ARITHMETIC MEANS AND THE TAUBERIAN CONSTANT .474541.

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1. Introduction.

Let Σu_n be a series of complex terms satisfying the Tauberian condition $\limsup |nu_n| < \infty$. Let $s_n = u_0 + u_1 + \cdots + u_n$ denote the sequence of partial sums of Σu_n, and let

$$(1.1)\qquad M_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) u_k$$

denote the arithmetic mean transform. The Kronecker formula

$$(1.2)\qquad M_n - s_n = \frac{1}{n+1} \sum_{k=0}^{n} k u_k,$$

which follows from (1.1), implies that the formula

$$(1.3)\qquad \limsup_{n\to\infty} |M_n - s_{p_n}| \leqq B \limsup |nu_n|$$

holds when $p_n = n$ and $B = 1$.

The questions with which we are concerned are the following where in one case we assume that Σu_n has bounded partial sums, and in the other case we do not make this assumption. How much can we reduce the constant B in (1.3) if, instead of requiring that $p_n = n$, we allow p_n to be the optimum sequence that can be selected after the series Σu_n has been given? It was shown in [3, Theorem 5.4] that B can be reduced to $\log 2 = .69315$, and no further, if we require that p_n be a function of n alone so that p_n must be independent of the terms of Σu_n. Moreover (1.3) holds when $p_n = [n/2]$ and $B = .69315$. It was also shown in [3, Theorem 9.2] that B can be reduced to .56348 by choosing p_n to be the most favorable one of the two integers $[3n/8]$ and $[5n/8]$, the choice being allowed to depend upon the terms of the series Σu_n. Finally, it was shown in [4, Theorem 14.3] that, even when Σu_n is assumed to have bounded partial sums, B cannot be reduced below the constant B_0 which is the unique number satisfying the equation

$$e^{-(\pi/2) B_0} = B_0. \tag{1.4}$$

The numerical value of B_0 is

$$B_0 = .474541 \tag{1.41}$$

and, without further explanation even in statements of theorems, B_0 will always stand for this constant.

While more precise statements are given in terms of the following definitions, and still more precise results are obtained in later sections, it is our main purpose to show that the constant B of (1.3) can in fact be reduced to B_0. Corresponding problems, in which one seeks information about constants C for which

$$\limsup_{n\to\infty} |s_n - M_{q_n}| \leqq C \limsup_{n\to\infty} |n u_n| \tag{1.5}$$

is attainable by choice of q_n, are much simpler and are solved in [3].

2. Definitions and statements of results.

The four following definitions differ in that boundedness of s_n is assumed in the first and third but not the second and fourth, and that the sign $<$ appears in the first two while the sign $\leqq$ appears in the last two.

Definition 2.1. *Let a positive number B have property P_1 if to each series Σu_n, for which $0 < \limsup |n u_n| < \infty$ and for which s_n is bounded, corresponds a sequence p_n such that*

$$\limsup_{n\to\infty} |M_n - s_{p_n}| < B \limsup_{n\to\infty} |n u_n|. \tag{2.11}$$

Definition 2.2. *Let a positive number B have property P_2 if to each series Σu_n for which $0 < \limsup |n u_n| < \infty$ corresponds a sequence p_n such that*

$$\limsup_{n\to\infty} |M_n - s_{p_n}| < B \limsup_{n\to\infty} |n u_n|. \tag{2.21}$$

Definition 2.3. *Let a positive number B have property P_1' if to each series Σu_n, for which lim sup $|n u_n| < \infty$ and for which s_n is bounded, corresponds a sequence p_n such that*

$$\limsup_{n\to\infty} |M_n - s_{p_n}| \leqq B \limsup_{n\to\infty} |n u_n|. \tag{3.31}$$

Definition 2.4. *Let a positive number B have property P_2' if to each series Σu_n for which lim sup $|n u_n| < \infty$ corresponds a sequence p_n such that*

$$\limsup_{n\to\infty} |M_n - s_{p_n}| \leqq B \limsup_{n\to\infty} |n u_n|. \tag{2.41}$$

It is known [4, Theorem 14.3] that each constant B less than B_0 fails to have property P_1' and hence also fails to have property P_1. We shall complete this result in Theorem 5.1 by showing that B_0 (and hence also each greater number), has property P_1 and hence has also property P_1'. Combining these results gives the following theorem.

