FACTORISABLE REPRESENTATIONS OF CURRENT GROUPS AND THE ARAKI-WOODS IMBEDDING THEOREM

BY

K. R. PARTHASARATHY and K. SCHMIDT

University of Bombay, India and University of London, England (1)

I. Introduction

In order to study the current commutation relations of quantum field theory, Araki and Woods [2] and Araki [1] introduced the notion of current groups and factorisable representations of such groups. Araki and Woods [2] and Streater [5] established that such representations admit a natural imbedding in a symmetric Fock space $\exp H$ over a Hilbert space H. If G is a locally compact group, then a suitable space of Borel functions on a Borel space with values in G is made into a group under pointwise multiplication. This is called a current group of G. Araki [1] established that the factorisable representations of the current group are based on certain cocycle valued measures. In this paper we show the existence of a measure on the Borel space over which the current group is constructed, relative to which a cocycle valued density exists. This yields a certain natural topology for the current group under which the factorisable representation is continuous.

In order to take into account all the cocycles of first order in the construction of factorisable representations it turns out that projective representations should also be considered. Finally, the Araki-Woods imbedding is explicitly constructed in terms of the cocycles. At this stage it may be worth remarking that our methods differ very much from that of Araki and Woods. We rely more on measure theory and not at all on lattice theory.

2. Araki functions

Throughout this paper H with or without suffixes will always stand for a complex separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let G be a fixed locally compact second countable group with identity element e. By a representation of G in H we shall always mean a continuous homomorphism of G into the group $\mathcal{U}(H)$ of unitary operators on H

(1) The present research was done at the University of Manchester, England.

with the weak (or equivalently strong) operator topology. R, $\mathbf{C}$ and $\mathcal{J}$ will always stand for the additive group of real numbers, the additive group of complex numbers and the multiplicative group of complex numbers of modulus unity.

We recall a few definitions from [4].

Definition 2.1. A continuous function s: $G\times G\to R$ is called an *additive multiplier* if the following conditions hold:

(1) $s(e, g) = s(g, e) = 0$ for all $g\in G$

(2) $s(g_1, g_2)+s(g_2^{-1}, g_1^{-1}) = 0$ for all $g_1, g_2\in G$

(3) $s(g_1g_2, g_3)+s(g_1, g_2) = s(g_1, g_2g_3)+s(g_2, g_3)$.

An additive multiplier is called *trivial* if there exists a measurable function a: $G\to R$, such that

$$s(g, h) = a(gh)-a(g)-a(h) \text{ for all } g, h\in G.$$

Two additive multipliers are called *equivalent* if their difference is trivial.

Definition 2.2. Let X be any set. A function K: $X\times X\to\mathbf{C}$ is called a *positive definite kernel* if, for every positive integer n and every choice of elements $x_1, x_2, ..., x_n$ in X and complex numbers $a_1, a_2, ..., a_n$,

$$\sum_{i,j} a_i\bar{a}_j K(x_i, x_j)\geqslant 0. \tag{2.1}$$

K is said to be *conditionally positive definite* if (2.1) holds whenever $\Sigma_i a_i=0$. A continuous function ϕ: $G\to\mathbf{C}$ is called *positive definite* if the kernel $K(g, h)=\phi(gh^{-1})$ is positive definite on $G\times G$ and $\phi(e)=1$. If s is an additive multiplier on $G\times G$, then ϕ is said to be *conditionally s-positive definite* if $\phi(e)=0$ and the kernel $K(g, h)=\phi(gh^{-1})+is(g, h^{-1})$ is conditionally positive definite. In the special case when $s\equiv 0$, we say that ϕ is *conditionally positive definite.*

In order to study factorisable projective representations of current groups, we have to make a detailed analysis of additive multiplier valued measures and conditionally positive definite function valued measures. To this end we introduce the following definitions inspired by the work of Araki [1].

Definition 2.3. Let $(T, \mathcal{S})$ be a standard Borel space. A function S: $\mathcal{S}\times G\times G\to R$ is called an *Araki multiplier* if the following conditions hold:

(1) for every fixed $(g_1, g_2)\in G\times G$, $S(\cdot, g_1, g_2)$ is a totally finite signed measure on $\mathcal{S}$;

(2) for every fixed $A\in\mathcal{S}$, the function $S(A, \cdot, \cdot)$ is an additive multiplier on $G\times G$.

For a given Araki multiplier S, a function ϕ: $\mathcal{S}\times G\to\mathbf{C}$ is called an *Araki S-function* if the following conditions hold:

(1) for every fixed $g\in G$, $\phi(\cdot, g)$ is a totally finite complex valued measure on $\mathcal{S}$;

(2) for every fixed $A\in\mathcal{S}$, $\phi(A,\cdot)$ is a conditionally $S(A,\cdot,\cdot)$-positive definite function on G.

If $S\equiv 0$, an Araki S-function will be simply called an *Araki function*.

We choose and fix a standard Borel space $(T, \mathcal{S})$ and a pair (S, ϕ) of an Araki multiplier and an Araki S-function. We now define a kernel K_ϕ on the space $(\mathcal{S}\times G)\times(\mathcal{S}\times G)$ by the equation

$$K_\phi(A, g; B, h) = \phi(A\cap B, gh^{-1}) - \phi(A\cap B, g) - \phi(A\cap B, h^{-1}) + iS(A\cap B, g, h^{-1}) \quad (2.2)$$

for all $A, B\in\mathcal{S}$ and $g, h\in G$. We shall analyse the properties of ϕ by studying the kernel K_ϕ.

LEMMA 2.1. *The kernel K_ϕ defined by (2.2) is positive definite in the space $(\mathcal{S}\times G)\times(\mathcal{S}\times G)$.*

Proof. Let $A_1, A_2, \dots, A_n\in\mathcal{S}$, $g_1, g_2, \dots, g_n\in G$ and $a_1, a_2, \dots, a_n$ be n complex numbers. Let $B_1, B_2, \dots, B_m$ be the atoms of the ring generated by $A_1, A_2, \dots, A_n$. Let

$$\chi(i,k) = \begin{cases} 1 & \text{if } B_k\subset A_i \\ 0, & \text{otherwise}, \quad i=1,2,\dots,n, \quad k=1,2,\dots,m. \end{cases}$$

Then equation (2.2) and an easy computation show that

$$\sum_{i,j=1}^{n} a_i\bar{a}_j K_\phi(A_i, g_i; A_j, g_j) = \sum_{k=1}^{m}\Big\{\sum_{i,j=1}^{n} a_i\chi(i,k)\overline{a_j\chi(j,k)}\,[\phi(B_k, g_i g_j^{-1}) - \phi(B_k, g_i) - \phi(B_k, g_j^{-1}) + iS(B_k, g_i, g_j^{-1})]\Big\}. \quad (2.3)$$

Since ϕ is an Araki S-function, for every fixed $B\in\mathcal{S}$, the kernel $\phi(B, gh^{-1})$ is conditionally $S(B,\cdot,\cdot)$-positive definite on $G\times G$. Hence by corollary 1 to Lemma 2.2 of [4], the kernel $\phi(B, gh^{-1}) - \phi(B, g) - \phi(B, h^{-1}) + iS(B, g, h^{-1})$ is positive definite on $G\times G$. This shows that every term within the curly brackets in (2.3) is non-negative. Hence K_ϕ is positive definite. This completes the proof of the lemma.

