\to+\infty$ for every $1\leq i\leq k$. Then $\{\alpha_1,\dots,\alpha_k\}$ is an orthonormal subset of T_pH. Let V be the span of $\{\alpha_1,\dots,\alpha_k\}$ in T_pH and let $B_p=\exp_p(V)$. It is routine to show that B_p is complete and totally geodesic with $B_p(\infty)=X$, which shows that H^* is closed in H.

We show next that H^* is a convex subset of H. Let p, q be any two points of H^*, and let r be an interior point of the geodesic segment joining p to q. Let B_p, B_q be the complete totally geodesic submanifolds of H such that $p\in B_p, q\in B_q$ and $B_p(\infty)=B_q(\infty)=X$. By the Sandwich lemma there exists a number $c>0$ and an isometric, totally geodesic imbedding $F: B_p\times[0,c]\to H$ such that $F(b,0)=b$ for all $b\in B_p$ and $F(B_p\times\{c\})=B_q$. Choose a point $p'\in B_p$ such that $q=F(p',c)$. Let $\gamma_1:[0,c]\to B_p$ be the constant speed geodesic of H such that $\gamma_1(0)=p$ and $\gamma_1(c)=p'$. Let $\gamma(s)=(\gamma_1(s),s)$, a geodesic in $B_1\times[0,c]$. Then $\sigma(s)=(F\circ\gamma)(s)$ is that geodesic in H with $\sigma(0)=p$ and $\sigma(c)=q$. Hence $r=\sigma(s_0)$ for some $s_0\in(0,c)$. Therefore, $r\in B_{s_0}=\{F(p^*,s_0):p^*\in B_p\}$. Since F is a totally geodesic isometric imbedding it follows that $B_t=\{F(p^*,t):p^*\in B_p\}$ is a complete, totally geodesic submanifold of H for every $t\in[0,c]$. Since $d(p^*,B_t)=t$

for every $t \in [0,c]$ it follows that $B_t(\infty)=B_p(\infty)=X$ for every $t \in [0,c]$. Hence $r \in B_{s_0} \subseteq H^*$, which proves that H^* is a convex subset of H. Finally, we conclude that $H^*=H$ since $L(D)=H(\infty)$ and H^* is closed, convex and invariant under D.

We now define a parallel foliation N in H. For each $p \in H$ let $N(p)=T_p(B_p)$, where B_p is the unique complete totally geodesic submanifold of H with $p \in B_p$ and $B_p(\infty)=X$. Then $N(p)$ has the same dimension for each p by Lemma 2.1.

We show first that N is a C^∞ distribution in H. Fix a complete, totally geodesic submanifold B of H such that $B(\infty)=X$ and let $B^\perp$ denote the normal bundle of B in H. By Lemma 3.1 of [3] the map $\exp: B^\perp \to H$ is a diffeomorphism. Using the two sublemmas of Lemma 2.1, it is easy to prove the following

LEMMA 2.4. *Let $\xi \in B^\perp$ be arbitrary. Then there exists a C^∞ normal vector field V on B such that* (1) *$V(p)=\xi$, where p is the point of attachment of ξ,* (2) *$\nabla_v V=0$ for any vector v tangent to B,* (3) *$B'=\{\exp_q(V(q)) : q \in B\}$ is a complete totally geodesic submanifold of H with $B'(\infty)=X$.*

Now we construct C^∞ unit vector fields $E_1, \ldots, E_r$, where r is the codimension of B in H, such that E_i is normal to B and $\nabla_v E_i=0$ for each vector v tangent to B and each i. We then define a diffeomorphism $F: B \times \mathbf{R}^r \to H$ given by $F(p,\xi)=\exp_p(\sum_{i=1}^r \xi_i E_i(p))$, where $\xi=(\xi_1, \ldots, \xi_r)$. By Lemma 2.4 the submanifolds B_q, $q \in H$ coincide with the sets $F(B,\xi)$, $\xi \in \mathbf{R}^r$. It follows that the distribution N is C^∞.

We now prove that the distribution N defined above is parallel; that is for every locally defined C^∞ vector field Z with values in N and for every vector v tangent to H the vector $\nabla_v Z$ is tangent to N. Let Z be such a vector field and let v be a vector tangent to H at p. If $v \in N(p)$ then $\nabla_v Z \in N(p)$ since B_p is totally geodesic in H. It suffices to prove that $\nabla_v Z \in N(p)$ if $v \in N(p)^\perp \subseteq T_p(H)$.

Let $v \in N(p)^\perp$ be given. Let $\{\xi_1, \ldots, \xi_k\}$ be an orthonormal basis for $N(p)$ and let $\xi_i(t)$, $1 \leq i \leq k$, denote the vector field along the geodesic γ_v obtained by parallel translation of $\xi_i=\xi_i(0)$ in H along γ_v. We show first that $\xi_i(t) \in N(\gamma_v t)$ for every t and every i. Fix a number t. Let B^* denote $B_{\gamma_v t}$ and let $P_{B^*}: H \to B^*$ denote the orthogonal projection. For each integer i let $\alpha_i(s)$ denote the unit speed geodesic with initial velocity ξ_i and let $\beta_i(s)=P_{B^*}(\alpha_i s)$. By Sublemma 2 of Lemma 2.1, it follows that β_i is a unit speed geodesic of H and α_i, β_i bound a flat strip in H. In particular $\beta_i'(0)$ is the parallel translate in H of $\alpha_i'(0)=\xi_i$ along γ_v. It follows that $\xi_i(t)=\beta_i'(0) \in N(\gamma_v t)$.

Let Z be a C^∞ vector field with values in N that is defined in a neighborhood of p. By the preceding paragraph $\{\xi_1(t), \ldots, \xi_k(t)\}$ is an orthonormal basis of $N(\gamma_v t)$ for every t. Hence, we can find C^∞ functions $f_1, \ldots, f_k$ defined in a neighborhood of zero in $\mathbf{R}$

such that $(Z\circ\gamma_v)(t)=\Sigma_{i=1}^k f_i(t)\,\xi_i(t)$. Therefore, $\nabla_v Z=\Sigma_{i=1}^k f_i'(0)\,\xi_i(0)\in N(p)$ since the vector fields ξ_i are parallel along γ_v. Hence, the distribution N is parallel in H. The proof of the proposition is now completed in the manner indicated earlier.

Section 3. The main result

THEOREM. *Let H be a Hadamard manifold with no Euclidean de Rham factor. Let G be a closed connected nonidentity subgroup of $I(H)$ whose normalizer in $I(H)$ satisfies the duality condition. Let $A=\mathrm{Fix}(G)\cap H(\infty)$. Then there exists a Riemannian product decomposition $H=H_1\times H_2$ such that*

(1) H_1 is a symmetric space of noncompact type and $G=I_0(H_1)$,

(2) $H_2(\infty)=A$.

By Fix(G) we mean $\{p\in\bar{H}=H\cup H(\infty):\varphi p=p$ for all $\varphi\in G\}$. The proof of the theorem requires several lemmas and for the sake of clarity we only state the lemmas now and prove them later. Let G satisfy the hypotheses of the theorem.

LEMMA 3.1. *G is a semisimple Lie group with trivial center, and G contains no compact normal subgroup except the identity.*

LEMMA 3.2. *G can be expressed as a direct product $G_1\times\ldots\times G_m$, where each subgroup G_i is noncompact, connected, simple, normal in G and closed in $I(H)$.*

LEMMA 3.3. *Let $\mathfrak{g}$ denote the Lie algebra of G, regarded as the set of Killing vector fields on H. Let $x\in H(\infty)$ be a point such that $\|X\|^2$ is nonincreasing on every geodesic in x for every $X\in\mathfrak{g}$. Then $x\in\mathrm{Fix}\,(G)\cap H(\infty)$.*

We remark that if X is an element of the abstract Lie algebra $\mathfrak{g}$, then X may be regarded as a Killing vector field on H whose flow transformations are $\{\exp(tX)\}$. Moreover, if $X=Y+Z$ in the abstract Lie algebra $\mathfrak{g}$, then $X(p)=Y(p)+Z(p)$ for all $p\in H$ since $X(p)=dp(X)$, where $p:G\to H$ is given by $p(g)=g(p)$. Regarding $X\in\mathfrak{g}$ as a Killing vector field on H it follows from Lemma 5.3 of [3] that $\|X\|^2$ is a C^∞ convex function on H. Furthermore, by Proposition 9.8 of [3] we see that if $\|X\|^2$ is nonincreasing on one geodesic belonging to x, then $\|X\|^2$ is nonincreasing on every geodesic belonging to x.

