A non-standard ideal of a radical Banach algebra of power series

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1. Introduction

We will be concerned here with certain radical Banach algebras of power series. Let $\mathbf{C}[[z]]$ be the algebra of formal power series over the complex field $\mathbf{C}$. We say that a sequence of positive reals $\{w(n)\}$ is a *radical algebra* weight provided the following hold:

$$w(0)=1 \text{ and } 0<w(n)\leq 1 \text{ for all } n\in\mathbf{Z}^+; \tag{1.1}$$

$$w(m+n)\leq w(m)\,w(n) \text{ for all } m,n\in\mathbf{Z}^+; \tag{1.2}$$

$$\lim_{n\to\infty} w(n)^{1/n}=0. \tag{1.3}$$

If these conditions hold, it is routine to check that

$$l^1(w(n))\equiv\left\{y=\sum_{n=0}^{\infty}y(n)\,z^n:\sum_{n=0}^{\infty}|y(n)|\,w(n)<\infty\right\}$$

is both a subalgebra of $\mathbf{C}[[z]]$ and a radical Banach algebra with identity adjoined. Conditions (1.1) and (1.2) make $l^1(w(n))$ into a Banach algebra; Condition (1.3) is needed to give it exactly one maximal ideal. The norm is defined in the natural way: $\|y\|=\Sigma_{n=0}^{\infty}|y(n)|\,w(n)$. We shall generally refer to $l^1(w(n))$ as a radical Banach algebra. The multiplication is given by the usual multiplication of formal power series. The reader is referred to [3], [4], and [8] for background material on such algebras. Besides $l^1(w(n))$, there are obvious closed ideals in $l^1(w(n))$:

$$M(n)\equiv\left\{\sum_{k=0}^{\infty}y(k)\,z^k\in l^1(w(n)):y(0)=y(1)=\ldots=y(n-1)=0\right\} \tag{1.4}$$

for $n=1, 2, \ldots$, and, of course, the zero ideal. Such closed ideals are referred to as *standard* ideals. Any other closed ideals are referred to as *non-standard* ideals. It has been an open question for some time whether or not there exists any radical algebra weight $\{w(n)\}$ such that $l^1(w(n))$ contains a non-standard ideal. The problem seems to go back to Šilov (see [6, p. 189]). An erroneous solution [6, Theorem 5, p. 205] appears in the literature (see [8, Proposition 2.1] for a specific discussion of the error). Interest has also been focused on the quotient algebras $(l^1(w(n))/\underset{\sim}{I})$ where $\underset{\sim}{I}$ is a closed ideal since these algebras are representative of all radical Banach algebras with power series generators [1]. In this paper we will show how to construct radical algebra weights $\{w(n)\}$ such that $l^1(w(n))$ contains non-standard ideals. The radical algebra weights we use will be semi-multiplicative in the sense of [11, Definition 2.1]. Briefly, these are weights where $w(m+n)$ actually equals $w(m)w(n)$ for many indices. We will give specific details of the construction in Section 2. Weights similar to these have also been shown to possess pathological multiplier algebras [2], and there may well be a connection between the two problems.

Before discussing the strategy of the construction, we remark that for very well-behaved weights there are positive results which ensure that all the closed ideals are standard. If $\ln w$ is a concave sequence, then a theorem of S. Grabiner [3, Theorem 4.1] implies that every non-zero ideal contains a power of z, and, since a non-zero closed ideal is standard if and only if it contains a power of z [3, Lemma 4.5], this certainly implies that all closed ideals for such a weight are standard. A weaker requirement is that the weight $\{w(n)\}$ be *star-shaped* [10, Definition 3.1]. Essentially, this means that the region below the graph of $y=\ln w$ is illuminated by the origin; equivalently, $\{w(n)^{1/n}\}$ is a decreasing sequence. In this case also, all closed ideals of $l^1(w(n))$ are standard provided that $w(n)^{1/n}$ is $O(1/n^a)$ for some $a>0$ [10, Corollary 3.6]. Since semi-multiplicative weights are, in a qualitative sense, as far away from star-shaped weights as possible (while still satisfying condition (1.2)) it is reasonable to look here for algebras with non-standard ideals.

Let $A=l^1(w(n))$ in the following. To construct a non-standard ideal it will suffice to find an element $x=\sum_{n=1}^{\infty} x(n)z^n$ with $x(1)\neq 0$ such that the closed ideal $\overline{Ax}$ generated by x is properly contained in the unique maximal ideal $M(1)$. Let T be the operator of right translation on A, so that the action of T on a power series in A is simply multiplication by z. It is an easy exercise to show that $\overline{Ax}$ is the closed linear span of the translates of x. Hence, x generates a non-standard ideal provided that $\overline{\operatorname{span}}\{T^k x\}_{k=0}^{\infty}$ is properly contained in $M(1)$. To demonstrate this, it will suffice to show that $z\notin \overline{\operatorname{span}}\{T^k x\}_{k=0}^{\infty}$. This will follow provided there exists $\varepsilon>0$ such that

$$\left\| z - \sum_{s=0}^{N} a(s)\, T^s x \right\| > \varepsilon \tag{1.5}$$

for all choices of N and finite sequences of coefficients $\{a(s)\}_{s=0}^{N}$. The elaborate nature of the construction we shall give is dictated by the fact that (1.5) must hold for all choices of coefficients. If x fails to generate a non-standard ideal, condition (1.5) fails, and the approximation of z by a linear combination of translates of x is possible. Suppose, for some sequences $\{a_k(s)\}_{s=0}^{N_k}$ that

$$\lim_{k\to\infty} \sum_{s=0}^{N_k} a_k(s)\, T^s x = z. \tag{1.6}$$

Let s be fixed. We outline the determination of $\lim_{k\to\infty} a_k(s)$ as follows (see [10, Chapter 2], for more detail). First, let $\{c(n)\}$ be the unique sequence satisfying the equation

$$\sum_{n=0}^{\infty} c(n)\, z^n \cdot x = z \quad \text{in } \mathbf{C}[[z]]. \tag{1.7}$$