Theorem 2.5. *The constant B_0 is the least number B having property P_1, and is the least number having property P_1'.*

It is known [4, Theorem 14.4] that B_0 fails to have property P_2. We shall complete this result in Theorem 4.3 by showing that B_0 has property P_2' and hence that each number B greater than B_0 has both properties P_2 and P_2'. Combining these results gives the following theorem.

Theorem 2.6. *The constant B_0 does not have property P_2 but is the greatest lower bound of numbers B having property P_2, and is the least number having property P_2'.*

3. Preliminary estimates.

We now obtain some consequences of the assumption that $R > 0$, that

$$\lambda = e^{-\pi R}, \tag{3.1}$$

and the Σu_n is a series for which lim sup $|n u_n| \leqq 1$ and, for an infinite set of values of n,

$$|M_n - s_k| \geqq R \qquad \lambda n \leqq k \leqq n. \tag{3.11}$$

Supposing n has a fixed value such that $2R\lambda n > 100$ and (3.1) holds, we put

$$s_k = M_n + R_k e^{i\theta_k} \qquad 0 \leqq k \leqq n \tag{3.12}$$

where $R_k \geqq 0$ and, at least when $k \geqq \lambda n$, θ_k varies slowly with k in the sense that it never makes unnecessarily large jumps of multiples of 2π. Then, when $\lambda n < k \leqq n$, we have $R_k \geqq R$ and the law of cosines gives

$$(3.13)\qquad \begin{aligned} |s_k - s_{k-1}|^2 &= R_{k-1}^2 + R_k^2 - 2R_{k-1}R_k \cos(\theta_k - \theta_{k-1}) \\ &= (R_{k-1} - R_k)^2 + 2R_{k-1}R_k[1 - \cos(\theta_k - \theta_{k-1})] \\ &\geqq 2R^2[1 - \cos(\theta_k - \theta_{k-1})] = 4R^2 \sin^2 \tfrac{1}{2}(\theta_k - \theta_{k-1}) \end{aligned}$$

and hence

$$(3.14)\qquad \sin \tfrac{1}{2}|\theta_k - \theta_{k-1}| \leqq |s_k - s_{k-1}|/2R = |u_k|/2R.$$

Letting δ_n denote the maximum of $|k u_k|$ for $\lambda n \leqq k \leqq n$, we see that $\lim\sup \delta_n \leqq 1$ and hence that we can choose a sequence ε_n' such that $\varepsilon_n' > 0$, $\varepsilon_n' \to 0$, and $\delta_n < 1 + \varepsilon_n'$. Then

$$(3.15)\qquad \sin \tfrac{1}{2}|\theta_k - \theta_{k-1}| \leqq (1 + \varepsilon_n')/2Rk \qquad \lambda n \leqq k \leqq n.$$

Letting ε_n be defined by

$$(3.16)\qquad 1 + \varepsilon_n = (1 + \varepsilon_n')\frac{\sin^{-1}[(1 + \varepsilon_n')/2Rn\lambda]}{(1 + \varepsilon_n')/2R_n\lambda},$$

we see that $\varepsilon_n > 0$, $\varepsilon_n \to 0$, and, when $\lambda n \leqq k \leqq n$,

$$(3.17)\qquad |\theta_k - \theta_{k-1}| \leqq 2(1 + \varepsilon_n)(1 + \varepsilon_n')^{-1} \sin \tfrac{1}{2}|\theta_k - \theta_{k-1}|$$

so that

$$(3.18)\qquad |\theta_k - \theta_{k-1}| \leqq (1 + \varepsilon_n)/Rk \qquad \lambda n \leqq k \leqq n.$$

Hence, when $\lambda n \leqq p \leqq q \leqq n$,

$$(3.2)\qquad \begin{aligned} |\theta_q - \theta_p| &\leqq \sum_{k=p+1}^{q} |\theta_k - \theta_{k-1}| \leqq \frac{1 + \varepsilon_n}{R} \sum_{k=p+1}^{q} \frac{1}{k} \\ &\leqq (1 + \varepsilon_n) R^{-1}(\log q - \log p). \end{aligned}$$

Let

$$(3.21)\qquad \alpha(n) = n e^{-\pi R/(1+\varepsilon_n)}, \qquad \beta(n) = n e^{-\pi R/2(1+\varepsilon_n)}.$$

