SOME PROPERTIES OF CONTINUED FRACTIONS.

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§ 1. Introduction.

Let

$$\{a_1, a_2, \ldots\} \tag{1}$$

be an infinite normal continued fraction, $a_1, a_2, \ldots$ being integers with $a_1 \geqq 0$, $a_k \geqq 1$ $(k = 2, 3, \ldots)$.

The consecutive convergents of (1) are denoted by $\frac{P_0}{Q_0}, \frac{P_1}{Q_1}, \frac{P_2}{Q_2}, \ldots$, where $\frac{P_0}{Q_0}$ has the usual symbolic sense $\frac{1}{0}$, and where the irreducible fraction $\frac{P_k}{Q_k}$ $(k \geqq 1)$ has the value of the continued fraction $\{a_1, a_2, \ldots, a_k\}$. We have:

$$\begin{cases} P_n = a_n P_{n-1} + P_{n-2}\,(n \geqq 2), \quad P_1 = a_1, \; P_0 = 1; \\ Q_n = a_n Q_{n-1} + Q_{n-2}\,(n \geqq 2), \quad Q_1 = 1, \; Q_0 = 0; \\ P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n+1}\,(n \geqq 1). \end{cases} \tag{2}$$

For $a_{n+1} \geqq 2$ $(n \geqq 1)$ the fractions

$$\frac{b P_n + P_{n-1}}{b Q_n + Q_{n-1}} \qquad (b = 1, 2, \ldots, a_{n+1} - 1) \tag{3}$$

are the interpolated fractions of (1). For $b = 1$ and $b = a_{n+1} - 1$ the fractions (3) are the extreme interpolated fractions between $\frac{P_{n-1}}{Q_{n-1}}$ and $\frac{P_{n+1}}{Q_{n+1}}$. The following theorems are well-known [1]:

1. *Is* α *a positive irrational number, then each convergent* $\frac{P}{Q}$ $(Q \geqq 1)$ *of* $\alpha = \{a_1, a_2, \ldots\}$ *satisfies the inequality:*

$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{Q^2}. \tag{4}$$

2. *Is* $\frac{P}{Q}$ *an irreducible fraction with* (4), *then* $\frac{P}{Q}$ *is either a convergent or an interpolated fraction of the irrational* α.
3. *Given a positive irrational* α, *then apart from the convergents, at most the two extreme fractions* $\frac{P}{Q}$, *interpolated between* $\frac{P_{n-1}}{Q_{n-1}}$ *and* $\frac{P_{n+1}}{Q_{n+1}}$ $(n \geqq 1)$, *satisfy* (4).

This last theorem is a theorem of Fatou [2], of which a proof has been given by Koksma [3].

In the present paper we extend the theorem of Fatou, by giving sufficient conditions, on which the first extreme interpolated fraction ($b = 1$) or the last one ($b = a_{n+1} - 1$) satisfy (4). This is expressed in

Theorem I. *Let* α *be a positive irrational number, and let* $\{a_1, a_2, \dots\}$ *be its continued fraction.*

If $a_{n+1} \leqq a_n + 1$ $(n \geqq 1,\ a_{n+1} \geqq 2)$, *then the fraction* $\frac{P}{Q} = \frac{P_n + P_{n-1}}{Q_n + Q_{n-1}}$ *satisfies* (4).

If $a_{n+1} \leqq a_{n+2} + 1$ $(n \geqq 1,\ a_{n+1} \geqq 2)$, *then the fraction* $\frac{P}{Q} = \frac{(a_{n+1} - 1) P_n + P_{n-1}}{(a_{n+1} - 1) Q_n + Q_{n-1}}$ *satisfies* (4).

We shall prove this theorem in § 2.

Furthermore: If e denotes an even integer, and o an odd integer, then all irreducible fractions $\frac{P_k}{Q_k}$ $(k \geqq 0)$ are of the three classes $\left[\frac{o}{e}\right]$, $\left[\frac{e}{o}\right]$ and $\left[\frac{o}{o}\right]$. As far as we know the distribution of the convergents with respect to these classes has never been considered before. In § 3 we give some lemmae and theorems, concerning this distribution.