LEMMA 3.4. *Let K be a maximal compact subgroup of G. If K has a unique fixed point in H, then H is a symmetric space of noncompact type and $G=I_0(H)$.*

LEMMA 3.5. *Let K be a maximal compact subgroup of G, and let $F=\mathrm{Fix}\,(K)\cap H$. Then* $\mathrm{Fix}\,(K)\cap H(\infty)=\mathrm{Fix}\,(G)\cap H(\infty)=F(\infty)$.

We remark that F is a closed totally geodesic submanifold of H by Theorem 5.1 of [23, p. 59]. Hence, it makes sense to speak of $F(\infty)$ as a subset of $H(\infty)$. We are now ready to prove the theorem. We first consider the case that $A = \mathrm{Fix}\,(G) \cap H(\infty)$ is empty. By Lemma 3.5 it follows that K has a unique fixed point in H, and by Lemma 3.4 we see that H is a symmetric space of noncompact type and $G = I_0(H)$. The theorem is proved in the case that A is empty.

Now suppose that A is nonempty. If φ is an arbitrary element of D, the normalizer of G in $I(H)$, then φ leaves invariant the set A. By Lemma 3.5 $A = F(\infty)$, where $F = \mathrm{Fix}\,(K) \cap H$ is a closed totally geodesic submanifold of H. By Proposition 2.2, there exists a Riemannian product decomposition $H = H_1 \times H_2$ such that $H_2(\infty) = A$. It remains to show that H_1 is a symmetric space of noncompact type and $G = I_0(H_1) \cong I_0(H_1) \times \{1\} \subseteq I(H)$.

Since $G \subseteq I_0(H) = I_0(H_1) \times I_0(H_2)$ we may write $g = g_1 \times g_2$ for each $g \in G$, where $g_i \in I_0(H_i)$ for $i = 1, 2$. We assert that $G \subseteq I_0(H_1) \times \{1\}$. If this were not the case, then we could find an element $g = g_1 \times g_2 \in G$ such that $g_2 \neq \{1\}$. Let $x \in H_2(\infty) = A$ be given arbitrarily and let $\gamma(t) = (p_1, \gamma_2(t))$ be a geodesic belonging to x, where γ_2 is a geodesic of H_2. By hypothesis $g \circ \gamma = (g_1(p_1), g_2 \circ \gamma_2)$ is asymptotic to γ since g fixes x. Therefore $g_2 \circ \gamma_2$ is asymptotic to γ_2 and g_2 fixes x. By the argument used in the proof of Proposition 2.3 of [11], we see that g_2 is a Clifford translation of H_2 and by Theorem 1 of [29] H_2 admits an Euclidean de Rham factor, contradicting our hypothesis that H has no Euclidean de Rham factor. Therefore $G \subseteq I_0(H_1) \times \{1\}$.

Regarding G as a subgroup of $I_0(H_1)$ it follows that $\mathrm{Fix}\,(G) \cap H_1(\infty) = \{\mathrm{Fix}\,(G) \cap H(\infty)\} \cap H_1(\infty) = H_2(\infty) \cap H_1(\infty)$ is empty. As soon as we prove that the normalizer of G in $I(H_1)$ satisfies the duality condition in H_1, we will conclude by Lemmas 3.4 and 3.5 that H_1 is a symmetric space of noncompact type and $G = I_0(H_1)$. This will complete the proof of the theorem except for the proofs of the supporting lemmas.

The subgroup $I(H_1) \times I(H_2)$ has finite index in $I(H)$ by the discussion of section 1. If D denotes the normalizer of G in $I(H)$, then $D^* = D \cap \{I(H_1) \times I(H_2)\}$ has finite index in D. Therefore D^* satisfies the duality condition in H by the proposition of Appendix I since D satisfies the duality condition. If $D_1^* = p_1(D^*)$, where $p_1 : D^* \to I(H_1)$ is the projection homomorphism, then D_1^* satisfies the duality condition in H_1 by the discussion of section 1. Moreover D_1^* normalizes $G \subseteq I_0(H_1)$ since D^* normalizes G regarded as a subgroup of $I(H)$. It follows that the normalizer of G in $I(H_1)$ satisfies the duality condition.

We now prove the Lemmas 3.1 through 3.5.

Proof of Lemma 3.1. By Theorem 2.4 of [11] every element in the center of G is a Clifford translation of H. If $I(H)$ admitted any nonidentity Clifford translations then H would admit an Euclidean de Rham factor by Theorem 1 of [29], contrary to our hypothesis on H. Therefore the center of G is trivial. Similarly one argues as in the proof of Proposition 2.5 of [11] to show that G is semisimple.

We show that G admits no compact normal subgroups except the identity. Let N be a compact normal subgroup of G. If N is discrete then N lies in the center of G since $t \mapsto \exp(tX)\varphi\exp(-tX)$ is a curve that lies in N for all $\varphi \in N$ and all $X \in \mathfrak{g}$, the Lie algebra of G. Since G is centerless, we need only consider the case that N is a Lie group of positive dimension. It suffices to prove that N_0, the connected component of N that contains the identity, is trivial.

Let D^* denote the normalizer of N_0 in $I(H)$. The first step is to prove that D^* satisfies the duality condition. Assuming this fact for the moment we show that N_0 fixes every point of H and hence is the identity. Since N is compact it follows from a theorem of E. Cartan [21, p. 75] that N has a fixed point p in H. The set of points S fixed by N_0 is nonempty, closed, convex in H and invariant under D^*. Since D^* satisfies the duality condition, it follows that $L(D^*)=H(\infty)$. Therefore the unit vectors at p tangent to a geodesic from p to a point of $D^*(p)$ are dense in the unit sphere of $T_p(H)$. It now follows from the properties of S listed above that $S=H$ and hence $N_0=\{1\}$.

We prove that D^* satisfies the duality condition. Observe that N_0 is also normal in G and hence its Lie algebra $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$. A semisimple Lie algebra $\mathfrak{g}$ has only finitely many ideals; $\mathfrak{g}$ has only finitely many simple ideals (whose direct sum is $\mathfrak{g}$) and every ideal of $\mathfrak{g}$ is a direct sum of simple ideals of $\mathfrak{g}$ [21, pp. 121–122]. Every element of D, the normalizer of G in $I(H)$, induces an inner automorphism of $\mathfrak{g}$ that permutes the ideals of $\mathfrak{g}$. Therefore the subgroup $\tilde{D}$ of D whose elements act as the identity permutation on the ideals of $\mathfrak{g}$ has finite index in D. Hence $\tilde{D}$ satisfies the duality condition since D satisfies the duality condition by hypothesis. If φ is any element of $\tilde{D}$ then $I_\varphi : \mathfrak{g} \to \mathfrak{g}$ leaves invariant $\mathfrak{g}_0$ and it is routine to show that φ normalizes N_0, the unique connected Lie subgroup of G with Lie algebra $\mathfrak{g}_0$. Hence $\tilde{D} \subseteq D^*$ and D^* satisfies the duality condition.

Proof of Lemma 3.2. The result follows by means of well known arguments from Lemma 3.1.

Proof of Lemma 3.3. Let γ be a geodesic belonging to x with $p=\gamma(0)$. Let $X \neq 0$ in $\mathfrak{g}$ be an arbitrary Killing vector field. By hypothesis the convex function $\|X\|^2$ is nonincreasing on γ. Define a C^∞ variation $F:\mathbf{R}\times\mathbf{R}\to H$ by $F(s,t)=\exp(tX)(\gamma_{px}s)$. Note that $F(s,0)=\gamma_{px}s$ for all $s\in\mathbf{R}$. The curves $t\mapsto F(s,t)$ are integral curves of X and hence $\|X(F(s,t))\|\equiv\|X(F(s,0))\|=f(s)$, which is independent of t. By hypothesis $f(s)=\|X(\gamma_{px}s)\|$ is nonincreasing in s.

Fix $t\neq 0$ and define $\gamma_t(s)=F(s,t)$. Then $\|X(\gamma_t s)\|=f(s)$ for all (s,t). The curve $u\mapsto F(s,u)$, $0\leq u\leq t$, has constant speed $f(s)$ by the previous paragraph. Therefore the curve has length $|t|f(s)$ and since it joins $\gamma_{px}s$ to $\gamma_t s$ we conclude that $d(\gamma_{px}s,\gamma_t s)\leq |t|f(s)\leq|t|f(0)$ for all $s\geq 0$. It follows that γ_{px} and γ_t are asymptotic geodesics for each $t\in\mathbf{R}$. Hence $\exp(tX)$ fixes x for each $t\in\mathbf{R}$ and $X\in\mathfrak{g}$ since $\gamma_t=\exp(tX)\circ\gamma_{px}$.