We call $\{c(n)\}$ the *associated* sequence [10, Definition 2.1]. In general, $\Sigma_{n=0}^{\infty} c(n) z^n$ is only a formal power series and not an element of A as a consequence of the considerations in [9]. Furthermore, $\{c(n)\}$ can be explicitly calculated:

$$c(0) = x(1)^{-1}$$

and

$$c(n) = -x(1)^{-1} \sum_{k=0}^{n-1} c(k)\, x(1+n-k). \tag{1.8}$$

Let $\{e_n^*\}_{n=0}^{\infty}$ be the dual basic sequence. By this, we mean that $e_n^*(z^m) = \delta_{mn}$. Define

$$\chi_n^* = \sum_{k=0}^{n} c(k)\, e_{1+n-k}^*. \tag{1.9}$$

It then follows that $\chi_n^* \in A^*$ and $\chi_n^*(T^m x) = \delta_{mn}$. If we fix s_0, and apply $\chi_{s_0}^*$ to both sides of (1.6), we obtain

$$\lim_{k\to\infty} \chi_{s_0}^* \left(\sum_{s=0}^{N_k} a_k(s)\, T^s x \right) = \chi_{s_0}^*(z).$$

Thus,

$$\lim_{k\to\infty} a_k(s_0) = c(s_0). \tag{1.10}$$

Therefore, the sequences of coefficients, $\{a_k(s)\}_{s=0}^{N_k}$, converge *pointwise* to the associated sequence as k tends to infinity ($N_k \to \infty$ as $k \to \infty$). We conclude that, if $\Sigma_{s=0}^{N}$ $a(s)\,T^s x$ is sufficiently close to z, initial terms of $\{a(s)\}$ are very close to initial terms of $\{c(s)\}$. With our choice of x, certain initial terms of $\{c(s)\}$ are very large. Also, it can be shown that finite sequences $\{c(s)\}_{s=0}^{N}$ satisfy (1.5) with $a(s)$ replaced by $c(s)$ (see [5]) for the types of weights we are considering. Our proof will proceed by contradiction. We shall take a very rapidly increasing sequence $\{n(k)\}$ with $n(1)=1$ and build a semi-multiplicative weight which has large 'drops' at $n(k)$. By this we mean that $w(n(k))$ is very much smaller than the previous terms. In between, the weight will be constructed inductively [11, Definition 2.2] using the semi-multiplicative condition

$$w(tn(k)+j) = w(n(k))^t\, w(j), \tag{1.11}$$

if $tn(k)+j<n(k+1)$. We will let $x(1)=1$ and choose $x=\Sigma_{k=1}^{\infty} x(n(k))\, z^{n(k)}$ where $|x(n(k))|\, w(n(k))=2^{-k+1}$ for $k=2,3,\ldots$. The rapid increase of the sequence $\{n(k)\}$ will ensure that there is minimal overlap between the terms in $\Sigma_{s=0}^{N} a(s)\, T^s x$. We will then show that if (1.5) fails for $\varepsilon=1$, anytime a term $|a(k)|$ is relatively large, there must exist $m>k$ such that $|a(m)|$ is also relatively large (see Proposition 3.18). Since some initial terms must be large as a consequence of (1.10), this will produce the desired contradiction because all approximations we are using are based upon finite linear combinations. Hence (1.5) must hold and the given x must generate a non-standard ideal.

We would like to thank the referee for noting that in our example, $M(1)$ is a radical Banach algebra with one generator, z, and with a proper closed ideal not contained in the closed ideal $\overline{Az^2}$, generated by z^2. This seems to be the first example of this kind.

In the next section we will give the precise quantitative rules of construction. We finally note that the questions we are considering here can be rephrased in the language of operator theory. Note that one can equivalently consider the unilateral weighted shift operator R on l^p which is similar to the right shift operator T on the weighted space $l^p(w(n))$. One says that R is *strictly cyclic* if $l^p(w(n))$ is an algebra (see [7, Proposition 32, p. 96]; this is stated for $p=2$ but this restriction is not essential). One says that an operator is *unicellular* if its closed invariant subspaces are totally ordered. A unilateral weighted shift operator R on l^p is unicellular if and only if the only closed invariant subspaces it possesses are the obvious ones. Hence, the question whether or not there exists a quasinilpotent strictly cyclic unilateral weighted shift R which is not unicellular [7, Question 19, second variation, p. 105] is equivalent to the question whether or not there exists some radical Banach algebra of power series, $l^p(w(n))$, which has a non-standard ideal. Of course, we are considering the case $p=1$ throughout this paper.

2. Semi-multiplicative weights

We will construct a semi-multiplicative weight *inductively* in the sense of [11, Definition 2.2]. This simply means that we first fix a strictly increasing sequence of positive integers $\{n(k)\}$. For simplicity we will use $n(1)=1$. We then let $w(0)=1$, and choose $w(1)$ so that $0<w(1)\leq 1$. If $w(0), w(1), \ldots, w(m)$ have all been defined we consider the following two possible cases: (i) $n(k)<m+1<n(k+1)$; and (ii) $m+1=n(k+1)$. In case (i), take $t \in \mathbf{N}$ and $j \in \{0, \ldots, n(k)-1\}$ so that $m+1=tn(k)+j$, and set

$$w(tn(k)+j) = w(n(k))^t\, w(j). \tag{2.1}$$

In case (ii), we choose $w(n(k+1))$ sufficiently small (but positive) so that

$$w(n(k+1)) < w(n(k))^{(n(k+1)+n(k))/n(k)}. \tag{2.2}$$

We also say that $\{w(n)\}$ is the semi-multiplicative weight *generated* by $\{w(n(k))\}$. Note that the sequence $\{w(n(k))^{1/n(k)}\}$ is decreasing. We shall repeatedly use the following combinatorial result, proved in [11, Lemma 2.4 and Lemma 2.5].