LEMMA 2.2. *Let ϕ be an Araki S-function. Then there exists a Hilbert space H spanned by vectors $Y(A, g, h)$, $A\in\mathcal{S}$, $g, h\in G$ such that the inner product $\langle\cdot,\cdot\rangle$ satisfies the equation*

$$\langle Y(A,g_1,h_1),\, Y(B,g_2,h_2)\rangle = \phi(A\cap B, h_1^{-1}g_1g_2^{-1}h_2) - \phi(A\cap B, h_1^{-1}g_1g_2^{-1})$$
$$- \phi(A\cap B, g_1g_2^{-1}h_2) + \phi(A\cap B, g_1g_2^{-1}) + i[S(A\cap B, h_1^{-1}g_1g_2^{-1}, h_2) \tag{2.4}$$
$$- S(A\cap B, g_1g_2^{-1}, h_2)]$$

for all $A, B\in \mathcal{S}, g_1, g_2, h_1, h_2 \in G$.

Proof. By Lemma 2.1 and Kolmogorov's theorem on stochastic processes, we can consider K_ϕ as the covariance function of a certain complex Gaussian stochastic process $\chi(A,g)$ with mean zero and "time variable" $(A,g)\in \mathcal{S}\times G$. In the standard notation of probability theory,

$$EX(A,g) = 0 \quad \text{for all } A\in\mathcal{S},\, h\in G,$$

$$EX(A,g)\overline{X(B,h)} = K_\phi(A,g,B,h) \quad \text{for all } A, B\in \mathcal{S}, \text{ and } g,h\in G.$$

We now write
$$Y(A,g,h) = X(A,h^{-1}g) - X(A,g). \tag{2.5}$$

A straightforward computation now shows that $E\,Y(A,g_1,h_1)\,\overline{Y(A,g_2,h_2)}$ is precisely the right hand side of (2.4). If H is defined as the expected mean square completion of the linear span of the random variables $Y(A,g,h)$, $A\in\mathcal{S}$, $g,h\in G$, and the inner product is defined by covariance, H becomes a Hilbert space. The separability of H is an immediate consequence of the fact that $\mathcal{S}$ is countably generated, G is second countable and ϕ is an Araki S-function. This completes the proof of the lemma.

Given an Araki S-function, we construct the Hilbert space H according to Lemma 2.2 and denote by $H(A)$ the closed linear span of all the elements $Y(A,g,h)$ as g and h vary over G. Let $P(A)$ be the projection onto the subspace $H(A)$. With these notations we have the following lemma:

LEMMA 2.3. *The map* $A\to P(A)$ *is a projection valued measure on* $(T,\mathcal{S})$. *Further*

$$P(A)\,Y(B,g,h) = Y(A\cap B,g,h) \quad \text{for } A,B\in\mathcal{S},\ g,h\in G. \tag{2.6}$$

Proof. Formula (2.4) implies that $Y(A,g_1,h_1)$ and $Y(B,g_2,h_2)$ are orthogonal as soon as A and B are disjoint. Hence $H(A)$ and $H(B)$ are orthogonal whenever A and B are disjoint. Since ϕ is an Araki S-function, (2.4) also implies that for a sequence of disjoint sets $A_1, A_2, \ldots, \in \mathcal{S}$,

$$\|Y(\textstyle\bigcup_i A_i, g, h) - \sum_{i=1}^{\infty} Y(A_i,g,h)\|^2 = 0.$$

This implies that $H(\bigcup_i A_i) = \oplus_i H(A_i)$. Hence $P(\cdot)$ is a projection valued measure. Since $Y(B,g,h) = Y(B\cap A,g,h) + Y(B\cap A',g,h)$ where A' is the complement of A, (2.6) is proved. This completes the proof of the lemma.

In the Hilbert space H defined by Lemma 2.2, we define the map U_g for $g \in G$ by the equation

$$U_{g_1} Y(A, g, h) = Y(A, g g_1^{-1}, h) \tag{2.7}$$

for all $A \in \mathcal{S}$, $g_1, g, h \in G$. Formula (2.4) implies that U_{g_1} is a well defined isometry on the set of all elements $Y(A, g, h)$, $A \in \mathcal{S}$, $g, h \in G$. Hence U_{g_1} can be extended to an isometry on H. Since the range of U_{g_1} is everywhere dense it follows that U_{g_1} is a unitary operator for every g_1. Further $U_{g_1} U_{g_2} = U_{g_1 g_2}$ for all $g_1, g_2 \in G$. Since ϕ is an Araki S-function, the map $g \to U_g$ is weakly continuous and hence a unitary representation. Further, (2.6) and (2.7) imply that

$$U_{g_1} P(A) Y(B, g, h) = Y(A \cap B, g g_1^{-1}, h) = P(A) U_{g_1} Y(B, g, h). \tag{2.8}$$

We shall write
$$P(A) U_g P(A) = U(A, g) \quad \text{for all } g \in G,\ A \in \mathcal{S}. \tag{2.9}$$

Equations (2.8) and (2.9) imply that U_g and $P(A)$ commute for all $g \in G$ and $A \in \mathcal{S}$. Further $U(A, g)$ vanishes on $H(A')$ and restricted to the subspace $H(A)$ yields a unitary representation for the group G. We also observe that $U(T, g) = U_g$. Further

$$U(A, g_1) Y(B, g, h) = Y(A \cap B, g g_1^{-1}, h). \tag{2.10}$$

We now write
$$\delta(A, h) = Y(A, e, h). \tag{2.11}$$

Then (2.5), (2.10) and (2.11) imply that

$$U(A, g)\delta(A, h) = \delta(A, gh) - \delta(A, h). \tag{2.12}$$

In other words on the subspace $H(A)$, the map $g \to U(A, g)$ is a representation and $g \to \delta(A, g)$ is a cocycle of the first order for that representation. In this context we refer to [1] and [4]. Further $\delta(A, g)$ satisfies the equation

$$P(B)\delta(A, g) = \delta(A \cap B, g). \tag{2.13}$$

For any measure μ on $(T, \mathcal{S})$ and any finite or countable cardinal n, we shall denote by $L_2(\mu, n)$ the direct sum of n copies of the Hilbert space $L_2(\mu)$. In the notation of direct integrals of Hilbert spaces, we may write

$$L_2(\mu, n) = \int \mathbf{C}^n \, d\mu$$

where $\mathbf{C}^n$ is the n dimensional complex Hilbert space if n is finite and the space of square summable sequences if n is infinite. If now we apply the Hahn-Hellinger theorem for the projection valued measure $P(A)$, we may assume that

$$H = \underset{n=\infty,1,2\ldots}{\oplus} \int \mathbf{C}^n d\mu_n$$

where μ_n are mutually orthogonal measures with disjoint supports B_n such that $\Sigma\mu_n$ is a totally finite measure and $P(A)$ is simply multiplication by the indicator function χ_A of the set A. Since U_g and $P(A)$ commute for all $g \in G$ and $A \in \mathcal{S}$, we may by a standard application of a result of Von Neumann and Fubini's Theorem assume that U_g restricted to $\int \mathbf{C}^n d\mu_n$ is multiplication by an n-dimensional matrix $V_n(t, g)$, $t \in T$ and for every fixed t and n, $g \to V_n(t, g)$ is an n-dimensional unitary representation of G. A similar application of Fubini's theorem and an argument similar to the one in the proof of Theorem 3.1 in [4] yield the equation $\delta(A, g) = \Sigma_n \int_A \delta_n(t, g) d\mu_n$ where $\delta_n(t, g)$ is a continuous cocycle for the representation $g \to V_n(t, g)$. We now write $\mu = \Sigma_n \mu_n$, $H_t = \mathbf{C}^n$, $V(t, g) = V_n(t, g)$ whenever t belongs to the support B_n of μ_n. Then we have in the notation of direct sum of Hilbert spaces,

$$H = \int H_t d\mu(t),$$

$$U(A, g) = \int \chi_A(t) V(t, g)\, d\mu(t),$$

$$\delta(A, g) = \int \chi_A(t)\, \delta(t, g)\, d\mu(t),$$

$$P(A) = \int \chi_A(t)\, I_t\, d\mu(t),$$

where I_t is the identity operator in H_t. Further $\delta(t, g)$ is a cocycle for $V(t, g)$. We can now summarise all our discussion in the form of a theorem.