The elements $\exp(tX):t\in\mathbf{R}$, $X\in\mathfrak{g}$ all fix x by the work above and cover a neighborhood U of the identity in G. Since U generates G, it follows that G fixes x.

Proof of Lemma 3.4. We show first that $L(G)=H(\infty)$. Let D denote the normalizer of G in $I(H)$, and let $x\in H(\infty)$ be given. Since D satisfies the duality condition we may choose a sequence $\{\varphi_n\}\subseteq D$ so that $\varphi_n p\to x$ as $n\to+\infty$. The groups $\varphi_n K\varphi_n^{-1}$ are maximal compact subgroups of G. The maximal compact subgroups of G are all conjugate by elements of G [21, p. 218] and hence we can choose $g_n\in G$ so that $\varphi_n K\varphi_n^{-1}=g_n Kg_n^{-1}$ for every n. Let $p\in H$ be the fixed point of K. If $h_n=g_n^{-1}\varphi_n$ then $h_n Kh_n^{-1}=K$ for every n and it follows that K fixes each point $h_n(p)$. Hence $h_n(p)=p$ and $g_n(p)=\varphi_n(p)$ for every n since K has a unique fixed point in H by hypothesis. Finally $x=\lim_{n\to\infty}\varphi_n(p)=\lim_{n\to\infty}g_n(p)$, which shows that $L(G)=H(\infty)$ since x was arbitrarily chosen.

Now let H^* be the orbit $G(p)$. Clearly K is the stability group of G at p and hence H^* is diffeomorphic to the coset space G/K. Relative to the G-invariant metric induced from H the closed submanifold H^* is in fact a Riemannian symmetric space of noncompact type by the discussion of section 1. To conclude that $H^*=H$ it suffices to prove that $\dim H^*=\dim H$. The fact that G acts effectively and transitively on $H=G/K$ will then imply by [21, p. 207] that $G=I_0(H)$. G acts effectively on G/K by Lemma 3.1 since the kernel of the obvious map $G\to I_0(G/K)$ is a closed normal subgroup of G that is contained in K.

We prove that $\dim H^*=\dim H$ by a contradiction argument. Suppose that $\dim H^*=k<\dim H$, and let $V=T_p(H^*)\subseteq T_p(H)$. Let $V^\perp$ denote the orthogonal complement of V in $T_p(H)$. Define disjoint closed subsets C, C^* in $H(\infty)$ as follows: Let $C=\{\gamma_v(\infty):v\in V\}$ and let $C^*=\{\gamma_v(\infty):v\in V^\perp\}$. We assume that v is a unit vector in the definitions above. Clearly C, C^* are both nonempty since $V, V^\perp$ are both nonempty. We assert that C, C^* are both invariant under G. Obviously V and $V^\perp$ are both invariant under $dK=\{\varphi_* : \varphi \in K\}$ since K fixes p. It follows that K leaves both C and C^* invariant. By Theorem 4.5 of [11], $G(x)=K(x)$ for every x in $L(G)=H(\infty)$. Hence G leaves both C and C^* invariant.

Now choose $x\in C^*$ arbitrarily. We assert that H^* is contained in the horosphere $L(p, x)$. Assuming for the moment that this has been established, we derive a contradiction and conclude that $\dim H=\dim H^*$. Let $z=\gamma_{px}(-\infty)$. Since $L(G)=H(\infty)$ we can choose a sequence $\{g_n\}\subset G$ such that $g_n p\to z$, which implies that $\sphericalangle_p(x, g_n p)\to\pi$. However, the fact that $H^*\subseteq L(p,x)$ means that $\sphericalangle_p(x, g_n p)\leq\pi/2$ for every n by the law of cosines [16, p. 57].

We now show that $H^*=G(p)\subseteq L(p,x)$. Let $g\in G$ be given and let $t\mapsto g_t$ be a C^∞ curve in G with $g_0=\{1\}$ and $g_1=g$. Let $\sigma(t)=g_t(p)$. Now $\sphericalangle_{\sigma(t)}(V(\sigma t, x), \sigma'(t))=\sphericalangle_p(V(p, g_t^{-1}x), (g_t^{-1})_*\sigma'(t))=\pi/2$ for every t since $g_t^{-1}x\in C^*$ and $(g_t^{-1})_*\sigma'(t)\in T_p(H^*)=V$. Let $F(t)=(f\circ\sigma)(t)$, where f is the Busemann function at x such that $f(p)=0$. (For a definition and properties of Busemann functions see section 3 of [16] and section 2 of [14].) Finally $F'(t)=(f\circ\sigma)'(t)=\langle\sigma'(t), \operatorname{grad} f(\sigma t)\rangle=-\langle\sigma'(t), V(\sigma t, x)\rangle\equiv 0$ by the observation above and Proposition 3.5 of [16]. Thus $F(t)\equiv F(0)=f(p)=0$ and it follows that $\sigma(t)\in L(p,x)$ for every t. In particular $g(p)=\sigma(1)\in L(p,x)$, which proves that $H^*=G(p)\subseteq L(p,x)$. This completes the proof of Lemma 3.4.

Proof of Lemma 3.5. We show first that $F(\infty)=\operatorname{Fix}(K)\cap H(\infty)$. The set F is a closed totally geodesic submanifold of H by Theorem 5.1 of [23, p. 59]. One may see directly that if p, q are distinct points of F, then the entire maximal geodesic γ_{pq} is contained in F. It follows that $F(\infty)\subseteq\operatorname{Fix}(K)\cap H(\infty)$. Now let $x\in\operatorname{Fix}(K)\cap H(\infty)$ be given. The set F is nonempty by a theorem of E. Cartan. Fix a point $p\in F$ and let $\varphi\in K$ be chosen arbitrarily. Then $\varphi(\gamma_{px})=\gamma_{\varphi p\varphi x}=\gamma_{px}$, and hence φ fixes every point of γ_{px} since φ fixes p. It follows that $x\in F(\infty)$, which proves that $F(\infty)=\operatorname{Fix}(K)\cap H(\infty)$. This set is empty if and only if F is a single point in H.

We show that $\operatorname{Fix}(G)\cap H(\infty)=\operatorname{Fix}(K)\cap H(\infty)$. First we reduce the problem to the case that G is simple. Let $G=G_1\times\ldots\times G_m$ be the direct product decomposition of Lemma 3.2. If D_i is the normalizer of G_i in $I(H)$, then D_i satisfies the duality condition by the argument used in the last part of the proof of Lemma 3.1. If $p_i: G\to G_i$ is the natural projection homomorphism, then $K_i=p_i(K)$ is a maximal compact subgroup of G_i and K is the direct product $K=K_1\times\ldots\times K_m$. Observe that $\operatorname{Fix}(K)\cap H(\infty)=\bigcap_{i=1}^m \operatorname{Fix}(K_i)\cap H(\infty)$ and $\operatorname{Fix}(G)\cap H(\infty)=\bigcap_{i=1}^m \operatorname{Fix}(G_i)\cap H(\infty)$. It therefore suffices to prove that $\operatorname{Fix}(G_i)\cap H(\infty)=\operatorname{Fix}(K_i)\cap H(\infty)$ for every i.

By the previous paragraph we need only consider the case that G is a noncompact, connected, closed, simple subgroup of $I(H)$ whose normalizer D in $I(H)$ satisfies the duality condition. Clearly $\mathrm{Fix}(G)\cap H(\infty)\subseteq \mathrm{Fix}(K)\cap H(\infty)$ since $K\subseteq G$. Now let $x\in \mathrm{Fix}(K)\cap H(\infty)$ be given. We shall use Lemma 3.3 to show that $x\in \mathrm{Fix}(G)\cap H(\infty)$, which will complete the proof of Lemma 3.5.

Let $\mathfrak{k}$ denote the Lie algebra of K, and let $\mathfrak{P}$ denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ relative to the Killing form B. By the discussion of section 1, $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{P}$ is a Cartan decomposition of $\mathfrak{g}$; in particular, B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{P}$. Choose a point $p\in F=\mathrm{Fix}(K)\cap H$. By the work above $\gamma_{px}(\mathbf{R})\subseteq F$ and hence $X\equiv 0$ on the maximal geodesic γ_{px} for every $X\in\mathfrak{k}$. To prove that $x\in \mathrm{Fix}(G)\cap H(\infty)$ it now suffices by Lemma 3.3 and the remarks following its statement to show that $\|X\|^2$ is nonincreasing on γ_{px} for every $X\in\mathfrak{P}$. In fact we need only show that $\|X\|^2$ is bounded above on $\gamma_{px}[0,\infty)$ for every $X\in\mathfrak{P}$ since $\|X\|^2$ is a convex function on H.