LEMMA 2.3. *Let $\{n(k)\}$ be an increasing sequence of positive integers. Let $\{w(n)\}$ be constructed* inductively *via* (2.1) *and* (2.2). *Then for all, $t, s \in \mathbf{Z}^+$*

$$w(t+s) \leq w(t)\, w(s).$$

If, additionally, the decreasing sequence $\{w(n(k))^{1/n(k)}\}$ tends to zero, then $\lim_{s\to\infty} w(s)^{1/s}=0$ *and $\{w(n)\}$ is a* radical algebra *weight.*

The proof of Lemma 2.3 is given in [11] and the result is easily seen to be valid even if the sequence is finite (in this case only the first part of the lemma applies) and the strict inequality in (2.2) is replaced by 'less than or equal'. In order to ensure that $l^1(w(n))$ contains a non-standard ideal we shall have to impose some additional conditions on $\{n(k)\}$ and $\{w(n(k))\}$. Qualitatively, these are conditions which make $\{n(k)\}$ a very lacunary sequence (much more so than was needed in [11]) and force even more rapid decrease on $\{w(n(k))\}$. We shall also choose $\{x(n(k))\}$ as we go along and then let $x=\Sigma_{k=1}^{\infty} x(n(k))\, z^{n(k)}$. Since there are several conditions we must impose and since they will be repeatedly referred to later we shall label them alphabetically:

(A) Pick $n(1)=1, n(2)\geqslant 4$, and, if $n(1), n(2), \ldots, n(j)$ have been chosen, pick $n(j+1)$ so that

$$n(j+1)>n(j)\,(n(j)+1), \quad j=2,3,\ldots.$$

This is the only condition on $\{n(j)\}$, and we assume that it holds in the following. Before we proceed further we need some notational simplification.

Definition 2.4. Let $n_1=1$ and $n_j=n(j)-1$ for $j=2,3,\ldots$.

It easily follows that $n_2\geqslant 3$. Also, if $j\geqslant 2$, then

$$n_{j+1}>n(j)\,(n(j)+1)-1,$$

and so the sequence $\{n_j\}$ is also strictly increasing.

(B) Note that $w(0), w(1), \ldots,\ w(n(k+1)-1)$ are defined as soon as $w(n(1))$, $w(n(2)), \ldots,\ w(n(k))$ are chosen (as a consequence of (2.1)). We will impose four conditions on $w(n(k+1))$ which will ensure that it is sufficiently small by comparison with the previous terms (condition (2.2) and three others). Hence there is no requirement that $w(n(k+1))$ be sufficiently large, *only that it be sufficiently small.* Before listing the four conditions ((B1)–(B4)) we note:

(C) Let $x(n(1))=x(1)=1$. If $k\geqslant 2$ and if $w(n(1)),\ w(n(2)), \ldots,\ w(n(k))$ have been chosen, choose $x(n(k))$ to satisfy

$$|x(n(k))|\, w(n(k)) = 2^{-k+1}.$$

We now list the four following restrictions on $\{w(n(k))\}$:

If $k\in\mathbf{Z}^+$, and if $w(n(1)), w(n(2)), \ldots, w(n(k))$ have been chosen, then $w(0), w(1), \ldots, w(n(k+1)-1)$ are determined and $x(n(1)), x(n(2)), \ldots, x(n(k))$ have been chosen using (C). The first n_{k+1} terms of the associated sequence $\{c(n)\}_{n=0}^{n_{k+1}-1}$ are determined via (1.7) and it follows that

$$\sum_{n=0}^{n_{k+1}-1} c(n)\, z^n\cdot x = z+d(n_{k+1}+1)\, z^{n_{k+1}+1}+\ldots.$$

Then $x(n(k+1))$ will be chosen to satisfy $|x(n(k+1))|\,w(n(k+1))=2^{-k}$ as soon as $w(n(k+1))$ is chosen. Observe the relation (*)

$$c(n_{k+1}) = -\left(\sum_{j=2}^{k} c(n_{k+1}-n_j)\,x(n(j))\right) - x(n(k+1)), \tag{*}$$

which follows from (1.8). We choose $w(n(k+1))$ to satisfy

$$0 < w(n(k+1)) < w(n(k))^{(n(k+1)+n(k))/n(k)}, \tag{B1}$$

$$0 < w(n(k+1)) < 2^{-k} w(n(k))^{t+1}, \tag{B2}$$

where $t = n_{k+1}/n_k$ and $\{n_i\}$ was defined in Definition 2.4,

$$0 < \frac{w(n(k+1))}{w(n(k+1)-1)} < (2^{-k})^{n_{k+2}/n_{k+1}} (2n_{k+2})^{-n_{k+2}} (2^{n_{k+3}} n_{k+3})^{-n_{k+3}}, \tag{B3}$$

and due to (*) we can moreover choose $w(n(k+1))$ so small that $|x(n(k+1))|$ is sufficiently large to ensure that

$$|c(n_{k+1})| > \tfrac{1}{2}|x(n(k+1))|, \tag{B4}$$

and we shall suppose this has been done.

We will *henceforth* assume that $\{n(k)\}$ has been chosen according to (A) and then $\{w(n)\}$ and x have been constructed according to conditions (B) and (C) above. We have then that $\{w(n)\}$ is a radical algebra weight and that $x = \Sigma_{k=1}^{\infty} x(n(k))\, z^{n(k)}$ is an element of $l^1(w(n))$. Let $A = l^1(w(n))$, as before. The remainder of this paper will be devoted to proving that $\overline{Ax}$ is a non-standard ideal using the strategy outlined in the introduction. We shall also write x as $\Sigma_{i=1}^{\infty} x(i)\, z^i$ when this is notationally more convenient. We now define some combinatorial quantities.

Definition 2.5. Let $\{n_k\}_{k=1}^{\infty}$ be as in Definition 2.4. Let $v(0)=1$, $v(n_1)=v(1)=1$ and $v(n_k) = |x(n(k))|^{-1} = 2^{k-1} w(n(k))$ for $k \in \mathbf{Z}^+$ and $k \geq 2$. Let $\{v(n)\}$ be the semi-multiplicative weight generated by $\{v(n_k)\}$.