THEOREM 2.1. *Let G be a locally compact second countable group and $(T, \mathcal{S})$ a standard Borel space. Let further (S, ϕ) be a pair consisting of an Araki multiplier S on $\mathcal{S} \times G \times G$ and an Araki S-function ϕ on $\mathcal{S} \times G$. Then there exists a complex separable Hilbert space H, a projection valued measure $A \to P(A)$ on $\mathcal{S}$, a continuous unitary representation $g \to U_g$ of G in H and a continuous function $g \to \delta(g)$ on G with values in H satisfying the following conditions:*

(a) *$U_g \delta(h) = \delta(gh) - \delta(g)$ for all $g, h \in G$;*

(b) *The subspaces $H(A) = P(A)(H)$ are invariant under all the U_g;*

(c) *For every $A \in \mathcal{S}$, $g, h_1, h_2 \in G$,*

$$\langle P(A)\, U_g\, \delta(h_1), \delta(h_2)\rangle = \phi(A, h_1^{-1} g^{-1} h_2) - \phi(A, h_1^{-1} g^{-1}) - \phi(A, g^{-1} h_2) + \phi(A, g^{-1}) + i[S(A, h_1^{-1} g^{-1}, h_2) - S(A, g^{-1}, h_2)]. \quad (2.14)$$

Further the Hilbert space H can be written as a direct integral $\int H_t d\mu(t)$ of Hilbert spaces H_t with respect to a totally finite measure μ on $\mathcal{S}$, where the family $\{H_t, t \in T\}$ satisfies the following:

(a′) *For every t, there exists a unitary representation $g \to V(t, g)$ of G in H_t such that $U_g = \int V(t, g) d\mu(t)$;*

(b′) *The projection valued measure $A \to P(A)$ is given by*

$$P(A) = \int \chi_A(t)\, I_t d\mu(t)$$

where I_t is the identity operator in H_t;

(c′) *For every t, there exists a continuous map $g \to \delta(t, g)$ from G into H_t such that*

$$V(t, g)\delta(t, h) = \delta(t, gh) - \delta(t, g) \quad \text{for all } g, h \in G \tag{2.15}$$

and

$$\delta(g) = \int \delta(t, g) d\mu(t) \quad \text{for all } g \in G.$$

The measure μ satisfying (a′), (b′), (c′) and (a), (b), (c) is determined uniquely upto equivalence. The map $t \to (V(t, \cdot), \phi(t, \cdot))$ is determined upto unitary equivalence a.e.(μ).

Conversely given a totally finite measure μ and a triplet $(H_t, V(t, \cdot), \delta(t, \cdot))$ for every t such that (2.15) is fulfilled and the direct integrals $\int H_t d\mu(t)$, $\int V(t, \cdot) d\mu(t)$ and $\int \delta(t, \cdot) d\mu(t)$ are well defined, there exists a pair (S, ϕ) consisting of an Araki multiplier S and an Araki S-function ϕ such that the triple $(U_g, P(A), \delta)$ defined by (a′), (b′), (c′) satisfies (a), (b) and (c).

If S', ϕ' is another pair satisfying the same properties then $S'(A, \cdot, \cdot) - S(A, \cdot, \cdot)$ is a trivial additive multiplier for every fixed $A \in \mathcal{S}$ and $\mathrm{Re}\, \phi = \mathrm{Re}\, \phi'$.

Proof. The only part that remains to be proved is the converse. This follows by a straightforward calculation if we put $\phi(A, g) = -\frac{1}{2}\langle P(A)\delta(g), \delta(g)\rangle$ and $S(A, g, h) = \mathrm{Im}\, \langle P(A)\delta(g^{-1}), \delta(h)\rangle$ and observe that $\mathrm{Im}\, \langle \delta(t, g^{-1}), \delta(t, h)\rangle$ is an additive multiplier for every $t \in T$. (See [4], Theorem 2.1). This completes the proof of the theorem.

In the case $S = 0$, the statement of Theorem 2.1 is considerably simplified and something more can be said about an Araki function. In fact we have the following theorem.

Theorem 2.2. *Let G be a locally compact second countable group and $(T, \mathcal{S})$ be a standard Borel space. Suppose that ϕ is an Araki function on $\mathcal{S} \times G$. Then there exist maps ψ: $T \times G \to \mathbf{C}$ and ξ: $\mathcal{S} \times G \to R$ such that the following properties are satisfied:*

(1) *For every fixed $t \in T$, $\psi(t, \cdot)$ is a conditionally positive definite continous function on G;*

(2) *For every fixed $g \in G$, $\psi(\cdot, g)$ is a measurable function on T;*

(3) *For every fixed* A, $\xi(A,\cdot)$ *is a continuous homomorphism of* G *into the real line;*

(4) *For every fixed* g, $\xi(\cdot, g)$ *is a totally finite countably additive function on* $\mathcal{S}$*;*

(5) *For all* $A \in \mathcal{S}$ *and* $g \in G$, $\phi(A, g) - i\xi(A, g) = \int_A \psi(t, g)\, d\mu(t)$ *for some totally finite measure* μ *on* $\mathcal{S}$*;*

(6) *There exists a family of Hilbert spaces* H_t, $t \in T$, *unitary representations* $g \to V(t, g)$ *of* G *in* H_t *and cocycles* $g \to \delta(t, g)$ *for the representation* $V(t,\cdot)$ *such that the Hilbert space* $H = \int H_t d\mu(t)$ *and the direct integrals* $\int V(t,\cdot)\, d\mu(t)$ *and* $\int \delta(t,\cdot)\, d\mu(t)$ *are all defined, and for all* $t \in T$, $A \in \mathcal{S}$, $g_1, g_2 \in G$,

$$\psi(t, g_1 g_2) - \psi(t, g_1) - \psi(t, g_2) = \langle \delta(t, g^{-1}), \delta(t, g_2) \rangle;$$

(7) *The measure* μ *is unique upto equivalence and the triple* $[H_t, V(t,\cdot), \delta(t,\cdot)]$ *is determined upto unitary equivalence for almost all* $t(\mu)$.