We simplify typography by understanding that t is an abbreviation for $[t]$, the greatest integer in t, in any symbol or equation in which t should be an integer. Putting $p = \beta(n)$ and $q = k$ in (3.2) gives, when $\beta(n) \leqq k \leqq n$),

$$(3.22)\qquad |\theta_k - \theta_{\beta(n)}| \leqq \frac{\pi}{2} + \frac{1 + \varepsilon_n}{R}(\log k - \log n)$$

and hence

$$(3.23)\qquad |\theta_k - \theta_{\beta(n)}| \leqq \pi/2 \qquad \beta(n) \leqq k \leqq n.$$

Putting $q = \beta(n)$ and $p = k$ in (3.2) gives, when $\alpha(n) \leqq k \leqq \beta(n)$

$$(3.24) \qquad |\,\theta_{\beta(n)} - \theta_k\,| \leqq (1 + \varepsilon_n)\, R^{-1} (\log n - \log k) - \frac{\pi}{2}$$

and hence

$$(3.25) \qquad |\,\theta_{\beta(n)} - \theta_k\,| \leqq \pi/2 \qquad \alpha(n) \leqq k \leqq \beta(n).$$

The formulas (3.23) and (3.25) imply, because of (3.12), that all of the points s_k in the whole range $\alpha(n) \leqq k \leqq n$ lie in the closed half-plane which has its edge passing through the point M_n and which includes and is bisected by the half-line extending from M_n onwards through $s_{\beta(n)}$.

Without changing our notation we suppose that, with n fixed as above, the elements of the sequence $s_0, s_1, s_2, \ldots$ and all quantities determined by it are translated and rotated in the complex plane so that $M_n = 0$ and $\theta_{\beta(n)} = \pi/2$. Then (3.12), (3.11) and (3.25) imply that

$$(3.3) \qquad s_k = R_k\, e^{i\theta_k} \qquad \alpha(n) \leqq k \leqq n$$

where $R_k \geqq R$ and $0 \leqq \theta_k \leqq \pi$. It follows that, when $\alpha(n) \leqq k \leqq n$, the imaginary part Im s_k of s_k is greater than or equal to Im s_k' where

$$(3.31) \qquad s_k' = R\, e^{i\theta_k} \qquad \alpha(n) \leqq k \leqq n.$$

If we set, when $\beta(n) \leqq k \leqq n$,

$$(3.4) \qquad \varphi_k - \frac{\pi}{2} = \frac{\pi}{2} + \frac{(1 + \varepsilon_n)}{R} (\log k - \log n)$$

and, when $\alpha(n) \leqq k < \beta(n)$,

$$3.41) \qquad \frac{\pi}{2} - \varphi_k = \frac{1 + \varepsilon_n}{R} (\log n - \log k) - \frac{\pi}{2},$$

then comparison with (3.22) and (3.23) shows that $|\,\theta_k - \pi/2\,| \leqq |\,\varphi_k - \pi/2\,|$ when $\alpha(n) \leqq k \leqq n$ and hence that Im $s_k' \geqq$ Im z_k where

$$(3.42) \qquad z_k = R\, e^{i\varphi_k} \qquad \alpha(n) \leqq k \leqq n.$$

From (3.4) and (3.41) we obtain

$$(3.43) \qquad \varphi_k = \pi + \frac{1 + \varepsilon_n}{R} \log \frac{k}{n} \qquad \alpha(n) \leqq k \leqq n.$$

We have also

$$(3.44) \qquad \text{Im } s_k \geqq \text{Im } z_k \geqq 0 \qquad \alpha(n) \leqq k \leqq n.$$

Our next step is to estimate the sum V_n defined by

$$V_n = \frac{1}{n+1} \sum_{k=a(n)}^{n} z_k . \tag{3.5}$$

This can be put in the form

$$V_n = -\frac{Rn}{n+1} \sum_{k=0}^{n} f_n\left(\frac{k}{n}\right)\frac{1}{n} \tag{3.51}$$

where

$$f_n(x) = 0, \qquad 0 \leqq x < e^{-\pi R/(1+\varepsilon_n)}, \tag{3.52}$$

$$f_n(x) = e^{i(1+\varepsilon_n)R^{-1}\log x}, \qquad e^{-\pi R/(1+\varepsilon_n)} \leqq x \leqq 1. \tag{3.53}$$