We must verify that the analogue of (2.2) holds, but if $k \in \mathbf{Z}^+$ and $k \geq 2$, then (B2) implies that

$$\begin{aligned} v(n_{k+1}) &= |x(n(k+1))|^{-1} \\ &= 2^k w(n(k+1)) \\ &< w(n(k))^{t+1}, \end{aligned}$$

where $t=n_{k+1}/n_k$. Thus

$$v(n_{k+1})<(2^{k-1}w(n(k)))^{t+1}$$
$$=v(n_k)^{(n_{k+1}+n_k)/n_k}.$$

Since the case $k=1$ is trivial we have verified the analogue of (2.2):

$$v(n_{k+1})<v(n_k)^{(n_{k+1}+n_k)/n_k}. \tag{2.6}$$

The other values $\{v(n)\}$ are, of course, defined using the analogue of (2.1). Namely, if $tn_k+j<n_{k+1}$ for some $k\in\mathbf{Z}^+$, then

$$v(tn_k+j)=v(n_k)^t\,v(j). \tag{2.7}$$

It is an immediate consequence of Lemma 2.3 and (2.6) that $v(t+s)\leq v(t)\,v(s)$ for all t, $s\in\mathbf{Z}^+$. We *emphasize* that $\{v(n)\}$ is not equal, or even similar to, the left shift of $\{w(n)\}$. Rather, it is the semi-multiplicative weight generated by the left shift of the *subsequence* $\{2^{k-1}w(n(k))\}_{k=2}^{\infty}$ together with $v(n_1)=1$. Since we will generally work with the reciprocal of v, we make the following definition.

Definition 2.8. Let $\Gamma(n)=v(n)^{-1}$ for all non-negative integers n. We call $\Gamma(n)$ the *character* of n.

Since $\{v(n)\}$ satisfies (1.2) we immediately obtain the following.

PROPOSITION 2.9. *For all* $t,s\in\mathbf{Z}^+$, $\Gamma(t+s)\geq\Gamma(t)\,\Gamma(s)$.

As a very important addendum, we note that one can find the character of m, $\Gamma(m)$, by repeatedly applying the division algorithm. First pick r so that $n_r\leq m<n_{r+1}$ and then divide:

$$\begin{aligned} m &= t_r n_r+R_r, \quad \text{where } t_r\in\mathbf{Z}^+ \text{ and } R_r<n_r\\ R_r &= t_{r-1}n_{r-1}+R_{r-1}, \quad \text{where } t_{r-1}\geq 0 \text{ and } R_{r-1}<n_{r-1}\\ &\vdots\\ R_2 &= t_1 n_1+0, \end{aligned} \tag{2.10}$$

where $R_j=m-\Sigma_{i=j}^{r} t_i n_i$ is the remainder at each stage (recall that $n_1=1$). Then it easily follows using Definition 2.5 and condition (2.7) that

$$\Gamma(m)=|x(n(2))|^{t_2}\,|x(n(3))|^{t_3}\ldots|x(n(r))|^{t_r} \tag{2.11}$$

(recall that $x(n(1))=1$). We will repeatedly use this fact about the character of m in the next section. The motivation for defining such a combinatorially involved quantity Γ is that, if it were possible to approximate z closely by a linear combination of translates of x, certain initial coefficients in the combination would have to be similar to terms appearing in (2.11). This is best appreciated by explicit computation (see [5] or [9, Lemma 4.1]).

3. Property (M) and non-standard ideals

We would like to incorporate $\Gamma(k)$ into a recursive condition. We will need the fact that $\Gamma(k)\,w(k)$ is very large when k is larger than $n_j(n_j+1)$ and less than n_{j+1}.

LEMMA 3.1. *Let $j\in\mathbf{Z}^+$ with $j\geq 2$ and suppose that $n_j(n_j+1)\leq k<n_{j+1}$. Then*

$$\Gamma(k)\,w(k)\geq (2k)^k\,(2^{n_{j+2}}n_{j+2})^{n_{j+2}-k}.$$

Proof. We digress for a moment to explain the concept behind the proof. Recall that condition (A) implies that $n(j+1)>n(j)\,(n(j)+1)$ since $j\geq 2$. Hence $n_{j+1}>n(j)\,(n(j)+1)-1$, and from this it follows that $n_{j+1}>n_j(n_j+2)$. Integers k with $n_j(n_j+1)\leq k<n_{j+1}$ will thus always exist, and this will be extremely important later on, in particular at the end of the proof of Lemma 3.14. We are essentially considering a partition here of the form: $\{\dots, n_j, n_j(n_j+1), n_{j+1}, \dots\}$. Furthermore, if k is sufficiently larger than n_j (namely $k\geq n_j(n_j+1)$) but smaller than n_{j+1}, this will force $v(k)$ to be substantially smaller than $w(k)$. Suppose now that we have the representation

$$\Gamma(k)=|x(n(2))|^{t_2}|\,x(n(3))|^{t_3}\dots|x(n(j))|^{t_j},$$

where $t_jn_j\leq k<(t_j+1)\,n_j$, as in (2.10) and (2.11). This means that $t_j\geq n_j+1=n(j)$ since $k\geq n_j(n_j+1)$. Although $t_j n_j<n_{j+1}$ (since $k<n_{j+1}$), it may happen that $(t_j-1)\,n(j)$ is larger than n_{j+1}.

Now we go to the proof. Let $\bar{w}$ be the semi-multiplicative weight generated by the finite sequence $\{w(n(i))\}_{i=1}^{j}$. An application of Lemma 2.3 shows that $\bar{w}(s+t)\leq\bar{w}(s)\,\bar{w}(t)$ for all, $s,t\in\mathbf{Z}^+$, although $\lim_{s\to\infty}\bar{w}(s)^{1/s}\neq 0$. Then

$$\begin{aligned}\Gamma(k)\,w(k)&=|x(n(2))|^{t_2}\dots|x(n(j))|^{t_j}\,w(k)\\&\geq|x(n(j))|^{t_j}\,w(k)\\&=|x(n(j))^{t_j}\;\bar{w}(k)\end{aligned}$$

$$\begin{aligned}&\geqslant |x(n(j))|^{t_j}\,\bar{w}((t_j+1)\,n_j)\\&=|x(n(j))|^{t_j}\,\bar{w}(t_j(n(j)-1)+n_j)\\&=|x(n(j))|^{t_j}\,\bar{w}(t_jn(j)-t_j+n_j)\\&\geqslant |x(n(j))|^{t_j}\,\bar{w}(t_j\,n(j)-n(j)+n_j),\end{aligned}$$

since $t_j \geqslant n(j)$. Thus

$$\begin{aligned}\Gamma(k)\,w(k)&\geqslant |x(n(j))|^{t_j}\bar{w}((t_j-1)\,n(j)+n_j)\\&=|x(n(j))|^{t_j}\,w(n(j))^{t_j-1}\,w(n_j)\\&=(2^{-j+1})^{t_j}\left[\frac{w(n(j)-1)}{w(n(j))}\right].\end{aligned}$$