Finally: By means of the results of § 2 and § 3 we prove in § 4 and § 5 the following theorems:

Theorem II. *For* $k \geqq 1$ *there exist infinitely many fractions* $\frac{P}{Q}$ *of each of the three classes* $\left[\frac{o}{e}\right]$, $\left[\frac{e}{o}\right]$ *and* $\left[\frac{o}{o}\right]$, *satisfying*

$$\left|\alpha - \frac{P}{Q}\right| < \frac{k}{Q^2}, \tag{5}$$

regardless of the values of the real irrational number α.

Theorem III. *For $k < 1$ there exist irrational numbers, everywhere dense on the real axis, for which* (5) *is satisfied by only a finite number of fractions of a given one of the three classes.*

The Theorems II and III are theorems of W. T. Scott. His proofs, however, are not based upon the theory of continued fractions, but depend on geometric properties of elliptic modular transformations [4].

§ 2. Proof of Theorem I.

1.
$$\frac{P_n + P_{n-1}}{Q_n + Q_{n-1}} = \frac{(a_n + 1) P_{n-1} + P_{n-2}}{(a_n + 1) Q_{n-1} + Q_{n-2}},$$

and

$$\alpha = \{a_1, a_2, \ldots, a_{n-1}, \gamma\} = \frac{\gamma P_{n-1} + P_{n-2}}{\gamma Q_{n-1} + Q_{n-2}}$$

with

$$\gamma = \{a_n, a_{n+1}, \ldots\}.$$

Then

$$\left| \frac{\gamma P_{n-1} + P_{n-2}}{\gamma Q_{n-1} + Q_{n-2}} - \frac{(a_n + 1) P_{n-1} + P_{n-2}}{(a_n + 1) Q_{n-1} + Q_{n-2}} \right| = \frac{1 + a_n - \gamma}{(\gamma Q_{n-1} + Q_{n-2}) \{(a_n + 1) Q_{n-1} + Q_{n-2}\}}, \tag{6}$$

as follows from (2) and $\gamma < a_n + 1$.

Now we prove that, given the first inequality of Theorem I, we have:

$$\frac{1 + a_n - \gamma}{\gamma Q_{n-1} + Q_{n-2}} < \frac{1}{(a_n + 1) Q_{n-1} + Q_{n-2}}. \tag{7}$$

The inequality (7) can be written as

$$\{(1 + a_n)^2 - \gamma (2 + a_n)\} Q_{n-1} < (\gamma - a_n) Q_{n-2}. \tag{8}$$

The right side of (8) is positive. The left side of (8) is negative on account of

$$(1 + a_n)^2 - \gamma (2 + a_n) < 0,$$

where the last inequality follows from the assumption $a_{n+1} \leqq a_n + 1$ and

$$\gamma > a_n + \cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2}}} \geqq a_n + \frac{1}{a_n + 2} = \frac{(1 + a_n)^2}{a_n + 2}.$$

From (6) and (7) the first part of our theorem follows.

2. $$\alpha = \{a_1, a_2, \ldots, a_n, \beta\} = \frac{\beta P_n + P_{n-1}}{\beta Q_n + Q_{n-1}}$$

with

$$\beta = \{a_{n+1}, a_{n+2}, \ldots\}.$$

Then

$$\left|\frac{\beta P_n + P_{n-1}}{\beta Q_n + Q_{n-1}} - \frac{(a_{n+1} - 1) P_n + P_{n-1}}{(a_{n+1} - 1) Q_n + Q_{n-1}}\right| = \frac{1 - a_{n+1} + \beta}{(\beta Q_n + Q_{n-1}) \{(a_{n+1} - 1) Q_n + Q_{n-1}\}}, \tag{9}$$

on account of $\beta > a_{n+1}$ and (2).