The proof involves several steps and the fact that G is a simple group is crucial, particularly in steps 1 and 3.

Step 1. For any point $q\in F=\mathrm{Fix}(K)\cap H$ the orbit $G(q)$ is a Riemannian symmetric space of the noncompact type relative to the metric induced from H. Moreover, there exists a positive number $\lambda=\lambda(q)$ such that $\|X\|^2(q)=\lambda B(X,X)$ for every $X\in\mathfrak{P}$.

Step 2. If D denotes the normalizer of G in $I(H)$ and N the normalizer of K in D, then N leaves the manifold F invariant.

Step 3. Let $X\in\mathfrak{P}$ and $\varphi\in N$ be arbitrary. Then $\|X\|^2(\varphi q)=\|X\|^2(q)$ for all $q\in F$.

Assuming for the moment that these facts have been verified, we complete the proof of Lemma 3.5. Let $X\in\mathfrak{P}$, $x\in \mathrm{Fix}(K)\cap H(\infty)$ and $p\in F$ be chosen. We show that $\|X\|^2$ is bounded above on $\gamma_{px}[0,\infty)$, which will prove the lemma. We assume also that $X\neq 0$. The remainder of the proof is still quite detailed so we present an outline now. Since $L(D)=H(\infty)$ we may choose a sequence $\{\varphi_n\}\subseteq D$ so that $\varphi_n p\to x$. Next we show that there exists a sequence $\{Z_n\}\subseteq\mathfrak{P}$ such that $\|Z_n(p)\|=\|Z_n(\varphi_n p)\|=1$ for every n. Passing to a subsequence we let $\{Z_n\}$ converge to $Z\in\mathfrak{P}$. We show that $\|Z(p)\|=1$ and $\|Z(\gamma_{px}t)\|\leqslant 1$ for all $t\geqslant 0$. Returning now to the arbitrarily chosen element $X\in\mathfrak{P}$ we use Step 1 to conclude that $\|X(q)\|=\alpha\|Z(q)\|$ for all $q\in F$, where $\alpha=\|X(q)\|/\|Z(q)\|=\{B(X,X)/B(Z,Z)\}^{1/2}$ is positive and does not depend on $q\in F$. Finally $\|X(\gamma_{px}t)\|=\alpha\|Z(\gamma_{px}t)\|\leqslant\alpha$ for all $t\geqslant 0$ since $\gamma_{px}(\mathbf{R})\subseteq F$. This will complete the proof of Lemma 3.5.

We begin the proof of the assertions outlined above. Since D satisfies the duality condition it follows that $L(D)=H(\infty)$ and hence we may choose a sequence $\{\varphi_n\}\subseteq D$ such that $\varphi_n p\to x$ as $n\to\infty$.

We construct a sequence $\{\xi_n\}\subset N$ such that $\varphi_n(p)$ lies in the orbit $G(\xi_n p)$ for every n. The maximal compact subgroups of G are all conjugate by elements of G and hence we can choose a sequence $\{g_n\}\subseteq G$ so that $\varphi_n K\varphi_n^{-1}=g_n K g_n^{-1}$. If $\xi_n=g_n^{-1}\varphi_n$ then ξ_n lies in N, the normalizer of K in D. Moreover $\varphi_n=g_n\xi_n$ so that $\varphi_n(p)$ lies in $G(\xi_n p)$.

Next we construct a sequence $\{Z_n\}\subseteq\mathfrak{P}$ so that $\|Z_n(p)\|=\|Z_n(\varphi_n p)\|=1$ for every n. By Steps 1 and 2 each orbit $G(\xi_n p)$ is a symmetric space of noncompact type relative to the metric induced from H. Moreover the map $g(\xi_n p)\mapsto gK$ is a diffeomorphism of $G(\xi_n p)$ onto G/K for every n since K is the subgroup of G that fixes $\xi_n p$. Now let γ_n be the unit speed geodesic of $G(\xi_n p)$ such that $\gamma_n(0)=\xi_n p$ and $\gamma_n(t_n)=\varphi_n p$, where t_n is the distance in $G(\xi_n p)$ between $\xi_n p$ and $\varphi_n p$. Since $G(\xi_n p)$ is isometric to G/K with a suitable G-invariant metric it follows from [21, p. 173] that $\gamma_n(t)=\exp(tZ_n)(\xi_n p)$ for some $Z_n\in\mathfrak{P}$ with $\|Z_n(\xi_n p)\|=1$. Using Step 3 and the fact that Z_n has constant length along integral curves, we see that $\|Z_n(\varphi_n p)\|=\|Z_n(\xi_n p)\|=\|Z_n(p)\|=1$ for every n.

Since $\|Z_n(p)\|=1$ for every n we may pass to a subsequence if necessary so that $\{Z_n\}$ converges to $Z\in\mathfrak{P}$ with $\|Z(p)\|=1$. We show next that $\|Z(\gamma_{px}t)\|\leqslant 1$ for all $t\geqslant 0$. Let σ_n be the unit speed geodesic in H with $\sigma_n(0)=p$ and $\sigma_n(s_n)=\varphi_n p$, where $s_n=d(p,\varphi_n p)\to+\infty$ as $n\to\infty$. Since $\|Z_n\|^2$ is a convex function in H, we obtain $\|Z_n(\sigma_n t)\|\leqslant\max\{\|Z_n(\sigma_n 0)\|,\|Z_n(\sigma_n s_n)\|\}=1$ for all $t\in[0,s_n]$. Since $\varphi_n p\to x$ it follows that $\sigma_n t\to\gamma_{px}t$ as $n\to\infty$ for every $t\in\mathbf{R}$. Since $Z_n\to Z$ it follows that

$$\|Z(\gamma_{px}t)\|=\lim_{n\to\infty}\|Z_n(\sigma_n t)\|\leqslant 1\quad\text{for every } t\geqslant 0.$$

We consider now the arbitrarily chosen element $X\in\mathfrak{P}$. We may assume that $X(p)\neq 0$ for otherwise $X\equiv 0$ on F and in particular on γ_{px} by Step 1. Since $\|Z(p)\|=1$ we also conclude from Step 1 that $\|X(q)\|=\alpha\|Z(q)\|$ for every $q\in F$, where $\alpha=\|X(q)\|/\|Z(q)\|=\{B(X,X)/B(Z,Z)\}^{1/2}$ is a positive constant. Finally, $\|X(\gamma_{px}t)\|=\alpha\|Z(\gamma_{px}t)\|\leqslant\alpha$ for all $t\geqslant 0$ since $\gamma_{px}(\mathbf{R})\subseteq F$. By earlier remarks this completes the proof of Lemma 3.5 except for the verification of Steps 1 through 3.

We verify Step 1. For any point $q\in H$ the orbit $G(q)$ is diffeomorphic to the coset space G/K^*, where K^* is the subgroup of G that fixes q. If q is a point of F then $K^*\supseteq K$, and it follows that $K^*=K$ since K^* is always compact and K is a maximal compact subgroup of G by hypothesis. The elements of G are isometries of H and hence act as isometries of $G(q)$ relative to the metric Q induced from H. If Q also denotes the metric on G/K induced by the diffeomorphism $g(q)\mapsto gK$, then Q is a G-invariant metric on G/K. By [21, p. 173] the coset space G/K equipped with the metric Q is a symmetric space of noncompact type. Hence $G(q)$ with the metric Q is a symmetric space of noncompact type for every point $q\in F$.

The fact that G is simple and the discussion of section 1 (or more precisely Theorem 8.2.9 of [30, p. 238]) show that any G-invariant metric on the coset space G/K is induced by the inner product λB on $\mathfrak{P}$ for some positive number λ, where B is the Killing form of $\mathfrak{g}$. Let $q \in F$ be given and let $\lambda = \lambda(q) > 0$ be chosen so that $G(q)$ with the metric induced from H is isometric to G/K with the metric induced from λB on $\mathfrak{P}$. The vectors tangent to $G(q)$ at q are precisely the vectors $X(q)$ with $X \in \mathfrak{P}$. Therefore $\|X\|^2(q) = \lambda B(X, X)$ for every $X \in \mathfrak{P}$, which completes the proof of Step 1.