Proof. Putting $S = 0$ in (2.14), $P(A)\delta(g) = \delta(A, g)$, we obtain from property (c) of Theorem 2.1,

$$\langle \delta(A, g_1^{-1}), \delta(A, g_2) \rangle = \phi(A, g_1 g_2) - \phi(A, g_1) - \phi(A, g_2).$$

We then construct μ, H_t, $V(t,\cdot)$ and $\delta(t,\cdot)$ according to the same theorem. Then we have

$$\delta(A,g) = \int_A \delta(t,g)\, d\mu(t)$$

and

$$\phi(A, g_1 g_2) - \phi(A, g_1) - \phi(A, g_2) = \int_A \langle \delta(t, g_1^{-1}), \delta(t, g_2) \rangle\, d\mu(t). \tag{2.16}$$

Since $\phi(A, e) = 0$ and $\phi(A, g^{-1}) = \overline{\phi(A, g)}$, it follows that

$$\operatorname{Re} \phi(A, g) = -\tfrac{1}{2} \int_A \langle \delta(t, g), \delta(t, g) \rangle\, d\mu(t).$$

We write

$$\alpha(A, g) = \operatorname{Im} \phi(A, g), \tag{2.17}$$

and

$$\alpha(A, g) = \xi(A, g) + \eta(A, g),$$

where for each fixed g, $\xi(\cdot, g)$ and $\eta(\cdot, g)$ are the singular and absolutely continuous parts of $\alpha(\cdot, g)$ with respect to the measure μ. Now suppose that g_1 and g_2 are any two elements fixed in G. Let $C \subset T$ be a set such that

$$\xi(B, g_1) - \xi(B, g_2) - \xi(B, g_1 g_2) = 0$$

for all Borel sets $B \subset C$, and $\mu(C') = 0$. Then by (2.16) and (2.17),

$$\xi(A, g_1 g_2) - \xi(A, g_1) - \xi(A, g_2) = \alpha(A \cap C', g_1 g_2) - \alpha(A \cap C', g_1) - \alpha(A \cap C', g_2)$$
$$= \operatorname{Im} \int_{A \cap C'} \langle \delta(t, g_1^{-1}), \delta(t, g_2) \rangle \, d\mu(t) = 0.$$

In other words $\xi(A, \cdot)$ is a homomorphism from G into R. Since $\eta(A, g)$ is measurable in g, it follows that $\xi(A, \cdot)$ is measurable. Hence it is a continuous homomorphism for every fixed A. Now write

$$\tilde{\phi}(A, g) = \phi(A, g) - i\xi(A, g).$$

Then $\tilde{\phi}$ is also an Araki function which satisfies the equation (2.16) and $\operatorname{Re} \tilde{\phi} = \operatorname{Re} \phi$.

Since $\eta(\cdot, g)$ is absolutely continuous with respect to μ, we can construct the Radon-Nykodym derivative $f(\cdot, g) = \dfrac{d\eta(\cdot, g)}{d\mu(\cdot)}$. Then we have from (2.16) and the uniqueness of the Radon-Nykodym derivative

$$\operatorname{Im} \langle \delta(t, g_1^{-1}), \delta(t, g_2) \rangle = f(t, g_1 g_2) - f(t, g_1) - f(t, g_2) \quad \text{a.e. } t[\mu],$$

for every $g_1, g_2 \in G$. By Theorem 2.1 of [4] we know that the left hand side is an additive multiplier for every t. By applying Fubini's theorem and using the fact that Haar measurable representations are continuous, we can without loss of generality assume that $f(t, \cdot)$ is continuous for every t. Once again applying Theorem 2.1 of [4], we obtain that the function $\psi(t, g) = -\frac{1}{2}\langle \delta(t, g), \delta(t, g) \rangle + i f(t, g)$ satisfies (1), (2) and (6). Further

$$\tilde{\phi}(A, g) = \operatorname{Re} \phi(A, g) + i\eta(A, g) = \int_A \psi(t, g) \, d\mu(t).$$

Hence (5) is also satisfied. Property (7) is already contained in Theorem 2.1. This completes the proof.

3. Factorisable families of positive definite functions

We start with a few definitions and lemmas.

Definition 3.1. Let $(T, \mathcal{S})$ be a standard Borel space. A function $M: \mathcal{S} \to \mathbf{C}$ is called a *nonatomic complex valued multiplicative measure* if the following conditions hold: (1) $0 < |M(A)| \leq 1$ for all $A \in \mathcal{S}$; (2) $M(\emptyset) = 1$; (3) $M(\bigcup_{i=1}^{\infty} A_i) = \prod_1^{\infty} M(A_i)$ for any sequence $\{A_n\}$ of disjoint sets from $\mathcal{S}$; (4) for every single point set $\{t\}$, $t \in T$, $M(\{t\}) = 1$.

Throughout the rest of the paper, by a multiplicative measure we shall always mean a nonatomic complex valued one. We shall now show that every multiplicative measure is the exponential of an additive measure.

LEMMA 3.1. *Let M be a multiplicative measure on (T, S). Then there exists a unique nonatomic complex valued totally finite measure m such that*

$$M(A) = \exp m(A) \quad \text{for all } A \in S.$$

Proof. First of all we observe that for any disjoint sequence A_n from S, the infinite product $\prod M(A_n)$ converges to $M(\bigcup A_n)$ in whatever order we write the sequence and hence $\Sigma_n |M(A_n) - 1| < \infty$. Let

$$\alpha(A) = \sup_{B \subset A,\, B \in S} |M(B) - 1|.$$

Then α is a monotonic countably subadditive function on S. Let A_n be a sequence in S, decreasing to a single point set $\{t\}$, $t \in T$. Then $B_i = A_i - \{t\}$ decreases to the empty set ϕ. For every i, we choose $C_i \subset B_i - B_{i+1}$ such that

$$|1 - M(C_i)| \geq \alpha(B_i - B_{i+1}) - 2^{-i}.$$

Then
$$\sum_{i=1}^{\infty} \alpha(B_i - B_{i+1}) \leq \sum_{i=1}^{\infty} |1 - M(C_i)| + 1.$$

Since the C_i's are disjoint the right hand side of the above inequality is finite and therefore

$$\lim_{n\to\infty} \alpha(A_n) = \lim_{n\to\infty} \alpha(B_n) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \alpha(B_k - B_{k+1}) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} |1 - M(C_k)| + 2^{-n+1} = 0.$$

Thus $\alpha(A_n)$ decreases to zero whenever A_n decreases to a single point set. Since (T, S) is standard we may assume without loss of generality that T is the closed unit interval with its usual Borel structure. By the discussion above it follows that for every $t \in T$, there exists a neighbourhood $N(t)$ of t such that $\alpha(N(t)) < \frac{1}{2}$. Using the compactness of T, we can select a finite number of neighbourhoods $N_1, N_2, \ldots, N_k$ such that they cover T and $\alpha(N_i) < \frac{1}{2}$ for all i. We write $E_i = N_i - \bigcup_{j=1}^{i-1} N_j$ and define m by

$$m(A) = \sum_{i=1}^{k} \log M(A \cap E_i)$$

where log stands for the principal branch of the logarithm. Clearly m is a measure satisfying the conditions of the lemma. The total finiteness of m follows from the fact that the E_i's cover T. The uniqueness of m is obvious. This completes the proof of the lemma.