Now we prove, given the second inequality of Theorem I:

$$\frac{\beta - a_{n+1} + 1}{\beta Q_n + Q_{n-1}} < \frac{1}{(a_{n+1} - 1) Q_n + Q_{n-1}}. \tag{10}$$

The inequality (10) can be reduced to:

$$Q_n \{(a_{n+1} - 1)^2 - \beta (a_{n+1} - 2)\} > Q_{n-1} (\beta - a_{n+1}).$$

This inequality holds when

$$(a_{n+1} - 1)^2 - \beta (a_{n+1} - 2) > \beta - a_{n+1}, \tag{11}$$

on account of $Q_n > Q_{n-1}$.

The exactness of (11) follows from

$$a_{n+1} - 1 \leqq a_{n+2} \text{ and } \beta < a_{n+1} + \frac{1}{a_{n+2}} \leqq a_{n+1} + \frac{1}{a_{n+1} - 1}.$$

From (9) and (10) the second part of Theorem I follows.

§ 3. Properties of convergents of continued fractions.

Evidently the class of $\frac{P_n}{Q_n}$ $(n \geqq 2)$ is determined by the classes of $\frac{P_{n-2}}{Q_{n-2}}$ and $\frac{P_{n-1}}{Q_{n-1}}$, and by a_n (even or odd), as follows from (2). We have the following lemma's.

Lemma 1. *Two consecutive convergents* $\frac{P_{n-1}}{Q_{n-1}}$ *and* $\frac{P_n}{Q_n}$ $(n \geqq 1)$ *are not of the same class.*

This follows from the last relation of (2).

Lemma 2. *If* a_n *is an even integer, the class of* $\frac{P_n}{Q_n}$ *is the same as that of* $\frac{P_{n-2}}{Q_{n-2}}$.

Proof. For a_n even we have

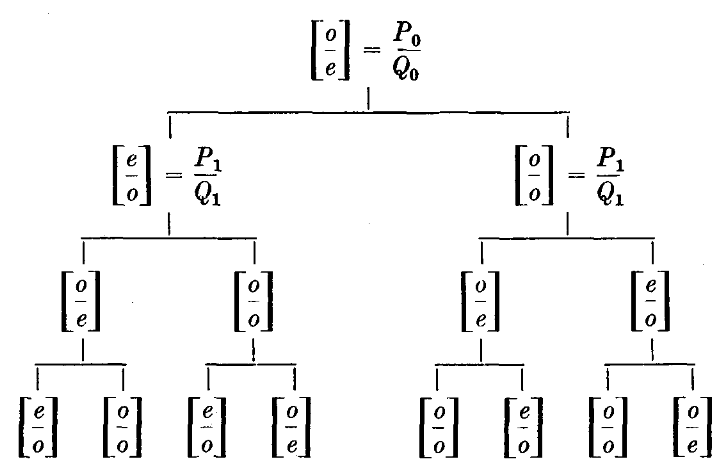
$$P_n \equiv P_{n-2}, \quad Q_n \equiv Q_{n-2} \pmod 2, \text{ as follows from (2).}$$

Lemma 3. *If* a_n *is an odd integer, three consecutive convergents* $\frac{P_{n-2}}{Q_{n-2}}$, $\frac{P_{n-1}}{Q_{n-1}}$ *and* $\frac{P_n}{Q_n}$ *are of different classes.*

This follows from (2).

Remarks. 1. By choosing suitable continued fractions it is easily seen that in the sequence $\frac{P_n}{Q_n}$ $(n = 0, 1, \ldots)$ each permutation of the three classes can occur.

2. For the sake of convenience we give here a scheme, from which the classes of the consecutive convergents of every continued fraction can be read off. Starting from some convergent $\frac{P_{n-1}}{Q_{n-1}}$, the class of which is represented by one of the symbols of the n-th row in the scheme, we can find the class of $\frac{P_n}{Q_n}$ by going to right if a_n is odd, and to left if a_n is even $(n \geqq 1)$.