To verify Step 2, let $\varphi \in N$, $q \in F$ and $k \in K$ be given. Then $k(\varphi q) = \varphi k^*(q) = \varphi q$, where $k^* = \varphi^{-1} k \varphi \in K$. Hence N leaves F invariant.

We conclude with the proof of Step 3. Let $X \in \mathfrak{P}$, $\varphi \in N$ and $q \in F$ be given. By Steps 1 and 2 the orbits $G(q)$ and $G(\varphi q)$ are both symmetric spaces of noncompact type relative to the metric induced from H. Moreover $G(q)$ and $G(\varphi q) = \varphi G(q)$ are isometric since φ is an isometry of H. The proof of Step 1 shows that there exist positive constants λ_1, λ_2 such that $G(q)$ and $G(\varphi q)$ are isometric to the coset space G/K with G-invariant metrics induced respectively from the inner products $\lambda_1 B, \lambda_2 B$ on $\mathfrak{P}$. It follows that $\lambda_1 = \lambda_2$ since the coset spaces G/K with the metrics $\lambda_1 B$ and $\lambda_2 B$ are isometric. Therefore, $\|X\|^2(q) = \lambda_1 B(X, X) = \lambda_2 B(X, X) = \|X\|^2(\varphi q)$ by Step 1 for every $X \in \mathfrak{P}$. This completes the proof of Step 3 and of Lemma 3.5.

Section 4. Applications

In this section, we apply the main theorem of section 3 to obtain a variety of results concerning the structure of isometry groups, lattices, characterizations of symmetric spaces and other questions.

Structure of isometry groups

PROPOSITION 4.1. *Let H be an arbitrary Hadamard manifold such that $I(H)$ satisfies the duality condition. Then there exist Hadamard manifolds H_0, H_1, H_2, two of which may have dimension zero, such that*

(1) *H is isometric to the Riemannian product $H_0 \times H_1 \times H_2$,*
(2) *H_0 is a Euclidean space,*
(3) *H_1 is a symmetric space of noncompact type,*
(4) *$I(H_2)$ is discrete but satisfies the duality condition.*

Remark. This and the next two results should be compared to Corollary 4.2 of [5]. This result also greatly strengthens Proposition 2.5 of [11].

Proof. We begin by writing H as a Riemannian product $H_0 \times H^*$, where H_0 is the Euclidean de Rham factor of H and H^* is the Riemannian product of all nonEuclidean de Rham factors of H. Then $I(H)=I(H_0)\times I(H^*)$ since every isometry of H leaves invariant the foliation of H induced by the Euclidean de Rham factor H_0. Let $p: I(H)\to I(H^*)$ denote the projection homomorphism. By the discussion of section 1, $I(H^*)=p(I(H))$ satisfies the duality condition since $I(H)$ satisfies the duality condition.

If $I(H^*)$ is discrete then we set $H_2=H^*$ and the proposition is proved for the case that the factor H_1 is missing. Suppose now that $I(H^*)$ is not discrete and let $G=I_0(H^*)$. By the main theorem there exists a Riemannian product decomposition $H^*=H_1\times H_2$ such that H_1 is a symmetric space of noncompact type, $G=I_0(H_1)$ and $H_2(\infty)=\mathrm{Fix}\,(G)\cap H^*(\infty)$ To complete the proof of proposition it remains only to show that $I(H_2)$ is discrete and satisfies the duality condition.

The group $I(H_2)$ is discrete since $I_0(H^*)=I_0(H_1)\times\{1\}$. To show that $I(H_2)$ satisfies the duality condition we first observe that $I(H_1)\times I(H_2)$ is a subgroup of finite index in $I(H^*)$ by the discussion of section 1 since every element of $I(H^*)$ permutes the foliations of H^* induced by the de Rham factors of H^*. Therefore $I(H_1)\times I(H_2)$ satisfies the duality condition by the proposition of Appendix I. If $p: I(H_1)\times I(H_2)\to I(H_2)$ is the projection homomorphism, then $I(H_2)$ satisfies the duality condition by the discussion of section 1. This completes the proof of the proposition.

PROPOSITION 4.2. *Let H be an arbitrary Hadamard manifold with no Euclidean de Rham factor. Let $G\subseteq I(H)$ be a subgroup whose normalizer D in $I(H)$ satisfies the duality condition. Then either*

(1) *G is discrete*

or

(2) *there exist Hadamard manifolds H_1, H_2 such that*

(a) *H is isometric to the Riemannian product $H_1\times H_2$,*

(b) *H_1 is a symmetric space of noncompact type,*

(c) $(\bar{G})_0=I_0(H_1)$,

(d) *there exists a discrete subgroup $B\subseteq I(H_2)$ whose normalizer in $I(H_2)$ satisfies the duality condition such that $I_0(H_1)\times B$ is a subgroup of $\bar{G}$ of finite index in $\bar{G}$.*

Proof. Let $G\subseteq I(H)$ be a subgroup whose normalizer D in $I(H)$ satisfies the duality condition. Suppose that G is not discrete. Then D also normalizes the closed connected group $G^*=(\bar{G})_0\subseteq I(H)$. By applying the main theorem to G^* we obtain a Riemannian product decomposition $H=H_1\times H_2$ such that H_1 is a symmetric space of noncompact type

type and $G^*=I_0(H_1)$. The proof is complete. Assertion (2d) follows routinely from assertion (2c). Let $\tilde{G}$ be a finite index subgroup of $\bar{G}$ that leaves invariant the foliations of H induced by H_1, H_2, and let $B=p_2(\tilde{G})$, where $p_2:\tilde{G}\to I(H_2)$ is the projection homomorphism.

PROPOSITION 4.3. *Let H be an irreducible Hadamard manifold, and let $G\subseteq I(H)$ be a subgroup that satisfies the duality condition. Then either G is discrete or $(\bar{G})_0=I_0(H)$ and H is a symmetric space of noncompact type.*

A Riemannian manifold N is said to be reducible if it is the Riemannian product of two manifolds of positive dimension. N is irreducible if it is not reducible. The proposition above follows immediately from Proposition 4.2.

Lattices

For any Hadamard manifold H a group $\Gamma\subseteq I(H)$ is a lattice if the quotient space H/Γ is a smooth Riemannian manifold of finite volume. The lattice Γ is uniform or nonuniform according to whether H/Γ is compact or noncompact. A lattice Γ is reducible if the manifold H/Γ has a finite cover that is reducible as a Riemannian manifold. A lattice Γ is irreducible if it is not reducible. The next result greatly strengthens Theorem 4.2 of [15].

PROPOSITION 4.4.(1) *Let H be an arbitrary Hadamard manifold, and let Γ be an irreducible lattice in H that does not contain Clifford translations. Then either*

(1) *$I(H)$ is discrete, Γ has finite index in $I(H)$ and H is irreducible*

or

(2) *H is isometric to the Riemannian product of a Euclidean space with a symmetric space of noncompact type.*

Proof. An isometry φ of a Hadamard manifold H is a Clifford translation if the displacement function $d_\varphi: p\to d(p, \varphi p)$ is constant in H. See [29] or section 2 of [11] for further facts about Clifford translations.

By the discussion of section 1 any lattice satisfies the duality condition and hence $I(H)$ satisfies the duality condition. We suppose first that $I(H)$ is discrete and prove the remaining assertions of (1). In this case it is easy to see that Γ must have finite index in

(1) See added in proof.

$I(H)$. The quotient space H/Γ has finite volume and hence the quotient space $H/I(H)$, a manifold with singularities, must also have finite volume. Since $I(H)$ is discrete this is possible only if Γ has finite index in $I(H)$. For a more detailed argument see the proof of Proposition 2.2 of [15].

We show next that H is irreducible if $I(H)$ is discrete. Suppose that H is reducible and write $H=H_1\times\ldots\times H_k$, a Riemannian product where $k\geq 2$ and each H_i is irreducible. Note that no manifold H_i is a Euclidean space since $I(H)$ is discrete. Let Γ^* be the subgroup of Γ consisting of those elements φ in Γ that preserve the foliations of H induced by the de Rham factors H_i. Each element φ in Γ^* can be written $\varphi=\varphi_1\times\ldots\times\varphi_k$, where $\varphi_i\in I(H_i)$. Let $p_i:\Gamma^*\to I(H_i)$ be the obvious projection homomorphism, and let Γ_i^* denote $p_i(\Gamma^*)$ for each $1\leq i\leq k$. The group Γ^* is a lattice since it has finite index in Γ. Hence Γ^* satisfies the duality condition and by the discussion of section 1, Γ_i^* satisfies the duality condition for each $1\leq i\leq k$. If Γ_i^* is not discrete for some i, then H_i is a symmetric space of noncompact type by Proposition 4.3, contradicting the hypothesis that $I(H)$ is discrete. Therefore, Γ_i^* is discrete for every i and it follows by Proposition 2.2 of [15] that Γ^* is reducible. This implies that Γ is reducible, contradicting our hypothesis. Therefore H is irreducible if $I(H)$ is discrete.