LEMMA 3.2. *Let X be an arbitrary topological space and K: $S \times X \times X \to \mathbf{C}$ have the following properties:*

(1) $K(\cdot, x, y)$ *is a multiplicative measure on* $\mathcal{S}$ *for every* $x, y \in X$;

(2) $K(A, \cdot, \cdot)$ *is a continuous positive definite kernel on* $X \times X$ *for every* $A \in \mathcal{S}$.

Then there exists a unique function $K': \mathcal{S} \times X \times X \to \mathbf{C}$ *satisfying the following properties:*

(1′) $K'(\cdot, x, y)$ *is a totally finite nonatomic complex valued measure on* $\mathcal{S}$ *for every* $x, y \in X$;

(2′) $K'(A, \cdot, \cdot)$ *is a conditionally positive definite kernel on* $X \times X$ *whose real part is continuous;*

(3′) $K(A, x, y) = \exp K'(A, x, y)$ *for all* $A \in \mathcal{S}$, $x, y \in X$.

Proof. For every fixed $x, y \in X$, we construct K' according to Lemma 3.1 so that (1′) and (3′) are fulfilled. To prove conditional positive definiteness we consider any r points $x_1, x_2, ..., x_r$ in X and a Borel set A in T. Now choose a sequence of finite measurable partitions $\{A_{nk}, 1 < k \leqslant n\}$ of A such that, for all $1 \leqslant i, j \leqslant r$,

$$\lim_{n\to\infty} \sup_{1\leqslant k\leqslant n} \operatorname{Var} K'(A_{nk}, x_i, x_j) = 0$$

where Var stands for variation. Since $|e^x - 1 - x| \leqslant 3x^2$ for all $|x| \leqslant 1$, we have

$$\lim_{n\to\infty} \left| K'(A, x_i, x_j) - \sum_{k=1}^{n} [K(A_{nk}, x_i, x_j) - 1] \right| \leqslant \lim_{n\to\infty} \sum_{k=1}^{n} \left| K'(A_{nk}, x_i, x_j) - K(A_{nk}, x_i, x_j) + 1 \right|$$

$$\leqslant 3 \operatorname{Var} K'(A, x_i, x_j) \lim_{n\to\infty} \sup_{k} |K'(A_{nk}, x_i x_j)| = 0.$$

Property (2) of K implies that for all constants $a_1, a_2, ..., a_r$ such that $\sum_{i=1}^r a_i = 0$, $\sum_{i,j,k} a_i \bar{a}_j [K(A_{nk}, x_i, x_j) - 1] \geqslant 0$. Hence $\sum_{k,j} a_i \bar{a}_j K'(A, x_i, x_j) \geqslant 0$. In other words $K'(A, \cdot, \cdot)$ is conditionally positive definite. Since $\operatorname{Re} K'(A, x, y) = \log |K(A, x, y)|$, it is automatically continuous. This concludes the proof of the lemma.

LEMMA 3.3. *Let* X *be any topological space and* $K: X \times X \to \mathbf{C}$ *be a positive definite kernel such that* Re K *is continuous. Then* K *is continuous.*

Proof. Let $Z(x)$, $x \in X$ be a complex Gaussian stochastic process with mean zero and covariance function $K(x, y)$. Then

$$E\, Z(x)\overline{Z(y)} = K(x, y).$$

We have
$$E|Z(x) - Z(y)|^2 = K(x, x) + K(y, y) - 2 \operatorname{Re} K(x, y).$$

Since $K(x, x)$ is real for all x, the right side of the above equation is continuous by hypothesis. This implies that $Z(x)$ is mean square continuous in x. Hence K is continuous. This completes the proof of the lemma.

We now go back to the group G and recall a definition from [4].

Definition 3.2. A measurable function $\sigma: G \times G \to \mathcal{J}$ is called a *multiplier* if

(1) $\sigma(e, g) = \sigma(g, e) = 1$ for all $g \in G$,

(2) $\sigma(g_1, g_2 g_3)\, \sigma(g_2, g_3) = \sigma(g_1 g_2, g_3)\, \sigma(g_1, g_2)$ for all $g_1, g_2, g_3 \in G$.

A function $\Phi: G \to \mathbf{C}$ is called *σ-positive definite* for a multiplier σ if (1) $\Phi(e) = 1$; (2) the kernel $K(g, h) = \Phi(gh^{-1})\sigma(g, h^{-1})$ is positive definite on $G \times G$.

Definition 3.3. A family $(\sigma(A, \cdot, \cdot), \Phi(A, \cdot, \cdot))$, $A \in \mathcal{S}$ of multipliers $\sigma(A, \cdot, \cdot)$ on $G \times G$ and $\sigma(A, \cdot, \cdot)$-positive definite functions $\Phi(A, \cdot)$ on G is said to be factorisable if (1) for every $g, h \in G$, $\sigma(\cdot, g, h)$ and $\Phi(\cdot, g)$ are multiplicative measures on $\mathcal{S}$; (2) $\sigma(A, g, h) = \bar{\sigma}(A, h^{-1}, g^{-1})$ for all $A \in \mathcal{S}$, $g, h \in G$.

THEOREM 3.1. *Let G be a locally compact second countable group and $(T, \mathcal{S})$ be a standard Borel space. Let $\{\sigma(A, \cdot, \cdot), \Phi(A, \cdot), A \in \mathcal{S}\}$ be a factorisable family of multipliers $\sigma(A, \cdot, \cdot)$ on $G \times G$ and $\sigma(A, \cdot, \cdot)$-positive definite functions $\Phi(A, \cdot)$ on G. Then there exists a map $\alpha: \mathcal{S} \times G \to \mathcal{J}$ such that $\alpha(A, \cdot)$ is a Borel function on G for every $A \in \mathcal{S}$, $\alpha(\cdot, g)$ is a multiplicative measure for every $g \in G$ and*

$$\Phi(A, gh)\sigma(A, g, h)\alpha(A, g)\alpha(A, h) = \exp[\phi(A, gh) + iS(A, g, h)] \quad \text{for all } A \in \mathcal{S}, g, h \in G, \tag{3.1}$$

where S is an Araki multiplier and ϕ is an Araki S-function. If $\sigma \equiv 1$, we can choose $\alpha \equiv 1$ and $S \equiv 0$.

Proof. We consider the function

$$K(A, g, h) = \Phi(A, gh^{-1})\sigma(A, g, h^{-1}).$$

Then K satisfies all the conditions of Lemma 3.2. Hence there exists a function $K'(A, g, h)$ satisfying the properties of Lemma 3.2. In particular,

$$\Phi(A, gh^{-1})\sigma(A, g, h^{-1}) = \exp K'(A, g, h). \tag{3.2}$$

We now put

$$\alpha(A, g) = |\Phi(A, g)|/\Phi(A, g),$$

$$\phi(A, g) = \operatorname{Re} K'(A, g, e),$$

$$S(A, g, h) = \operatorname{Im}[K'(A, g, h^{-1}) - K'(A, g, e) - K'(A, e, h^{-1})].$$

A straightforward computation shows that the identity (3.1) holds good. By property (2′) of Lemma 3.2, $\phi(A, g)$ is continuous in g. Since $K'(A, g, h)$ is conditionally positive definite for fixed A, $K'(A, g, h) - K'(A, g, e) - K'(A, e, h)$ is positive definite (cf. Lemma 2.2 in [4]). Since its real part is continuous it follows from Lemma 3.3 that $S(A, g, h)$ is continuous in g and h. Further (3.2) implies that

$$\exp iS(A,g,h)=\frac{\alpha(A,g)\,\alpha(A,h)}{\alpha(A,gh)}\,\sigma(A,g,h)\quad \text{for all } A\in\mathcal{S}, g,h\in G.$$

S is an additive measure in A. The right hand side of the above equation is a multiplicative measure in A and multiplier in g, h. The uniqueness of the logarithmic measure in Lemma 3.1 implies that S is an Araki multiplier.