In the following we give a formula which indicates the class of the n-th convergent of an arbitrary continued fraction.

Let C^1 denote the cycle $\left(\left[\frac{o}{e}\right], \left[\frac{o}{o}\right], \left[\frac{e}{o}\right]\right)$, and C^{-1} the inverse cycle $\left(\left[\frac{o}{e}\right], \left[\frac{e}{o}\right], \left[\frac{o}{o}\right]\right)$.

Let $[a]$ be the class of an arbitrary convergent $\frac{P_k}{Q_k}$ of $\{a_1, a_2, \ldots, a_k, a_{k+1}, \ldots\}$; this element has a fixed place in the scheme. Now we denote by D (C^1 or C^{-1}), *the cycle of* $[a]$, *that is the cycle of the corresponding consecutive classes of* $\frac{P_{k+1}}{Q_{k+1}}, \frac{P_{k+2}}{Q_{k+2}}, \ldots,$ *if the partial quotients* $a_{k+1}, a_{k+2}, \ldots$ *would be odd.* (In the scheme the cycle is found by going from $[a]$ to right.) It is evident that *variation of cycle is possible only by an odd number of consecutive even a's.*

Let $S_l^D[a]$ $(l \geqq 0)$ denote an operator, applied on the element $[a]$ in D, such that:

$$\begin{cases} S_l^D[a] = [a] \text{ for even } l; \\ S_l^D[a] = \text{the element in } D \text{ preceding } [a], \text{ for odd } l. \end{cases}$$

Let $T_l^D[a]$ $(l \geqq 0)$ denote an operator, applied on the element $[a]$ in D, such that:

$$\begin{cases} T_l^D[a] = [a] \text{ if } l \equiv 0 \pmod 3; \\ T_l^D[a] = \text{the element in } D, \text{ following } [a], \text{ if } l \equiv 1 \pmod 3; \\ T_l^D[a] = \text{the element in } D, \text{ preceding } [a], \text{ if } l \equiv 2 \pmod 3. \end{cases}$$

Let A_n be the continued fraction:

$$(12) \qquad A_n = \{\underbrace{a_1, a_2, \ldots, a_{k_1}}_{\text{even } a\text{'s}}, \underbrace{a_{k_1+1}, \ldots, a_{k_1+k_2}}_{\text{odd } a\text{'s}}, \ldots, \underbrace{a_{k_1+k_2+\cdots+k_{2n-1}+1}, \ldots, a_{k_1+k_2+\cdots+k_{2n}}}_{\text{odd } a\text{'s}}\}.$$

If the first a is odd, then we put $k_1 = 0$; if the last a is even, then we put $k_{2n} = 0$.

Now we shall prove

Theorem 1. *The class of the last convergent of* A_n *equals*

$$(13) \qquad \psi(n) = \prod_{i=0}^{n-1} T_{k_{2i+2}}^{C(-1)^{k_1+k_3+\cdots+k_{2i+1}}} S_{k_{2i+1}}^{(-1)^{k_1+k_3+\cdots+k_{2i-1}}} \left[\frac{o}{e}\right],$$

where $\prod T S$ *means*

$$T_{k_{2n}} S_{k_{2n-1}} \left[\cdots \left[T_{k_4} S_{k_3} \left[T_{k_2} S_{k_1} \left[\frac{o}{e}\right]\right]\right]\right], \text{ and } k_{-1} = 0;$$

and where $T_{k_{2i+2}} S_{k_{2i+1}}[a]$ *is the element* $T_{k_{2i+2}}[S_{k_{2i+1}}[a]]$.