Using condition (B3) with $k+1$ replaced by j, and noting that $t_j < n_{j+1}/n_j$ since $k < n_{j+1}$, we see that

$$\begin{aligned}\Gamma(k)\,w(k)&>(2n_{j+1})^{n_{j+1}}(2^{n_{j+2}}n_{j+2})^{n_{j+2}}\\&\geqslant (2k)^k(2^{n_{j+2}}n_{j+2})^{n_{j+2}-k},\end{aligned}$$

and the lemma is proved.

Unfortunately, we are not able to do induction on $\Gamma(k)$ alone since $\Gamma(k)\,w(k)$ is rather erratic just after n_{j+1}. For example, $\Gamma(n_{j+1})\,w(n_{j+1})$ equals $|x(n(j+1))|\times w(n(j+1)-1)$, which is large, but $\Gamma(n(j+1))\,w(n(j+1))$ equals $|x(n(j+1))|\,w(n(j+1))$, which is small. We need to incorporate $\Gamma(k)$ with another quantity which will 'dampen' this erratic nature of $\Gamma(k)$. We first require a preliminary definition, motivated by Lemma 3.1.

Definition 3.3. If $k \in \mathbf{Z}^+$ with $k \geqslant n_2$, we define $r=r(k)$ to be the unique natural number satisfying

$$n_r(n_r+1) \leqslant k < n_{r+1}(n_{r+1}+1). \tag{3.4}$$

A few remarks are in order. It follows from Definition 2.4 and condition (A) that: $n_2 \geqslant 3 \geqslant n_1(n_1+2)$ and if $r \in \mathbf{Z}^+$, with $r>1$, then $n_{r+1} > (n(r)\,(n(r)+1)-1) \geqslant n_r(n_r+2)$. Hence for all $r \in \mathbf{Z}^+$

$$n_{r+1}(n_{r+1}+1) > n_{r+1} \geqslant n_r(n_r+2). \tag{3.5}$$

If k is much larger than n_{j+1} and close to n_{j+2} then $r(k)$ will be $j+1$. If k is only moderately larger than n_{j+1} then $r(k)$ will be j. This double partition of $\mathbf{Z}^+$

$(\{\dots, n_r(n_r+1),\ n_{r+1}(n_{r+1}+1), \dots\}$ and $\{\dots, n_j,\ n_{j+1}, \dots\})$ will be a constant theme throughout the rest of the paper. We are interested in the quantity

$$\frac{(2^{n_{r+2}} n_{r+2})^{n_{r+2}-k}}{w(k)}. \tag{3.6}$$

where $r=r(k)$, for several reasons. First, the numerator is rather large whether k is slightly smaller or slightly larger than n_j; the numerator will be large when $\Gamma(k)\,w(k)$ is small. Second, it contains large powers of two which will be useful in cancelling terms such as $|x(n(j+1))|\,w(n(j+1))$. Finally, the numerator can be related to $\Gamma(k)\,w(k)$ by using Lemma 3.1. This suggests that we may be able to do recursion on the quantity

$$\max\left\{(1/2k)^k\,\Gamma(k), \frac{(2^{n_{r+2}} n_{r+2})^{n_{r+2}-k}}{w(k)}\right\}. \tag{3.7}$$

We thus define

Definition 3.8. Let $\{a(s)\}_{s=0}^{N}$ be a finite set of coefficients and let $k\in\mathbf{Z}^+$ with $n_2\leqslant k\leqslant N$. We say $a(k)$ has *property* (M) provided

$$|a(k)|\geqslant \max\left\{(1/2k)^k\,\Gamma(k), \frac{(2^{n_{r+2}} n_{r+2})^{n_{r+2}-k}}{w(k)}\right\},$$

where $r=r(k)$ is as in Definition 3.3.

Clearly $|a(k)|$ is rather large if $a(k)$ has property (M). We next have two lemmas which show that property (M), under suitable hypotheses, is a recursive condition.

LEMMA 3.9. *Suppose that $k\in\mathbf{Z}^+$ with $k\geqslant n_j$ where $j\geqslant 2$ and that $n(j)+k<n(j+1)$. Suppose that $\{a(s)\}_{s=0}^{N}$, where $N\geqslant k$, is a finite set of coefficients, and let $y=\Sigma_{s=0}^{N} a(s)\,T^s x$. Suppose that*

$$\|z-y\|<1.$$

Then if $a(k)$ has property (M) *there exists $m\in\mathbf{Z}^+$ with $m>k, m<n_{j+1}$ and $m\leqslant N$ such that $a(m)$ also has property* (M).

Proof. We remark that for reasons of notational convenience we may refer in this lemma and the following lemma to terms $a(h)$ where $h>N$. It is to be understood *such terms are zero.* We also remark that the conclusion of the lemma forces k to actually be less than N.

To obtain a contradiction, suppose first that for all $p \in \{n(1), n(2), \ldots, n(j-1)\}$ that

$$|a(k)|\,x(n(j))| \geq 2j|a(n(j)+k-p)|\,|x(p)|.$$

Writing $y=\Sigma_{s=0}^{N} a(s)\,T^s x$ as $\Sigma_{i=1}^{\infty} y(i)\,z^i$, we see that

$$\begin{aligned} |y(n(j)+k)| &\geq |a(k)|\,|x(n(j))| - \sum_{i=1}^{j-1} |a(n(j)+k-n(i))|\,|x(n(i))| \\ &\geq \tfrac{1}{2}|a(k)||x(n(j))|. \end{aligned}$$

Thus,

$$\begin{aligned} \|z-y\| &\geq |y(n(j)+k)|\,w(n(j)+k) \\ &\geq \tfrac{1}{2}|a(k)|\,|x(n(j))|\,w(n(j))\,w(k) \end{aligned}$$

since $n(j)+k<n(j+1)$ and $\{w(n)\}$ is semi-multiplicative. Using condition (C), we see that the above is

$$\begin{aligned} &= \tfrac{1}{2}|a(k)|w(k)\,2^{-j+1} \\ &\geq 2^{-j}(2^{n_{r+2}}n_{r+2})^{n_{r+2}-k}, \end{aligned}$$

using property (M), where $r=r(k)$. Since $r \geqslant j-1, k<n_{j+1}$, and $j<n_{j+1}$ the above is $\geqslant n_{j+1} \geqslant 1$, contradicting the hypothesis that $\|z-y\|<1$.