Suppose now that $I(H)$ is not discrete. By Proposition 4.1, there exists a Riemannian product decomposition $H=H_0\times H_1\times H_2$, where H_0 is a Euclidean space, H_1 is a symmetric space of noncompact type and $I(H_2)$ is a discrete group that satisfies the duality condition. The proposition will be proved when we show that H_2 has dimension zero. Suppose that H_2 has positive dimension, and let Γ^* be the finite index subgroup of Γ that leaves invariant the foliations of H corresponding to the factors H_0, H_1, H_2. Let $p_i:\Gamma^*\to I(H_i)$ be the corresponding projection homomorphisms for $i=0$, 1, 2. Since $I(H_2)$ is discrete it follows that $\Gamma_2^*= p_2(\Gamma^*)$ is also discrete. If $p=p_0\times p_1$: $\Gamma^*\to I(H_0\times H_1)$, then $\Gamma_1^*=p(\Gamma^*)$ is discrete by Theorem 4.1 of [15] since Γ contains no Clifford translations. Therefore, Γ^* is reducible by Proposition 2.2 of [15] and hence Γ is reducible, contradicting our hypothesis. Therefore, H_2 has dimension zero and the proposition is proved.

PROPOSITION 4.5. *Let H be a reducible Hadamard manifold with no Euclidean de Rham factor, and let Γ be a lattice in H. If Γ is irreducible then H is a symmetric space of noncompact type.*

Proof. Let Γ and H be as described above. Since H has no Euclidean de Rham factor the lattice Γ contains no Clifford translations by Theorem 1 of [29]. It now follows immediately from the previous result that H is a symmetric space of noncompact type.

PROPOSITION 4.6.(2) *Let H be a reducible symmetric space with no Euclidean de Rham factor. Let H^* be a reducible Hadamard manifold with no Euclidean de Rham factor. Let Γ be an irreducible uniform lattice in H, and let Γ^* be a uniform lattice in H^*. If Γ and Γ^* are isomorphic as groups, then H/Γ and H^*/Γ^* are isometric as manifolds if one multiples the metric of H or H^* by a suitable positive constant.*

Remark. The hypothesis that Γ^* be a *uniform* lattice is really unnecessary. For topological reasons it follows that if two lattices are isomorphic as groups then either both are uniform or both are nonuniform. Moreover, the spaces on which they act as isometries have the same dimension.

Proof. We observe first that Γ^* has no Clifford translations by Theorem 1 of [29] since H^* has no Euclidean de Rham factor. We show next that Γ^* acts irreducibly on H^*, and it will then follow from Proposition 4.4 that H^* is a symmetric space of noncompact type. Finally, the rigidity theorem of Mostow will say that H/Γ and H^*/Γ^* are isometric if one multiplies the metric of H or H^* by a suitable positive constant.

Suppose that Γ^* is a reducible lattice. Then we can find a finite index subgroup Γ^{**} of Γ^* that is a direct product of subgroups Γ_1^* and Γ_2^*. Let $\theta: \Gamma^* \to \Gamma$ be an isomorphism. If $\tilde{\Gamma}=\theta(\Gamma^{**})$ and $\tilde{\Gamma}_i=\theta(\Gamma_i^*)$ for $i=1,2$, then $\tilde{\Gamma}$ has finite index in Γ and $\tilde{\Gamma}$ is the direct product of $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$. The lattice $\tilde{\Gamma}$ has trivial center since any central element of $\tilde{\Gamma}$ would be a Clifford translation by Proposition 2.3 of [11], contradicting the hypothesis that H has no Euclidean de Rham factor. Therefore, by Theorem 2 of [25] or Theorem 2 of [18] we conclude that $H/\tilde{\Gamma}$ is the Riemannian product of two manifolds with fundamental groups isomorphic to $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$. This contradicts the hypothesis that Γ (hence also $\tilde{\Gamma}$) is an irreducible lattice in H. Therefore, Γ^* must be an irreducible lattice in H^*, which completes the proof of the proposition in the manner indicated above. We use the form of the rigidity theorem stated in [26] on p. 133 or on p. 187.

PROPOSITION 4.7. *Let H be a reducible Hadamard manifold with no Euclidean de Rham factor. Let $\Gamma \subseteq I(H)$ be a lattice that preserves the factors of the de Rham decomposition of H. Then the following conditions are equivalent:*

(1) *Γ is irreducible,*

(2) *if $H=H_1 \times H_2$ is any decomposition of H as a Riemannian product and if*

(2) See added in proof.

$p_i:\Gamma\to I(H_i)$ are the corresponding projection homomorphisms for $i=1,2$, then the kernel of $p_i=\{1\}$ for $i=i,2$,

(3) *H is a symmetric space of noncompact type. If $H=H_1\times H_2$ is any decomposition of H as a Riemannian product and if $p_i:\Gamma\to I(H_i)$ are the corresponding projection homomorphisms for $i=1,2$, then $\overline{p_i(\Gamma)}\supseteq I_0(H_i)$ for $i=1,2$.*

Remarks. (i) The existence of the projections p_1, p_2 in conditions (2) and (3) is a consequence of the fact that if an isometry φ of H preserves the foliations induced by the de Rham factors of H then it preserves the foliations induced by any Riemannian product decomposition of H. If H is a Riemannian product $H_1\times H_2$, then each H_i must be a Riemannian product of certain de Rham factors of H for $i=1,2$.

(ii) If Γ is irreducible then H is a symmetric space of noncompact type by Proposition 4.5. The Proposition now becomes a known result; see for example the corollary of [28, p. 40] or Corollaries 5.21 and 5.23 of [27, pp. 86–87]. For convenience, we include a short proof in Appendix II that uses the main theorem of section 3.

Characterizations of symmetric spaces

The next three results give solutions to problems 2, 3 and 5 posed at the end of [11].

PROPOSITION 4.8. *Let H be an irreducible Hadamard manifold such that $I(H)$ satisfies the duality condition. Then either $I(H)$ is discrete or H is a symmetric space of noncompact type.*

Proof. This follows immediately from Proposition 4.3.

PROPOSITION 4.9. *Let H be a Hadamard manifold such that $I_0(H)$ satisfies the duality condition. Then H is the Riemannian product of a Euclidean space H_0 with a symmetric space H_1 of noncompact type (either H_0 or H_1 may be absent).*

Proof. By Proposition 4.1 we can write H as a Riemannian product $H_0\times H_1\times H_2$, where H_0 is a Euclidean space, H_1 is a symmetric space of noncompact type and $I(H_2)$ is a discrete group. Clearly $I_0(H)=I_0(H_0)\times I_0(H_1)\times\{1\}$ and $I_0(H)$ can satisfy the duality condition only if the factor H_2 has dimension zero. This completes the proof.

PROPOSITION 4.10. *If $I(H)$ is noncompact and acts minimally on $H(\infty)$, then either $I(H)$ is discrete or H is a Euclidean space or H is a rank 1 symmetric space of noncompact type.*

Proof. To say that $I(H)$ acts minimally on $H(\infty)$ means that every orbit of $I(H)$ in $H(\infty)$ is dense in $H(\infty)$. By Theorem 5.7 of [11] H is irreducible and by Proposition 4.7 of [11] the group $I(H)$ satisfies the duality condition. The result now follows from Proposition 4.1 and Propositions 4.12 and 4.13 of [11].

Other applications

All results in this section have dealt with isometry groups $G\subseteq I(H)$ whose normalizers D in $I(H)$ satisfy the duality condition. In some cases one may obtain similar results if one replaces the duality condition by the weaker condition that $L(D)=H(\infty)$. However, in general one must then place additional restrictions on G. We give two examples. The first is clearly a variation of the main theorem of section 3.