Proof. For $n = 1$ and (12) is

$$A = \{\underbrace{a_1, a_2, \ldots, a_{k_1}}_{\text{even } a\text{'s}}, \underbrace{a_{k_1+1}, \ldots, a_{k_1+k_2}}_{\text{odd } a\text{'s}}\}.$$

Then in this case we have:

1. If $k_1 = 0$, the class of A is $T_{k_2}^{C^1}\left[\frac{o}{e}\right] = \left[\frac{o}{e}\right], \left[\frac{o}{o}\right]$ or $\left[\frac{e}{o}\right]$ for $k_2 \equiv 0, 1$ or $2 \pmod 3$, respectively.

2. If k_1 is even and $\neq 0$, then the class of A is

$$T_{k_2}^{C^1} S_{k_1}^{C^1} \begin{bmatrix} o \\ - \\ e \end{bmatrix} = T_{k_2}^{C^1} \begin{bmatrix} o \\ - \\ e \end{bmatrix} = \begin{bmatrix} o \\ - \\ e \end{bmatrix}, \begin{bmatrix} o \\ - \\ o \end{bmatrix} \text{ or } \begin{bmatrix} e \\ - \\ o \end{bmatrix}$$

for $k_2 \equiv 0$, 1 or 2 (mod 3), resp.

3. If k_1 is odd, then the class of A is

$$T_{k_2}^{C^{-1}} S_{k_1}^{C^1} \begin{bmatrix} o \\ - \\ e \end{bmatrix} = T_{k_2}^{C^{-1}} \begin{bmatrix} e \\ - \\ o \end{bmatrix} = \begin{bmatrix} e \\ - \\ o \end{bmatrix}, \begin{bmatrix} o \\ - \\ o \end{bmatrix} \text{ or } \begin{bmatrix} o \\ - \\ e \end{bmatrix}$$

for $k_2 \equiv 0$, 1 or 2 (mod 3), resp.

The exactness of the assertions 1, 2, and 3 follow from the fact, that $\frac{P_0}{Q_0}$ is of the class $\begin{bmatrix} o \\ - \\ e \end{bmatrix}$, and from Lemma 3. So our theorem is true for $n = 1$. Let the assertion hold for n, then we shall prove the exactness for the case $n + 1$. We distinguish two subcases:

A. $k_{2n+2} = 0$, so the last a's are even.

Then from (13) and our assumption we see that the class of A_n is

$$S_{k_{2n+1}}^{C^{(-1)^{k_1+k_3+\cdots+k_{2n-1}}}} [\psi(n)] = \psi(n+1).$$

For, if the class of the last convergent of A_n is $\psi(n)$, then the class $\psi(n+1)$ of the last convergent of A_{n+1} (with $k_{2n+2} = 0$) is $\psi(n)$ for k_{2n+1} is even, and the class preceding $\psi(n)$ in the cycle $C^{(-1)^{k_1+k_3+\cdots+k_{2n-1}}}$ for k_{2n+1} odd, that is in both cases

$$S_{k_{2n+1}}^{C^{(-1)^{k_1+k_3+\cdots+k_{2n-1}}}} [\psi(n)] = \psi(n+1).$$

B. $k_{2n+2} \neq 0$, so the last a's are odd.

Now the class of A_{n+1} is

$$T_{k_{2n+2}}^{C^{(-1)^{k_1+k_2+\cdots+k_{2n+1}}}} S_{k_{2n+1}}^{C^{(-1)^{k_1+k_3+\cdots+k_{2n-1}}}} [\psi(n)] = \psi(n+1),$$

as follows by a similar argument as in A. This completes the proof.

Let n be a fixed positive integer. Each continued fraction $\{a_1, a_2, \ldots, a_n\}$ belongs to a class which is determined by a fixed succession of n symbols e or o. The number of such classes, over which all continued fractions $\{a_1, a_2, \ldots, a_n\}$ can be distributed, is 2^n. Let $K_1, K_2, \ldots, K_{2^n}$ be these classes. Now $\frac{P_n}{Q_n}$ (the value of the continued fraction) belongs to one of the classes $\left[\frac{o}{e}\right]$, $\left[\frac{o}{o}\right]$ and $\left[\frac{e}{o}\right]$, mentioned above. Now we put the question how the classes $K_1, K_2, \ldots, K_{2^n}$ are distributed among the classes $\left[\frac{o}{e}\right]$, $\left[\frac{o}{o}\right]$ and $\left[\frac{e}{o}\right]$ The following theorem gives the answer.