If $\sigma\equiv 1$, the function $\Phi(A, gh^{-1})$ itself satisfies the conditions of Lemma 3.2 and (3.2) becomes $\Phi(A, gh^{-1})=\exp K'(A, g, h)$. The uniqueness of the logarithm in Lemma 3.1 implies the existence of a function $\phi(A, g)$ such that $K'(A,g,h)=\phi(A, gh^{-1})$ where ϕ is conditionally positive definite for every A. Further the real part of ϕ is continuous for every fixed A. Hence by Lemma 2.2 in [4] and Lemma 3.3, $\phi(A,gh^{-1})-\phi(A,g)-\phi(A,h^{-1})$ is continuous as a function of g and h for fixed A. If we take a continuous function $f(g)$ which vanishes outside a compact set and whose integral over G with respect to a right invariant Haar measure is unity, it follows that $\int [\phi(A, gh^{-1})-\phi(A, g)-\phi(A, h^{-1})]$ $f(h)dh$ is continuous in g. Since $\int \phi(A, h^{-1})f(h)dh$ is constant and $\int \phi(A, gh^{-1})f(h)dh=$ $\int \phi(A, h^{-1})f(hg)dh$ is continuous in g it follows that $\phi(A, g)$ is continuous in g for every $A\in\mathcal{S}$. This completes the proof of the theorem.

4. Factorisable representations of current groups

Let H be a complex separable Hilbert space and $\mathcal{U}(H)$ be the group of all unitary operators with the weak (or equivalently strong) operator topology. Let G be a locally compact second countable group. A Borel mapping $g\to W_g$ from G into $\mathcal{U}(H)$ is called a *multiplier representation with multiplier* σ if

$$W_{g_1}W_{g_2}=\sigma(g_1, g_2)\,W_{g_1g_2},\quad \text{for all } g_1, g_2\in G,\ W_e=I,$$

where I is the identity operator in H. W is an ordinary representation if $\sigma\equiv 1$. A triplet (W, x, σ) where W is a multiplier representation with multiplier σ and x is a unit vector in H is called a *cyclic multiplier representation* if the vectors $\{W_g x, g\in G\}$ span H. Two multiplier representations $W^{(1)}$ and $W^{(2)}$ in Hilbert spaces $H^{(1)}$ and $H^{(2)}$ respectively are said to be *projectively equivalent* if there exists a unitary isomorphism U: $H^{(1)}\to H^{(2)}$ and a Borel function a: $G\to\mathcal{J}$ such that $U\,W_g^{(1)}U^{-1}=a(g)\,W_g^{(2)}$. If $a\equiv 1$ we shall say that $W^{(1)}$ and $W^{(2)}$ are *unitarily equivalent*. Two cyclic multiplier representations $(W^{(1)}, x_1, \sigma_1)$ and $(W^{(2)}, x_2, \sigma_2)$ are said to be projectively (unitarily) equivalent, if $W^{(1)}$ and $W^{(2)}$ are projectively (unitarily) equivalent and the vectors x_1 and x_2 correspond under the equivalence.

Definition 4.1. Let $(W^{(i)}, x_i, \sigma_i)$, $i=1, 2$ be two cyclic multiplier representations acting in Hilbert spaces H_i, $i=1, 2$ respectively. Their convolution denoted by $(W^{(1)}, x_1, \sigma_1)*$

$(W^{(2)}, x_2, \sigma_2)$ is the multiplier representation $W^{(1)} \otimes W^{(2)}$ restricted to the cyclic subspace generated by $x_1 \otimes x_2$ in $H_1 \otimes H_2$. The multiplier of the convolution is clearly $\sigma_1 \sigma_2$.

Remark. For any cyclic multiplier representation (W, x, σ) of G, the function $\langle W_g x, x\rangle$ is called its expectation value. Then it is clear that $\langle W_g x, x\rangle$ is σ-positive definite. The expectation value of the convolution of two cyclic multiplier representations is the product of the expectation values of the individual multiplier representations.

Definition 4.2. Let (T, S) be a standard Borel space and for every $A \in S$, let $(W^A, x_A, \sigma(A, \cdot, \cdot))$ be a cyclic multiplier representation of G. The family $\{W^A, x_A, \sigma(A, \cdot, \cdot), A \in S\}$ is said to be *factorisable* if

(1) for every sequence $\{A_n\}$ of sets in S descending to a single point set,

$$\lim \langle W_g^{A_n} x_{A_n}, x_{A_n}\rangle = 1,$$

$$\lim \sigma(A_n, g, h) = 1$$

uniformly on the compact sets of G and $G \times G$ respectively;

(2) for every $A \in S$ and any finite measurable partition of A into sets $A_1, A_2, ..., A_k$, the cyclic representations $(W^A, x_A, \sigma(A, \cdot, \cdot))$ and $(W^{A_1}, x_{A_1}, \sigma(A_1, \cdot, \cdot)) * ... * (W^{A_n}, x_{A_n}, \sigma(A_n, \cdot, \cdot))$ are unitarily equivalent.

LEMMA 4.1. *Let (T, S) be a standard Borel space, G be connected and $\{W^A, x_A, \sigma(A, \cdot, \cdot), A \in S\}$ be a factorisable family of cyclic multiplier representations of G. Then there exists another factorisable family $\{\hat{W}^A, \hat{x}_A, \hat{\sigma}(A, \cdot, \cdot), A \in S\}$, such that (1) for each A, $(W^A, x_A, \sigma(A, \cdot, \cdot))$ and $(\hat{W}^A, \hat{x}_A, \hat{\sigma}(A, \cdot, \cdot))$ are projectively equivalent, (2) the functions $\hat{\Phi}(A, g) = \langle \hat{W}_g^A \hat{x}_A, \hat{x}_A\rangle$ are $\hat{\sigma}(A, \cdot, \cdot)$-positive definite for every A, and (3) the family $\{\hat{\sigma}(A, \cdot, \cdot), \Phi(A, \cdot), A \in S\}$ is factorisable in the sense of Definition 3.3. If $\sigma \equiv 1$, then we can put $W^A = \hat{W}^A$. Conversely, every factorisable family $\{\sigma(A, \cdot, \cdot), \Phi(A, \cdot), A \in S\}$ yields a factorisable family $\{W^A, x_A, \sigma(A, \cdot, \cdot), A \in S\}$ by the equation $\Phi(A, g) = \langle W_g^A x_A, x_A\rangle$.*