PROPOSITION 4.11. *Let H be an arbitrary Hadamard manifold, and let $G\subseteq I(H)$ be a closed connected nonidentity Lie subgroup. Assume that $L(D)=H(\infty)$, where D is the normalizer of G in $I(H)$. If there exists a point $p\in H$ such that the orbit $G(p)$ is a totally geodesic submanifold of H, then there exists a Riemannian product decomposition $H=H_1\times H_2$ such that H_1 is homogeneous and G is a transitive subgroup of $I_0(H_1)\times\{1\}$.*

Remark. If $G=I_0(H)\neq 1$ then we can show also that $G=I_0(H_1)\times\{1\}$ and $I(H_2)$ is a discrete group such that $L(I(H_2))=H_2(\infty)$.

Proof. Let $p\in H$ be a point such that $B=G(p)$ is a totally geodesic submanifold of H. Since G is closed in $I(H)$ it follows that B is closed as a subset of H and hence B is complete as a Riemannian manifold. The result now follows from Corollary 2.3.

PROPOSITION 4.12. *Let $G\subseteq I(H)$ be a subgroup whose center A is nontrivial and contains no parabolic isometries. If the normalizer D of G in $I(H)$ satisfies the condition $L(D)=H(\infty)$ then*

(1) *A consists of Clifford translations of H,*

(2) *There exists a Riemannian product decomposition $H=H_1\times H_2$ such that H_1 is a Euclidean space of positive dimension and $A\subseteq I(H_1)\times\{1\}$ is a group of translations of H_1 such that the quotient space H_1/A is compact.*

Remark. If one requires D to satisfy the duality condition and deletes the hypothesis that A have no parabolic elements, then assertion (1) above becomes Theorem 2.4 of [11]. An isometry φ of H is parabolic if the displacement function $d_\varphi : p\mapsto d(p, \varphi p)$ has no minimum in H. If d_φ has a positive or zero minimum then φ is respectively hyperbolic or elliptic.

Proof. Let $\bar{A}$ denote the closure of A in $I(H)$. Then $\bar{A}$ is an abelian Lie subgroup of $I(H)$. By Theorem 1′ of [18] there exists a flat, totally geodesic submanifold B of H such that $\varphi(B)=B$ for all $\varphi\in\bar{A}$ and the quotient space $B/\bar{A}$ is compact. The latter assertion implies that there exists a compact set $C\subseteq B$ such that B is the union of the sets $\varphi(C)$, $\varphi\in\bar{A}$. If C has diameter $\leq R$ then for any geodesic γ of B and any point $p\in C$ we can find a sequence $\{\varphi_n\}\subseteq\bar{A}$ such that $d(\varphi_n p, \gamma(n))\leq R$ for every positive integer n. It follows that $\varphi_n(p)\to\gamma(\infty)$ as $n\to\infty$, which shows that $L(\bar{A})=B(\infty)$. The group D normalizes G and hence also normalizes both A and $\bar{A}$. We may now apply Corollary 2.3 to conclude that there exists a Riemannian product decomposition $H=H_1\times H_2$ such that $H_1(\infty)=B(\infty)$ and $\bar{A}\subseteq I(H_1)\times\{1\}$. Moreover B is one of the leaves of the foliation of H induced by H_1 so that H_1 is a Euclidean space. The quotient space $H_1/\bar{A}$ is compact since $B/\bar{A}$ is compact. It will follow that H_1/A is compact once we show that $\bar{A}$ consists of Clifford translations. This step will also complete the proof of the proposition.

From Proposition 2.3 of [11] and the fact that $\bar{A}$ is abelian with $L(\bar{A})=B(\infty)$ we see that the displacement function $d_\varphi: p\to d(p, \varphi p)$ is a constant function on B for every $\varphi\in\bar{A}$. Hence each element $\varphi\in\bar{A}$ acts as a (Clifford) translation on B and since $\bar{A}\subseteq I(H_1)\times\{1\}$ it follows immediately that $\bar{A}$ consists of Clifford translations of H.

Appendix I

We prove the following result stated in section 1:

PROPOSITION. *Let H be a Hadamard manifold and let $D\subseteq I(H)$ be a subgroup that satisfies the duality condition. If D^* is a subgroup of D with finite index in D, then D^* satisfies the duality condition.*

Proof. We first need the following

LEMMA. *Let $D\subseteq I(H)$ satisfy the duality condition, and let $\tilde{D}$ be a normal subgroup of D. Then $\Omega(\tilde{D})\subseteq T_1H$ is invariant under $\{\varphi_*:\varphi\in D\}$.*

Proof. The set $\Omega(\tilde{D})$ is defined in the introduction. Let $v\in\Omega(\tilde{D})$ and $\varphi\in D$ be given. We show that $(\varphi)_* v\in\Omega(\tilde{D})$. By definition we can choose sequences $\{\psi_n\}\subset\tilde{D}$, $\{t_n\}\subseteq\mathbf{R}$ and $\{v_n\}\subset T_1H$ such that $v_n\to v$, $t_n\to+\infty$ and $(\psi_n)_* T_{t_n} v_n\to v$ as $n\to+\infty$. Here $\{T_t\}$ denotes the geodesic flow. Define $\tilde{\psi}_n=\varphi\psi_n$ for every n. Then $\tilde{\psi}_n=\alpha_n\varphi$, where $\alpha_n=$

$\varphi\psi_n\varphi^{-1}\in\tilde{D}$ for every n. Then $(\tilde{\psi}_n)_* T_{t_n} v_n=(\varphi)_*\{(\psi_n)_* T_{t_n} v_n\}\to(\varphi)_* v$ as $n\to+\infty$. On the other hand, $(\tilde{\psi}_n)_* T_{t_n} v_n=(\alpha_n)_*\{(\varphi)_* T_{t_n} v_n\}=(\alpha_n)_* T_{t_n}\{(\varphi)_* v_n\}$ for every n. Since $(\varphi)_* v_n\to(\varphi)_* v$ and $(\alpha_n)_* T_{t_n}\{(\varphi)_* v_n\}\to(\varphi)_* v$, where $\{\alpha_n\}\subseteq\tilde{D}$, it follows that $(\varphi)_* v\in\Omega(\tilde{D})$. The lemma is proved.

We now prove the proposition. Let $D\subseteq I(H)$ satisfy the duality condition, and let D^* be a subgroup of finite index. Let $\xi_1,\dots,\xi_k$ be elements in D such that D is the union of the left cosets $\xi_i D^*$, $1\le i\le k$. If $D^{**}=\bigcap_{i=1}^{k}\xi_i D^*\xi_i^{-1}$, then D^{**} is a normal subgroup of D and has finite index in D. Since $D^*\supseteq D^{**}$ it suffices to prove that D^{**} satisfies the duality condition. This is equivalent to proving that $\Omega(D^{**})=T_1 H$ by the discussion of the introduction.

Let $A=\{v\in T_1 H:(\varphi_n)_* T_{t_n} v\to v$ as $n\to+\infty$ for some sequence $\{\varphi_n\}\subseteq D$ and some sequence $\{t_n\}\subset\mathbf{R}$ with $t_n\to+\infty\}$. Since D satisfies the duality condition, it follows by the argument in [15, p. 464] that A is dense in $T_1 H$. It suffices now to show that $A\subseteq\Omega(D^{**})$ since $\Omega(D^{**})$ is a closed subset of $T_1 H$.

Let $v\in A$ be given and choose sequences $\{\varphi_n\}\subseteq D$ and $\{t_n\}\subset\mathbf{R}$ such that $t_n\to+\infty$ and $(\varphi_n)_* T_{t_n} v\to v$ as $n\to+\infty$. By passing to a subsequence, we can find an element $\alpha\in D$ and a sequence $\{\tilde{\varphi}_n\}\subseteq D^{**}$ such that $\varphi_n=\alpha\tilde{\varphi}_n$ for every n. Hence $(\alpha)_*(\tilde{\varphi}_n)_* T_{t_n} v\to v$ or $(\tilde{\varphi}_n)_* T_{t_n} v\to(\alpha^{-1})_* v$ as $n\to+\infty$. To show that $v\in\Omega(D^{**})$ it suffices to show that $(\alpha^{-1})_* v\in\Omega(D^{**})$ by the lemma above. It will then follow that $A\subseteq\Omega(D^{**})$ since v is arbitrary. Let $O\subseteq T_1 H$ be any neighborhood of $(\alpha^{-1})_* v$ and let $v_n=(\tilde{\varphi}_n)_* T_{t_n} v$ for every n. Choose a positive integer N so that $v_n\in O$ for all $n\ge N$. Note that $v_n=(\tilde{\varphi}_n\tilde{\varphi}_N^{-1})_* T_{(t_n-t_N)} v_N$ for every $n\ge N$. Hence $[(\tilde{\varphi}_n\tilde{\varphi}_N^{-1})_* T_{(t_n-t_N)}(O)]\cap O$ is nonempty for every $n\ge N$. Since $t_n-t_N\to+\infty$ as $n\to+\infty$ and the neighborhood O is arbitrary, it follows that $(\alpha^{-1})_* v\in\Omega(D^{**})$.