Theorem 2. *If A_n, B_n and C_n are the numbers of those classes of $K_1, K_2, \ldots, K_{2^n}$, belonging to $\left[\frac{o}{e}\right]$, $\left[\frac{o}{o}\right]$ and $\left[\frac{e}{o}\right]$ respectively, then we have:*

$$A_n = \tfrac{2}{3}\{2^{n-1} - (-1)^{n-1}\}; \quad B_n = C_n = \tfrac{1}{3}\{2^n - (-1)^n\}.$$

Proof. Obviously

$$A_p + B_p + C_p = 2^p \quad (p = 1, 2, \ldots, n). \tag{14}$$

From the lemma's 2 and 3 it follows:

$$\begin{cases} A_p = B_{p-1} + C_{p-1} \\ B_p = C_{p-1} + A_{p-1} \\ C_p = A_{p-1} + B_{p-1} \end{cases} \quad (p = 2, 3, \ldots, n). \tag{15}$$

From (14) and (15) it follows: $A_p + A_{p-1} = 2^{p-1}$ $(p = 2, 3, \ldots, n)$. For $n = 1$ it is $A_1 = 0$, $B_1 = C_1 = 1$; hence the assertion of the theorem is true for $n = 1$. By induction (we assume the theorem to be true for $n - 1$), we find

$$A_n = 2^{n-1} - A_{n-1} = 2^{n-1} - \tfrac{2}{3}\{2^{n-2} - (-1)^{n-2}\} = \tfrac{2}{3}\{2^{n-1} - (-1)^{n-1}\},$$

and

$$B_n = C_n = 2^{n-1} - \tfrac{1}{3}\{2^{n-1} - (-1)^{n-1}\} = \tfrac{1}{3}\{2^n - (-1)^n\}.$$

§ 4. Proof of Theorem II.

Without loss of generality we may assume that α is a positive irrational number. From the theory of the continued fractions it is known, that the convergents $\frac{P_n}{Q_n}$ $(n = 1, 2, \ldots)$ of $\alpha = \{a_1, a_2, \ldots\}$ satisfy the inequality

$$\left|\alpha - \frac{P_n}{Q_n}\right| < \frac{k}{Q_n^2} \quad (k \geqq 1).$$

If in the development $\{a_1, a_2, \ldots\}$ *infinitely many odd integers* occur, then for this α Theorem II is proved, as follows from Lemma 3.

Now we assume that in $\{a_1, a_2, \ldots\}$ from a certain $n = n_0 \geqq 1$ *only even* partial quotients occur. Then from that index n_0 the convergents are of two classes, either $\left[\frac{o}{o}\right], \left[\frac{e}{o}\right]$, or $\left[\frac{o}{o}\right], \left[\frac{o}{e}\right]$ or $\left[\frac{o}{e}, \frac{e}{o}\right]$, so that the inequality (5) can be satisfied by an infinite number of fractions of each of two classes.

The extreme interpolated fractions

$$\frac{P_n + P_{n-1}}{Q_n + Q_{n-1}} \quad \text{and} \quad \frac{(a_{n+1} - 1)P_n + P_{n-1}}{(a_{n+1} - 1)Q_n + Q_{n-1}}$$

are of the remaining class. It is easily seen that the condition $a_{n+1} \leqq a_{n+2} + 1$ occurs infinitely many times, regardless of the considered $\alpha = \{a_1, a_2, \ldots\}$, so that (5) holds with infinitely many fractions $\frac{(a_{n+1} - 1)P_n + P_{n-1}}{(a_{n+1} - 1)Q_n + Q_{n-1}}$, as follows from Theorem I. This completes the proof of Theorem II.

§ 5. Proof of Theorem III.