Thus the sum in (3.51) is a Riemann sum for the function $f_n(x)$, and it is easy to show that $\lim (V_n - V'_n) = 0$ where V'_n is the corresponding Riemann sum for $\lim f_n(x)$. Therefore

$$V_n = o(1) - R \int_{e^{-\pi R}}^{1} e^{iR^{-1}\log x}\, dx. \tag{3.54}$$

Evaluating the integral by use of the formula

$$\int_a^b e^{ik\log x}\, dx = \int_a^b x^{ik}\, dx = \frac{x^{1+ik}}{1+ik}\Bigg]_a^b = \frac{e^{(1+ik)\log x}}{1+ik}\Bigg]_a^b, \tag{3.55}$$

gives

$$V_n = o(1) + \frac{R^2}{R^2+1}(1+e^{-\pi R})(-R+i). \tag{3.56}$$

Using (3.44), (3.5), and (3.56) gives

$$\operatorname{Im} \frac{1}{n+1} \sum_{k=a(n)}^{n} s_k \geqq o(1) + \frac{R^2}{R^2+1}(1+e^{-\pi R}). \tag{3.57}$$

We shall use also the simple result in the following lemma.

Lemma 3.6. *If* Σu_n *is a series for which* $\limsup |n u_n| < \infty$, *then*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{\log n} (s_k - M_n) = 0. \tag{3.61}$$

To prove this, we choose H such that $|n u_n| \leqq H$ when $n = 1, 2, 3, \ldots$ and find that

$$|s_k| \leqq |u_0| + \sum_{j=1}^{k} H j^{-1} \leqq |u_0| + (1+H)\log(k+1) \tag{3.61}$$

and

(3.62) $$|M_n| \leqq \max_{0 \leqq k \leqq n} |s_k| \leqq |u_0| + (1 + H) \log (n + 1).$$

Use of the crude inequality

(3.63) $$|s_k - M_n| \leqq |s_k| + |M_n| \leqq 2[|u_0| + (1 + H) \log (n + 1)]$$

then leads to (3.61).

4. Sequences that may be unbounded.

We now prove the following lemma.

Lemma 4.1. *If* $R > B_0$ *and if* Σu_n *is a series for which* $\limsup |n u_n| \leqq 1$, *then there is a sequence* p_n *such that* $e^{-\pi R} n \leqq p_n \leqq n$ *and*

(4.11) $$\limsup_{n \to \infty} |M_n - s_{p_n}| < R.$$

Suppose this lemma is false. Then, for some $R > B_0$, there must be a series Σu_n satisfying the assumptions set forth in the first sentence of section 3. Hence we may use all of the results of section 3. Let $h > 1$, and suppose that n is fixed so large that $|k u_k| \leqq h$ when $k \geqq \log n$. Since $\operatorname{Im} s_{\alpha(n)} \geqq 0$, we obtain the inequalities

(4.2) $$\operatorname{Im} s_k \geqq \operatorname{Im} z_k, \qquad \log n \leqq k < \alpha(n),$$

by defining z_k over $\log n \leqq k < \alpha(n)$ so that

(4.21) $$z_{\alpha(n)-1} = -ih/\alpha(n)$$

and

(4.22) $$z_k - z_{k-1} = ih/k, \qquad \log n \leqq k < \alpha(n) - 1.$$

Letting

(4.23) $$W_n = \frac{1}{n+1} \sum_{k=1+\log n}^{\alpha(n)-1} z_k$$

we find that

(4.24) $$\begin{aligned} W_n &= + \frac{1}{n+1} \sum_{k=1+\log n}^{\alpha(n)-1} \Big[z_{\alpha(n)-1} - \sum_{j=k+1}^{\alpha(n)-1} (z_j - z_{j-1})\Big] \\ &= o(1) - \frac{ih}{n+1} \sum_{k=1+\log n}^{\alpha(n)-1} \sum_{j=k+1}^{\alpha(n)} \frac{1}{j} \\ &= o(1) - \frac{ih}{n+1} [\alpha(n) - \log n] = o(1) - ihe^{-\pi R}. \end{aligned}$$

Using (4.2), (4.23), and (4.24) gives

$$\text{Im}\,\frac{1}{n+1}\sum_{k=1+\log n}^{a(n)-1} s_k \geqq o(1) - h e^{-\pi R}. \tag{4.25}$$