Proof. Suppose $\{W^A, x_A, \sigma(A, \cdot, \cdot), A \in S\}$ is factorisable. Let $\Phi(A, g) = \langle W_g^A x_A, x_A\rangle$. By the argument of Lemma 5.6 in [4], $|\Phi(A, \cdot)|^2$ is positive definite and continuous for every A. We shall now prove that $\Phi(A, g)$ does not vanish at any point. First of all we note that

$$\Phi(T, g) = \Phi(A \cup A', g) = \Phi(A, g)\Phi(A', g)$$

Thus, to prove our claim it is enough to show that $\Phi(T, g)$ does not vanish anywhere. $|\Phi(T, g)|^2$ is continuous and positive definite. The set $N = \{g: \Phi(T, g) \neq 0\}$ is an open subset of G. Let g_1, g_2 be any two points in N. Then by Lemma 3.1 the multiplicative measures $\Phi(A, g_i)$, $i=1, 2$ are exponentials of non atomic additive measures. Hence there exists a sequence of finite measurable partitions $\{A_{nk}, 1 \leqslant k \leqslant n\}$ of T such that

$$\lim_{n\to\infty} \sup_{i=1,2} \sup_{k} [1-|\Phi(A_{nk}, g_i)|]=0.$$

Hence
$$\lim_{n\to\infty} \sum_{k=1}^{n} [1-|\Phi(A_{nk}, g_i)|^2]<\infty, \quad i=1,2. \tag{4.1}$$

The positive definiteness and non-negativity of $|\Phi(A, g)|^2$ for every A and Lemma 3.6 in [3] imply that

$$1-|\Phi(A, g_1 g_2)|^2 \leqslant 2[(1-|\Phi(A, g_1)|^2)+(1-|\Phi(A, g_2)|^2)]$$

for every A. Now (4.1) implies

$$\lim_{n\to\infty} \sum_{k=1}^{n} (1-|\Phi(A_{nk}, g_1 g_2)|^2)<\infty.$$

Hence
$$|\Phi(T, g_1 g_2)|^2=\lim_{n\to\infty} \prod_{k} |\Phi(A_{nk}, g_1 g_2)|^2 \neq 0.$$

This shows that N is an open subgroup of G and hence N is closed. Since G is connected, $N=G$. This proves the claim. We now put

$$\hat{W}_g^A=\frac{|\langle W_g^A x_A, x_A\rangle|}{\langle W_g^A x_A, x_A\rangle} W_g^A.$$

Changing $\sigma(A, \cdot, \cdot)$ accordingly into $\hat{\sigma}(A, \cdot, \cdot)$ and putting $\hat{x}_A=x_A$, we get another factorisable family $\{\hat{W}^A, \hat{x}_A, \hat{\sigma}(A, \cdot, \cdot)\}$, which, by the Remark after Definition 4.1, satisfies all the required properties. For $\sigma\equiv 1$ the result is obvious. The converse follows from the one to one correspondence between σ-positive definite functions and cyclic multiplier representations with multiplier σ (up to unitary equivalence). This completes the proof of the lemma.

COROLLARY 4.1. *Let $\{W^A, x_A, \sigma(A, \cdot, \cdot), A\in \mathcal{S}\}$ be a factorisable family of cyclic multiplier representations of a connected locally compact second countable group G. Then there exists another factorisable family $\{\hat{W}^A, \hat{x}_A, \hat{\sigma}(A, \cdot, \cdot), A \in \mathcal{S}\}$ such that* (1) *for each A, $(W^A, x_A, \sigma(A, \cdot, \cdot))$ and $(\hat{W}^A, \hat{x}_A, \hat{\sigma}(A, \cdot, \cdot))$ are projectively equivalent;* (2) *there exists an Araki multiplier $S(A, \cdot, \cdot)$ and an Araki S-function $\phi(A, \cdot)$ such that*

$$\langle \hat{W}_g^A \hat{x}_A, \hat{x}_A\rangle=\exp \phi(A, g)$$
$$\hat{\sigma}(A, g, h)=\exp iS(A, g, h)$$

for all $A\in \mathcal{S}$, $g, h \in G$.

Proof. Define $\hat{W}^A$ and $\hat{x}_A$ as in Lemma 4.1. An application of Theorem 3.1 gives the result.

Consider the standard Borel space (T, S) and the group G. Let $F(T, G)$ be the group of all measurable maps γ: $T \rightarrow G$ which take only finitely many values. $F(T, G)$ is considered as a group under pointwise multiplication with identity $\tilde{e}$. We shall call it the *weak current group* of G over T. For any $A \in S$, we shall denote by $F(A, G)$ the subgroup of all maps γ which are equal to e outside A. For any measurable partition $A_1, A_2, ..., A_n$ of A, $F(A, G) = \prod_1^n F(A_i, G)$ in the sense of direct products. We further define a homomorphism π_A: $F(T, G) \rightarrow F(A, G)$ by the equation

$$(\pi_A \gamma)(t) = \gamma(t) \quad \text{if } t \in A,$$
$$= e \quad \text{otherwise.}$$

Finally we introduce the functions χ^A, $A \in S$, $g \in G$ by

$$\chi^A(t) = g \quad \text{if } t \in A,$$
$$= e \quad \text{if } t \in A.$$

Then any function $\gamma \in F(T, G)$ can be written as a product $\prod_{i=1}^n \chi_{g_i}^{A_i}$ where $\{A_i, 1 \leqslant i \leqslant n\}$ is a measurable partition of T.

Suppose $\{W^A, x_A, \sigma(A, \cdot, \cdot), A \in S\}$ is a factorisable family of representations of G. For any two elements $\gamma_1, \gamma_2 \in F(T, G)$ we write

$$\tilde{\sigma}(\gamma_1, \gamma_2) = \prod_{i,j} \sigma(A_i \cap B_j, g_i, h_j)$$

where $\gamma_1 = \prod_{i=1}^m \chi_{g_i}^{A_i}, \gamma_2 = \prod_{j=1}^n \chi_{h_j}^{B_j}$. Then it is easy to verify that $\tilde{\sigma}$ is a multiplier for the group $F(T, G)$. If now we define

$$\Psi(\gamma) = \prod_{i=1}^n \langle W_{g_i}^{A_i} x_{A_i}, x_{A_i} \rangle$$

whenever $\gamma = \prod_{i=1}^n \chi_{g_i}^{A_i}$, then Ψ is a $\tilde{\sigma}$-positive definite function on $F(T, G)$. Hence there exists a cyclic multiplier representation $(\tilde{W}, \tilde{x}, \tilde{\sigma})$ for the group $F(T, G)$. Further $\tilde{W}$ has the property

$$\langle \tilde{W}_{\chi_g^A} \tilde{x}, \tilde{x} \rangle = \langle W_g^A x_A, x_A \rangle \quad \text{for all } A \in S, g \in G.$$

This leads us to the following natural definition:

Definition 4.3. Let $F(T, G)$ be the weak current group of G over T. A cyclic multiplier representation $(\tilde{W}, \tilde{x}, \tilde{\sigma})$ is said to be *factorisable* if the maps $g \rightarrow W_g^A$ where $W_g^A = \tilde{W}_{\chi_g^A}$

and the functions $\sigma(A, g, h) = \tilde{\sigma}(\chi_g^A, \chi_h^A)$ have the property that $(W^A, \tilde{x}, \sigma(A, \cdot, \cdot))$ is a factorisable family of representations of G.

With this definition and Theorem 3.1 we now have the following theorem.