Hence there exists $p \in \{n(1), n(2), \ldots, n(j-1)\}$ such that

$$|a(k)|\,|x(n(j))| < 2j|a(n(j)+k-p)|\,|x(p)|. \tag{3.10}$$

Let $m=(n(j)+k-p)$. Then $m \geqslant n(j)+k-n(j-1)>k$. Also $m \leqslant N$ since clearly $a(m) \neq 0$. Note that $m \leqslant n_{j+1}$ since $n(j)+k<n(j+1)$ and $p \geqslant 1$. Then we see that

$$\begin{aligned} |a(m)| &> \frac{|a(k)|\,|x(n(j))|}{2j|x(p)|} \\ &\geq \left(\frac{1}{2k}\right)^k \frac{\Gamma(k)|\,x(n(j))|}{2j|x(q+1)|}, \end{aligned}$$

using the fact that $a(k)$ has property (M) and letting $q+1=p$. But $j<n_j<k$ and $\Gamma(k)|\,x(n(j))|=\Gamma(n_j+k)$ as a consequence of the considerations in (2.10) and (2.11) since $n_j+k<n_{j+1}$. The above is then

$$\begin{aligned} &\geq \left(\frac{1}{2k}\right)^{k+1} \frac{\Gamma(n_j+k)}{|x(q+1)|} \\ &= \left(\frac{1}{2k}\right)^{k+1} \frac{\Gamma(n_j+k)}{\Gamma(q)} \end{aligned}$$

whether $q=0$ or $q=n_i$ where $i\geqslant 2$. Since $q\leqslant n_j+k$, Proposition 2.9 implies that the above is

$$\begin{aligned}&\geqslant (1/2m)^m\,\Gamma(n_j+k-q)\\ &=(1/2m)^m\,\Gamma(n(j)+k-p)\\ &=(1/2m)^m\,\Gamma(m).\end{aligned}$$

Hence

$$|a(m)|\geqslant (1/2m)^m\,\Gamma(m). \tag{3.11}$$

It remains to show that $|a(m)|\geqslant (2^{n_{r+2}}n_{r+2})^{n_{r+2}-m}/w(m)$. There are two cases: (i) $m\geqslant n_j(n_j+1)$ and (ii) $m<n_j(n_j+1)$. In case (i) Lemma 3.1 implies

$$\Gamma(m)\,w(m)\geqslant (2m)^m\,(2^{\,n_{j+2}}n_{j+2})^{n_{j+2}-m}$$

since $m<n_{j+1}$. But then (3.11) and the fact that $r(m)=r=j$ imply that

$$|a(m)|\geqslant \frac{(2^{n_{r+2}}n_{r+2})^{n_{r+2}-m}}{w(m)}. \tag{3.12}$$

Thus, (3.11) and (3.12) together imply that $a(m)$ also has property (M).

In case (ii) $m<n_j(n_j+1)$, so $r(m)=r(k)=j-1$. Using (3.10) we have that

$$|a(k)|\,|x(n(j))|\,w(n(j)+k)<2j|a(n(j)+k-p)|\,|x(p)|\,w(n(j)+k).$$

Since the weight $\{w(n)\}$ is semi-multiplicative,

$$|a(k)|\,w(k)|\,x(n(j))|\,w(n(j))<2j|a(n(j)+k-p)|\,w(n(j)+k-p)\,|x(p)|\,w(p).$$

Using condition (C) and the fact that $m=n(j)+k-p$, we see that

$$|a(k)|w(k)\,2^{-j+1}<2j|a(m)|\,w(m).$$

Thus,

$$|a(m)|\,w(m)>(2^jj)^{-1}\,(2^{n_{j+1}}n_{j+1})^{n_{j+1}-k}$$

since $a(k)$ has property (M). Since $j\leqslant n_{j+1}$ and $m\geqslant k+1$ it follows that

$$|a(m)|\,w(m)>(2^{n_{j+1}}n_{j+1})^{n_{j+1}-m}.$$

Then since $r=r(k)=r(m)=j-1$

$$|a(m)| \geq \frac{(2^{n_{r+2}} n_{r+2})^{n_{r+2}-m}}{w(m)}. \tag{3.13}$$

Then (3.11) and (3.13) together imply that $a(m)$ also has property (M) in this case. This completes the proof of the lemma.

We note in passing that the hypothesis of Lemma 3.9 could be weakened from $\|z-y\|<1$ to

$$|y(n(j)+k)|\, w(n(j)+k) < 2^{-j}(2^{n_{r+2}} n_{r+2})^{n_{r+2}-k},$$

where $r=r(k)$. Only minor changes in the proof are needed.

The other alternative to Lemma 3.9 is handled by the next lemma.

LEMMA 3.14. *Suppose that $k \in \mathbf{Z}^+$ with $n_j \leq k < n_{j+1}$ where $j \geq 2$ but $n(j)+k \geq n(j+1)$. Suppose that $\{a(s)\}_{s=0}^N$, where $N \geq k$, is a finite set of coefficients, and let $y=\Sigma_{s=0}^N a(s) T^s x$. Suppose that*

$$\|z-y\| < 1.$$

Then, if $a(k)$ has property (M), *there exists $m \in \mathbf{Z}^+$ with $m > n_{j+1} > k$ and $m \leq N$ such that $a(m)$ also has property* (M).

Proof. Again, the conclusion here forces k to actually be less than N. Since $n(j)+k \geq n(j+1)$, we obtain $n_j+k \geq n_{j+1} \geq n_j(n_j+2)$ using (3.5). Thus $k \geq n_j(n_j+1)$ and $r=r(k)=j$.