After the transformation (depending upon n) by which we made $M_n = 0$ in section 3, we have

$$M_n = \frac{1}{n+1}\left[\sum_{k=0}^{\log n} + \sum_{k=1+\log n}^{a(n)-1} + \sum_{k=a(n)}^{n}\right] s_k = 0. \tag{4.26}$$

But from (4.26) we obtain, by use of (3.61), (4.25), and (3.57),

$$\text{Im}\, M_n \geqq o(1) + \frac{R^2}{R^2+1}(1+e^{-\pi R}) - h e^{-\pi R}. \tag{4.27}$$

Since (4.27) holds for each $h > 1$, it must hold also when $h = 1$ and we obtain

$$\text{Im}\, M_n \geqq o(1) + \frac{R^2 - e^{-\pi R}}{R^2+1}. \tag{4.28}$$

Our hypothesis that $R > B_0$ implies that $R^2 > e^{-\pi R}$; therefore (4.28) contradicts (4.26) and Lemma 4.1 is proved.

With the aid of the preceding lemma, we prove the following theorem which shows that B_0 has property P_2' of Definition 2.4 and which moreover gives some information about the sequence p_n which may in some sense or other be as precise as is obtainable.

Theorem 4.3. *If* Σu_n *is a series for which lim sup* $|n u_n| < \infty$, *then there is a sequence* p_n *such that*

$$B_0^2 n \leqq p_n \leqq n \tag{4.31}$$

and

$$\limsup_{n\to\infty} |M_n - s_{p_n}| \leqq B_0 \limsup_{n\to\infty} |n u_n|. \tag{4.32}$$

In case lim sup $|n u_n| = 0$ we can, as (1.2) shows, attain the desired conclusion immediately by taking $p_n = n$. We can therefore assume that lim sup $|n u_n| = h > 0$. Since division of each term of Σu_n by h results in division of M_n, s_k and lim sup $|n u_n|$ by the same constant h, we can and shall assume that $h = 1$. Lemma 4.1 provides, corresponding to each $R > B_0$, a sequence $q(R, n)$ such that $e^{-\pi R} n \leqq$ $\leqq q(R, n) \leqq n$ and

$$\limsup_{n\to\infty} |M_n - s_{q(R,n)}| < R. \tag{4.4}$$

Choose $p(R, n)$ such that $|M_n - s_k|$ attains its minimum over the interval $n \exp(-\pi R) \leqq k \leqq n$ when $k = p(R, n)$. Then obviously

$$\limsup_{n \to \infty} |M_n - s_{p(R, n)}| < R. \tag{4.42}$$

If k lies in the interval $e^{-\pi R} n \leqq k \leqq n$ but outside the shorter interval $B_0^2 n = n \exp(-B_0 \pi) \leqq k \leqq n$, then (where we put $\alpha = e^{-\pi R} n$ and $\beta = n \exp(-B_0 \pi)$ to simplify subscripts)

$$|s_k - s_\beta| \leqq \sum_{j=k+1}^{\beta} |u_j| \leqq \sum_{k=\alpha}^{\beta} |u_j| \tag{4.43}$$

$$\leqq o(1) + \sum_{k=\alpha}^{\beta} j^{-1} = o(1) + \pi(R - B_0).$$

It follows that if we choose p_n such that $|M_n - s_k|$ attains its minimum over the shorter interval $B_0^2 n \leqq k \leqq n$ when $k = p_n$, then

$$\limsup |s_{p(R, n)} - s_{p_n}| \leqq \pi(R - B_0). \tag{4.44}$$

From this and (4.42) we obtain, for each $R > B_0$,

$$\limsup_{n \to \infty} |M_n - s_{p_n}| < R + \pi(R - B_0). \tag{4.45}$$

Since the left member is now independent of R, we conclude that

$$\limsup_{n \to \infty} |M_n - s_{p_n}| \leqq B_0 \tag{4.46}$$

and complete the proof of Theorem 4.3.

5. Bounded sequences.

The program of this section is in some respects similar to that of the preceding. We do not use a preliminary lemma. The following theorem shows that B_0 has property P_1 of Definition 2.1 and, in fact, gives a very much sharper result.