If $k < 1$, then each fraction $\frac{P}{Q}$, satisfying (5), is *either a convergent or an extreme interpolated fraction* of $\alpha = \{a_1, a_2, \ldots\}$, on account of § 1, 2.

A. Now consider $\alpha = m + \sqrt{m^2 + 1} = \{2m, 2m, \ldots\}$ (m integer and $\geqq 1$). The convergents of this continued fraction are alternately of the classes $\left[\frac{e}{o}\right]$ and $\left[\frac{o}{e}\right]$. We prove in 1 and 2 that from a certain n for $k < \frac{m}{\sqrt{1 + m^2}}$ the extreme interpolated fractions, which are of the remaining class, do not satisfy (5).

1. Obviously $\alpha = \frac{\alpha P_n + P_{n-1}}{\alpha Q_n + Q_{n-1}}$. Hence

$$(16) \qquad \left|\frac{\alpha P_n + P_{n-1}}{\alpha Q_n + Q_{n-1}} - \frac{P_n + P_{n-1}}{Q_n + Q_{n-1}}\right| = \frac{\alpha - 1}{(\alpha Q_n + Q_{n-1})(Q_n + Q_{n-1})} =$$

$$= \frac{(\alpha - 1)\left(\frac{Q_n}{Q_{n-1}} + 1\right)}{\left(\alpha \frac{Q_n}{Q_{n-1}} + 1\right)} \cdot \frac{1}{(Q_n + Q_{n-1})^2}.$$

It is well-known that

$$\frac{Q_n}{Q_{n-1}} = \underbrace{\{2m, 2m, \ldots, 2m\}}_{n-1 \text{ partial quotients}},$$

so that

$$\lim_{n\to\infty} \frac{Q_n}{Q_{n-1}} = \alpha,$$

and

$$\lim_{n\to\infty} \frac{(\alpha-1)\left(\frac{Q_n}{Q_{n-1}}+1\right)}{\alpha\frac{Q_n}{Q_{n-1}}+1} = \frac{\alpha^2-1}{\alpha^2+1} = \frac{m}{\sqrt{1+m^2}}.$$

Thus, for $k < \frac{m}{\sqrt{1+m^2}}$, we see that from a certain n the right side of (16) is greater than $\frac{k}{(Q_n+Q_{n-1})^2}$.

2. Furthermore

17) $$\left|\alpha - \frac{(a_{n+1}-1)P_n + P_{n-1}}{(a_{n+1}-1)Q_n + Q_{n-1}}\right| = \left|\frac{\alpha P_{n+1}+P_n}{\alpha Q_{n+1}+Q_n} - \frac{P_{n+1}-P_n}{Q_{n+1}-Q_n}\right| =$$

$$= \frac{\alpha+1}{(\alpha Q_{n+1}+Q_n)(Q_{n+1}-Q_n)} = \frac{(\alpha+1)\left(\frac{Q_{n+1}}{Q_n}-1\right)}{\alpha\frac{Q_{n+1}}{Q_n}+1}\cdot\frac{1}{(Q_{n+1}-Q_n)^2}.$$

Now

$$\lim_{n\to\infty} \frac{(\alpha+1)\left(\frac{Q_{n+1}}{Q_n}-1\right)}{\alpha\frac{Q_{n+1}}{Q_n}+1} = \frac{\alpha^2-1}{\alpha^2+1} = \frac{m}{\sqrt{1+m^2}}.$$

From this it follows that from a certain n for $k < \frac{m}{\sqrt{1+m^2}}$ the right side of (17) is greater than $\frac{k}{(Q_{n+1}-Q_n)^2}$.