THEOREM 4.1. *Let G be a connected, locally compact and second countable group. Let (T, S) be a standard Borel space and let $F(T, G)$ the weak current group of G over T. Let $(\tilde{W}, \tilde{x}, \tilde{\sigma})$ be a factorisable cyclic multiplier representation of $F(T, G)$ in a complex separable Hilbert space H. Then there exists a projectively equivalent factorisable representation $(\tilde{W}', \tilde{x}', \tilde{\sigma}')$ such that for every $A \in S$, $g, h \in G$*

$$\langle W'^A_{gh} \tilde{x}', \tilde{x}' \rangle \, \sigma'(A,g,h) = \exp\,[\phi(A, gh) + iS(A,g,h)] \tag{4.2}$$

where
$$W'^A_g = \tilde{W}'_{\chi_g^A}, \quad \sigma'(A,g,h) = \tilde{\sigma}'(\chi_g^A, \chi_h^A),$$

S is an Araki multiplier and ϕ is an Araki S-function.

Conversely, given an Araki multiplier S and an Araki S-function ϕ one can construct a factorisable representation $(\tilde{W}', \tilde{x}', \tilde{\sigma}')$ of $F(T, G)$ satisfying the above equations. For the converse to hold, G need not be connected.

If $\tilde{\sigma} = 1$, we can replace projective equivalence by unitary equivalence, choose $\tilde{\sigma}' \equiv 1$, $S \equiv 0$ and ϕ to be an Araki function.

Remark 1. The above theorem together with Theorems 2.1 and 2.2 gives a complete description of all factorisable multiplier representations of weak current groups when G is connected.

Remark 2. If we consider the Araki multiplier S and the Araki S-function ϕ satisfying (4.2) and use Theorem 2.1 we obtain a measure μ on (T, S), representations V^t of G in Hilbert spaces H_t and cocycles $\delta(t, \cdot)$ connected with the pair (S, ϕ). If we write for any $\gamma \in F(T, G)$,

$$\Delta(\gamma) = \{\delta(t, \gamma(t)), t \in T\} \in \int H_t \, d\mu(t),$$

$$U(A, g) = \int \chi_A(t) V(t, g) \, d\mu(t),$$

$$U_\gamma = \prod_{i=1}^{n} U(A_i, g_i),$$

whenever $\gamma = \prod_{i=1}^n \chi_{g_i}^{A_i}$, we obtain

$$U_{\gamma_1} \Delta(\gamma_2) = \Delta(\gamma_1 \gamma_2) - \Delta(\gamma_1).$$

In other words $\gamma \to U_\gamma$ is a unitary (ordinary) representation of $F(T, G)$ and Δ is a cocycle for this representation. This cocycle determines the factorisable representation of $F(T, G)$ associated with the pair (A, ϕ).

Remark 3. The identification $\gamma \to \Delta(\gamma)$, where Δ is the cocycle described in Remark 2, defines a metric d in $F(T, G)$ by the equation

$$d(\gamma_1, \gamma_2) = \left[\int \|\delta(t, \gamma_1(t)) - \delta(t, \gamma_2(t))\|^2 \, d\mu(t)\right]^{\frac{1}{2}}.$$

We call the completion of $F(T, G)$ under this metric the *full current group* of G over T and denote it by $\Gamma(T, G)$. It is clear that the factorisable representation of $F(T, G)$ associated with Δ extends uniquely to a continuous representation of $\Gamma(T, G)$. Then the extended map $\gamma \to \Delta(\gamma)$ is a continuous cocycle on $\Gamma(T, G)$.

Remark 4. Suppose $(\tilde{W}, \tilde{x}, \tilde{\sigma})$ is a factorisable representation of $F(T, G)$ and (S, ϕ) the associated pair of Araki multiplier and Araki S-function.

For any $\gamma = \prod_{i=1}^n \chi_{g_i}^{A_i}$, where $\{A_i, 1 \leqslant i \leqslant n\}$ is a partition of T, we define

$$\tilde{\phi}(\gamma) = \sum_{i=1}^n \phi(A_i, g_i).$$

If $\gamma_1 = \prod_{i=1}^m \chi_{g_i}^{A}$ and $\gamma_2 = \prod_{j=1}^n \chi_{h_j}^{B_j}$ where $\{A_i, 1 \leqslant i \leqslant m\}$ and $(B_j, 1 \leqslant j \leqslant n\}$ are partitions of T, we define

$$\tilde{S}(\gamma_1, \gamma_2) = \sum_{i,j} S(A_i \cap B_j, g_i, h_j).$$

Let Δ be the cocycle defined in Remark 2. $\tilde{\phi}$ is a conditionally S-positive definite function on $F(T, G)$. $\tilde{\phi}$, $\tilde{S}$ and Δ have natural extensions to the complete current group. Let $H = \int H_t \, d\mu(t)$ be the Hilbert space where the representation $\gamma \to U_\gamma$ and the cocycle Δ are defined. We construct the symmetric Fock space $\exp H$ over H. We write

$$x(\gamma) = [\exp \tilde{\phi}(\gamma)] \exp \Delta(\gamma^{-1}).$$

By Theorem 2.1 and the definition of Δ,

$$\langle \Delta(\gamma_1), \Delta(\gamma_2) \rangle = \tilde{\phi}(\gamma_1^{-1}\gamma_2) - \tilde{\phi}(\gamma_1^{-1}) - \tilde{\phi}(\gamma_2) + i\tilde{S}(\gamma_1^{-1}, \gamma_2).$$

Then
$$\langle x(\gamma_1), x(\gamma_2) \rangle = \exp \tilde{\phi}(\gamma_1 \gamma_2^{-1}) + i\tilde{S}(\gamma_1, \gamma_2^{-1}).$$

We define the map

$$W_{\gamma_1} : x(\gamma) \to x(\gamma\gamma_1^{-1}) \exp i\tilde{S}(\gamma, \gamma_1^{-1}).$$

This is an isometry on the set $\{x(\gamma), \gamma \in F(T, G)\}$. Hence it can be extended to a unitary operator on the closed linear span $\tilde{H}$ of this set. Then,

$$W_{\gamma_1} W_{\gamma_2} = \exp [-i\tilde{S}(\gamma_1, \gamma_2)] W_{\gamma_1 \gamma_2},$$

$$\langle W_\gamma x(\tilde{e}), x(\tilde{e}) \rangle = \exp \overline{\phi(\gamma)},$$

where $\tilde{e}$ is the identity element of $F(T, G)$. If we change the inner product in H to its conjugate, we get a factorisable multiplier representation associated with the pair (S, ϕ). This is the Araki-Woods imbedding theorem (cf. [2] and [5]).

References

[1]. Araki, H., Factorisable representations of current algebra. *RIMS–41*, reprint, Kyoto (1969).

[2]. Araki, H. & Woods, E. J., Complete Boolean algebras of type I factors. *Publications of RIMS*, Kyoto University, Series A, 2 (1966), No. 2, 157–242.

[3]. Parthasarathy, K. R., Infinitely divisible representations and positive definite functions on a compact group. *Commun. math. Phys.*, 16 (1970), 148–156.

[4]. Parthasarathy, K. R. & Schmidt, K., Infinitely divisible projective representations, cocycles and Levy-Khinchine-Araki formula on locally compact groups, *Research Report* 17 (1970), *Manchester-Sheffield School of Probability and Statistics.*

[5]. Streater, R. F., Current commutation relations, continuous tensor products, and infinitely divisible group representations. *Rendiconti di Sc. Int. di Fisica "E. Fermi"*, XI, (1969), 247–263.

Received April 16, 1970