Theorem 5.1. *If Σu_n is a series for which $0 < \limsup |n u_n| < \infty$ and s_n is bounded, then there exist a constant B^*, which depends only upon the diameter D of the set of points in the sequence s_n and which is less than B_0, and a sequence p_n such that $n \exp(-\pi B^*) \leqq p_n \leqq n$ and*

$$\limsup_{n \to \infty} |M_n - s_{p_n}| \leqq B^* \limsup_{n \to \infty} |n u_n|. \tag{5.11}$$

We have not encumbered the statement of this theorem with a prescription for determination of B^*; the prescription is given in the two sentences following (5.61). Assuming that the theorem is false, we conclude that, when $R = B^*$, there is a series Σu_n satisfying the assumptions set forth in the first sentence of section 3. As in the preceding section we make free use of the notation and formulas of section 3, including the sequence z_k defined over $\alpha(n) \leqq k \leqq n$ in (3.42). Using the diameter D, we obtain improved estimates of M_n by defining z_k over $0 \leqq k < \alpha(n)$ in a manner different from that in (4.21) and (4.22). From (3.44), which shows that $\operatorname{Im} s_n \geqq 0$ for some values of n, we conclude that $\operatorname{Im} s_n \geqq -D$ for every n. Let $h > 1$ and suppose that n is fixed so large that $|k u_k| \leqq h$ when $k \geqq \log n$. We obtain the inequalities

$$\text{(5.2)} \qquad \operatorname{Im} s_k \geqq \operatorname{Im} z_k \qquad 0 \leqq k < \alpha(n)$$

by defining z_k so that

$$\text{(5.21)} \qquad z_{\alpha(n)-1} = -ih/\alpha(n),$$

$$\text{(5.22)} \qquad z_k - z_{k-1} = ih/k, \qquad \alpha(n)e^{-D/h} \leqq k < \alpha(n) - 1,$$

and

$$\text{(5.23)} \qquad z_k = -iD, \qquad 0 \leqq k < \alpha(n)e^{-D/h}.$$

To simplify formulas, let

$$\text{(5.3)} \qquad a = \alpha(n)e^{-D/h}, \qquad b = \alpha(n) - 1$$

and

$$\text{(5.31)} \qquad X_n = \frac{1}{n+1}\sum_{k=0}^{a-1} z_k, \qquad Y_m = \frac{1}{n+1}\sum_{k=a}^{b} z_k.$$

Then

$$\text{(5.32)} \qquad X_n = o(1) - iDe^{-nR}e^{-D/h},$$

and

$$\text{(5.33)} \qquad Y_n = \frac{1}{n+1}\sum_{k=a}^{b}\left[z_b - \sum_{j=k+1}^{b}(z_j - z_{j-1})\right]$$

$$= o(1) - \frac{ih}{n+1}\sum_{k=a}^{b}\sum_{j=k+1}^{b}\frac{1}{j} = o(1) - \frac{ih}{n+1}\sum_{k=a}^{b-1}\sum_{j=k+1}^{b}\frac{1}{j}$$

$$= o(1) - \frac{ih}{n+1}\sum_{j=a+1}^{b}\sum_{k=a}^{j-1}\frac{1}{j} = o(1) - \frac{ih}{n+1}\sum_{j=a+1}^{b}\left(1 - \frac{a}{j}\right) =$$

$$= o(1) - ih(n+1)^{-1}[(b-a) - a(\log b - \log a)]$$

$$= o(1) - ie^{-\pi R}[h - he^{-D/h} - De^{-D/h}].$$

Hence

$$\text{(5.4)} \qquad \operatorname{Im} \frac{1}{n+1} \sum_{k=0}^{a(n)-1} z_k = o(1) - he^{-\pi R} + he^{-\pi R} e^{-D/h}.$$

But on account of (5.2) the left member of

$$\text{(5.5)} \qquad \operatorname{Im} \frac{1}{n+1} \sum_{k=0}^{a(n)-1} s_k \geqq o(1) - e^{-\pi R} + e^{-\pi R} e^{-D}$$

is greater than or equal to the right member of (5.4) for each $h > 1$ and hence (5.5) holds. Combining (3.57) and (5.5) gives

$$\text{(5.6)} \qquad \operatorname{Im} M_n = \operatorname{Im} \frac{1}{n+1} \sum_{k=0}^{n} s_k \geqq o(1) + F(R, D)$$

where

$$\text{(5.61)} \qquad F(R, D) = \frac{R^2 - e^{-\pi R}}{R^2 + 1} + e^{-\pi R} e^{-D}.$$

The difference $R^2 - e^{-\pi R}$ is positive when $R > B_0$ and, since $F(0, D) < 0$, it follows that for each fixed positive D there is a constant $B_1 = B_1(D)$ less than B_0 such that $F(B_1 D) = 0$ and $F(R, B) > 0$ when $R > B_1$. Let B^* be chosen such that $B_1 < B^* < B_0$. Then $F(B^*, D) > 0$ and this contradicts the formula (5.6) which holds when $R = B^*$ and the sequence $s_0, s_1, \ldots$ is, for each n, translated and rotated so that (among other things) $M_n = 0$. This completes the proof of Theorem 5.1.