To obtain a contradiction suppose first that for all $p \in \{n(1), n(2), \ldots, n(j)\}$ that

$$|a(k)|\,|x(n(j+1))| \geq 2j|a(n(j+1)+k-p)|\,|x(p)|.$$

Note here we are using $n(j+1)$ rather than $n(j)$ as in Lemma 3.9. Again, writing $y=\Sigma_{s=0}^N a(s) T^s x$ as $\Sigma_{i=1}^\infty y(i) z^i$ we see that

$$\begin{aligned} |y(n(j+1)+k)| &\geq |a(k)|\,|x(n(j+1))| - \sum_{i=1}^{j} |a(n(j+1)+k-n(i))|\,|x(n(i))| \\ &\geq \tfrac{1}{2}|a(k)|\,|x(n(j+1))|. \end{aligned}$$

Thus,

$$\begin{aligned} \|z-y\| &\geq |y(n(j+1)+k)\,| w(n(j+1)+k) \\ &\geq \tfrac{1}{2}|a(k)|\,|x(n(j+1))\,| w(n(j+1))\, w(k) \end{aligned}$$

since $n(j+1)+k$ is much less than $n(j+2)$ and the weight $\{w(n)\}$ is semi-multiplicative. Using condition (C), we see that the above is

$$=\tfrac{1}{2}\,|a(k)|\,w(k)\,2^{-j}$$
$$\geq 2^{-j-1}(2^{n_{j+2}}n_{j+2})^{n_{j+2}-k}$$

using property (M). Since $k<n_{j+2}$ and $j+1<n_{j+2}$ the above is $\geq n_{j+2}\geq 1$, and this contradicts the hypothesis that $\|z-y\|<1$.

Hence there exists $p\in\{n(1),n(2),\ldots,n(j)\}$ such that

$$|a(k)|\,|x(n(j+1))|<2j|a(n(j+1)+k-p)|\,|x(p)|. \tag{3.15}$$

Let $m=n(j+1)+k-p$. Then $m>n(j+1)>n_{j+1}>k$ since $k\geq n_j(n_j+1)=n_j n(j)>p$ (recall $j\geq 2$). It is important here that $k\geq n_j(n_j+1)$; $k\geq n_j$ is not a strong enough assertion. Clearly $m\leq N$ since $a(m)\neq 0$. The remainder of the proof is rather routine and similar in spirit to the second case in Lemma 3.9. We see that

$$\begin{aligned}|a(m)| &> \frac{|a(k)||x(n(j+1))|}{2j|x(p)|}\\ &\geq \left(\frac{1}{2k}\right)^{k}\frac{\Gamma(k)|x(n(j+1))|}{2j|x(q+1)|},\end{aligned}$$

using the fact that $a(k)$ has property (M) and letting $q+1=p$. But $j<n_j\leq k$ and $\Gamma(k)\,|x(n(j+1))|=\Gamma(n_{j+1}+k)$ as a consequence of the considerations in (2.10) and (2.11) since $n_{j+1}+k$ is much less than n_{j+2}. The above is then

$$\begin{aligned}&\geq\left(\frac{1}{2k}\right)^{k+1}\frac{\Gamma(n_{j+1}+k)}{|x(q+1)|}\\ &=\left(\frac{1}{2k}\right)^{k+1}\frac{\Gamma(n_{j+1}+k)}{\Gamma(q)}\end{aligned}$$

whether $q=0$ or $q=n_i$, where $i\geq 2$. Since $q\leq n_{j+1}+k$, Proposition 2.9 implies that the above is

$$\begin{aligned}&\geq(1/2m)^m\,\Gamma(n_{j+1}+k-q)\\ &=(1/2m)^m\,\Gamma(n(j+1)+k-p)\\ &=(1/2m)^m\,\Gamma(m).\end{aligned}$$

Hence

$$|a(m)|\geq(1/2m)^m\,\Gamma(m). \tag{3.16}$$

Also (3.15) implies that

$$|a(k)|\,|x(n(j+1))|\,w(n(j+1)+k)<2j|a(n(j+1)+k-p)|\,|x(p)|w(n(j+1)+k).$$

Since the weight $\{w(n)\}$ is semi-multiplicative,

$$|a(k)|\,w(k)\,|x(n(j+1))|\,w(n(j+1))<2j|a(n(j+1)+k-p)|\,w(n(j+1)+k-p)|x(p)|\,w(p).$$

Using condition (C) and the fact that $m=n(j+1)+k-p$ we see that

$$|a(k)|\,w(k)\,2^{-j}<2j|a(m)|\,w(m).$$

Thus,

$$|a(m)|\,w(m)>(2^{j+1}j)^{-1}(2^{n_{j+2}}n_{j+2})^{n_{j+2}-k}$$

since $a(k)$ has property (M). Since $j+1<n_{j+2}$ and $m\geqslant k+1$ it follows that

$$|a(m)|\,w(m)>(2^{n_{j+2}}n_{j+2})^{n_{j+2}-m}.$$

It is clear that $m>n_j(n_j+1)$. Since $m\leqslant 2n_{j+1}<n_{j+1}(n_{j+1}+1)$, it follows that $r(m)=r(k)=r=j$ also. Then the above inequality implies that

$$|a(m)|\geqslant\frac{(2^{n_{r+2}}n_{r+2})^{n_{r+2}-m}}{w(m)}. \tag{3.17}$$

As an aside we remark that here is the reason why we need two lemmas. Here $r(k)=r(m)$, which is directly the result of condition (A). This would not necessarily be the case if $n(j)+k<n(j+1)$. Hence, Lemma 3.9 cannot be dispensed with.

Finally (3.16) and (3.17) together imply that $a(m)$ has property (M) also. This completes the proof of the lemma.

We remark, as in the case of Lemma 3.9, that the hypothesis in Lemma 3.14 could be weakened from $\|z-y\|<1$ to

$$|y(n(j+1)+k)|\,w(n(j+1)+k)<2^{-j-1}(2^{n_{r+2}}n_{r+2})^{n_{r+2}-k},$$

where $r=r(k)$. Combining Lemma 3.9 with Lemma 3.14, we obtain the following main recursion result.