B. Obviously the convergents of $\beta = \{2m+1, 2m, 2m, \ldots\}$ are of the classes $\left[\frac{o}{o}\right]$ and $\left[\frac{o}{e}\right]$. It is easily seen, that there are not infinitely many fractions $\frac{P}{Q}$ of the class $\left[\frac{e}{o}\right]$, satisfying:

$$\left|\beta - \frac{P}{Q}\right| < \frac{k}{Q^2} \quad \left(k < \frac{m}{\sqrt{1+m^2}}\right).$$

For, otherwise, since $\beta = 1 + \alpha$, there would be infinitely many fractions $\frac{P-Q}{Q}$ of the class $\left[\frac{o}{o}\right]$, satisfying

$$\left|\alpha - \frac{P-Q}{Q}\right| < \frac{k}{Q^2} \quad \left(k < \frac{m}{\sqrt{1+m^2}}\right),$$

and this is impossible on account of A.

C. The convergents of $\gamma = \frac{1}{1+\alpha} = \{0, 2m+1, 2m, 2m, \ldots\}$ are of the classes $\left[\frac{o}{o}\right]$ and $\left[\frac{e}{o}\right]$. It is excluded that there are infinitely many fractions $\frac{P}{Q}$ of the class $\left[\frac{o}{e}\right]$, satisfying

$$\left|\gamma - \frac{P}{Q}\right| < \frac{k}{Q^2} \quad \left(k < \frac{m}{\sqrt{1+m^2}}\right).$$

Otherwise we should have $\left(\text{since the sequence } \frac{P}{Q} \text{ has to limit } \frac{1}{1+\alpha}\right)$:

$$\left|1 + \alpha - \frac{Q}{P}\right| < \frac{k(1+\alpha)}{PQ} = \frac{k}{P^2} \cdot \frac{P(1+\alpha)}{Q} < \frac{k_1}{P^2}$$

with $k_1 < \frac{m}{\sqrt{1+m^2}}$, for infinitely many fractions $\frac{Q}{P}$, which of course are of the class $\left[\frac{e}{o}\right]$, and this is impossible, as follows from B.

D. From A, B and C it follows that in the case $k < 1$, (given one of the three classes) there exist at least one irrational for which (5) is satisfied by only a finite number of fractions of this class.

The exactness of Theorem III now is a consequence of A, B and C, and the following facts:

1. The numbers $\frac{a\alpha + 2b}{2c\alpha + d}$ (a, b, c and d integers with $ad - 4bc = \pm 1$) lie everywhere dense on the real axis for each irrational α.

2. The fractions $\frac{aP + 2bQ}{2cP + dQ}$ (a, b, c, d, P and Q integers with $ad - 4bc = \pm 1$) are of the same class as $\frac{P}{Q}$.

3. Given $k < 1$, and given some class, there exists an integer m and a corresponding irrational λ (α, β or γ), such that the inequality

$$\left|\lambda - \frac{P}{Q}\right| < \frac{k}{Q^2} \quad \left(k < \frac{m}{\sqrt{1+m^2}}\right)$$

can be satisfied by only a finite number of fractions $\frac{P}{Q}$ of the given class.

If we assume that the inequality

$$\left|\frac{a\lambda + 2b}{2c\lambda + d} - \frac{p}{q}\right| < \frac{k}{q^2}$$

holds for infinitely many fractions $\frac{p}{q}$ of that class, then we should also have for these $\frac{p}{q}$:

$$\left|\lambda - \frac{pd - 2bq}{aq - 2cp}\right| < \frac{k}{(aq - 2cp)^2}\left|(2c\lambda + d)\left(2c\frac{p}{q} - a\right)\right|.$$

Now

$$\lim_{q\to\infty}\left|(2c\lambda + d)\left(2c\frac{p}{q} - a\right)\right| = |ad - 4bc| = 1.$$

Hence, for $k < k_1 < \frac{m}{\sqrt{1+m^2}} < 1$, there would be infinitely many fractions $\frac{P}{Q} = \frac{pd - 2bq}{aq - 2cp}$ $\left(\text{of the same class as } \frac{p}{q}\right)$, for which

$$\left|\lambda - \frac{P}{Q}\right| < \frac{k_1}{Q^2},$$

and this is impossible.

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