Appendix II

Proof of Proposition 4.7. (1)$\Rightarrow$(3) By Proposition 4.5 H is a symmetric space of noncompact type. Now let H be decomposed into a Riemannian product $H_1\times H_2$, and let $p_i:\Gamma\to I(H_i)$ be the corresponding projection homomorphisms for $i=1,2$. Neither of the groups $\Gamma_i=p_i(\Gamma)$ can be discrete for $i=1,2$ for otherwise Γ would be reducible by Theorem 4.1 of [15] and Proposition 2.2 of [15]. Therefore $G_i=(\bar{\Gamma}_i)_0$ is a closed connected Lie subgroup of $I(H_i)$ of positive dimension for $i=1,2$.

We show that $G_i=I_0(H_i)$ for $i=1,2$ and we consider only the case $i=1$. By the discussion of section 1, Γ_1 satisfies the duality condition since Γ does. Since Γ_1

normalizes G_1 it follows from the main theorem of section 3 that H_1 splits as a Riemannian product $H_\alpha \times H_\beta$ such that $G_1 = I_0(H_\alpha)$. It suffices to show that $H_\alpha = H_1$. Suppose instead that H_β has positive dimension and let $p_\alpha : \Gamma \to I(H_\alpha)$ and $p_\beta : \Gamma \to I(H_\beta)$ be the projection homomorphisms. The group $\Gamma_\beta = p_\beta(\Gamma) = p_\beta(\Gamma_1)$ satisfies the duality condition in H_β and is discrete since $(\bar{\Gamma}_i)_0 = I_0(H_\alpha)$. If we let H^* denote the Riemannian product $H_\alpha \times H_2$ and let $p^* : \Gamma \to I(H^*)$ denote the projection homomorphism, then by applying Theorem 4.1 of [15] and Proposition 2.2 of [15] to the Riemannian product decomposition $H = H_\beta \times H^*$ we see that Γ is reducible, contradicting our hypothesis. Therefore H_β has dimension zero and $(\bar{\Gamma}_1)_0 = I_0(H_\alpha) = I_0(H_1)$.

The assertion (2)$\Rightarrow$(1) follows immediately from the definition of irreducibility of a lattice. It remains only to prove that (3)$\Rightarrow$(2). Let H, H_1, H_2, p_1 and p_2 be as in the statement of (2), and let N_1, N_2 denote the kernels of p_1, p_2 respectively. We show only that $N_1 = \{1\}$. By definition N_1 is a subgroup of $\Gamma \cap I(H_2)$ and hence is discrete in $I(H_2)$. Since N_1 is normal in Γ it follows that $N_1 = p_2(N_1)$ is normal in $\Gamma_2 = p_2(\Gamma)$. By hypothesis $\bar{\Gamma}_2 \supseteq I_0(H_2)$ and hence $I_0(H_2)$ normalizes N_1. For each X in the Lie algebra of $I_0(H_2)$ and each $\varphi \in N_1$ the curve $t \to \exp(tX)\,\varphi \exp(-tX)$ lies in N_1 and hence must be constant since N_1 is discrete. Therefore N_1 and $I_0(H_2)$ commute and by Proposition 2.3 of [11] the group N_1 consists of Clifford translations. However, H_2 has no Euclidean de Rham factor since H does not and therefore H_2 cannot admit nonidentity Clifford translations by Theorem 1 of [29]. It follows that $N_1 = \{1\}$ and similarly we see that $N_2 = \{1\}$.

Added in proof. (1) Recently, we have proved that if H is a Hadamard manifold with Euclidean de Rham factor H_0 of dimension $k \geq 1$ then for every lattice Γ in H the subgroup of Clifford translations of Γ is free abelian of rank $k \geq 1$. In particular if case (1) of Proposition 4.4 does not hold then in case (2) H must be isometric to a symmetric space of noncompact type.

(2) Proposition 4.6 remains true even if one omits the hypothesis on H^* concerning reducibility and the nonexistence of a Euclidean de Rham factor. Details will appear elsewhere. Gromov has proved an even more general result.

References

[1] BALLMANN, W., *Einige neue resultate über Mannigfaltigkeiten negativer Krümmung.* Dissertation, Univ. of Bonn, 1978.

[2] — *Einige neue resultate über Mannigfaltigkeiten negativer Krümmung.* Bonner Math. Schriften, Vol. 113, 1978.

[3] BISHOP, R. & O'NEILL, B., Manifolds of negative curvature. *Trans. Amer. Math. Soc.*, 145 (1969), 1–49.

[4] BOREL, A., Compact Clifford-Klein forms of symmetric spaces. *Topology*, 2 (1963), 111–122.
[5] — Density properties for certain subgroups of semi-simple groups without compact components. *Ann. of Math.*, 72(1) (1960), 179–188.
[6] CARTAN, E., *Leçons sur la géométrie des espaces de Riemann*. Paris, Gauthier-Villars, 1946.
[7] — Sur une classe remarquable d'espaces de Riemann. *Bull. Soc. Math. France*, 54 (1926), 214–264 and 55 (1927), 114–134.
[8] — Sur certaines formes Riemanniennes remarquables des géométries à group fondamental simple. *Ann. École Norm.*, 44 (1927), 345–467.
[9] — Les espaces Riemanniens symétriques. *Verh. Int. Math. Kongr.*, I, *Zürich* (1932), 152–161.
[10] CHEN, S., Duality condition and Property (S). *Pacific J. Math.*, 98 (1982), 313–322.
[11] CHEN, S. & EBERLEIN, P., Isometry groups of simply connected manifolds of nonpositive curvature. *Illinois J. Math.*, 24 (1) (1980), 73–103.
[12] DANI, S. G., A simple proof of Borel's density theorem. *Math. Z.*, 174 (1980), 81–94.
[13] EBERLEIN, P., Geodesic flows on negatively curved manifolds, I. *Ann. of Math.*, 95 (1972), 492–510.
[14] — Geodesic flows on negatively curved manifolds, II. *Trans. Amer. Math. Soc.*, 178 (1973), 57–82.
[15] — Lattices in spaces of nonpositive curvature. *Ann. of Math.*, 111 (1980), 435–476.
[16] EBERLEIN, P. & O'NEILL, B., Visibility manifolds. *Pacific J. Math.*, 46 (1973), 45–109.
[17] GOTO, M. & GOTO, M., Isometry groups of negatively pinched 3-manifolds. *Hiroshima Math. J.*, 9 (2) (1979), 313–319.
[18] GROMOLL, D. & WOLF, J., Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of nonpositive curvature. *Bull. Amer. Math. Soc.*, 77 (1971), 545–552.
[19] HEINTZE, E., Personal communication.
[20] — *Mannigfaltigkeiten negativer Krümmung*. Habilitationschrift, University of Bonn, 1976.
[21] HELGASON, S., *Differential geometry and symmetric spaces*. Academic Press, New York, 1962.
[22] KOBAYASHI, S., Fixed points of isometries. *Nagoya Math. J.*, 13 (1958), 63–68.
[23] — *Transformation groups in differential geometry*. Springer-Verlag, New York, 1972.
[24] KOBAYASHI, S. & NOMIZU, K., *Foundations of differential geometry, Vol. 1*. J. Wiley and Sons, New York, 1963, pp. 179–193.
[25] LAWSON, H. B. & YAU, S.-T., Compact manifolds of nonpositive curvature. *J. Differential Geom.*, 7 (1972), 211–228.
[26] MOSTOW, G. D., *Strong rigidity of locally symmetric spaces*. Annals of Math. Studies, number 78. Princeton University Press, Princeton, New Jersey, 1973.
[27] RAGHUNATHAN, M. S., *Discrete subgroups of Lie groups*. Springer-Verlag, New York, 1972.
[28] SHIMIZU, H., On discontinuous groups operating on the product of upper half planes. *Ann. of Math.*, 77 (1) (1963), 33–71.
[29] WOLF, J., Homogeneity and bounded isometries in manifolds of negative curvature. *Illinois J. Math.*, 8 (1964), 14–18.
[30] — *Spaces of constant curvature*. 2nd edition, published by the author. Berkeley, California, 1972.

Received February 2, 1981