PROPOSITION 3.18. *Let $\{a(s)\}_{s=0}^{N}$ be a finite set of coefficients, and let $y=\Sigma_{s=0}^{N}a(s)\,T^s x$. Suppose that $k\in\mathbf{Z}^+$ with $n_2\leqslant k\leqslant N$ and that*

$$\|z-y\|<1.$$

If $a(k)$ has property (M), *then there exists $m \in \mathbf{Z}^+$ with $m>k$ and $m \leq N$ such that $a(m)$ also has property* (M).

Since there is only a finite set of coefficients occurring in the hypothesis of Proposition 3.18, if some $a(k), n_2 \leq k \leq N$, had property (M) we could continue to apply the proposition and find an infinite chain of coefficients $a(m_i)$, where $k<m_1<m_2<m_3 \ldots$, each with property (M). This is a contradiction since the supply of coefficients would be exhausted. Thus we immediately obtain the following corollary.

COROLLARY 3.19. *Let $\{a(s)\}_{s=0}^N$ be a finite set of coefficients, and let $y=\Sigma_{s=0}^N a(s)\, T^s x$. Suppose that*

$$\|z-y\|<1.$$

If $k \in \mathbf{Z}^+$ and $k \geq n_2$, then $a(k)$ does not *have property* (M).

We are finally able to prove our major result.

THEOREM 3.20. *Suppose $\{n(k)\}$ has been chosen according to condition* (A). *Suppose then that $\{w(n)\}$ and x have been constructed according to conditions* (B) *and* (C). *Then the closed ideal generated by x is a non-standard ideal in $l^1(w(n))$.*

Proof. Suppose instead that x generates a *standard* ideal. Then for any *fixed* $j \in \mathbf{Z}^+$ with $j \geq 2$, we can pick a finite set of coefficients $\{a(s)\}_{s=0}^N$, where N is sufficiently large and $\|z-\Sigma_{s=0}^N a(s)\, T^s x\|$ is sufficiently small and less than one so that $|c(n_j)|<2|a(n_j)|$. This follows since, for fixed j, $a(n_j)$ will tend to $c(n_j)$ and $c(n_j) \neq 0$ (recall the discussion concerning (1.10) of the introduction). Using condition (B4) for the first time, we see that

$$|c(n_j)|>\tfrac{1}{2}|x(n(j))|.$$

Thus,

$$\begin{aligned} |a(n_j)| &> \tfrac{1}{2}|c(n_j)| \\ &> \tfrac{1}{4}|x(n(j))|. \end{aligned}$$

Hence, we have that

$$|a(n_j)| \geq (1/2n_j)^{n_j}\Gamma(n_j), \tag{3.21}$$

since $\Gamma(n_j)=|x(n(j))|$ and $n_j \geqslant 2$. Also,

$$\begin{aligned}|a(n_j)|\,w(n_j) &> \tfrac{1}{4}|\,x(n(j))|\,w(n_j)\\ &= \tfrac{1}{4}|x(n(j))|w(n(j)-1)\\ &= \frac{\tfrac{1}{4}w(n(j)-1)\,2^{-j+1}}{w(n(j))},\end{aligned}$$

by condition (C). Then applying condition (B3) with $k+1$ replaced by j (recall j is fixed and $j \geqslant 2$) we see that the above is

$$> 2^{-j-1}(2^{j-1})^{n_{j+1}/n_j}(2n_{j+1})^{n_{j+1}}(2^{n_{j+2}}n_{j+2})^{n_{j+2}}.$$

Since $j+1 \leqslant n_{j+1}$ the above is

$$\begin{aligned}&\geqslant (2^{n_{j+2}}n_{j+2})^{n_{j+2}}\\ &\geqslant (2^{n_{r+2}}n_{r+2})^{n_{r+2}-n_j},\end{aligned}$$

since $r=r(n_j)=j-1$. This implies that

$$|a(n_j)| \geqslant \frac{(2^{n_{r+2}}n_{r+2})^{n_{r+2}-n_j}}{w(n_j)}. \tag{3.22}$$

Together, (3.21) and (3.22) imply that $a(n_j)$ has property (M). Since $j \geqslant 2$, this contradicts Corollary 3.19. Hence x must generate a non-standard ideal, and the result follows.

References

[1] Allan, G. R., Commutative Banach algebras with power series generators, in *Radical Banach algebras and automatic continuity, Proceedings Long Beach 1981*, Springer-Verlag, 975.

[2] Bade, W. G., Dales, H. G. & Laursen, K. B., Multipliers of radical Banach algebras of power series. *Memoirs of the American Mathematical Society*. To appear.

[3] Grabiner, S., Weighted shifts and Banach algebras of power series. *Amer. J. Math.*, 97 (1975), 16–42.

[4] Laursen, K. B., Ideal structure in radical sequence algebras, in *Radical Banach algebras and automatic continuity, Proceedings Long Beach 1981*, Springer-Verlag, 975.

[5] McClure, J. P., Nonstandard ideals and approximations in primary weighted l^1-algebras. Preprint.

[6] Nikolskii, N. K., Selected problems of weighted approximation and spectral analysis. *Proc. Steklov Inst. Math.*, 120 (1974) (AMS translation, 1976).

[7] SHIELDS, A. L., Weighted shift operators and analytic function theory, in *Topics in operator theory* (ed. C. Pearcy) (AMS Surveys No. 13, 1974).

[8] THOMAS, M. P., Closed ideals and bi-orthogonal systems in radical Banach algebras of power series. *Proc. Edinburgh Math. Soc.*, 25 (1982), 245–257.

[9] — Closed ideals of $l^1(w_n)$ when $\{w_n\}$ is star-shaped. *Pacific J. Math.*, 105 (1983), 237–255.

[10] — Approximation in the radical algebra $l^1(w_n)$ when $\{w_n\}$ is star-shaped, in *Radical Banach algebras and automatic continuity, Proceedings Long Beach 1981,* Springer-Verlag, 975.

[11] — A non-standard closed subalgebra of a radical Banach algebra of power series. *J. London Math. Soc.* To appear.

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