Our proof of Theorem 5.1 suggests very strongly that, to put matters roughly, if the diameter D is large then for some series Σu_n the constant B^* of (5.11) can be only a little less than B_0. Examples given in [4] show that this is true. In any case, Theorem 2.5 shows that there is no fixed constant B^* less than B_0 such that (5.11) holds for all finite diameters.

6. Theorems on limit points.

We now prove the two following theorems on approximation to limit points of the sequence M_n by limit points of the sequence s_n.

Theorem 6.1. *If Σu_n is a series such that* $\limsup |nu_n| < \infty$ *and s_n is bounded, then there is a constant B^* less than B_0 such that to each limit point ζ_M of the sequence M_n corresponds a limit point ζ_s of the sequence s_n such that*

(6.11) $$|\zeta_M - \zeta_s| \leqq B^* \limsup_{n\to\infty} |n u_n|.$$

Theorem 6.2. *The constant B_0 is the least constant with the following property. If Σu_n is a series for which* $\limsup |n u_n| < \infty$, *then to each limit point ζ_M of the sequence M_n corresponds a limit point ζ_s of the sequence s_n such that*

(6.21) $$|\zeta_M - \zeta_s| \leqq B_0 \limsup_{n\to\infty} |n u_n|.$$

Theorem 6.1 follows very easily from Theorem (5.1). Assuming that ζ_M is a imit point of the sequence M_n, we choose a sequence $n_1, n_2, n_3, \ldots$ such that $M(n_k) \to \zeta_M$. Restricting the n in the left member of (5.11) to values in this sequence, we see that the corresponding bounded sequence s_{p_n} must have a limit point ζ_s for which (6.11) holds. To prove Theorem 6.2, we note that Theorem 4.3 and the argument used above imply that B_0 has the property in question. It was shown in [4] that no constant B less than B_0 has the property, and thus Theorem 6.2 is proved.

7. Conclusion.

Theorem 6.2 solves, for arithmetic mean transforms, the problem analogous to a problem proposed by Hadwiger [5] for Abel power series transforms of series. The problem of Hadwiger really consists of two parts of which the more difficult problem is the following. Let Σu_n be a series for which $\limsup |n u_n| < \infty$. Let $\sigma(t) = \Sigma t^k u_k$ denote the Abel transform, and let ζ_A represent a limit point of this transform, that is, a number ζ_A such that $\sigma(t_n) \to \zeta_A$ for some sequence t_n such that $0 < t_n < 1$ and $t_n \to 1$. The problem is to determine the least constant C_0 such that to each limit point ζ_A of $\sigma(t)$ corresponds a limit point ζ_s of s_n such that

(7.1) $$|\zeta_A - \zeta_s| \leqq C_0 \limsup_{n\to\infty} |n u_n|.$$

Hadwiger [5] showed that $.4858 \leqq C_0 \leqq 1.0160$.

It seems that the exact value of C_0 has, despite the fact that several authors have studied Hadwiger's problems and their generalizations, never even been conjectured. It was shown in [1] and [2] that $C_0 \leqq .9680448$, and that the latter constant has several optimal properties related to the problem. The fundamental fact that $C_0 < .9680448$ was proved in [3] where it was shown that $C_0 \leqq .838381$. By use of Theorem 4.3 and the method of [3; section 10] we can show that $C_0 \leqq .749439$, but the author does not expect this smaller upper bound to turn out to be the value of C_0. It is hoped that the methods of [4] and this paper will be helpful in extending our results from arithmetic mean transforms M_n to Abel transforms and perhaps to more or less general classes of transforms. However, it is expected that the computations involved in such extensions will be far from trivial. In any case, the best constant C_0 in (7.1) remains undetermined.

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